Chapter 2

Fermion masses in the Standard Model and beyond

False facts are highly injurious to the progress of science, for they often endure long; but false views, if supported by some evidence, do little harm, for every one takes a salutary pleasure in proving their falseness.

Charles Darwin [12], The descent of man

Flavour symmetries provide new ways to describe the apparent structure in the quark and lepton masses. To be able to appreciate this, we first study the way elementary-fermion masses are generated in the original Standard Model. In the first section of this chapter, we discuss fermion masses in the Standard Model with only one generation. We will see that the quark and charged lepton masses are generated straightforwardly, but that neutrino masses are quite challenging to the theorist already. In the next section, we extend the analysis to the familiar three generation Standard Model, counting how many new degrees of freedom are hidden in the fermion masses and mixing. In the two sections that follow, we describe the working of family symmetries. We include two relatively simple models: the Froggatt–Nielsen model, that explains the hierarchy among the generations in section 2.3 and the model of Altarelli and Feruglio, that reproduces the tribimaximal mixing pattern in section 2.4. Lastly, section 2.5 presents the conclusions of the chapter.

2.1 The one family Standard Model

In this section, we describe how the Standard Model can accommodate masses for the quarks and leptons in case there is a rather minimal number of them. We discuss a situation where there are only two quarks, one of the up-type and one of the down-type, that we simply call up and down. We also assume the existence of only one charged lepton, dubbed the electron and one neutrino, that we refer to as such.

As always, all information is contained in the Lagrangian. The most general $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge invariant Lagrangian with only renormalisable operators reads

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_K + \mathcal{L}_{\text{gauge}} + \mathcal{L}_Y + V_{\text{Higgs}}. \tag{2.1}$$

Here, $\mathcal{L}_K$ are the kinetic terms for the quarks, the leptons and the Higgs field. The demand of invariance under *local* symmetry transformations, requires the appearance of gauge bosons in covariant derivatives. Their own kinetic terms and self-interactions are given in the second part of the Lagrangian $\mathcal{L}_{\text{gauge}}$. Kinetic and gauge terms are very well known since the original formulation of the Standard Model and we do not modify them in this thesis, except for the fact that we discuss a gauge group different from the Standard Model's in chapter 4. Even there, the extension is straightforward.

The last term in 2.1 is the potential for the Higgs field. If there is only one Higgs field, this is also very well-known. It is the famous Mexican hat potential, where the Higgs field drifts away to its minimum that is not at the origin, thereby breaking the electroweak symmetry. The value of the Higgs field at the minimum is called the vacuum expectation value or vev. This is schematically represented in figure 2.1. In case of more than one Higgs field, the potential might become more involved. In chapter 5 we study the most general potential for a three Higgs fields that transform together as a triplet of the flavour symmetry A_4 and the different vacuum expectation values these fields can be in. In the remainder of this chapter, we simply assume the existence of some Higgs potential that gives non-zero vevs for one or more Higgses and focus on the last term we did not discuss yet, the Yukawa interactions $\mathcal{L}_Y$.

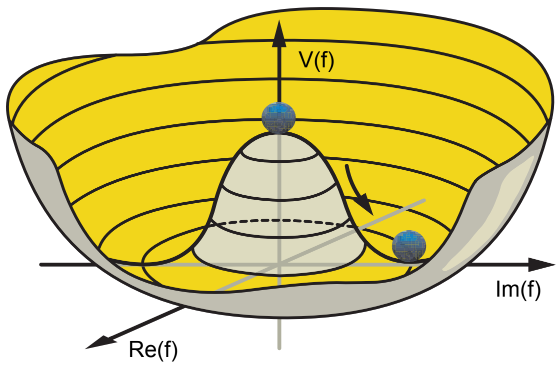

Figure 2.1: *A cartoon of the Higgs potential and its non-zero vacuum expectation value.*

2.1.1 Yukawa couplings

The terms in 2.1 that are of most importance for this chapter, are the terms in $\mathcal{L}_Y$, the Yukawa interactions between the Higgs fields and the quarks or leptons that eventually give rise to mass terms for the latter. To appreciate these, we first turn to elementary particles *below* the electroweak symmetry breaking (EWSB) scale. These particles and their relevant quantum numbers - the electromagnetic charge and the representation of the colour gauge group - are given in table 2.1

Field	symbol	$(SU(3)_C, U(1)_{\text{em}})$
up quark	u	(3, $\frac{2}{3}$)
down quark	d	(3, -$\frac{1}{3}$)
neutrino	ν	(1, 0)
electron	e	(1,-1)

Table 2.1: The quarks and leptons below the EWSB scale and their representations under the relevant gauge group.

Mass terms are constructed as quadratic terms in the fermion fields. They contain a spinor ψ that represents an incoming fermion as well as a barred spinor $\bar{\psi}$ that represents an outgoing fermion as shown in figure 2.2. Below the electroweak scale, fermion masses read

$$\begin{aligned} \mathcal{L}_{\text{mass}} &= m_u \bar{u}_L u_R + m_d \bar{d}_L d_R + m_e \bar{e}_L e_R + \text{h.c.} \\ &= m_u \bar{u} u + m_d \bar{d} d + m_e \bar{e} e \ . \end{aligned} \tag{2.2}$$

Note that in the definition of $\bar{\psi} = \psi^\dagger \gamma_0$, there is a complex conjugate. Therefore, a spinor $\bar{\psi}$ has the opposite quantum numbers as ψ. Thus, for instance, $\bar{u}$ is in the representation $(\bar{3}, -2/3)$ of

Figure 2.2: *A propagating fermion with a mass insertion according to equation (2.2)*

$SU(3) \times U(1)_{\text{em}}$. This ensures that all terms in the Lagrangian (2.2) are singlets of the colour times electromagnetic gauge group. In the term $m_u \bar{u}u$, we have $\bar{3} \times 3 \ni 1$ for colour and $-2/3 + 2/3 = 0$ for electric charge, etc. We did not include neutrino masses for reasons that are explained shortly.

Now we move to the Standard Model above the EWSB scale. This is a chiral theory, meaning that left-handed and righthanded fields are no longer treated on equal footing. In the left hand side of table 2.2, we repeat the content of table 2.1, this time taking the left- or righthandedness of the fields into account. On the right hand side, we add the Standard Model fields that correspond to the fields on the left. Standard Model fields have quantum numbers of $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$. The colour group is the same as below the electroweak scale. The representations under $SU(2)_L$ are such that lefthanded fields are in the doublet representation, while righthanded fields are in the singlet representation. We normalize hypercharge such that the electric charge is given by $Q = I_3 + Y$, where I_3 is +(-)$\frac{1}{2}$ for the upper (lower) component of an $SU(2)_L$-doublet and zero for righthanded fields.

Field		$(SU(3)_C, U(1)_{\text{em}})$	Field		$(SU(3)_C, SU(2)_L, U(1)_Y)$
RH up quark	u_R	(3, $\frac{2}{3}$)	RH up quark	u_R	(3,1,$\frac{2}{3}$)
LH up quark	u_L	(3, $\frac{2}{3}$)	LH quark doublet	Q_L	(3,2,$\frac{1}{6}$)
LH down quark	d_L	(3, -$\frac{1}{3}$)			
RH down quark	d_R	(3, -$\frac{1}{3}$)	RH down quark	d_R	(3,1, -$\frac{1}{3}$)
(RH neutrino)	(ν_R)	(1,0)	(RH neutrino)	(ν_R)	(1,1,0)
LH neutrino	ν_L	(1,0)	LH Lepton doublet	L_L	(1,2,$-\frac{1}{2}$)
LH electron	e_L	(1,-1)			
RH electron	e_R	(1,-1)	RH electron	e_R	(1,1,-1)

Table 2.2: Elementary fermions below (left) and above (right) the scale of electroweak symmetry breaking. The righthanded neutrino is printed in grey to stress that its existence is uncertain as explained in the text.

Table 2.2 mentions a righthanded neutrino in grey. Indeed a non-particle physicist that would see a version of table 2.2 without it, would probably immediately add it to 'complete the symmetry' of the table, where for every lefthanded field in the left half of the table, there is also a righthanded field and for every doublet on the right, there are two singlets. In the original Standard Model, however, the righthanded neutrino is absent. The reason is simple: it has never been observed. This is a consequence of the fact that it is a singlet under the complete Standard Model gauge group. This means that, barring gravity, it cannot interact with any of the other particles, perfectly hiding its possible existence. For now we assume that there are no righthanded neutrinos and discuss the masses of the other particles of table 2.2. Later in this section, we explore the new physics possibilities that the inclusion of a righthanded neutrino offers.

In passing, we note that after electroweak symmetry breaking, neutrinos of any handedness are singlets under the residual $SU(3)_C \times U(1)_{\text{em}}$ gauge group. This means that the only way in which neutrinos can interact is by the fact that they are part of a doublet *above* the electroweak scale. There is thus a huge gap between the energy that neutrinos normally have and the energy scale above which they can interact. This explains the claim made in section 1.2.1 that neutrinos can traverse lightmonths of lead without ever interacting.

The analogue of equation (2.2) above the electroweak scale reads[1]

$$\mathcal{L}_{\text{mass}} = y_u \begin{pmatrix} \bar{Q}_L^u & \bar{Q}_L^d \end{pmatrix} \begin{pmatrix} H_u^0 \\ H_u^- \end{pmatrix} u_R + y_d \begin{pmatrix} \bar{Q}_L^u & \bar{Q}_L^d \end{pmatrix} \begin{pmatrix} H_d^+ \\ H_d^0 \end{pmatrix} d_R + y_e \begin{pmatrix} \bar{L}_L^\nu & \bar{L}_L^e \end{pmatrix} \begin{pmatrix} H_d^+ \\ H_d^0 \end{pmatrix} e_R + \text{h.c.} \tag{2.3}$$

Here H_d and H_u are Higgs fields with quantum numbers $(1, 2, +1/2)$ and $(1, 2, -1/2)$ respectively. Note that the Higgs fields are required by gauge invariance, as no terms with only left- and righthanded quarks or leptons can give an SM singlet. This is why direct mass terms (with a dimensionful coupling constant m_x) are forbidden and we have only indirect mass terms from interactions with the Higgs field. The coupling constants, the Yukawa couplings y_x are dimensionless. In the minimal Standard Model, only one independent Higgs field can be used as H_d and H_u can be related via $H_u = i\sigma_2 H_d^*$. In many extensions of the Standard Model, including the minimal supersymmetric Standard Model, this identification is not allowed and two separate Higgs fields are required. In this chapter, we use both H_d and H_u, keeping in mind that the two fields might be related .

After the neutral components of the Higgs fields develop vacuum expectation values of respectively v_{H_d} and v_{H_u}, the Higgs fields can be expanded around these minima

$$H_d = \frac{1}{\sqrt{2}} \begin{pmatrix} h_d^+ \\ v_{H_d} + h_d^0 \end{pmatrix}, \quad H_u = \frac{1}{\sqrt{2}} \begin{pmatrix} v_{H_u} + h_u^0 \\ h_u^- \end{pmatrix} . \tag{2.4}$$

The factor $\sqrt{2}$ in (2.4) is just a convention to have both components conveniently normalized. In the minimal Standard Model, obviously, $v_{H_d} = v_{H_u}$ as the vev can be chosen real. In a two Higgs doublet model, the quadratic sum of the two vevs equals 'the' electroweak vacuum expectation value v_{ew}. The ratio of the two vacuum expectation values is an important parameter called $\tan\beta$.

$$v_{H_u}^2 + v_{H_d}^2 = v_{\text{ew}} = 246 \text{ GeV}^2 , \qquad \tan\beta = \frac{v_{H_u}}{v_{H_d}} . \tag{2.5}$$

The Higgs fields are complex $SU(2)$-doublets, so they have four real components each. If H_d and H_u are unrelated, this gives in total eight real components; if they are related as in the Standard Model, the number is only four. Three components correspond to Goldstone bosons that give mass to the W^+, W^- and Z bosons. In the Standard Model, these are the two charged components for the Ws and the imaginary part A of the expansion around the vev for the neutral Z. This leaves only one Higgs boson h.

