Color factors in QCD

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1 Preface

1.1 About the LPSC

The present report has been written during a five week internship at the Laboratoire de Physique Subatomique et de Cosmologie (LPSC) in Grenoble. The LPSC has about 200 employees and is an important fundamental research institute in France that has also a couple of international collaborations.

It is a union of the IN2P3 (L'Institut national de physique nucléaire et de physique des particules), ST2I (Sciences et technologies de l'information et de l'ingénierie), CNRS (Centre national de la recherche scientifique), UJF (Université Joseph Fourier) and the INPG (Grenoble Institute of Technology).

Major fields of research are particle physics and fundamental symmetries, astroparticle physics and cosmology, nuclear physics and its applications to energy and medicine, and development of particle accelerators.

1.2 Proceeding

First we have a mathematical look at the $SU(N)$ group, its generators and Lie algebra. That is necessary because $SU(3)$ is the gauge group of QCD (cf. Sec. 3).
After this there is an introduction to color factors and Feynman rules before we calculate the color factors for some examples.
Finally we provide an algorithm, and also its implementation in Mathematica, that calculates color factors.

2 Group SU(N)

The special unitary group $SU(N)$ is a subgroup of the unitary group $U(N)$. While the unitary group consists of all unitary $N \times N$ matrices, the special unitary group consists only of the special $N \times N$ unitary matrices that have determinant 1. Therefore we have two conditions for a group element $A \in SU(N)$:

1. Unitary: $A^\dagger A = 1$ ($A^\dagger$ means the adjoint matrix of A, that is the conjugate transposed of A)
2. Determinant: $\det A = 1$

A matrix A of the $SU(N)$ has N^2 complex entries A_{ij} and can be described by the $2N^2$ real values $\Re(A_{ij})$ and $\Im(A_{ij})$. In fact the number of independent real parameters that determine the matrix is lower. The first condition leads to N^2 equations for the matrix entries and together with the second condition we obtain $2N^2 - N^2 - 1 = N^2 - 1$ independent real parameters called α_1, ..., α_{N^2-1}.

We have now found a parametrization $h : \mathbb{R}^{N^2-1} \longrightarrow SU(N)$ for the group elements of $SU(N)$ that is continuous and has a continuous inverse function. The inverse function h^{-1} is a homeomorphism and can be considered as a (N^2-1)-dimensional chart.

The $SU(N)$ group is also a (N^2-1)-dimensional differentiable manifold and therefore a so called Lie group [5].

2.1 Generators of the SU(N)

Let D be a representation of the group. Then we can parameterize the representation as well with a parameter set $\alpha = \alpha_1$, ..., α_{N^2-1} and furthermore we choose the identity matrix for the value of D at $\alpha = 0$. In a neighborhood of the identity we make a Taylor expansion of D and we obtain:

$$D(\alpha) = D(0) + \frac{\partial}{\partial \alpha_a} D(\alpha)|_{\alpha=0} \alpha_a + ... \text{ [5].}$$

with Einstein summation convention that is always used in the following without mentioning.

By using the definition of some generators $T^a := -i \frac{\partial}{\partial \alpha_a} D(\alpha)|_{\alpha=0}$ we get:

$$D(\alpha) = 1 + iT^a \alpha_a + ... \,.$$

If α is almost 0 it is sufficient to take only the linear term. But more general the representation could be realized by

$$D(\alpha) = \lim_{k \to \infty} \left(1 + \frac{iT^a \alpha_a}{k} \right)^k = e^{iT^a \alpha_a}$$

which is the reason to call the T^a generators. They form a basis of the tangent space at the unit matrix: $T_1 SU(N)$.

The conclusions from above were quite general and now if we consider the $SU(N)$ again we could ask the question 'how can we get generators for this group?'. In the following we will see that the hermitian and traceless matrices are generators of the $SU(N)$.

- T^a hermitian $\Rightarrow D(\alpha) = e^{i\alpha_a T^a}$ unitary:
  Proof:
  $$D(\alpha)^\dagger D(\alpha) = e^{-i(T^a)^\dagger \alpha_a} e^{iT^a\alpha_a} = e^{-iT^a\alpha_a} e^{iT^a\alpha_a} = 1$$
- T^a traceless and hermitian $\Rightarrow \det(e^{i\alpha_a T^a}) = 1$:
  Proof:
  A linear combination $H := \alpha_a T^a$ of the traceless and hermitian generators is traceless and hermitian again. H is diagonalizable because it is hermitian and so it is possible to choose a basis S in which H is diagonal. Now the determinant can be calculated by:
  $$\begin{aligned} \det(e^{iH}) &= \det(e^{iS^{-1}HS}) \\ &= \prod_k e^{(iS^{-1}HS)_{kk}} \\ &= e^{i\mathrm{Tr}(S^{-1}HS)} = e^{i\mathrm{Tr}(H)} = e^0 = 1\ . \end{aligned}$$
  Here we have used that building the determinant and the trace of a matrix are independent from the basis.

