# Solution to Problem Set 4

October 2017

## Pb 1. 20 pts.

There are many ways of doing this problem but the easiest would be

$$\begin{aligned}\hat{a}^\dagger |\alpha\rangle =& \hat{a}^\dagger \hat{D}(\alpha) |0\rangle = \hat{a}^\dagger \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)|0\rangle = \hat{a}^\dagger e^{-\alpha^*\alpha/2} e^{\alpha\hat{a}^\dagger}|0\rangle \\ =& \left(\frac{\partial}{\partial\alpha} + \frac{\alpha^*}{2}\right) e^{-\alpha^*\alpha/2} e^{\alpha\hat{a}^\dagger}|0\rangle = \left(\frac{\partial}{\partial\alpha} + \frac{\alpha^*}{2}\right)|\alpha\rangle\,,\end{aligned} \tag{1}$$

where we used the Zassenhauss formula (see full expression in Wikipedia or any advanced quantum textbook), which reduces to

$$e^{\hat{X}+\hat{Y}} = e^{\hat{X}} e^{\hat{Y}} e^{-[\hat{X},\hat{Y}]/2}, \tag{2}$$

if $[X, Y]$ commutes with both $X$ and $Y$. The displacement operator can then be expressed as

$$\hat{D}(\alpha) = e^{-\alpha^*\alpha/2} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}}. \tag{3}$$

## Pb 2. 80 pts.

For this problem, I highly recommended two books for reference.

- Sean Carroll *Geometry and Spacetime, an Introduction to General Relativity.* This may be the standard textbook for GR class next semester. You will find a lot of good examples and exercises about Lorentz Indices in the first two chapters.
- John Baez etc. *Gauge Fields, Knots and Gravity.* A very cute book tells about gauge fields and gravity as an aspect of differential geometry. You can learn a lot about the second problem in first two parts.

In this problem, we use Greek letter $\alpha, \beta, \gamma, \cdots, \mu, \nu, \cdots$ for spacetime Lorentzian indices $0, 1, 2, 3$ and Latin letter $i, j, k, \cdots$ for spatial indices $1, 2, 3$. Still we set $c = 1$.

### a) Maxwell Equation with Lorentz Indices

In this problem we look at the covariant formulation of classical electromagnetism. As defined in the problem statement, the action $S$ is

$$S = \int d^4x\mathcal{L} = -\frac{1}{g^2}\int d^4x F_{\mu\nu}F^{\mu\nu} = -\frac{1}{g^2}\int d^4x F_{\mu\nu} g^{\mu\gamma} g^{\nu\kappa} F_{\gamma\kappa}, \tag{4}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -F_{\nu\mu}, \tag{5}$$

and we're interested in finding the equations of motion for the electromagnetic 4-potential $A^\mu = (\phi, \vec{A})$, which can be determined from the Euler-Lagrange equations for fields,

$$\begin{aligned}0 = \partial_\beta \left( \frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)} \right) - \frac{\partial \mathcal{L}}{\partial A_\alpha} &= -\,\partial_\beta \left( \frac{g^{\mu\gamma} g^{\nu\kappa}}{g^2} \frac{\partial(F_{\mu\nu} F_{\gamma\kappa})}{\partial(\partial_\beta A_\alpha)} \right) \\ &= -\,\partial_\beta \left( \frac{g^{\mu\gamma} g^{\nu\kappa}}{g^2} \left[ \frac{\partial F_{\mu\nu}}{\partial(\partial_\beta A_\alpha)} F_{\gamma\kappa} + F_{\mu\nu} \frac{\partial F_{\gamma\kappa}}{\partial(\partial_\beta A_\alpha)} \right] \right) \\ &= -\,\partial_\beta \left( \frac{g^{\mu\gamma} g^{\nu\kappa}}{g^2} \left[ \left(\delta^\beta_\mu \delta^\alpha_\nu - \delta^\beta_\nu \delta^\alpha_\mu\right) F_{\gamma\kappa} + \left(\delta^\beta_\gamma \delta^\alpha_\kappa - \delta^\beta_\kappa \delta^\alpha_\gamma\right) F_{\mu\nu} \right] \right) \\ &= -\,\frac{1}{g^2} \partial_\beta \left( \left(g^{\beta\gamma} g^{\alpha\kappa} - g^{\alpha\gamma} g^{\beta\kappa}\right) F_{\gamma\kappa} + \left(g^{\mu\beta} g^{\nu\alpha} - g^{\mu\alpha} g^{\nu\beta}\right) F_{\mu\nu} \right) \\ &= -\,\frac{2}{g^2} \partial_\beta \left(F^{\beta\alpha} - F^{\alpha\beta}\right) = -\frac{4}{g^2} \partial_\beta F^{\beta\alpha}, \end{aligned} \tag{6}$$