In a two Higgs doublet model, the Goldstone boson for the W^+ is formed from a certain linear combination of the charged components h_d^+ and h_u^- (or rather its conjugate), while the orthogonal combination becomes the physical charged Higgs. Typically, both vevs v_{H_d} and v_{H_u} are real. In that case, the Goldstone boson of the Z particle comes from a linear combination of the imaginary parts of h_d^0 and h_u^0, but not of the real parts. The other linear combination of the imaginary parts becomes a pseudoscalar Higgs, while the real parts of h_d^0 and h_u^0 mix to two scalar Higgs bosons.

Inserting the Higgs vevs of (2.4) into equation (2.3) reproduces equation (2.2) with $m_u = y_u v_{H_u}/\sqrt{2}$, $m_d = y_d v_{H_d}/\sqrt{2}$ and $m_e = y_e v_{H_d}/\sqrt{2}$. This vev insertion is shown with a cross in figure 2.3. Inserting the terms with the active Higgs bosons gives fermion-Higgs vertices.

$$\begin{aligned} \mathcal{L}_{\bar{f}fH} = &\frac{y_u}{\sqrt{2}} \left(\bar{u}_L h_u^0 u_R + \bar{d}_L h_u^- u_R \right) + \frac{y_d}{\sqrt{2}} \left(\bar{u}_L h_d^+ d_R + \bar{d}_L h_d^0 d_R \right) + \\ &\frac{y_e}{\sqrt{2}} \left(\bar{\nu}_L h_d^+ e_R + \bar{e}_L h_d^0 e_R \right) + \text{h.c.} \end{aligned} \tag{2.6}$$

[1] Alternative conventions can be found in the literature, where $y_{u,d,e}$ are given by the coefficients of the Hermitian conjugate of the main terms given in equation (2.3)

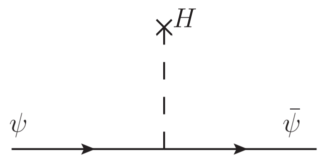

Figure 2.3: *A propagating fermion that gets a mass insertion by interacting with the Higgs field*

The absence of righthanded neutrinos in the spectrum of the Standard Model explains why there is no neutrino mass term in equations (2.2) and (2.3). Even if the righthanded neutrino has only trivial quantum numbers under the Standard Model gauge group, it would be needed to complete the fermion flow. In its absence, no coupling between the lefthanded neutrino in the lepton doublet and the (up-type) Higgs field can be constructed. At the time when the Standard Model was constructed, the neutrino was indeed thought to be massless. Its mass was only inferred much later by the observation of neutrino oscillations. Now we know that neutrinos have mass, we know that the content of this section cannot be the whole story.

In the next subsection we describe the way to extend the Standard Model with supersymmetry. In the six subsections that follow, we study the different possibilities to include neutrino masses in the Standard Model.

2.1.2 Fermion masses in supersymmetry

The Lagrangian that gives rise to fermion masses (2.3) contains elementary fermions and scalars (the Higgs fields). As mentioned in section 1.3.2, supersymmetry gives a boson for every fermion in the theory and vice versa. Those two states together (as well as one extra auxiliary field) form a supermultiplet or superfield. These are the building blocks of supersymmetric Lagrangians. In particular the superpotential $\mathcal{W}$ is relevant here. The superpotential is a holomorphic function of the superfields of the theory, meaning that it can contain the superfields, up to three of them, but not their Hermitian conjugates.

In the standard supersymmetry literature, it is customary not to reproduce the exact terms in (2.3), but terms that are basically its Hermitian conjugate, but then with $H^\dagger$ redefined to H, such that the Standard Model Higgs has negative hypercharge. In this case, the mass term for the up quark reads $y_u \bar{u}_R H_u \cdot Q_L$ and we need three superfields. Going from right to left, the first is a superfield that contains the quark doublet as fermionic component. Secondly there is a superfield with a Higgs doublet with hypercharge $+1/2$ as scalar component. The holomorphicity of the superpotential now explains the remark below equation (2.3). In the Standard Model, this Higgs field might be related to the Higgs field of the second term H_d via $H_u = i\sigma_2 H_d^*$, but in supersymmetry this is forbidden as it would render the superpotential non-holomorphic.

The third superfield is more problematic. It is the superfield that should give rise to $\bar{u}_R$. The bar implies complex conjugation, so having u_R as a fermionic component is not allowed. If instead we take its charge conjugate $(u_R)^c$ as an element, the corresponding supermultiplet does not need to be conjugated and is allowed in the superpotential. Due to the nature of charge conjugation, $(u_R)^c$ is itself a lefthanded field and can as such be written as $(u^c)_L$ – see for instance [13]. This has the extra advantage that all fermionic fields in the theory are now lefthanded, which allows them to be grouped together in grand unified multiplets. The best example is $SO(10)$ grand unification, where *all* Standard Model fermions are collected in a single 16-plet.

The generation of the superpotential terms for down quarks and electrons is similar to those for up quarks. In the minimal supersymmetric standard model (MSSM) the superpotential reads

$$\mathcal{W} = \mu \Phi_{H_u} \Phi_{H_d} - y_u \Phi_Q \Phi_{H_u} \Phi_{u^c} - y_d \Phi_Q \Phi_{H_d} \Phi_{d^c} - y_u \Phi_L \Phi_{H_d} \Phi_{e^c} \ . \tag{2.7}$$

We indicate supermultiplets with a capital Φ and a subscript that indicates the Standard Model component[2]. The first term in (2.7) gives rise to part of the Higgs potential. The other three terms reproduce the known fermion mass terms. Soft supersymmetry breaking terms are supposed to give additional contributions to the sfermion mass terms. These are terms that do not respect supersymmetry, but are added to the theory by hand to explain non-observation of sparticles so far. Note furthermore that in principle additional terms are possible in the superpotential (2.7). These are terms such as $\Phi_{u^c}\Phi_{d^c}\Phi_{d^c}$ or $\Phi_Q\Phi_{d^c}\Phi_L$ that violate respectively baryon number and lepton number and together can give rise to proton decay. They are absent however, if R-parity is imposed as an exact multiplicative symmetry.

Standard Model particles can be assigned R-parity $+1$, while sparticles (squarks, sleptons and Higgsinos) have -1. R-parity can be expressed in the spin, baryon and lepton quantum numbers. The lepton doublet has lepton number $+1$, while the anti-electron has -1; baryons, being made up of three quarks have baryon number $+1$, giving the individual quarks $+1/3$, while anti quarks have $-1/3$. R-parity is then defined as

$$R_p = (-1)^{3(B-L)+2s} \ . \tag{2.8}$$

A single subsection can never do justice to the rich phenomenology of supersymmetry and the MSSM. See for instance [14] for a more complete picture.

2.1.3 Dirac neutrinos

The most straightforward way to include neutrino masses is to allow the existence of righthanded neutrinos. Even if they are not observed themselves, their existence is motivated by the fact that they now allow the neutrinos that we do know to get a mass. This mass is of the same type as for the quarks and charged leptons and arises from the Yukawa interactions

$$\mathcal{L}_{\nu_D\text{-mass}} = y_\nu \begin{pmatrix} \bar{L}_L^\nu & \bar{L}_L^e \end{pmatrix} \begin{pmatrix} H_u^0 \\ H_u^- \end{pmatrix} \nu_R + \text{h.c.} \tag{2.9}$$

If neutrinos get a mass term according to this mechanism, they are called Dirac neutrinos. Dirac particles are not identical to their antiparticles, for which all charges are reversed. We see that the righthanded neutrino can *a priori* be a non-Dirac (or Majorana) particle as it does not seem to have any charges.

The righthanded neutrino might have a different type of charge than the ones mentioned in table 2.2 though. A candidate charge is lepton number that was introduced above. The Standard Model seems to respect lepton number (and baryon number as well) as accidental symmetries, but we might promote it to a symmetry that we demand to be explicitly conserved. Indeed equation (2.9) respects lepton number as well, as opposed to the alternatives we will see in the next sections[3].

Just like the other fermions, below the EWSB scale, neutrinos get an effective mass as in equation 2.2 and righthanded and lefthanded components have the same mass, given by $y_\nu v_{H_u}/\sqrt{2}$. The dimensionless parameters y_ν have very small values: 10^{-12} to 10^{-15} depending on the exact neutrino masses. According to the logic of section 1.3, one might wonder whether there is a reason for this 'unnaturally small' value.

In this scenario, the universe is filled with extra light degrees of freedom from the otherwise unobservable righthanded neutrinos. If precision cosmological observations might measure these, this will credit the scenario. If there are experiments that observe lepton number violation, for instance in neutrinoless double beta decay, the scenario is discredited.

[2] As all multiplets contain lefthanded fermions, the subscript L can be dropped to prevent cluttered notation.

[3] Actually, there is a rare, non-perturbative process in the Standard Model, called sphaleron interactions [15, 16]. In these interactions nine quarks can be converted to three antileptons and both baryon number and lepton number are violated. The difference $B-L$ is still conserved and this is thus a better candidate for an exact symmetry than L itself. Assigning a lepton number to the righthanded neutrino automatically also gives it a $B-L$ charge.

2.1.4 Majorana neutrinos

Fermions that are their own anti particles are called Majorana fermions. The quarks and charged leptons of the Standard Model clearly are not Majorana particles, as they have a charge that is the opposite for the anti particles. Below the electroweak scale, (lefthanded) neutrinos are singlets under the residual Standard Model group, so they are indeed a candidate to be Majorana particles. For Majorana fermions, a second type of mass term is allowed[4].

$$\mathcal{L}_{\nu_M\text{-mass}} = \frac{1}{2} m_\nu \bar{\nu}_L (\nu_L)^c + \text{h.c.} \tag{2.10}$$

Because the charge conjugate of the lefthanded neutrino is itself righthanded, the explicit addition of a righthanded neutrino is not needed. Above the EWSB scale, lefthanded neutrinos are part of the lepton doublet, that is in a non-trivial representation of the electroweak group and can therefore not be a Majorana spinor. In the remainder of this section, we give four mechanisms that reproduce equation (2.10) below the electroweak scale. One of these uses an effective dimension-5 operator; the other three are versions of the so-called seesaw mechanism.

2.1.5 The Weinberg operator

The fields $\bar{\nu}_L$ and $(\nu_L)^c$ that appear in equation (2.10) are singlets of the residual gauge group after electroweak symmetry breaking. Above this scale, we can form Standard Model singlets from their counterparts $\bar{L}_L$ and $(L_L)^c$ by multiplying these by H_u. The so-called Weinberg operator can now provide an effective Majorana mass for neutrinos.

$$\mathcal{L}_{\nu_M\text{-eff}} = \frac{f_\nu}{M_X} \left[\begin{pmatrix} \bar{L}_L^\nu & \bar{L}_L^e \end{pmatrix} \begin{pmatrix} H_u^0 \\ H_u^- \end{pmatrix} \right] \left[\begin{pmatrix} H_u^0 & H_u^- \end{pmatrix} \begin{pmatrix} (L_L^\nu)^c \\ (L_L^e)^c \end{pmatrix} \right] . \tag{2.11}$$

Here, M_X is a – presumably large – mass scale that appears because of the fact that this operator is non-renormalizable. After the Higgs field gets its vev, a neutrino mass is generated.

$$m_\nu = \frac{f_\nu}{2} \frac{(v_{H_u})^2}{M_X} . \tag{2.12}$$

Typically, M_X is much larger than the Higgs vev. In many models it is as high as the Grand Unified scale of section 1.3.3. This implies that neutrino masses are much below the electroweak scale for 'natural' values of the dimensionless parameter f_ν. This might explain why the neutrinos are much lighter than the quarks and charged leptons as shown in figure 1.16.

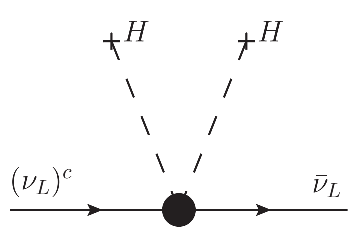

Figure 2.4: *The effective dimension 5 operator to generate a Majorana neutrino mass.*

The Weinberg operator is schematically given in figure 2.4. The 'blob' symbolizes the unknown physics behind the dimension-5 coupling. There are two ways to dissolve the blob using only 'normal' dimension-3 and -4 operators. These are given in figure 2.5.