2.2 The Lie algebra $\mathfrak{su}(N)$

A major advantage of using generators is that they establish a vector space $\mathfrak{su}(N)$. A basis $\{T^a\}$ of the hermitian and traceless matrices has to consists of $N^2 - 1$ linearly independent matrices because that is the dimension of this space.
Now it is possible to find another structure on $\mathfrak{su}(N)$ by adding the commutator of matrices as bilinear map from $\mathfrak{su}(N) \times \mathfrak{su}(N)$ to $\mathfrak{su}(N)$.

We consider the following definition of a Lie algebra [7]:
Let L be a vector space over the field k. L is a Lie algebra $:\Leftrightarrow$ There is a bilinear map $[\cdot,\cdot] : L \times L \longrightarrow L$ with: $[x, x] = 0$ for all $x \in L$ and $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$ (Jacobi identity).

The first attribute of the so called Lie bracket is obviously complied with the commutator of matrices and the second follows by expanding the commutators. So the $\mathfrak{su}(N)$ is a Lie algebra.

It is often useful to have another expression for the commutator $[T^a, T^b]$. In the following we will see that $i[T^a, T^b]$ is an element of the $\mathfrak{su}(N)$.

- $(i[T^a, T^b])^\dagger = -i[T^b, T^a] = i[T^a, T^b]$ because T^a and T^b are hermitian.
- $\mathrm{Tr}([T^a, T^b]) = \mathrm{Tr}(T^aT^b) - \mathrm{Tr}(T^bT^a) = 0$ because $\mathrm{Tr}(AB)$=$\mathrm{Tr}(BA)$

So $i[T^a, T^b]$ is a hermitian traceless matrix and therefore can be written as a linear combination of the generators:

$$i[T^a, T^b] = -f^{abc}T^c \Rightarrow [T^a, T^b] = if^{abc}T^c. \tag{2.1}$$

with some coefficients f^{abc} that are called structure constants and specify a Lie algebra. The structure constants depend on the basis of $\mathfrak{su}(N)$ and they are antisymmetric and real. In order to show this we use the anti-commutativity of the commutator and that the generators are hermitian:

$$\begin{aligned} if^{abc}T^c = [T^a, T^b] = -[T^b, T^a] = -if^{bac}T^c \\ \Rightarrow f^{abc} = -f^{bac}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} if^{abc}T^c = [T^a, T^b] = [T^b, T^a]^\dagger = -i(f^{bac})^*T^c \\ \Rightarrow f^{abc} = (f^{abc})^*. \end{aligned} \tag{2.3}$$

In summary as we can see in Fig. 1 we used the structure of the Lie group $SU(N)$ to obtain some generators that are elements of the tangent space of the identity matrix. The generators combined with the commutator of matrices establish the Lie algebra $\mathfrak{su}(N)$.

$$\begin{array}{ccc} \text{Lie Group} & \text{generators} & \text{real Lie algebra} \\ SU(N) & \longrightarrow T^a : e^{i\alpha_a T^a} \in SU(N) \longrightarrow & \mathfrak{su}(N) = \text{span}\{T^a\} \end{array}$$

Figure 1: Connection between Lie group and Lie algebra

2.3 The adjoint representation of $\mathfrak{su}(N)$

The so called adjoint representation of the $\mathfrak{su}(N)$ is generated by the structure constants. They can be used to define $(N^2-1) \times (N^2-1)$ matrices [5]:

$$F^a_{bc} := -if^{abc}. \tag{2.4}$$

It is possible to rewrite the Jacobi identity for the generators with $[T^a, [T^b, T^c]] = -f^{bcd}f^{ade}$ what leads to $\underbrace{f^{abd}f^{cde}}_{=-if^{abd}F^d_{ce}} + \underbrace{f^{bcd}f^{ade} + f^{cad}f^{bde}}_{=[F^a,F^b]_{ce}} = 0$. And therefore:

$$[F^a, F^b] = if^{abc}F^c. \tag{2.5}$$

2.4 SU(2) and SU(3)

$SU(2)$ and $SU(3)$ are important algebras in particle physics.
The standard generators of $SU(2)$ in physics are $T^a = \frac{1}{2}\sigma_a$ whereas σ_a ($a = 1, ..., 3$) are the Pauli matrices. For $SU(2)$ the structure constants of the algebra are equal to the Levi-Civita symbol $f^{ijk} = \epsilon_{ijk}$.