and we found two of the Maxwell's equations in vacuum (Gauss electric law & Ampere law),

$$\partial_\beta F^{\beta\alpha} = 0. \tag{7}$$

The electromagnetic tensor also satisfies the Bianchi property

$$\partial_{[\mu} F_{\nu\gamma]} = 0, \tag{8}$$

where [] is the alternating sum over all permutations. This is because partial derivatives commute (recall the definition of $F_\mu \nu$ contains derivative). This identity leads to the other two Maxwell equations (Faraday law & Gauss magnetic law). The Maxwell equations with sources are

$$\partial_\mu F^{\mu\nu} = J^\nu, \tag{9}$$

and it means that the source terms should come from

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{4}{g^2} J^\nu, \tag{10}$$

meaning the action should be modified as

$$S_1 \to -\frac{1}{g^2} \int d^4x F_{\mu\nu} F^{\mu\nu} - \frac{4}{g^2} \int d^4x A_\mu J^\mu. \tag{11}$$

To write Maxwell Equation explicitly, we may take advantages of anti-symmetric tensor $\epsilon$. First we want to define $E$ and $B$ by $F^{i0} = E^i$ and $F^{ij} = -\epsilon^{ijk} B_k$ and inversely we have $E^i = F^{i0}$ and $B^i = -\frac{\epsilon^{ijk}}{2} F_{jk}$.

Starting with (9), first we set $\nu = 0$ and $\mu = i = 1, 2, 3$, (9) becomes

$$\partial_i F^{i0} = \partial_i E^i = \rho \tag{12}$$

which is Gauss's Law for electric field. Then we set $\nu = j = 1, 2, 3$ and Ampere's Law for magnetic field appears

$$\partial_\mu F^{\mu j} = \partial_t F^{0j} + \partial_i F^{ij} = -\frac{\partial E^j}{\partial t} - \epsilon^{ijk} \partial_i B_k = -\frac{\partial E^j}{\partial t} + (\nabla \times \vec{B})^j = j^j \tag{13}$$

As we mention above, the other two equations are from Bianchi identity. For simplicity, we rewrite Bianchi identity as

$$\epsilon^{\mu\nu\alpha\beta} \partial_\nu F_{\alpha\beta} = \epsilon^{\mu\nu\alpha\beta} \partial_\nu (\partial_\alpha A_\beta - \partial_\beta A_\alpha) = 2\epsilon^{\mu\nu\alpha\beta} \partial_\alpha \partial_\beta A_\beta = 0 \tag{14}$$

Of course from (14) we will get an explicitly impression on why Bianchi identity holds for electromagnetic fields. For $\mu = 0$, (14) is the Gauss's Law for magnetic field. Noticing $\epsilon^{0\nu\alpha\beta} = \epsilon^{ijk}$, we have

$$0 = \epsilon^{ijk} \partial_i F_{jk} = \epsilon^{ijk} \epsilon_{jkl} \partial_i B^l = 2\delta^i_l \partial_i B^l = 2\nabla \cdot \vec{B} \tag{15}$$

For $\mu = i = i, j, k$, calculation is a little bit more complicated but we can still work out

$$0 = \epsilon^{i0jk} \partial_t F_{jk} + \epsilon^{ij0k} \partial_j F_{0k} + \epsilon^{ijk0} \partial_j F_{k0} = -\epsilon^{ijk} \partial_t F_{jk} - 2\epsilon^{ijk} \partial_j F_{k0} = 2\frac{\partial \vec{B}^{(i)}}{\partial t} + 2\nabla \times \vec{E}^{(i)} = 0 \tag{16}$$

Till now we have derived all four Maxwell equations explicitly.