[4]Some authors choose to define m_ν via the Hermitian conjugate of the main term in (2.10), i.e. $m_\nu \leftrightarrow m_\nu{}^*$. As the phase of m_ν is not observable, this is not a problem; all observables are the same in both conventions.

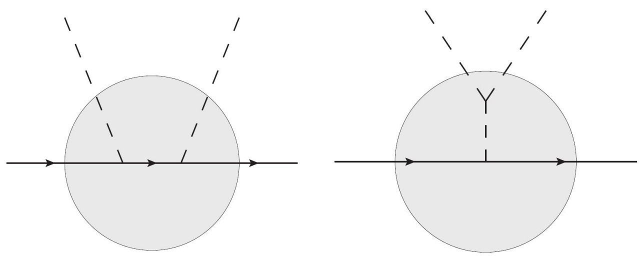

Figure 2.5: *The two ways to dissolve the effective Weinberg operator of figure 2.4*

Figure 2.5 contains intermediate particles: a fermion in the figure on the left and a boson in the figure on the right. These particles are assumed to be very heavy. In fact, the heavier the new particle is, the lighter is the lefthanded neutrino, just as on a seesaw in a children's playground: the higher one kid, the lower the other.

In all of the vertices of figure 2.5 two $SU(2)$ doublets meet. According to the group theory rule $2 \times 2 = 1 + 3$, the intermediate fermion or boson should thus be a singlet or a triplet. This can be used to classify the different seesaw mechanisms. An $SU(2)$-singlet fermion gives rise to the so-called type-I seesaw; an $SU(2)$-triplet boson to the seesaw of type-II and an $SU(2)$-triplet fermion to the type-III seesaw. Having an intermediate $SU(2)$-singlet boson is no option as can be seen from the right figure in 2.5. This would basically 'add nothing' to the fermion flow.

2.1.6 Type-I Seesaw

We first study the type-I seesaw, in which an intermediate $SU(2)$-singlet fermion appears in the diagram on the left of figure 2.5. The hypercharge of the field is calculated to be 0, giving it exactly the quantum numbers of the righthanded neutrino. The couplings between the lefthanded neutrino, the Higgs field and the righthanded neutrino are thus simply the Yukawa couplings of equation (2.9).

In section 2.1.3 we noticed that the righthanded neutrino might well be a Majorana particle, unless new exactly conserved charges like lepton number forbid this. If the righthanded neutrino is indeed a Majorana particle, a Majorana mass term analogous to equation (2.10) is also allowed. The mass might be very large as it does not have to be generated at the electroweak scale.

The (lefthanded) neutrino mass can be estimated from the diagram in figure 2.6. The two Yukawa interactions give a factor $\frac{1}{2}(y_\nu v_{H_2})^2$, while the propagator gives a factor $i/(\not{p} - M_M)$, that for low momenta can be approximated by $(-i)/M_M$, with M_M the righthanded neutrino Majorana mass.

$$m_\nu = \frac{{y_\nu}^2}{2} \frac{(v_{H_u})^2}{M_M} \; . \tag{2.13}$$

This is exactly of the form (2.12) if the high energy scale M_X and the Majorana mass scale M_M are related according to $M_X/f_\nu = M_M/y_\nu^2$.

The light neutrino mass of equation (2.13) can also be obtained from a more formal analysis. The total neutrino mass Lagrangian reads

$$\mathcal{L}_{\text{type-I}} = y_\nu \begin{pmatrix} \bar{L}_L^\nu \\ \bar{L}_L^e \end{pmatrix} \cdot \begin{pmatrix} H_u^0 \\ H_u^- \end{pmatrix} \nu_R + \frac{1}{2} M_M \overline{\nu_R} (\nu_R)^c + \text{h.c.} \tag{2.14}$$

After the Higgs fields obtain their vevs, this becomes

$$\begin{aligned} \mathcal{L}_{\mathcal{D}+\mathcal{M}} &= \frac{1}{2} m_D \overline{\nu_L} \nu_R + \frac{1}{2} m_D \overline{(\nu_R)^c} (\nu_L)^c + \frac{1}{2} M_M \overline{(\nu_R)^c} \nu_R + \text{h.c.} \\ &= \frac{1}{2} \begin{pmatrix} \overline{\nu_L} & \overline{(\nu_R)^c} \end{pmatrix} \begin{pmatrix} 0 & m_D \\ m_D & M_M \end{pmatrix} \begin{pmatrix} (\nu_L)^c \\ \nu_R \end{pmatrix} + \text{h.c.} \end{aligned} \tag{2.15}$$

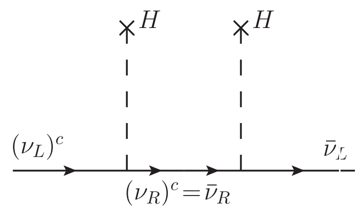

Figure 2.6: *The type-I seesaw.*

Here we used the spinor identity $\overline{\nu_L}\nu_R = \overline{(\nu_R)^c}(\nu_L)^c$. In the second line, we switched to a matrix notation. Finding the masses of the light and heavy neutrinos corresponds to finding the eigenvalues of this matrix.

$$m_{N,\,\nu} = \frac{M_M \pm \sqrt{1 + 4\frac{(m_D)^2}{M_M}}}{2} \simeq \begin{cases} M_M, & \text{For the heavy state } N. \\ -\frac{(m_D)^2}{M_M}, & \text{For the light state } \nu. \end{cases} \tag{2.16}$$

The last approximation is valid under the assumption that $M_M \gg m_D$ for reasons that are given above. The eigenstates ν and N of the mass matrix in (2.15) can be given in terms of the original states $(\nu_L)^c$ and ν_R

$$\begin{array}{cccccc} \nu & \propto & & (\nu_L)^c & + \frac{m_D}{M_M} & \nu_R \ , \\ N & \propto & \frac{m_D}{M_M} & (\nu_L)^c & + & \nu_R \ . \end{array} \tag{2.17}$$

We see that the light neutrino is almost entirely the (conjugate of) the old lefthanded neutrino, i.e. the neutrino that was part of the lepton doublet.

2.1.7 Type-II Seesaw

Instead of fermionic, the messenger can also be bosonic. In that case, the two Higgses first 'fuse' to the new boson; this boson then couples to the fermion flow. The two Higgses are doublets of $SU(2)_L$, so the new boson can a priori be a singlet or a triplet and the hypercharge should be +1. Only a triplet can generate a neutrino-neutrino coupling. This mechanism is known as the type-II seesaw and depicted in figure 2.7.

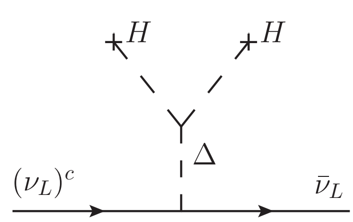

Figure 2.7: *The type-II seesaw.*

The bosonic triplet can be written as $\Delta = \left(\Delta^{++}, \quad \Delta^{+}, \quad \Delta^{0}\right)^T$. It gives rise to a mass term when the third (electrically neutral) component gets a vacuum expectation value v_Δ

$$\mathcal{L}_{II} = g_\nu \, \bar{\nu}_L \, (\nu_L)^c \, v_\Delta \ . \tag{2.18}$$

Neutrino masses are very small if the vev of Δ is very small. This is indeed plausible as can be seen from the combined potential of the doublet and triplet Higgs fields. We show the analysis for the case of the Standard Model with one doublet Higgs (so $H_u = i\sigma_2 H_d^*$ in equation (2.3)); the extension to multiple Higgs doublet models is straightforward.

$$V = V(H) + {M_T}^2 \Delta^\dagger \Delta + (\alpha H H \Delta + \alpha^* H^\dagger H^\dagger \Delta^\dagger) + \ldots \tag{2.19}$$

The first term is the normal Higgs potential $V(H) = \mu^2 H^\dagger H + \lambda (H^\dagger H)^2$; the next term is a mass term for the triplet Higgs and the last term is a cubic interaction between the doublet and triplet Higgses. Note that α is here a dimensionful parameter. The ellipsis contains quartic interaction terms with the triplet Higgs, like $H^\dagger H \Delta^\dagger \Delta$ or $(\Delta^\dagger \Delta)^2$, that are not relevant here, as their contribution is strongly suppressed with respect to those given in (2.19). If the doublet Higgs is sufficiently lighter than the triplet Higgs, the doublet obtains its vev $v_H = -\mu^2/2\lambda$ in the ordinary way. In terms of this vev and the one of the triplet, the potential now reads

$$V = V_0 + {M_T}^2 v_\Delta v_\Delta^* + \alpha\, {v_H}^2 v_\Delta + \alpha^* (v_H^*)^2 {v_\Delta}^* \ . \tag{2.20}$$

Demanding the first derivative with respect to v_Δ to be zero gives

$$\frac{\partial V}{\partial v_\Delta} = 0 \Rightarrow v_\Delta^* = (-)\alpha \frac{{v_H}^2}{{M_T}^2} \ . \tag{2.21}$$

This equation justifies the use of the word seesaw. The higher the scale of the triplet Higgs (or rather the presumably comparable scale of α and M_T), the lower the scale of v_Δ and hence the lighter the neutrinos. Indeed in many Grand Unified Theories, there is the relation

$$\alpha \simeq M_T \simeq M_R \ . \tag{2.22}$$

In that case, both the type-I and II seesaw predict neutrino masses of order ${v_H}^2/M_R$ and an analysis of neutrino masses should take into account both types of seesaw. We will see an example in the model of chapter 4.

2.1.8 Type-III Seesaw

Lastly, the intermediate particle can be a fermionic triplet that flows in the same channel as the fermionic singlet of the type-I seesaw. This type-III seesaw is sketched in figure 2.8.

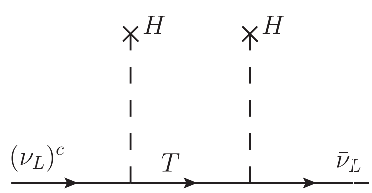

Figure 2.8: *The type-III seesaw. T denotes the fermionic triplet messenger.*

The seesaws of type-I, II and III are the only possibilities to generate neutrino masses with only renormalisable operators and only one intermediate messenger. Many suggestions exist in the literature of mechanisms that need more than one intermediate particle. They are known as the double seesaw, inverse seesaw, etc. See for instance [17] for a detailed discussion.

2.2 The three family Standard Model

In the previous section, we ignored the fact that there are three generations of quarks and leptons. In this section, we correct for this. We discuss the quarks sector first, then the lepton sector. The inclusion of three families basically amounts to adding generation labels to the fields in the mass Lagrangian (2.3): $Q_L \to Q_{Li}$; $u_R \to u_{Rj}$ and $d_R \to d_{Rj}$. Obviously, $i, j = 1, 2, 3$. To prepare the supersymmetric model of chapter 4 (where H_u and H_d are unrelated) and the models of chapters 5 (where there are three copies of H_d in the triplet representation of a family symmetry group), we also allow the Higgs fields to come in several copies: $H_d \to H_a^d$ and $H_u \to H_b^u$, where a and b run from 1 to respectively n_d and n_u. We denote their vevs as $v_a^d e^{i\omega_a}$ and $v_b^u e^{i\omega_b}$, where the phases indicate that the vevs can be complex. With three families, the old coupling constants y_u and y_d now become matrices in generation space and in case of multiple Higgs fields even a vector of matrices.