The standard generators of $SU(3)$ in physics are $T^a = \frac{1}{2}\lambda_a$ whereas λ_a ($a = 1, ..., 8$) are the Gell-Mann matrices [1,5].

2.5 Some useful relations

In this section we provide a list of useful relations for the Lie algebra $SU(N)$ which are in part implemented in the Mathematica code presented in Sec. 5.1.

$$\text{Antisymmetric structure constants: } [T^a, T^b] = if^{abc}T^c. \tag{2.6}$$

$$\text{Traceless: } \mathrm{Tr}(T^a) = 0. \tag{2.7}$$

$$\text{Normalization: } \mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta_{ab}. \tag{2.8}$$

$$\text{Symmetric structure constants: } \{T^a, T^b\} = \frac{1}{N}\delta_{ab} + d^{abc}T^c. \tag{2.9}$$

$$\text{Projection operator: } T^a_{ij}T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right). \tag{2.10}$$

A proof of equation (2.10) has been relegated to the appendix.

Using equations (2.6)-(2.10) one can easily derive the following rules:

$$f^{abc} = -2i\mathrm{Tr}(T^a[T^b, T^c]), \tag{2.11}$$

$$d^{abc} = 2\mathrm{Tr}(T^a\{T^b, T^c\}), \tag{2.12}$$

$$T^aT^b = \frac{1}{2}\left(\frac{1}{N}\delta_{ab} + (d^{abc} + if^{abc})T^c\right), \tag{2.13}$$

$$\mathrm{Tr}(T^aT^bT^c) = \frac{1}{4}(d^{abc} + if^{abc}), \tag{2.14}$$

$$\mathrm{Tr}(T^aT^bT^aT^c) = -\frac{1}{4N}\delta_{bc}, \tag{2.15}$$

$$f^{acd}f^{bcd} = N\delta_{ab}, \tag{2.16}$$

$$f^{acd}d^{bcd} = 0, \tag{2.17}$$

$$f^{ade}f^{bef}f^{cfd} = \frac{N}{2}f^{abc}. \tag{2.18}$$

2.6 Casimir operators

T^aT^a is a Casimir operator in the fundamental representation of $\mathfrak{su}(N)$. That means T^aT^a commutes with the generators of $SU(N)$ and therefore with all elements of the Lie algebra what we will see in the following.
Considering the (i,j)th entry of the Casimir we obtain with eq. (2.10):

$$\begin{aligned}
(T^aT^a)_{ij} = T^a_{ik}T^a_{kj} &= \frac{1}{2}\left(\delta_{ij}\delta_{kk} - \frac{1}{N}\delta_{ik}\delta_{kj}\right) \\
&= \frac{1}{2}\left(\delta_{ij}N - \frac{1}{N}\delta_{ij}\right) = \delta_{ij}\frac{N^2-1}{2N}
\end{aligned}$$

$$\Rightarrow (T^a T^a)_{ij} = \delta_{ij} C_F \text{ with } C_F := \frac{N^2-1}{2N}. \tag{2.19}$$

So $T^a T^a = C_F \mathbb{1}_N$ is a multiple of the unit matrix and commutes with the generators of $SU(N)$.

Analog we consider the Casimir operator in the adjoint representation and use eq. (2.16):

$$(F^c F^c)_{ab} \quad = \quad F^c_{ad} F^c_{db} = i^2 f^{cad} f^{cdb} = f^{acd} f^{bcd} = \delta_{ab} N$$

$$\Rightarrow (F^c F^c)_{ab} = \delta_{ab} C_A \text{ with } C_A := N. \tag{2.20}$$

That means $F^c F^c = C_A \mathbb{1}_{N^2-1}$ commutes with the generators in the adjoint representation.