## b) A glance at differential Geometry, Hodge Operator, and other properties on differential forms.

Now we turn to differential geometry. In that formalism the action $S_1$ takes the simpler form

$$S_2 = -\frac{1}{g^2}\int F \wedge (\star F) \equiv S_1, \tag{17}$$
$$A \equiv A_\mu dx^\mu \tag{18}$$
$$F \equiv F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{19}$$
$$d \equiv dx^\mu \partial_\mu \tag{20}$$

where the wedge product satisfies $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$ as stated in the problem.

Now we can find the equations of motion using $S_2$. We define the exterior derivative of p-form $X$ as

$$dX = (dX_{k_1\cdots k_p}) \wedge dx^{k_1} \wedge \cdots \wedge dx^{k_p} = \frac{1}{p!}\partial_{[k_1} X_{k_2\cdots k_{p+1}]} dx^{k_1} \wedge \cdots \wedge dx^{k_{p+1}}, \tag{21}$$

which can be found from the definition of $d$ given in the problem.

- Some Useful Properties of Differential Form

  Suppose we have a p-form $X \equiv X_I dx^I$, where $I$ represent multiple indices.

  - $d^2 X = 0$, this is because

  $$d(dX) = \partial_\mu \partial_\nu X_I dx^\mu \wedge dx^\nu \wedge dx^I = 0 \tag{22}$$

  - $d(X \wedge Y) = dX \wedge Y + (-1)^p X \wedge dY$, we can proof it as

  $$\begin{aligned} d(X \wedge Y) =& (\partial_\mu X_I Y_J + X_I \partial_\mu Y_J) dx^\mu \wedge dx^I \wedge dy^J \\ =& (\partial_\mu X_I Y_J) dx^\mu \wedge dx^I \wedge dy^J + (-1)^p (X_I \partial_\mu Y_J) dx^I \wedge dx^\mu \wedge dy^J \\ =& dX \wedge Y + (-1)^p X \wedge dY \end{aligned} \tag{23}$$

- Some Useful Properties of Hodge Operator. At first we will figure out $**$ is a kind of self dual.

  $$X = X_{k_1\cdots k_p} dx^{k_1} \wedge \cdots \wedge dx^{k_p} =\equiv X_I dx^I, \tag{24}$$
  $$\star X = \frac{\sqrt{|\det \mathrm{g}_{\mu\nu}|}}{p!} \epsilon_{k_{p+1}\cdots k_{d+1}}{}^{k_1\cdots k_p} X_{k_1\cdots k_p} dx^{k_{p+1}} \wedge \cdots \wedge dx^{k_{d+1}}, \tag{25}$$
  $$\star(\star X) = \frac{\sqrt{|\det \mathrm{g}_{\mu\nu}|}}{(d+1-p)!} \epsilon_{q_1\cdots q_p}{}^{k_{p+1}\cdots k_{d+1}} \frac{\sqrt{|\det \mathrm{g}_{\mu\nu}|}}{p!} \epsilon_{k_{p+1}\cdots k_{d+1}}{}^{k_1\cdots k_p} X_{k_1\cdots k_p} dx^{q_1} \wedge \cdots \wedge dx^{q_p}, \tag{26}$$

  where applying the Hodge star operator twice gives back a p-form, $d+1-(d+1-p) = p$. We will need the following identity

  $$\epsilon^{k_{p+1}\cdots k_{d+1} q_1 \cdots q_p} \epsilon_{k_{p+1}\cdots k_{d+1} k_1 \cdots k_p} = -(d+1-p)! \delta^{q_1}_{k_1} \cdots \delta^{q_p}_{k_p}, \tag{27}$$