2.2.1 Quark masses

The quark part of equation (2.3) becomes[5]

$$\mathcal{L}_{\text{Q mass}} = Y_{ijb}^u \bar{Q}_{Li} H_b^u u_{Rj} + Y_{ija}^d \bar{Q}_{Li} H_a^d d_{Rj} + \text{h.c.} \tag{2.23}$$

The crucial observation is that the fields here are given in the interaction basis and they do not correspond to mass eigenstates. If all Higgs fields obtain their vacuum expectation values, the Lagrangian contains mass terms and fermion-Higgs interactions.

$$\mathcal{L}_{\text{Q mass}} = (M^u)_{ij} \bar{u}_{Li} u_{Rj} + (M^d)_{ij} \bar{d}_{Li} d_{Rj} + \text{h.c.} \tag{2.24}$$

$$\begin{aligned} \mathcal{L}_{\bar{f}fH} = & \frac{Y_{ijb}^u}{\sqrt{2}} \left(\bar{u}_{Li}\, h_b^{u0}\, u_{Rj} + \bar{d}_{Li}\, h_b^{u-}\, u_{Rj} \right) + \\ & \frac{Y_{ija}^d}{\sqrt{2}} \left(\bar{u}_{Li}\, h_a^{d+}\, d_{Rj} + \bar{d}_{Li}\, h_a^{d0}\, d_{Rj} \right) + \text{h.c.} \end{aligned} \tag{2.25}$$

The mass matrices in the first Lagrangian are given by the expressions below. It is important that these are not diagonal in flavour space.

$$(M^u)_{ij} = \sum_b Y_{ijb}^u \frac{v_b^u e^{i\omega_b}}{\sqrt{2}} , \qquad (M^d)_{ij} = \sum_a Y_{ija}^d \frac{v_a^d e^{i\omega_a}}{\sqrt{2}} . \tag{2.26}$$

The same holds for the mass matrices of the Higgses. We have already seen this in the section about the one-family Standard Model, where for instance the neutral Goldstone boson can be a mixture of A_u and A_d.

A basis transformation related the weak interaction basis to a basis where the mass matrices are diagonal. To distinguish the mass basis, we put a hat on relevant fields and operators and use the letters $r, s, \ldots$ for the fermion family indices and $\alpha, \beta, \ldots$ for the Higgs copy indices. We focus on the mass terms first, leaving the diagonalization of the Higgs mass matrices to section 2.2.5.

The fermion fields in the mass basis are defined as

$$\begin{aligned} u_{Ri} &= V_{Rir}^u \hat{u}_{Rr}, \quad u_{Li} = V_{Lir}^u \hat{u}_{Lr} , \\ d_{Ri} &= V_{Rir}^d \hat{d}_{Rr}, \quad d_{Li} = V_{Lir}^d \hat{d}_{Lr} . \end{aligned} \tag{2.27}$$

Here $V_{L,R}^{u,d}$ are unitary matrices such that the mass matrices in the mass basis are diagonal

$$\begin{aligned} \hat{M}_{rs}^u &= (V_L^u)^\dagger{}_{ri}\, M_{ij}^u\, V_{RLjs}^u = \text{diag}(m_u, m_c, m_t) . \\ \hat{M}_{rs}^d &= (V_L^d)^\dagger{}_{ri}\, M_{ij}^d\, V_{Rjs}^d = \text{diag}(m_d, m_s, m_b) . \end{aligned} \tag{2.28}$$

For practical purposes, V_L^u (V_R^u) can be calculated as the matrix that has the normalized eigenvectors of $M^u M^{u\dagger}$ ($M^{u\dagger} M^u$) in its columns and idem in the down sector.

[5] Again, some authors choose to define the Yukawa couplings by the Hermitian conjugates of the terms in (2.23). Some formulas, such as those directly below equation (2.28) change, but all observables are the same.

The values on the diagonal of $\hat{M}^u$ and $\hat{M}^d$ are the quark masses. Experimentally, these are given by [6]

$$\begin{aligned} m_u &= 1.7 - 3.3\text{MeV}, & m_c &= 1.27^{+0.07}_{-0.09}\text{GeV}, & m_t &= 172.0 \pm 0.9 \pm 0.3\text{GeV}, \\ m_d &= 4.1 - 5.8\text{MeV}, & m_s &= 101^{+29}_{-21}\text{MeV}, & m_b &= 4.19^{+0.18}_{-0.06}\text{GeV}. \end{aligned} \tag{2.29}$$

The large uncertainties in the light (u, d, s) quarks is due to the fact that quarks only exist in hadrons and that most of the mass of a hadron is not in the constituent quarks, but due to QCD effects. It was mentioned in section 1.4.1 that quark masses vary with the energy they are observed at. The masses in (2.29) are evaluated at 2 GeV using the $\overline{\text{MS}}$ scheme for the u, d and s quark; the c and b mass are the running masses at the mass scale itself, again using the $\overline{\text{MS}}$ scheme and the top mass is from direct observations of top events.

2.2.2 The CKM matrix

The transformation to the mass basis implies that the weak interaction with the W boson is no longer diagonal. The coupling of the quarks in their mass basis to the W boson is governed by the famous Cabibbo-Kobayashi-Maskawa (CKM) matrix.

$$\begin{aligned} \mathcal{L}_{CC} =& \bar{u}_{Li}\gamma^\mu d_{Li} W^+_\mu + \text{h.c.} \\ =& \bar{\hat{u}}_{Lr}\gamma^\mu (V_{\text{CKM}})_{rs}\hat{d}_{Ls} W^+_\mu + \text{h.c.} \qquad V_{\text{CKM}} = (V^u_L)^\dagger V^d_L. \end{aligned} \tag{2.30}$$

The CKM matrix is the product of two unitary matrices, $(V^u_L)^\dagger$ and V^d_L and is as such unitary itself. A general 3×3 unitary matrix has nine real parameters. However, not all of these are observable, as some phases can be absorbed in the quark fields. All quarks can absorb a phase, except for one global phase. This removes five phases, leaving the CKM matrix with four real parameters. Three of these, θ^q_{12}, θ^q_{13} and θ^q_{23}, are mixing angles that control the mixing between the particles of two of the three generations and one, δ^q_{CP}, is a complex phase that gives rise to CP violation. It was this counting and the realization that CP violation in the quark sector can only occur in case of at least three generations that earned Makoto Kobayashi and Toshihide Maskawa [18] the Nobel Prize of 2008.

In this section, we discuss two parametrizations of the CKM matrix. The first one is in terms of the aforementioned three angles and one phase, the second in terms of the so-called Wolfenstein parameters. The Wolfenstein parametrization takes into account that the CKM matrix elements respect a hierarchy in which some of the terms are much larger than others.

In the standard parametrization [19], the CKM matrix is expressed as

$$\begin{aligned} V_{\text{CKM}} &= \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \cdot \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta^q_{\text{CP}}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta^q_{\text{CP}}} & 0 & c_{13} \end{pmatrix} \cdot \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta^q_{\text{CP}}} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta^q_{\text{CP}}} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta^q_{\text{CP}}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta^q_{\text{CP}}} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta^q_{\text{CP}}} & c_{13}c_{23} \end{pmatrix}. \end{aligned} \tag{2.31}$$

Here s_{ij} and c_{ij} are respectively the sine and the cosine of the mixing angles θ^q_{ij}. These three angles can be recovered from (2.31) by the following expressions

$$\sin\theta^q_{13} = |(V_{\text{CKM}})_{13}|\,, \quad \tan\theta^q_{12} = \frac{|(V_{\text{CKM}})_{12}|}{|(V_{\text{CKM}})_{11}|}\,, \quad \tan\theta^q_{23} = \frac{|(V_{\text{CKM}})_{23}|}{|(V_{\text{CKM}})_{33}|}\,. \tag{2.32}$$

The CP violating phase δ^q_{CP} can be recovered from the argument of the (1 3)-element of the CKM matrix. In practical calculations however, it is not always directly possible to eliminate the phases as described above and arrive at the parametrization (2.31). In that case, $\delta^q_{\rm CP}$ can be calculated via

$$\delta^q_{\rm CP} = -\arg\left(\frac{\frac{(V_{\rm CKM})^*_{11}(V_{\rm CKM})_{13}(V_{\rm CKM})_{31}(V_{\rm CKM})^*_{33}}{c_{12}c^2_{13}s_{13}c_{23}} + c_{12}s_{13}c_{23}}{s_{12}s_{23}}\right) . \tag{2.33}$$

The mixing angles and the CP-violating phase are input to calculate the Jarlskog invariant of CP-violation [20]

$$J_{\rm CP} = {\rm Im}\Big[V_{ij}V_{ji}V^*_{ii}V^*_{jj}\Big] = c_{12}\ c^2_{13}\ c_{23}\ s_{12}\ s_{13}\ s_{23}\ \sin\delta^q_{\rm CP}\ . \tag{2.34}$$

In the second term ij = 12, 13 or 23 and no summation is assumed.

The experimental determinations of the values of the absolute values of the elements of the CKM matrix are [6]

$$|V_{\rm CKM}| = \begin{pmatrix} 0.9742 \pm 0.00015 & 0.2253 \pm 0.0007 & 0.00347^{+0.00016}_{-0.00012} \\ 0.2252 \pm 0.0007 & 0.97345^{+0.00015}_{-0.00016} & 0.0410^{+0.0011}_{-0.0007} \\ 0.00862^{+0.00026}_{-0.00020} & 0.0403^{+0.0011}_{-0.0007} & 0.999152^{+0.000030}_{-0.000045} \end{pmatrix} . \tag{2.35}$$

This is calculated under the assumption that the CKM matrix is unitary (this is theoretically well motivated as described above) and that there are no more than three generations. The direct constraint on for instance the (3 3)-element is much weaker: $|V_{tb}| = 0.88 \pm 0.07$.

The angles that correspond to the data in (2.35) and the CP phase that can be calculated separately are

$$\theta^q_{12} = 13.0^\circ, \quad \theta^q_{13} = 0.199^\circ, \quad \theta^q_{23} = 2.35^\circ, \quad \delta^q_{\rm CP} = 68.9^\circ. \tag{2.36}$$

These data support the claim made in section 1.4.2: the CKM matrix is almost diagonal. Although the first mass eigenstate is not exactly the first interaction eigenstate, the difference is not large, etc. Only the mixing between the first and the second mass eigenstates is of medium order, with the mixing parameter θ^q_{12} equal to 13.0°, while the other two mixing angles are tiny. This explains that there is only a small sliver of red respectively blue in the first two circles of figure 1.17 and almost no yellow.

The standard parametrization uses θ^q_{12}, θ^q_{13}, θ^q_{23} and $\delta^q_{\rm CP}$ to parameterize the CKM-matrix. Some of these parameters are very small. The Wolfenstein parametrization [21] is an alternative parametrization, in which all parameters are between 0.1 and 1. The first parameter λ is $\sin\theta^q_{12} \approx 0.23$. This is slightly smaller than 1 and allows a power expansion of the CKM matrix in powers of λ. To good accuracy the (1 2)-element of the CKM matrix is equal to λ, since $\cos\theta^q_{13} \approx 1$. The (2 3)-element is more or less equal to λ^2; we define the deviation A via $\sin\theta^q_{23} = A\lambda^2$. The (1 3)-element of the CKM matrix is of order λ^3. The real and imaginary part ρ and η of the coefficient are defined as $\sin\theta^q_{13}\ e^{i\delta^q_{\rm CP}} = A\lambda^3(\rho + i\eta)$. This gives the CKM matrix to third order in λ

$$V_{\rm CKM} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\bar\rho - i\bar\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4). \tag{2.37}$$

The Wolfenstein parameters are given by[6]

$$\lambda = 0.2246 \pm 0.0011, \quad A = 0.832 \pm 0.017, \quad \bar\rho = 0.130 \pm 0.018, \quad \bar\eta = 0.341 \pm 0.013. \tag{2.38}$$

2.2.3 Lepton masses

In sections 2.1.3 and 2.1.4 the Dirac or Majorana nature of neutrinos was discussed. If neutrinos are Dirac particles, the theory of lepton masses in three generations is an exact copy of the theory

[6]For calculations to higher order in λ two parameters $\bar\rho$ and $\bar\eta$ are preferred over ρ and η; these are defined as

$$\sin\theta^q_{13}\ e^{i\delta^q_{\rm CP}} = A\lambda^3(\rho + i\eta) = \frac{A\lambda^3(\bar\rho + i\bar\eta)\sqrt{1-A^2\lambda^4}}{\sqrt{1-\lambda^2}[1 - A^2\lambda^4(\bar\rho + i\bar\eta)]}\ .$$

of quarks. The lepton analogue to the CKM matrix, often called the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, can be parameterized in the same way as the CKM matrix in (2.31). There are small, but important differences if neutrinos are Majorana particles as we assume in the remainder of this section and in fact in most of this thesis.