3 Color factors

3.1 QCD

Quantum Chromodynamics (QCD) is a field theory of strong interaction and describes the interaction of quarks and gluons. According to the electric charge in Quantum Electrodynamic (QED) there is a color charge in QCD but in contrary to QED there are three different color charges (called red(R), green(G) and blue(B)) in QCD. Each quark carries one color but exchange of gluons can change the color of a quark. Gluons are elementary particles that mediate strong force and they also carry a color charge what leads to the possibility of self interaction. The color charge of a gluon is a combination of color and anti-color.
There are eight different gluon types which form a $SU(3)$ octet [4]:

$$R\bar{G}, R\bar{B}, G\bar{R}, G\bar{B}, B\bar{R}, B\bar{G}, \frac{1}{\sqrt{2}}(R\bar{R} - G\bar{G}), \frac{1}{\sqrt{6}}(R\bar{R} + G\bar{G} - 2B\bar{B}).$$

The symmetric singlet state $\frac{1}{\sqrt{3}}(R\bar{R}+G\bar{G}+B\bar{B})$ does not exist because it cannot mediate color.

All observed particles do not carry a net color charge. They are white or colorless.

The coupling strength in QCD is $\frac{1}{2}c_1 c_2 \alpha_s$ where $\alpha_s = \frac{g^2}{4\pi}$, g is the strong coupling constant depending on energy [4] and c_1, c_2 are color coefficients. In QED the coupling strength is similar: $e_1 e_2 \alpha$ where e_1 and e_2 are the electric charges in units of the elementary charge e and $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant.

The term $\frac{1}{2}|c_1 c_2|$ is called color factor.

3.2 Feynman rules

The Feynman rules follow from the QCD Lagrangian. The free Lagrangian in QCD is

$$\mathfrak{L}_0 = \bar{q}(i\gamma^\mu \partial_\mu - m)q \text{ [4]} \tag{3.1}$$

where q is a quark color field and γ^μ are the Dirac matrices.
Demanding invariance of the QCD Lagrangian under $SU(3)$ transformation ($q \rightarrow e^{i\alpha_a T^a} q$) results in interaction terms:

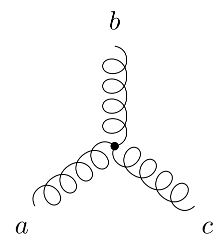
$$\mathfrak{L} = \bar{q}(i\gamma^\mu \partial_\mu - m)q - g(\bar{q}\gamma^\mu T^a q)G^a_\mu - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a \text{ [4]} \tag{3.2}$$

where G^a_μ are eight gluon fields that transform as

$$G^a_\mu \rightarrow G^a_\mu - \frac{1}{g}\partial_\mu \alpha_a - f^{abc}\alpha_b G^c_\mu$$

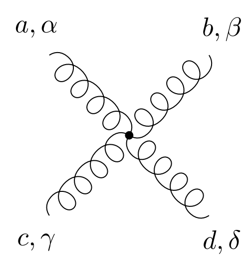
and $G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - gf^{abc}G^b_\mu G^c_\nu$ is the field strength tensor [4].
The first term of the Lagrangian corresponds to the quark propagator, the second to the quark-gluon vertex and the third term leads to the gluon propagator and the interaction terms three-gluon and four-gluon vertex. We can interpret the different terms by Feynman diagrams. That leads to the following Feynman rules taken from [3] where we focus on color terms.

Three-gluon vertex:



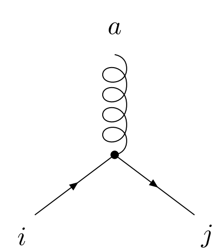
$$-gf^{abc}$$

Four-gluon vertex:



$$\begin{aligned}
&-ig^2 f^{xac} f^{xbd}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\gamma})\\
&-ig^2 f^{xad} f^{xbc}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta})\\
&-ig^2 f^{xab} f^{xcd}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma})
\end{aligned}$$

Quark-gluon vertex:



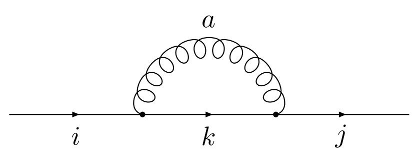
$$-igT^a_{ji}$$

4 Examples for Calculation of Color factors

In the following we consider some examples for an analytic calculation of color factors. All Feynman diagrams were generated with LaTeX and the LaTeX style file axodraw [9].

We start with one-loop corrections to the gluon and quark propagator and then consider three-point functions. Finally we calculate the color factors for some four-point functions. In order to calculate the color factors we make use of the Feynman rules (cf. Sec. 3.2) associated with the vertices. Equation numbers on top of equal signs refer to the relations provided in Sec. 2.5.
The Casimir operators C_F and C_A and the normalization condition $\mathrm{Tr}(T^aT^b) = T_F\delta_{ab}$ ($T_F = \frac{1}{2}$) can be considered as fundamental color factors.