  ... Using the identity above we have the following result

  $$\begin{aligned} \epsilon_{q_1\cdots q_p}{}^{k_{p+1}\cdots k_{d+1}} \epsilon_{k_{p+1}\cdots k_{d+1}}{}^{k_1\cdots k_p} =& (-1)^{d+1-p} \epsilon^{k_{p+1}}{}_{q_1\cdots q_p}{}^{k_{p+2}\cdots k_{d+1}} \epsilon_{k_{p+1}\cdots k_{d+1}}{}^{k_1\cdots k_p} \\ =& (-1)^{p(d+1-p)} \epsilon^{k_{p+1}\cdots k_{d+1}}{}_{q_1\cdots q_p} \epsilon_{k_{p+1}\cdots k_{d+1}}{}^{k_1\cdots k_p} \\ =& (-1)^{p(d+1-p)} g_{q_1\alpha_1} \cdots g_{q_p\alpha_p} \epsilon^{k_{p+1}\cdots k_{d+1}\alpha_1\cdots\alpha_p} g^{k_1\beta_1} \cdots g^{k_p\beta_p} \epsilon_{k_{p+1}\cdots k_{d+1}\beta_1\cdots\beta_p} \\ =& -(-1)^{p(d+1-p)} (d+1-p)! g_{q_1\alpha_1} \cdots g_{q_p\alpha_p} g^{k_1\beta_1} \cdots g^{k_p\beta_p} \delta^{\alpha_1}_{\beta_1} \cdots \delta^{\alpha_p}_{\beta_p} \\ =& -(-1)^{p(d+1-p)} (d+1-p)! \delta^{k_1}_{q_1} \cdots \delta^{k_p}_{q_p}, \end{aligned} \tag{28}$$

which allows us to simplify

$$\begin{aligned}
\star(\star X) &= -(-1)^{p(d+1-p)}\frac{|\det\, g_{\mu\nu}|}{p!}\delta^{k_1}_{q_1}\cdots\delta^{k_p}_{q_p}X_{k_1\cdots k_p}dx^{q_1}\wedge\cdots\wedge dx^{q_p} \\
&= -(-1)^{p(d+1-p)}\frac{|\det\, g_{\mu\nu}|}{p!}X_{k_1\cdots k_p}dx^{k_1}\wedge\cdots\wedge dx^{k_p} \\
&= -(-1)^{p(d+1-p)}\sqrt{|\det\, g_{\mu\nu}|}X = -(-1)^{p(d+1-p)}X,
\end{aligned} \tag{29}$$

where det $g_{\mu\nu} = -1$ in our case. Now we focus on the special case

$$\star(\star F) = -(-1)^4 F = -F. \tag{30}$$

Let's look at the Hodge operator more carefully. The Hodge star operator $(\star)$ maps the p-form $X$ to a $(d+1-p)$-form $\star X$ as follows. Therefore, it is easy for us to construct a $n-$form from two $p-$forms $A$ and $B$ as $A\wedge\star B$. And we have some such identity

$$\int A\wedge\star B = \int \star A\wedge B \tag{31}$$

## c) Maxwell equation in differential form

It is useful to realize that $F = dA$ and this is very easy to verify $F = F_{\mu\nu}dx^\mu\wedge dx^\nu = \frac{1}{2!}(\partial_\mu A_\nu - \partial_\nu A_\mu)dx^\mu\wedge dx^\nu = dA$. From (22), we can derive second pair of Maxwell equations.

$$dF = d(dA) = 0 \tag{32}$$

By action principle, we can extremize the action $S_2$ to get the first pair of Maxwell equation.

$$\begin{aligned}
0 = \delta_A S_2 &= -\frac{1}{g^2}\int \left[d(\delta A)\wedge(\star dA) + dA\wedge(\star d(\delta A))\right] \\
&= -\frac{2}{g^2}\int d(\delta A)\wedge(\star dA) \\
&= -\frac{2}{g^2}\int \left[d(\delta A\wedge(\star F)) + \delta A\wedge d(\star F)\right] \\
&= -\frac{2}{g^2}\int \delta A\wedge d(\star F),
\end{aligned} \tag{33}$$

where we used the identity, for a p-form $X$ and a q-form $Y$,

which is easy to show with what's given in the problem. We just found

$$d(\star F) = 0. \tag{34}$$

Note that in the presence of sources you would find

$$dF = 0, \tag{35}$$

$$d(\star F) = \star J. \tag{36}$$

Now we consider the action

$$S_3 = -\frac{1}{g^2}\int F\wedge(\star F) + \theta\int F\wedge F, \tag{37}$$

and extremize the action

$$0 = \delta_A S_3 = -\frac{2}{g^2}\int d(\delta A)\wedge(\star F) + 2\theta\int d(\delta A)\wedge F = -\frac{2}{g^2}\int \delta A\wedge d(\star F) + 2\theta\int \delta A\wedge dF, \tag{38}$$

but since $dF = d(dA) = 0$, the equations of motion are unchanged: $d(\star F) = 0$.