The lepton masses below the electroweak scale are given by

$$\mathcal{L}_{\text{Lmass}} = (M^e)_{ij}\bar{e}_{Ri}e_{Lj} + (M^\nu)_{ij}\bar{\nu}_{Li}(\nu_L)^c_j + \text{h.c.} \tag{2.39}$$

The symmetric neutrino mass matrix $(M^\nu)_{ij}$ might originate from one or more of the seesaw mechanisms. In model building, the seesaws of type I and II or a combination of these are very popular. Indeed, in chapter 4 we show that in many grand unified models, both seesaws are automatically present and their contributions to the neutrino masses are comparable. In case of interplay of type I and II seesaw, the neutrino mass matrix reads

$$(M^\nu)_{ij} = -M^D_{iI}(M^R)^{-1}_{IJ}(M^D)^T_{Jj} + M^L_{ij}. \tag{2.40}$$

Here M^D is the Dirac mass matrix, given by $M^D_{iI} = \sum_a Y^\nu_{iIa}v^u_a/\sqrt{2}$; $i = 1,2,3$ is the index that counts the number of normal lefthanded neutrinos. I counts the number of right handed neutrino species. It need not run from 1 to 3, as there may be fewer or more righthanded neutrinos. M^L is the contribution of the type-II seesaw equal to $g^\nu_{ij}v_\Delta$.

The charged lepton masses are diagonalized as in equations (2.27) and (2.28)

$$e_{Ri} = V^e_{Rir}\hat{e}_{Rr}\ , \quad e_{Li} = V^e_{Lir}\hat{e}_{Lr}\ , \tag{2.41}$$

$$\hat{M}^e_{rs} = (V^e_L)^\dagger{}_{ri}M^e_{ij}V^e_{Rjs} = \text{diag}(m_e, m_\mu, m_\tau)\ . \tag{2.42}$$

The masses of the the electron, the muon and tau lepton can be measured much better than the quark masses, as they can be measured directly in detectors instead of only in hadrons. They are given by

$$\begin{aligned} m_e &= 0.510998910 \pm 0.000000013\text{MeV}\ ,\\ m_\mu &= 105.658367 \pm 0.000004\text{MeV}\ ,\\ m_\tau &= 1776.82 \pm 0.16\text{MeV}\ . \end{aligned} \tag{2.43}$$

The neutrinos are diagonalized via a single unitary matrix U^ν instead of two (left and right) as in the case for all Dirac particles.

$$\hat{M}^\nu_{rs} = (U^\nu)^\dagger_{ri}(M^\nu)_{ij}(U^\nu)^*_{js}\ , \tag{2.44}$$

$$\nu_L = (U_\nu)^*\hat{\nu}_L\ . \tag{2.45}$$

Neutrino masses are very hard to measure. Neutrinos rarely interact and they are almost always highly relativistic, meaning that only a tiny fraction of their energy is in the rest mass. The only direct signs for neutrino masses are from neutrino oscillations. These are not sensitive to the neutrino masses themselves, but to the differences between the squares of two of them. The results of solar neutrino oscillations (neutrinos from nuclear fusion in the center of the sun) and atmospheric neutrino oscillations (neutrinos formed when cosmic rays collide with air particles in the outer atmosphere) are given in table 2.3.

The three neutrino mass eigenstates are generically denoted as ν_1, ν_2 and ν_3. This ordering does *not* always correspond to the ordering from lightest to heaviest. The solar and atmospheric mass differences imply two gaps between the mass states, one much larger than the other. The states ν_1 and ν_2 are defined as the states separated by the solar mass gap, with ν_1 the lightest of the pair.

$$\Delta m^2_{\text{sol}} \equiv \Delta m^2_{12} \equiv m_2^2 - m_1^2\ . \tag{2.46}$$

The third neutrino ν_3 can be heavier or lighter than the solar pair; it is separated from them by the larger atmospheric gap. In the former case, the neutrino ordering is called normal, because just like with charged leptons and quarks, the gap between the third and second family is larger than the one between the first and the second one. If ν_3 is the lightest neutrino, the ordering is called inverted.

parameter	Ref. [22–24] best fit (1σ)	3σ-interval	Ref. [25–28] best fit (1σ)	3σ-interval	
$\Delta m^2_{\rm sol}$ [$\times 10^{-5}$eV^2]	$7.58^{+0.22}_{-0.26}$	$6.99 - 8.18$	$7.59^{+0.20}_{-0.18}$	$7.09 - 8.19$	
$\Delta m^2_{\rm atm}$ [$\times 10^{-3}$eV^2]	$2.35^{+0.12}_{-0.09}$	$2.06 - 2.81$	$2.50^{+0.09}_{-0.16}$	$2.14 - 2.76$	Normal hierarchy
			$2.40^{+0.08}_{-0.09}$	$2.13 - 2.67$	Inverted hierarchy

Table 2.3: *Neutrino oscillation parameters from two independent global fits [22–24] and [25–28].*

The definition of $\Delta m^2_{\rm atm}$ in terms of neutrino masses is different for the two orderings; it is a positive quantity given by the difference of the mass squared of ν_3 and of the solar doublet neutrino closest to it.

$$\Delta m^2_{\rm atm} = \begin{cases} m_3^2 - m_1^2, & \text{Normal ordering.} \\ m_2^2 - m_3^2, & \text{Inverted ordering.} \end{cases} \tag{2.47}$$

The neutrino oscillation parameters only contain information about the differences of (squares of) neutrino masses, not about the magnitude of the masses themselves. In one scenario, the lightest neutrino may be almost massless. The other masses are now given approximately by $\sqrt{\Delta m^2_{\rm sol}}$ and $\sqrt{\Delta m^2_{\rm atm}}$ (normal ordering) or both very close to $\sqrt{\Delta m^2_{\rm atm}}$ (inverted ordering). In this case, the words normal ordering and inverted ordering are often replaced by normal hierarchy and inverted hierarchy. In an other scenario, the neutrinos are relatively heavy and the differences between the neutrino masses is small compared to the masses themselves. This is called a quasi degenerate spectrum. The sum of quasi degenerate neutrino masses is constrained by cosmological data. Although the different groups do not agree on the bound [6], most report a value near 1.0 eV, requiring that the individual neutrinos have masses smaller than approximately 0.3 eV. Information on the absolute mass scale of neutrinos may also be found in the endpoint of tritium beta decay by the Katrin collaboration [29] or by one of the groups looking for neutrinoless double beta decay [30–34]. A schematic representation of the four possibilities for the neutrino hierarchy and ordering is given in figure 2.9.

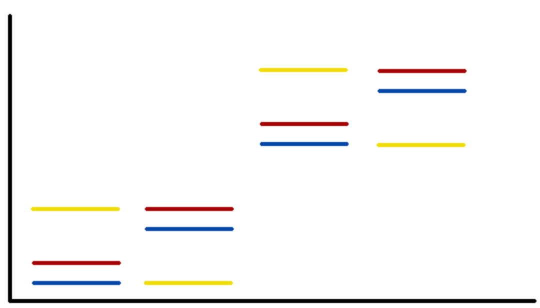

Figure 2.9: *Schematic representation of the normal hierarchy (NH), inverted hierarchy (IH) and quasi degenerate neutrinos with normal ordering (QD NO) and with inverted ordering (QD IO).*

2.2.4 The PMNS matrix

The Pontecorvo-Maki-Nakagawa-Sakata matrix parameterizes the flavour mixing in weak interactions of leptons. The PMNS matrix is defined as

$$V_{\rm PMNS} = (V^e_L)^\dagger V^\nu. \tag{2.48}$$

If neutrinos are Majorana particles, the PMNS matrix can be parameterized as

$$V_{\text{PMNS}} = \begin{pmatrix} V_{e1} & V_{e2} & V_{e3} \\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\ V_{\tau 1} & V_{\tau 2} & V_{\tau 3} \end{pmatrix} = R(\theta^l_{12}, \theta^l_{13}, \theta^l_{23}, \delta^l_{\text{CP}}) \times \begin{pmatrix} e^{i\varphi_1/2} & 0 & 0 \\ 0 & e^{i\varphi_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{2.49}$$

Here $R(\theta^l_{12}, \theta^l_{13}, \theta^l_{23}, \delta^l_{\text{CP}})$ is the part of the mixing matrix that depends on the mixing angles and the Dirac CP-violating phase. This is identical to the parametrization of the CKM matrix in (2.31). The matrix on the right contains extra CP-violating phases, the so-called Majorana phases.

The mixing angles θ^l_{ij} and Dirac phase δ^l_{CP} can be inferred from the PMNS matrix via formulas analogous to (2.32) and (2.33). The Majorana phases can be written in terms of an auxiliary angle $\delta_e = \arg[e^{i\delta^l_{\text{CP}}}(V_{\text{PMNS}})_{13}]$ as

$$\varphi_1 = 2 \arg[e^{i\delta_e}(V_{\text{PMNS}})^*_{11}] , \qquad \varphi_2 = 2 \arg[e^{i\delta_e}(V_{\text{PMNS}})^*_{12}] . \tag{2.50}$$

The angles of the PMNS matrix are totally different to those of the CKM matrix. In the quark sector all angles are small to very small. In the lepton sector, the angle θ^l_{23} – the atmospheric angle – is very large. Possibly it has exactly the right value (45°) to produce maximal mixing in the (2 3) sector. The solar angle or θ^l_{12} is also very large, but significantly smaller than maximal (45°). The third mixing angle, θ^l_{13} is sometimes called the reactor angle. It is much smaller than the other two lepton mixing angles. Until recently, experimental data were compatible with a vanishing value of the angle [22, 23, 25, 26]. For the newest results, this is (just) below the 3σ range [24, 27, 28]. At the other end of the range, the angle can be almost as large as the Cabibbo angle, the largest angle in the CKM matrix.

In table 2.4 we give the mixing angles according to the two global fits also used in table 2.3. In the last column, we mention the values according to the tribimaximal mixing pattern, first introduced by Harrison, Perkins and Scott in 2002 and alluded to in section 1.4.2 [8]; see also [35–40]. We see that this pattern indeed fits the data rather well, although the agreement was a lot stronger with older datapoints. The fit significantly deteriorated when evidence for non-zero θ_{13} was found, first as a slight hint in global fits to accommodate slightly conflicting data, later from a dedicated search at the Tokai to Kamioka (T2K) experiment [41].

	Ref. [22–24]		Ref. [25–28]			
parameter	best fit (1σ)	3σ-interval	best fit (1σ)	3σ-interval		TBM values
$\sin^2\theta^l_{12}$	$0.312^{+0.017}_{-0.016}$	$0.265 - 0.364$	$0.312^{+0.017}_{-0.015}$	$0.27 - 0.36$		$1/3$
$\sin^2\theta^l_{23}$	$0.42^{+0.08}_{-0.03}$	$0.34 - 0.64$	$0.52^{+0.06}_{-0.07}$	$0.39 - 0.64$	NH	$1/2$
			0.52 ± 0.06	$0.39 - 0.64$	IH	
$\sin^2\theta^l_{13}$	0.025 ± 0.007	$0.005 - 0.050$	$0.013^{+0.007}_{-0.005}$	$0.001 - 0.035$	NH	0
			$0.016^{+0.008}_{-0.006}$	$0.001 - 0.039$	IH	

Table 2.4: *Neutrino oscillation parameters from two independent global fits [22–24] and [25–28] and the values of the tribimaximal mixing pattern. In the second fit, separate results for normal hierarchy (NH) and inverted hierarchy (IH) are given.*

If the neutrino mixing matrix is exactly equal to the tribimaximal Harrison-Perkins-Scott matrix, the absolute values of its elements are given by

$$|U_{\text{TBM}}| = \begin{pmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ \sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \\ \sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \end{pmatrix} . \tag{2.51}$$

All three phases of the PMNS matrix are unknown at the moment. The phase δ^l_{CP} might be discovered together with the mixing angle θ_{13} in detectors near nuclear reactors [42–45]. The Majorana phases do not show up in oscillations, but might be inferred from nuclear decay processes. In particular, in neutrinoless double beta decay, a parameter $m_{\beta\beta}$ is probed that contains both phases.

$$\begin{aligned} m_{\beta\beta} &= \sum_{i=1-3} m_i (V_{\text{PMNS}})^2_{ei} \\ &= m_1 \cos^2\theta^l_{12}\cos^2\theta^l_{13} e^{i\varphi_1} + m_2\cos^2\theta^l_{13}\sin^2\theta^l_{12}e^{i\varphi_2} + m_3\sin^2\theta^l_{13}e^{-2i\delta} \, . \end{aligned} \tag{2.52}$$

2.2.5 Diagonalizing the Higgs sector

This section discusses the diagonalization of the Higgs sector, necessary if there is more than one Higgs boson present. We will see that if more than one Higgs boson is present, generally this gives rise to flavour changing neutral currents (FCNCs) that are experimentally very tightly constrained. In chapter 5 we use these FCNCs to test certain multi-Higgs models.