4.1 One-loop correction to the quark propagator

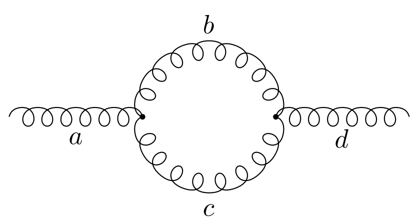


$$\begin{aligned} T^a_{ki}T^a_{jk} &\overset{2.10}{=} \frac{1}{2}\left(\delta_{kk}\delta_{ij} - \frac{1}{N}\delta_{ki}\delta_{jk}\right) \\ &= \frac{1}{2}\left(N\delta_{ij} - \frac{1}{N}\delta_{ij}\right) \\ &= \frac{1}{2}\left(N - \frac{1}{N}\right)\delta_{ij} = C_F\delta_{ij} \end{aligned}$$

After summation over the final color states and averaging over initial color states we get:

$$\frac{1}{N}\sum_{j=1}^{N}\frac{1}{2}\left(N - \frac{1}{N}\right)\delta_{ij} = \frac{1}{2N}\left(N - \frac{1}{N}\right)\delta_{ii} = \frac{1}{2}\left(N - \frac{1}{N}\right) = C_F$$

4.2 One-loop-gluon correction to the gluon propagator

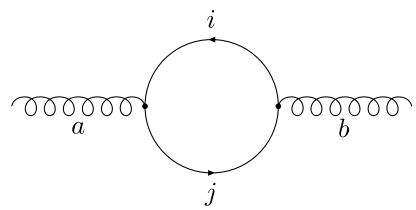


$$f^{abc}f^{dbc} \overset{2.16}{=} N\delta_{ab} = C_A\delta_{ab}$$

Summation and averaging over end and initial states results in:

$$\frac{1}{N^2-1}N\delta_{aa} = N = C_A$$

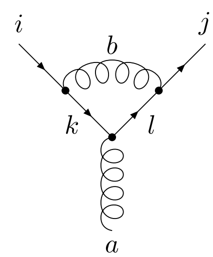
4.3 One-loop-quark correction to the gluon propagator



$$\begin{aligned} T^a_{ji}T^b_{ij} &= \mathrm{Tr}(T^aT^b) \\ &\overset{2.8}{=} \frac{1}{2}\delta_{ab} = T_F\delta_{ab} \end{aligned}$$

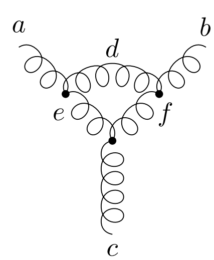
Here we have N^2-1 initial color states and we obtain the color factor $T_F = \frac{1}{2}$ after averaging and summation.

4.4 One-loop correction to the quark-gluon vertex



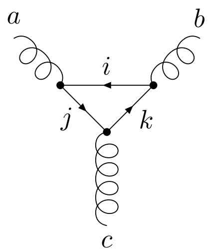
$$\begin{aligned} T^b_{ki}T^b_{jl}T^a_{lk} &\overset{2.10}{=} \frac{1}{2}\left(\delta_{kl}\delta_{ij} - \frac{1}{N}\delta_{ki}\delta_{jl}\right)T^a_{lk} \\ &= \frac{1}{2}\delta_{ij}T^a_{kk} - \frac{1}{2N}T^a_{ji} \\ &\overset{2.7}{=} -\frac{1}{2N}T^a_{ji} \end{aligned}$$

4.5 One-loop correction to the three-quark vertex 1



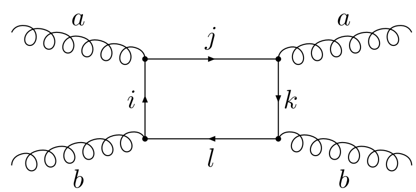
$$f^{ade}f^{efc}f^{dbf} = f^{ade}f^{cef}f^{bfd} \overset{2.18}{=} -\frac{N}{2}f^{abc}$$

4.6 One-loop correction to the three-quark vertex 2



$$T^a_{ji}T^b_{kj}T^c_{ik} = \mathrm{Tr}(T^aT^cT^b) \overset{2.14}{=} \frac{1}{4}(d^{acb} + if^{acb})$$