## d) A particle moving in fields

Now we consider the action

$$S_4 = -\frac{1}{g^2}\int F\wedge(\star F) + \theta\int F\wedge F + \frac{m}{2}\int dt\dot{X}^2 + q\int A = \int dtL, \tag{39}$$

The 4-current due to the point particle is $I^\mu = q(1,\dot{\vec{X}}) = q(1,\vec{V})$ (note: this is not a density), and thus $qA = qA_\mu dX^\mu = A_\mu(qdX^\mu/dt)dt = A_\mu I^\mu dt = q\phi dt - q\vec{A}\cdot\vec{V}$. First the canonical momentum of the point particle is

$$P^j = \frac{dL}{d\dot{X}^j} = m\dot{X}^j - qA^j \;\Rightarrow\; \vec{P} = m\dot{\vec{X}} - q\vec{A}, \tag{40}$$

where $j$ stands for spatial coordinates. The momentum of the particle is shifted by $q\hat{A}$. Secondly we find the equations of motion of the point particle

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{X}^j}\right) - \frac{\partial L}{\partial X^j} =& m\ddot{X}^j - q\frac{dA^j}{dt} - q\frac{d\phi}{dX^j} + q\frac{d}{dX^j}(\vec{A}\cdot\vec{V})\\ =& m\ddot{X}^j - q\frac{\partial A^j}{\partial t} - q\frac{d\phi}{dX^j} + q\frac{d}{dX^j}(\vec{A}\cdot\vec{V}) - q\frac{dX^i}{dt}\frac{dA^j}{dX^i}\\ =& m\ddot{X}^j - q\frac{\partial A^j}{\partial t} - q\frac{d\phi}{dX^j} + q\frac{d}{dX^j}(\vec{A}\cdot\vec{V}) - qV_i\frac{dA^j}{dX^i}.\end{aligned} \tag{41}$$

We find that

$$m\ddot{\vec{X}} = q\frac{\partial\vec{A}}{\partial t} + q\nabla\phi - q\nabla(\vec{A}\cdot\vec{V}) + q(\vec{V}\cdot\nabla)\vec{A} = -q\left[-\frac{\partial\vec{A}}{\partial t} - \nabla\phi + \vec{V}\times(\nabla\times\vec{A})\right] = -q\left[\vec{E} + \vec{V}\times\vec{B}\right] \tag{42}$$

It shouldn't be surprising that we found the Lorentz force. Then we extremize the action for the EM fields

$$0 = \delta_A S_4 = -\frac{1}{g^2}\int d(\delta A)\wedge(\star F) + \int\delta A\wedge(\star J) = -\frac{1}{g^2}\int\delta A\wedge d(\star F) + \int\delta A\wedge(\star J), \tag{43}$$

and we found

$$d(\star F) = \star J, \tag{44}$$

(where $J$ is a 1-form). Now define the current density use Dira delta functions,

$$J^\mu = q\left(\delta(\vec{x}-\vec{X}), \vec{V}\delta(\vec{x}-\vec{X})\right) \tag{45}$$

where $\vec{X}(t)$ is the path of the particle (such that $\int d\vec{x}A_\mu J^\mu = A_\mu I^\mu\big|_{x=X}$ which we used before). Then do the Hodge star operation and you get the Maxwell equations with sources.

## e) Chern-Simons term

Now we consider the action

$$S_5 = c_1\int A\wedge dA + c_2\int A\wedge A\wedge A \tag{46}$$

and extremize the action

$$\begin{aligned}0 = \delta_A S_5 =& c_1\int\delta A\wedge dA + c_1\int A\wedge d(\delta A) + 3c_2\int\delta A\wedge A\wedge A\\ =& 2c_1\int\delta A\wedge dA + 3c_2\int\delta A\wedge A\wedge A\end{aligned} \tag{47}$$

$$F := dA + \frac{3c_2}{2c_1}A\wedge A = 0 \tag{48}$$

Here the cubic term is related to the structure constant of gauge group. Notice that in general we have non-Abelian gauge theory (Yang -Mills Theory). For instance, usually we $c_2 = \frac{2}{3}c_1$ and we define field strength (in explict form)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{49}$$