We recall from equation (2.4) that the Higgs fields can be expanded around the vacuum expectation value according to

$$H^d_a = \frac{1}{\sqrt{2}}\begin{pmatrix} h^{d+}_a \\ v_a e^{i\omega_a} + \text{Re}\, h^{d0}_a + i\,\text{Im}\, h^{d0}_a \end{pmatrix} \, , \quad H^u_b = \frac{1}{\sqrt{2}}\begin{pmatrix} v_b e^{i\omega_b} + \text{Re}\, h^{u0}_b + i\,\text{Im}\, h^{u0}_b \\ h^{u-}_a \end{pmatrix} \, . \tag{2.53}$$

Now the interaction eigenstates $h_{u,d}$ mix to mass eigenstates of the Higgses ("physical Higgses") and a number of Goldstone bosons. As argued below equation (2.3), the states in H^d should thereby be compared with those in $-i\sigma_2(H^u)^*$. Indeed in both cases, the charged states are positive.

In the neutral sector the mass eigenstates are formed according to

$$\hat{h}_\alpha = U_{\alpha a} h_a \, . \tag{2.54}$$

The vector h_a holds all neutral components of the original Higgs fields. Due to the complex conjugation alluded to above, Im h^{u0}_b carry minus signs

$$h_a = \left(\text{Re}\, h^{u0}_1 \, \cdots \, \text{Re}\, h^{u0}_{n_u}, \, -\text{Im}\, h^{u0}_1 \, \cdots \, -\text{Im}\, h^{u0}_{n_u}, \, \text{Re}\, h^{d0}_1 \, \cdots \, \text{Re}\, h^{d0}_{n_d}, \, \text{Im}\, h^{d0}_1 \, \cdots \, \text{Im}\, h^{d0}_{n_d}\right) . \tag{2.55}$$

These $2(n_u+n_d)$ states give rise to the states in $\hat{h}_\alpha$: $2(n_u+n_d)-1$ physical Higgses and the Goldstone boson that gives a mass to the Z-boson.

$$\hat{h}_\alpha = \left(\hat{h}_1 \, \cdots \, \hat{h}_{2(n_u+n_d)-1}, \, \pi^0\right) . \tag{2.56}$$

In the charged sector we have likewise

$$\hat{h}^+_\alpha = S_{\alpha a} h^+_a \, . \tag{2.57}$$

The vector h^+_a holds the positive states from H^d and from $-i\sigma_2(H^u)^*$, while the vector $\hat{h}^+_\alpha$ contains the $n_u + n_d - 1$ positively charged scalars and the Goldstone boson that gives rise to the W^+ boson mass.

$$\begin{aligned} h^+_a &= \left((H^u)^+_1 \, \cdots h^{u+}_{n_u}, \, h^{d+}_1 \, \cdots \, h^{d+}_{n_d}\right) , \\ \hat{h}^+_\alpha &= \left(\hat{h}^+_1 \, \cdots \, \hat{h}^+_{n_u+n_d-1}, \, \pi^+\right) . \end{aligned} \tag{2.58}$$

The expressions in (2.54) and (2.57) are needed to rewrite the fermion-Higgs interactions, equation (2.25) and its lepton analogue, in the mass basis. To keep the discussion clear and the formulas short, we only discuss the quark case in this section. The lepton case is completely analogous.

In the mass basis the part of the Lagrangian which includes interactions with the neutral Higgses – the first and fourth term of equation (2.25) – becomes

$$\mathcal{L}_{Y,n} = \bar{\hat{d}}_r (R^d)^\alpha_{rs} h_\alpha \frac{1+\gamma_5}{2} \hat{d}_s + \bar{\hat{u}}_r (R^u)^\alpha_{rs} h_\alpha \frac{1+\gamma_5}{2} \hat{u}_s + \text{h.c.} \tag{2.59}$$

Here we defined the coupling tensors R^d and R^u according to

$$\begin{aligned}(R^d)^\alpha_{rs} &= \left[V^{d\dagger}_{Lri} \frac{1}{\sqrt{2}} (U^{\dagger(2n_u+a)\alpha} + iU^{\dagger(2n_u+n_d+a)\alpha}) Y^d_{ija} V^d_{Rjs} \right] , \\ (R^u)^\alpha_{rs} &= \left[V^{u\dagger}_{Lri} \frac{1}{\sqrt{2}} (U^{\dagger a\alpha} - iU^{\dagger(n_u+a)\alpha}) Y^u_{ija} V^u_{Rjs} \right] .\end{aligned} \tag{2.60}$$

In case the up-type and down-type Higgses are not separate, we can still use this formula with some small modifications. From the perspective of the down-type Higgs, there are no other Higgses, so $\text{Re}h^d_1$ is the first element of h_a in (2.55), so in R^d we should take $n_u = 0$. On the other hand, in R^u we should use $n_u = n_d$, because $-\text{Im}h^{u0}_1$ is the $(n_d+1)^{\text{th}}$ element of h_a.

The interactions with the charged Higgs in (2.25) become

$$\mathcal{L}_{Y,ch} = \bar{\hat{u}}_r (T^d)^\alpha_{rs} \hat{h}^+_\alpha \frac{1+\gamma_5}{2} \hat{d}_s + \bar{\hat{d}}_r (T^u)^\alpha_{rs} \hat{h}^-_\alpha \frac{1+\gamma_5}{2} \hat{u}_s + \text{h.c.} \tag{2.61}$$

The coupling tensors T are given by the following expressions that can also be used when there are no separate up-type Higgses ($n_u = 0$).

$$(T^d)^\alpha_{rs} = \left[V^{d\dagger}_{Lri} S^{\dagger(n_u+a)\alpha} Y^d_{ija} V^d_{Rjs} \right] , \qquad (T^u)^\alpha_{rs} = \left[V^{u\dagger}_{Lri} S^{T a\alpha} Y^u_{ija} V^u_{Rjs} \right] . \tag{2.62}$$

Expanding the Hermitian conjugate, the Lagrangian (2.59) can be written in terms of scalar and pseudoscalar couplings with non-chiral fermions.

$$\begin{aligned}\mathcal{L}_Y =& \Big(\bar{\hat{d}}_r \big((I^d)^\alpha_{r,s} + \gamma_5 (J^d)^\alpha_{r,s} \big) \hat{h}_\alpha \hat{d}_s + \bar{\hat{u}}_r \big((I^u)^\alpha_{r,s} + \gamma_5 (J^u)^\alpha_{r,s} \big) \hat{h}_\alpha \hat{u}_s \\ &+ \bar{\hat{u}}_r \big(F^\beta_{r,s} + \gamma_5 G^\beta_{r,s} \big) \hat{h}^+_\beta \hat{d}_s + \bar{\hat{d}}_r \big(F^{\beta *}_{r,s} - \gamma_5 G^{\beta *}_{r,s} \big) \hat{h}^-_\beta \hat{u}_s \Big) .\end{aligned} \tag{2.63}$$

The new coefficients are defined in the following way:

$$\begin{aligned}(I^{d,u})^\alpha_{r,s} &= \frac{1}{2} \Big((R^{d,u})^\alpha_{rs} + ((R^{d,u})^\alpha_{sr})^* \Big) , \\ (J^{d,u})^\alpha_{r,s} &= \frac{1}{2} \Big((R^{d,u})^\alpha_{rs} - ((R^{d,u})^\alpha_{sr})^* \Big) , \\ F^\beta_{r,s} &= \frac{1}{2} \Big(((T^d)^\beta_{rs})^* - ((T^u)^\beta_{sr})^* \Big) , \\ G^\beta_{r,s} &= \frac{1}{2} \Big(((T^d)^\beta_{rs})^* + ((T^u)^\beta_{sr})^* \Big) .\end{aligned} \tag{2.64}$$

The operators I, J, F and G determine whether flavour changing interactions are possible and what their strength is. Note that for particularly symmetric vevs of the Higgs fields, many of these operators are automatically zero, thus forbidding many flavour changing interactions or allowing them only if certain selection rules are met. We discuss some flavour changing interactions in chapter 5. The matrices I and J are important building blocks for the expressions there.

2.3 Fermion masses in family symmetric models

In section 1.4 family symmetries were introduced. We counted the number of free parameters and found that 20 or 22 of the parameters correspond to the fermion masses and mixings in respectively the neutrino-Dirac and neutrino-Majorana case. We also mentioned that some dimensionless parameters are remarkably small. The Yukawa couplings of the first generation y_u, y_d and y_e are examples we have seen in this chapter. All of these are of the order 10^{-5}, with the electron Yukawa even smaller. If neutrinos are Dirac particles, their Yukawa couplings are even more tiny, being approximately 10^{-13}. We have seen that the type-I seesaw provided a 'solution' to this. New physics had to be introduced as the properties of the righthanded neutrino are now significantly different from those of the lefthanded one, but the parameters could all be of 'natural' magnitude. Even the new energy scale introduced was consistent with expectations.

The type-I seesaw provides a good example of some general philosophies that are also behind models with flavour symmetries. In section 1.4 we mentioned that there does not *need* to be something like flavour symmetries. In principle, the mass sector of the Standard Model can be the result of some God playing dice with 20 or 22 available parameters. From a theoretical point of view, however, alternative theories are preferable if they either manage to have much fewer parameters or if the parameters they have are less finetuned, meaning that the observed physics follows from a larger range of parameters. Ideally, a model does both.

The key ingredients of models with flavour symmetries are the matrices of Yukawa couplings Y^x_{ij}, $x \in \{u, d, e, \nu\}$ and comparable parameters in the seesaws. The models were already introduced in the text of section 1.4. In this section, we study in some more mathematical rigour how we can force the Yukawa and seesaw parameters to reproduce the physics we observe.

The best way to introduce family symmetries is probably by going through a remarkably simple, but efficient example, the Froggatt–Nielsen model [9], that we discuss in the next section. The Froggatt–Nielsen model uses an Abelian flavour symmetry. The models that the other chapters of this thesis are based on, use non-Abelian symmetries. Furthermore, these symmetries are often global and discrete. We discuss their properties at the end of this section and give an typical example, the model of Altarelli and Feruglio [10,11], in the next section.

2.3.1 The Froggatt–Nielsen model

A first family symmetric model that we discuss is the Froggatt–Nielsen model, or rather a toy version of it. To focus only on the core issues we restrict the discussion to the charged lepton sector and ignore family mixing, i.e. we assume that only couplings diagonal in flavour space occur. Even in this simplified set up, there is a striking feature in the data, namely the huge hierarchy between the masses of the particles under consideration. The tau lepton is much heavier than the muon, which is itself again much heavier than the electron. The Froggatt–Nielsen mechanism provides an explanation to this by assigning different charges to the three fields.