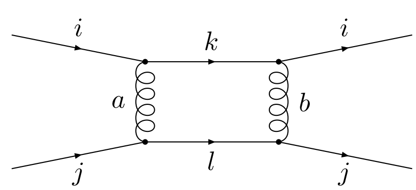
4.7 Process $g + g \to g + g$



$$\begin{aligned} T^a_{ji}T^b_{il}T^b_{lk}T^a_{kj} &\overset{2.10}{=} \frac{1}{4}\left(\delta_{jj}\delta_{ik} - \frac{1}{N}\delta_{ji}\delta_{kj}\right)\left(\delta_{ik}\delta_{ll} - \frac{1}{N}\delta_{il}\delta_{lk}\right) \\ &= \frac{1}{4}\left(N\delta_{ik} - \frac{1}{N}\delta_{ki}\right)\left(\delta_{ik}N - \frac{1}{N}\delta_{ik}\right) \\ &= \frac{1}{4}\delta^2_{ik}\left(N - \frac{1}{N}\right)^2 = \frac{1}{4}N\left(N - \frac{1}{N}\right)^2 = \frac{1}{4N}(N^2-1)^2 \\ &= C_AC_F^2 \end{aligned}$$

The color factor for this example is $\frac{1}{4N} = \frac{T_F^2}{C_A}$ because there are $(N^2-1)^2$ possible initial states for the two gluons.

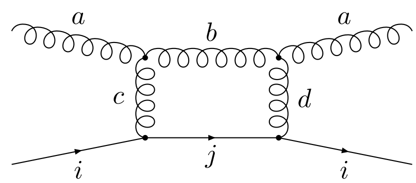
4.8 Process $q + q \to q + q$



$$\begin{aligned} T^a_{ki}T^b_{ik}T^a_{lj}T^b_{jl} &= \mathrm{Tr}(T^aT^b)\mathrm{Tr}(T^aT^b) \\ &= \frac{1}{4}\delta_{ab}\delta_{ab} = \frac{1}{4}(N^2-1) = \frac{1}{2}C_AC_F \end{aligned}$$

This means we get $\frac{1}{4N^2}(N^2-1) = \frac{T_FC_F}{C_A}$ after averaging.

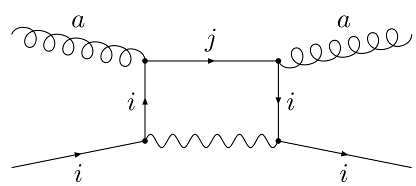
4.9 Process $g + q \rightarrow g + q$



$$\begin{aligned}
f^{abc} f^{abd} T^c_{ji} T^d_{ij} &= f^{cab} f^{dab} \mathrm{Tr}(T^c T^d) \overset{2.8}{=} \frac{1}{2} \delta_{cd} f^{cab} f^{dab} \\
&= \frac{1}{2} f^{cab} f^{cab} \overset{2.16}{=} \frac{1}{2} N \delta_{aa} = \frac{1}{2} N(N^2-1) = C_A^2 C_F
\end{aligned}$$

Averaging over the $N(N^2-1)$ initial states results in the color factor $\frac{1}{2} = T_F$.

4.10 Process $g + q \rightarrow q + \gamma$



$$T^a_{ji} T^a_{ij} = \mathrm{Tr}(T^a T^a) \overset{2.8}{=} \frac{1}{2} \delta_{aa} = \frac{1}{2}(N^2-1) = C_A C_F$$

We obtain the color factor $\frac{1}{2N} = \frac{T_F}{C_A}$ with $N(N^2-1)$ initial states.

5 Algorithm

The examples from above had only up to four vertices but if we consider Feynman diagrams with many vertices the term of generators and structure constants becomes very long. Therefore it is useful to implement a simple algorithm that computes color factors by applying specified replacement rules. Calculating color factors by applying Feynman rules for a squared matrix element always leads to paired color indices [1]. That is why the following algorithm is sufficient for calculating color factors of the square of amplitudes.

The general algorithm consists of two steps [1, 2].
`INPUT`: Expression of entries T^a_{ij} of the generators and structure constants f^{abc},d^{abc}
First step:
Replacement of three-gluon vertices by terms of the generators

$$f^{abc} \rightarrow -2i\mathrm{Tr}(T^a[T^b, T^c]) \tag{5.1}$$

$$d^{abc} \rightarrow 2\mathrm{Tr}(T^a\{T^b, T^c\}) \tag{5.2}$$

`OUTPUT`: String of entries T^a_{ij} of the generators

Second step:
Replacement of internal gluon lines by the gluon projection operator

$$T^a_{ij} T^a_{kl} \to \frac{1}{2} \left(\delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right) \tag{5.3}$$

`OUTPUT`: Expression of Kronecker Delta functions that can be simplified by using properties of the Kronecker Delta. Simplifying results in the color factor.