The Yukawa Lagrangian for our simplified model reads

$$\mathcal{L} = y_e \begin{pmatrix} \nu_e \\ e \end{pmatrix} \cdot \begin{pmatrix} H_d^+ \\ H_d^0 \end{pmatrix} e^c + y_\mu \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \cdot \begin{pmatrix} H_d^+ \\ H_d^0 \end{pmatrix} \mu^c + y_\tau \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \cdot \begin{pmatrix} H_d^+ \\ H_d^0 \end{pmatrix} \tau^c. \tag{2.65}$$

After the Higgs fields obtains it vev, this gives a diagonal mass matrix for the three leptons.

$$\mathcal{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} e & \mu & \tau \end{pmatrix} \begin{pmatrix} y_e v & & \\ & y_\mu v & \\ & & y_\tau v \end{pmatrix} \begin{pmatrix} e^c \\ \mu^c \\ \tau^c \end{pmatrix}. \tag{2.66}$$

The observation that the tau lepton, muon and electron have a huge hierarchy, now translates to $y_e \ll y_\mu \ll y_\tau$. Actual, a study of the data suggests

$$y_e : y_\mu : y_\tau \approx \lambda^5 : \lambda^2 : 1, \qquad \lambda = 0.2. \tag{2.67}$$

At this stage, λ is just a parameter with size 0.2. It appears because the data show that the logarithmic gaps between tau lepton and muon and between muon and electron are roughly 3:2 related. In chapter 4 the parameter λ is linked to the Cabibbo angle.

The central idea of Froggatt and Nielsen was that the huge mass gaps may be not a coincidence. They proposed a new $U(1)_{\rm FN}$ symmetry and to assign a new charge that differs over the three families as in table 2.5.

field	all doublets	e^c	μ^c	τ^c
FN-charge	0	5	2	0

Table 2.5: Frogatt and Nielsen's proposal for new charges for the charged leptons.

The Lagrangian (2.65) is now no longer a valid Lagrangian: no longer all terms are singlets of the full symmetry group. The first term for instance has $U(1)_{\rm FN}$-charge 0+0+5$\neq$ 0. This can be solved by introducing a new field, the so-called Froggatt–Nielsen messenger θ that has negative FN-charge as shown in table 2.6.

field	all doublets	e^c	μ^c	τ^c	θ
FN-charge	0	5	2	0	-1

Table 2.6: An update of table 2.5 that also includes the bosonic Froggatt–Nielsen messenger.

With the Froggatt–Nielsen messenger, it is possible to 'fix' the Lagrangian (2.65) by inserting appropriate powers of the messenger. This makes the first two terms in the Lagrangian non-renormalisable. We correct for this by dividing by appropriate powers of an assumedly high cut-off scale $M_{\rm FN}$. Like the three versions of the seesaw mechanism can dissolve the Weinberg operator, it is in principle possible to express the terms below in a combination of renormalisable operators, but here we stick to the effective operator approach.

$$\mathcal{L} = \frac{1}{M_{\rm FN}^5} y'_e \begin{pmatrix} \nu_e \\ e \end{pmatrix} \cdot \begin{pmatrix} H_d^+ \\ H_d^0 \end{pmatrix} \theta^5 e^c + \frac{1}{M_{\rm FN}^2} y'_\mu \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \cdot \begin{pmatrix} H_d^+ \\ H_d^0 \end{pmatrix} \theta^2 \mu^c + y'_\tau \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \cdot \begin{pmatrix} H_d^+ \\ H_d^0 \end{pmatrix} \tau^c \tag{2.68}$$

In the next step, the Froggatt–Nielsen-messenger θ acquires a vev just slightly below the cut-off scale $\langle\theta\rangle = \lambda\ M_{\rm FN}$, with $\lambda \approx 0.2$. For the exact mechanism of this symmetry breaking, there are several candidates. We present a supersymmetric case in which the $U(1)_{FN}$ is gauged such that θ gets its vev through a D-term. The corresponding potential is of the form

$$V_{D,\rm FN} = \frac{1}{2}(M_{\rm FI}^2 - g_{\rm FN}|\theta|^2 + ...)^2\ . \tag{2.69}$$

The gauge coupling constant of $U(1)_{\rm FN}$ is $g_{\rm FN}$ and $M_{\rm FI}^2$ denotes the contribution of the Fayet-Iliopoulos (FI) term. Dots in equation (2.69) represent e.g. terms involving the fields e^c and μ^c which are charged under $U(1)_{FN}$. These terms are however not relevant to calculate the VEV of the FN field and we omit them in the present discussion. $V_{D,FN}$ leads in the supersymmetric limit to

$$|\langle\theta\rangle|^2 = \frac{M_{\rm FI}^2}{g_{\rm FN}}\ . \tag{2.70}$$

This is identified as $\lambda^2 M_{\rm FN}^2$.

When both the Higgs and the FN-messenger have vacuum expectation values, the mass matrix is diagonal again.

$$\mathcal{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} e & \mu & \tau \end{pmatrix} \begin{pmatrix} y'_e\lambda^5 v & & \\ & y'_\mu\lambda^2 v & \\ & & y'_\tau v \end{pmatrix} \begin{pmatrix} e^c \\ \mu^c \\ \tau^c. \end{pmatrix}\ . \tag{2.71}$$

The mass matrix looks a lot like the one in (2.66), but with the exception that the hierarchy is 'factored out' in the factors λ^2 and λ^5. The three remaining dimensionless couplings y'_e, y'_μ and y'_τ can all be of the same order and more or less of order 1.

Concluding, we see that the Froggatt–Nielsen mechanism can give a natural explanation for the charged lepton hierarchy and that it allows all dimensionless parameters to be of the same order. The price to be paid is the introduction of a new symmetry $U_{\rm FN}$ and of a new field θ as well as allowing non-renormalisable operators in the mass Lagrangian.

In terms of cold parameter-counting, the Froggatt–Nielsen symmetry is certainly not a progress. The addition of the symmetry did not reduce the number of free parameters of the model. We started with three parameters y_e, y_μ and y_τ and we ended with more parameters. Not only the analogous y'_e, y'_μ and y'_τ, but also the parameter λ that sets the scale of the messenger vev and some discrete parameters that give the FN-charges as shown in table 2.6. The gain that we have is not in having fewer parameters, but in 'more natural' parameters. Whether the gain in naturalness is worth the excess complexity should probably be seen as a matter of personal taste.

Ultimately, it is up to experiments to confirm or falsify the model. For confirmation, detection of the Froggatt–Nielsen messenger is necessary as well as showing that it has the correct couplings to the other particles. The theory is falsified if θ is not observed in experiments that would have the ability to detect it if it were to exist. However, close inspection of the formulas in this section show that the exact scale at which the messenger should exist, is unknown. The mass or the vev of θ does not appear in any formula, only the ratio $\langle\theta\rangle/M_{\rm FN}$. This lack of exact predictivity makes falsification of the theory very hard.

2.3.2 Global, non-Abelian, discrete symmetries

The Froggatt–Nielsen mechanism proved very apt in explaining the hierarchy between the three copies of a certain particle type, for instance the charged leptons of the previous section. In chapter 1 we discussed a second striking feature of the mass sector, the mixing patterns. Two of the three angles of the CKM matrix are very close to vanishing, with θ^l_{13} as small as 0.2°. In the PMNS matrix, at least two of the three angles are large and it is remarkable that the mixing pattern can be well described by the simple fractions 1/3 and 1/2 in table 2.4. Such mixing patterns cannot be reproduced with (an adaptation of) the Froggatt–Nielsen mechanism. To describe those, we need a different type of symmetries.

When selecting symmetries, one has to make three choices: local or global; Abelian or non-Abelian and continuous or discrete. It turns out that global, non-Abelian, discrete symmetries make good candidates for flavour symmetries.

To start with the first choice: local or global. When a symmetry is made local or gauged, this introduces many new degrees of freedom, e.g. in new gauge bosons. These complicate the theory and often lead to new sources of flavour changing neutral currents, which have not been observed. A global symmetry is enough for the aim we have set, constraining the Yukawa couplings.

The second choice is if the symmetry has to be Abelian or non-Abelian. Schur's lemma dictates that every complex irreducible representation of an Abelian symmetry group is one-dimensional. To be able to describe mixing patterns like the tribimaximal one, it is highly preferred to have three-dimensional representations. We conclude that candidate family symmetries should be non-Abelian. Having said that, many models have a non-Abelian symmetry as the main symmetry, but need additional Abelian symmetries as secondary or auxiliary symmetry groups. For instance, the full symmetry group of the model of chapter 4 is

$$G_f = S_4 \times Z_4 \times U(1)_{FN} \times U(1)_R \ . \tag{2.72}$$

The group S_4 is indeed non-Abelian. It ensures bimaximal mixing, a mixing pattern that is comparable to tribimaximal. The additional Z_4 is further needed to prevent some unwanted couplings and separates the quark and the lepton sector as well as neutrinos and charged leptons within the latter. The Froggatt–Nielsen symmetry gives the quark and lepton mass hierarchies and the R-symmetry is a generalization of the better-known R-parity of supersymmetry. It separates supermultiplets with different functions (see for more details the next section).

Lastly, there is a choice between continuous and discrete symmetry groups. Although there exist many excellent models that use continuous groups, for obvious reasons mostly $SU(3)$ and $SO(3)$ (see for instance [46]), discrete groups are more popular. These have a richer choice in lower dimensional representations and the groups can be relatively small and simple (in the non-mathematical use of the word). Popular choices are the 12-element group A_4 and the 24-element S_4. For a review, see e.g. [47], that also contains a long list of references to the various models.

The prototype model of how a non-Abelian discrete symmetry can reproduce a given mixing pattern is the model of Altarelli and Feruglio [11]. We discuss this model in the next section, thereby also providing the general strategies of flavour symmetric model building.

2.4 The Altarelli–Feruglio model

2.4.1 Description of the model

The model of Altarelli and Feruglio [11] applies to the lepton sector of the Standard Model. The aim of the model is to produce a PMNS matrix that is exactly of the tribimaximal (TBM) form (2.51); until recently, this was in perfect accordance with all data and at the moment, it is still a very good approximation. To achieve TBM mixing, the lepton sector is required to be invariant not only under the symmetries of the Standard Model, but also under an additional horizontal symmetry. For this horizontal symmetry, the group A_4 is chosen. A_4 is a small discrete non-Abelian group, that is described in more detail below and in appendix 3.A. The important point here is that it has a three-dimensional representation. Actually, it is the smallest group with this property.

Much like in the Froggatt–Nielsen model, the Standard Model terms in themselves are not invariant under A_4 and the introduction of a new scalar field is required to save the symmetry. In the Froggatt–Nielsen model, this was the messenger θ; here three so-called flavons are used. Two of these flavons, φ_T and φ_S are triplets of A_4; a third ξ is a singlet.

In the next step, the flavons get a vacuum expectation value according to a specific pattern, thereby breaking the flavour symmetry. We show that the required patterns naturally appears from the minimization of a superpotential. In an earlier model [10], Altarelli and Feruglio showed that alternatively, this can be achieved in a set up with one extra dimension, where the fields are located at different branes.

The symmetry breaking by the vevs of the flavons is not complete. There are residual symmetries: Z_3 in the neutrino sector and Z_2 in the charged lepton sector. Naively, these are already the *tri* and *bi* of tribimaximal mixing. This breaking of the flavour symmetry into two different subgroups in the two lepton subsectors is the central point of flavour symmetric model building. It is now possible to write down the mass matrices of charged leptons and neutrinos. In the basis chosen by Altarelli and Feruglio, the charged lepton mass matrix is exactly diagonal. This implies $V_L^e = \mathbb{1}$, so the requirement that the PMNS matrix is equal to the tribimaximal one reduces to $V^\nu = V_{\text{TBM}}$. The neutrino mass matrix is indeed of the form that generates this.

In the following, we first give some properties of the group A_4 needed to understand the Altarelli–Feruglio model. Then we give the derivation of the tribimaximal mixing as described in words above. In this part, the required form of the flavon vevs is simply assumed; it is subsequently derived. We continue with a brief discussion on higher-order corrections to the model and quark masses. We finish with a conclusion in the form of a balance.