In order to improve performance of the algorithm we can also implement the following rules:

$$\mathrm{Tr}(T^a) = 0 \tag{5.4}$$

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2} \delta_{ab} \tag{5.5}$$

It is necessary to implement some rules for the delta functions.

5.1 Source Code of the Mathematica package color.m

The following source code is a Mathematica package. It can be loaded in a Mathematica notebook by entering `<<color.m`. Then all functions defined in the package can be used in this notebook as it will be described below.

The first part of the package is a header. The header is only comment and therefore not necessary for the program but it is helpful for documentation. The header format conforms with standard headers for packages [8].

```
(* :Title: color.m
Last change: 26.09.08
*)

(* :Context: color` *)

(* :Author: Vera Derya *)

(* :Summary: The "color" package can be used to calculate SU(N) color factors as they appear in QCD for example. *)

(* :Package Version: 1.0 *)

(* :Mathematica Version: 6.0 *)

(* :Keywords: SU(N), color, colorfactor, color factor, QCD *)

(* :Limitations: It is often necessary to Expand[] the expression to get the result. *)

(* :Examples:   TrT[a, b]
     TrT[a, b, a, c]
     Expand[f[a, b, c] f[b, a, d] T[c, j, i] T[d, i, j]]
     Expand[f[a, c, d] f[b, d, c]]
     Expand[f[a, e, d] f[b, c, e] TrT[a, d, c, b]]
*)
```

In the next part functions and variables of the package are declared with the command `::usage::`. Furthermore there is given a brief description of the meaning of the variables. These descriptions can be retrieved in a Mathematica notebook by entering `?name`. Here is a list of the functions that are used in the program:

- `f[a,b,c]` $\widehat{=} f^{abc}$
- `d[a,b,c]` $\widehat{=} d^{abc}$
- `T[a,i,j]` $\widehat{=} T^{a}_{ij}$
- `KDf[i,j]` $\widehat{=} \delta_{ij}$
- `KDad[a,b]` $\widehat{=} \delta_{ab}$
- `TrT[a1,...,an]` $\widehat{=} \mathrm{Tr}(T^{a1}...T^{an})$

`BeginPackage["color`"]` starts a context named `color``. All of the functions from above are declared in this context.

```
BeginPackage["color`"]
(* Note: All expressions are in Einstein summation convention *)

  N::usage = "SU(N)"
  T::usage = "T[a, i, j] means the (i, j)th entry of the SU(N)-generator matrix T^a. i,j=1,...,N and a=1,...,(N^2-1)
    Implemented rules:
      T[a, i, j]T[a, k, l] = 1/2 ( KDf[i, l] KDf[j, k] - 1/N KDf[i, j] KDf[k, l] )
      Tr(T^a) = 0
      Tr(T^a T^b) = 1/2 KDad[a, b]"

  f::usage = "f[a, b, c] is the structure constant that is defined by the commutator relation [T^a, T^b]=i f[a, b, c]T^c.
    Implemented rule: f[a, b, c] = -2i*Tr(T^a[T^b, T^c])"

  d::usage = "d[a, b, c] is the structure constant that is defined by the anti-commutator relation {T^a, T^b}=1/N KDad[a, b]+d[a, b, c]T^c
    Implemented rule: d[a, b, c] = 2*Tr(T^a{T^b, T^c})"

  KDf::usage = "KDf[i, j] is the Kronecker Delta function in the fundamental representation of SU(N). i,j=1,...,N
    Implemented rules:
      Attributes[KDf] = {Orderless}"

      KDf[i, i] = N
      KDf[i, j] KDf[j, k] = KDf[i, k]
      KDf[i, j]^2 = N
      KDf[i, j] T[a, i, k] = T[a, j, k]
      KDf[i, j] T[a, k, i] = T[a, k, j]
```

```
KDad::usage = "KDad[a, b] is the Kronecker Delta function
    in the adjoint representation of SU(N). a,b=1,...,(N
    ^2-1)
  Implemented rules:
    Attributes[KDad] = {Orderless}"

    KDad[a, a] = N^2 - 1
    KDad[a, b] KDad[b, c] = KDad[a, c]
    KDad[a, b]^2 = N^2-1
    KDad[a, b] T[a, i, j] = T[b, i, j]

TrT::usage = "TrT builds the trace of a product of
    generators T^a_1...T^a_n:
  TrT[a_1, a_2, ..., a_n] creates a string of T like T[a_1
      , i_1, i_2] T[a_2, i_2, i_3]...T[a_n, i_n, i_1] with
      auxiliary symbols i_1, ..., i_n."
```

The last part of the package begins with the command `Begin["`Private`"]` that starts a new private context `color`Private`` in order to avoid conflicts with names of internally used variables. That is the main part of the program because the functions are defined here. Relations (5.1) to (5.5) are implemented here.

```
Begin["`Private`"]

(* Definition for f: f[a,b,c] -> -2i*Trace(T^a[T^b,T^c])
    *)

  f /: f[a_, b_, c_] := Module[{i, k, l}, -2*I*(T[a, i, k
      ] T[b, k, l] T[c, l, i] - T[a, i, k] T[c, k, l] T[b,
      l, i])];

(* Definition for d: d[a,b,c] -> 2*Trace(T^a{T^b,T^c}) *)

  d /: d[a_, b_, c_] := Module[{i, k, l}, 2*(T[a, i, k] T[
      b, k, l] T[c, l, i] + T[a, i, k] T[c, k, l] T[b, l, i
      ])];

(* Definitions for T in order to reduce expressions to
    Kronecker Delta functions KDf and KDad *)

  T /: T[a_, i_, j_] T[a_, k_, l_] := 1/2 (KDf[i, l] KDf[j
      , k] - 1/N KDf[i, j] KDf[k, l] );
  T /: T[a_, i_, i_] := 0;
  T /: T[a_, i_, j_] T[b_, j_, i_] := 1/2 KDad[a, b];

(* Rules and Attributes for the Kronecker Delta functions
    KDf and KDad *)

  Attributes[KDf] = {Orderless};
  Attributes[KDad] = {Orderless};

  KDf /: KDf[i_, i_] := N;
  KDad /: KDad[a_, a_] := N^2 - 1;
```

```

    KDf /: KDf[i_, j_] KDf[j_, k_] := KDf[i, k];
    KDad /: KDad[a_, b_] KDad[b_, c_] := KDad[a, c];

    KDf /: KDf[i_, j_]^2 := N;
    KDad /: KDad[a_, b_]^2 := N^2-1;

    KDf /: KDf[i_, j_] T[a_, i_, k_] := T[a, j, k];
    KDf /: KDf[i_, j_] T[a_, k_, i_] := T[a, k, j];
    KDad /: KDad[a_, b_] T[a_, i_, j_] := T[b, i, j];

  (* Definition of the trace function TrT *)

    TrT[x__] := Expand[Module[{mnr = $ModuleNumber, i},
        Product[T[{x}[[j]], i[j + mnr], i[j + mnr + 1]], {j
        , 1, Length[{x}] - 1}]*
    T[{x}[[Length[{x}]]], i[Length[{x}] + mnr], i[mnr
         + 1]]]];

  End[]

Protect["color`*"];

EndPackage[]
```

A Projection operator

<u>Statement</u>: $T^a_{ij}T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)$

<u>Proof</u>:
The generators and the identity matrix build a basis of the hermitian $N \times N$ matrices. Let M be an arbitrary hermitian $N \times N$ matrix that can be represented by a linear combination: $M = m_0 \mathbb{1}_N + m_a T^a$ with $m_0, m_a \in \mathbb{R}$.
We obtain the coefficients m_0 and m_a by calculating the traces of M and $T^a M$:

$$\mathrm{Tr}(M) = m_0 \cdot N \Rightarrow m_0 = \frac{1}{N}\mathrm{Tr}(M) \tag{A.1}$$

$$\mathrm{Tr}(T^a M) = m_b \mathrm{Tr}(T^a T^b) = \frac{1}{2} m_b \delta_{ab} = \frac{1}{2} m_a \Rightarrow m_a = 2\mathrm{Tr}(T^a M) \tag{A.2}$$

$$\Rightarrow M = \frac{1}{N}\mathrm{Tr}(M)\mathbb{1}_N + 2\mathrm{Tr}(T^a M)T^a$$

Now we consider the (i,j)th entry of M:

$$M_{ij} = \frac{1}{N} M_{kk}\delta_{ij} + 2T^a_{kl} M_{lk} T^a_{ij} = M_{lk} \underbrace{\left(\frac{1}{N}\delta_{kl}\delta_{ij} + 2T^a_{kl}T^a_{ij}\right)}_{=\delta_{il}\delta_{jk}} \quad \forall M$$

$$\Rightarrow T^a_{ij}T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)$$

□

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