2.4.2 Group theory of A_4

The group A_4 is a discrete group with 12 elements. Details of its group theory are given in appendix 3.A. Here we just present the information needed to understand the Altarelli–Feruglio model. A_4 has three irreducible representations, a trivial singlet 1 and two non-trivial one-dimensional representations $1'$ and $1''$ as well as a triplet representation 3. All 12 elements can be represented as products of two generating elements S and T that satisfy the following relations, thereby defining the 'presentation' of the group

$$S^2 = (ST)^3 = T^3 = 1\,. \tag{2.73}$$

The elements S and T can be represented as $(1,1)$, $(1,\omega)$, $(1,\omega^2)$ for the three one-dimensional representations. Here $\omega = e^{2\pi i/3}$ is a cube root of unity. In the three dimensional representation, we have

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}\;; \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}\,. \tag{2.74}$$

It can easily be checked that for all representations, the requirements (2.73) are met. However, only in the three-dimensional case, this happens in a non-trivial way for the generator S. Generating all 12 elements of A_4 is only possible with the three dimensional representations, that are therefore called faithful. Different bases than (2.74) are possible. In particular, a basis of A_4 in which the generator S is diagonal is often used. This basis and its relations to the Altarelli–Feruglio basis (2.74) are given in appendix 3.A.

The elements S and T in themselves generate the two maximal subgroups of A_4, the Abelian Z_2 and Z_3 as is clear from equation (2.73). Like in all groups, multiplication of two representations of A_4 gives again a sum of A_4 representations. The multiplication rules are given by

$$\begin{aligned} &1\times 1 = 1, \quad 1\times 1' = 1', \quad 1\times 1'' = 1'', \\ &1'\times 1' = 1'', \quad 1'\times 1'' = 1, \quad 1''\times 1'' = 1', \\ &1\times 3 = 3, \quad 1'\times 3 = 3, \quad 1''\times 3 = 3, \\ &3\times 3 = 1 + 1' + 1'' + 3 + 3. \end{aligned} \tag{2.75}$$

If α is a singlet of one of the types 1, $1'$ and $1''$ and $(\beta_1, \beta_2, \beta_3)$ is a triplet, the explicit form of the products on the third line is given by $(\alpha\,\beta_1,\ \alpha\,\beta_2,\ \alpha\,\beta_3)$; $(\alpha\,\beta_3,\ \alpha\,\beta_1,\ \alpha\,\beta_2)$ and $(\alpha\,\beta_2,\ \alpha\,\beta_3,\ \alpha\,\beta_1)$ respectively. The elements of the last product can be given as a function of the elements of two triplets $\alpha = (\alpha_1,\ \alpha_2,\ \alpha_3)$ and $\beta = (\beta_1,\ \beta_2,\ \beta_3)$

$$\begin{array}{lcl} (\alpha\;\beta)_1 & = & \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2\,; \\ (\alpha\;\beta)_{1'} & = & \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_3\,; \\ (\alpha\;\beta)_{1''} & = & \alpha_1\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1\,; \\ (\alpha\;\beta)_{3;\mathrm{sym}} & = & \frac{1}{3}(2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2,\ -\alpha_1\beta_2 - \alpha_2\beta_1 + 2\alpha_3\beta_3,\ -\alpha_1\beta_3 + 2\alpha_2\beta_2 - \alpha_3\beta_1)\,; \\ (\alpha\;\beta)_{3;\mathrm{asym}} & = & \frac{1}{2}(\alpha_2\beta_3 - \alpha_3\beta_2,\ \alpha_1\beta_2 - \alpha_2\beta_1,\ \alpha_3\beta_1 - \alpha_1\beta_3)\,. \end{array} \tag{2.76}$$

From the third line in (2.75), we see that the product of three triplets $(3\times 3)_3\ \ \times 3$ can also give a singlet. This is possible with the first product of two triplets in the symmetric or antisymmetric combination. In the next subsection, we need the singlet of the combination $(\alpha\;\alpha\;\beta)$. This selects the symmetric combination for the first product.

$$(\alpha\;\alpha\;\beta)_1 = \big((\alpha\;\alpha)_{3;\mathrm{sym}}\times\beta\big)_1 = \frac{2}{3}(\alpha_1^2\beta_1 - \alpha_2\alpha_3\beta_1,\ \alpha_3^2\beta_2 - \alpha_1\alpha_2\beta_2,\ \alpha_2^2\beta_3 - \alpha_1\alpha_3\beta_3)\,. \tag{2.77}$$

2.4.3 The model

In the AF model, the symmetry group A_4 works in the family direction. We choose to put the three copies of the lepton doublet (holding the lefthanded electron + electron neutrino; muon + muon neutrino and tau + tau neutrino) together in a triplet L of A_4. The righthanded charged leptons (or rather their lefthanded antiparticles as explained in section 2.1.2) are not combined in a triplet; instead, they are assumed to be in the three different one dimensional representations: e^c in 1; μ^c in $1'$ and τ^c in $1''$. The Higgs fields (separate H^u and H^d because of the supersymmetric context) are family blind and thus in the trivial singlet representation.

There are three flavons: we want the triplet φ_T to couple only to the charged leptons and the triplet φ_S as well as the (trivial) singlet ξ to couple only to the neutrinos. The two triplets are indistinguishable at this moment: both are singlets of all Standard Model gauge groups and triplets of A_4, so we cannot use one of the already used symmetries to ensure this separation. Instead, a second, auxiliary, symmetry is invoked – the Abelian Z_3 in this case. The charges of all fields, as shown in table 2.7 are exactly such that all wanted couplings are guaranteed, while the unwanted couplings are forbidden.

Field	L	e^c	μ^c	τ^c	$H_{u,d}$	φ_T	φ_S	ξ
A_4	3	1	$1'$	$1''$	1	3	3	1
Z_3	ω	ω^2	ω^2	ω^2	1	1	ω	ω

Table 2.7: The A_4 and Z_3 representations of the fields in the Altarelli–Feruglio model.

The superpotential for the mass terms becomes

$$\begin{aligned}\mathcal{W} =&\frac{y_e}{\Lambda}e^c H_d(\varphi_T L)_1 + \frac{y_\mu}{\Lambda}\mu^c H_d(\varphi_T L)_{1''} + \frac{y_\tau}{\Lambda}\tau^c H_d(\varphi_T L)_{1'} + \\ &\frac{x_a}{\Lambda^2}\xi H_u H_u (LL)_1 + \frac{x_b}{\Lambda^2} H_u H_u (LL\varphi_S)_1 + \ldots\end{aligned} \tag{2.78}$$

Terms such as $y_e'/\Lambda\ e^c H_d(\varphi_S L)_1$ or $x_b'/\Lambda\ (LL\varphi_T)_1$ that would couple flavons to the 'wrong' sectors are indeed absent due to the extra Z_3 symmetry. We recall from equation (2.11) that the terms with two Ls give rise to a neutrino mass term, while the first three terms obviously lead to charged lepton masses. The dots at the end of equation (2.78) refer to terms suppressed by higher powers of Λ.

In [48] and [49] it was shown that models cannot reproduce maximal atmospheric mixing ($\theta_{23}^l = 45°$) if the flavour symmetry is exact. The flavour symmetry thus needs to be broken. This occurs when the flavons φ_T, φ_S and ξ develop vacuum expectation values in very specific directions in the A_4 space; the next subsection show how this follows from an analysis of their superpotential. The vev of φ_T is such that it is invariant under the T-generator of A_4 (equation 2.73). This means that terms in the Lagrangian that are dependent on φ_T (and φ_T only) will have a residual symmetry after A_4 is broken by φ_T taking its vacuum expectation value. This symmetry is the Abelian Z_3. As only the charged lepton mass terms depend on φ_T, we say that Z_3 is the residual symmetry in the charged lepton sector.

Similarly, φ_S gets a vacuum expectation value that is invariant under the S generator and the singlet ξ gets a constant vev, that does not break A_4. As a result, the neutrino sector has a residual Z_2 symmetry generated by S. Basically, the *tri* of the residual Z_3 in the charged lepton sector and the *bi* of the residual Z_2 for the neutrinos are the main ingredients of tribimaximal mixing. The explicit forms of the vacuum expectation values of the flavons are

$$\langle\varphi_T\rangle = v_T(1,\ 0,\ 0); \qquad \langle\varphi_S\rangle = v_S(1,\ 1,\ 1); \qquad \langle\xi\rangle = v_\xi. \tag{2.79}$$

The effective Lagrangian after the flavons and the Higgses obtain their vacuum expectation values is

$$\begin{aligned}\mathcal{L} =&\frac{v_d v_T}{\Lambda}(y_e e^c e + y_\mu \mu^c \mu + y_\tau \tau^c \tau) \\ &+\frac{x_a v_u^2 v_\xi}{\Lambda^2}(\nu_e\nu_e + 2\nu_\mu\nu_\tau) + \frac{x_b 2 v_u^2 v_S}{3\Lambda^2}(\nu_e\nu_e + \nu_\mu\nu_\mu + \nu_\tau\nu_\tau - \nu_e\nu_\mu - \nu_e\nu_\tau - \nu_\mu\nu_\tau) \\ &+\text{h.c.} + \ldots\end{aligned} \tag{2.80}$$

The charged lepton mass matrix is now diagonal, with the masses given by

$$m_e = y_e v_d \frac{v_T}{\sqrt{2}\Lambda}, \quad m_\mu = y_\mu v_d \frac{v_T}{\sqrt{2}\Lambda}, \quad m_\tau = y_\tau v_d \frac{v_T}{\sqrt{2}\Lambda}. \tag{2.81}$$

All masses are suppressed from the electroweak scale by a factor $\frac{v_T}{\Lambda}$ (and a factor $\cos\beta$ from v_d). The model so far does not explain the hierarchy in the masses, although it is possible to combine the model with the Froggatt–Nielsen model of section 2.3.1.

In the neutrino sector, the mass matrix reads

$$M_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} a+2b/3 & -b/3 & -b/3 \\ -b/3 & 2b/3 & a-b/3 \\ -b/3 & a-b/3 & 2b/3 \end{pmatrix}. \tag{2.82}$$

Here a and b are dimensionless parameters given by

$$a = \frac{2x_a v_\xi}{\Lambda}, \quad b = \frac{2x_b v_S}{\Lambda}. \tag{2.83}$$

The neutrino mass matrix (2.82) is diagonalized exactly by the tribimaximal mixing matrix (2.51)

$$\hat{M}_\nu = U^\dagger M_\nu U^* = \begin{pmatrix} a+b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a+b \end{pmatrix}, \qquad U = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}. \tag{2.84}$$

This is the main result of this section. Using a relatively limited amount of new ingredients, it is possible to exactly predict the neutrino mixing matrix.

A few comments are in order. Firstly on the ratio between the various flavon vevs and the cut-off scale. This ratio should be smaller than 1 in order to sensibly cut off the superpotential (2.78) after the given terms. A lower limit emerges from the mass of the tau lepton in equation (2.81), where the parameter y_τ should be smaller than 4π to be in the perturbative regime. If we assume that for all three flavons the vevs are of the same order of magnitude, we find

$$\frac{0.00080}{\cos\beta} \lesssim \frac{v_T}{\Lambda} \approx \frac{v_S}{\Lambda} \approx \frac{v_\xi}{\Lambda} \lesssim 1 . \tag{2.85}$$

Secondly, not only a neutrino mixing matrix, but also a relation between the neutrino masses is found. In [10] it was shown that the atmospheric and solar mass differences can be accommodated in the normal hierarchy and that there is an extra relation between the parameter of neutrinoless double beta decay and the third neutrino mass. This makes the model testable in future neutrino experiments. Thirdly, using the atmospheric mass difference, the absolute scale of Λ itself can be limited to

$$\Lambda < 1.8 \times 10^{15}\ \mathrm{GeV}. \tag{2.86}$$

The model described so far assumes that neutrinos are Majorana particles, but does not use one of the three seesaw mechanism to generate their masses. Instead, the effective Weinberg operator of section 2.1.5 is used. This seems reasonable, given that the couplings of fermions, Higgs fields and flavons are already effective operators due to the extra flavon insertions. Addition of a righthanded neutrino to the model is possible and a version of the model that uses the type-I seesaw can be written down, giving the same general conclusions, but differing in many details such as the relations between the neutrino masses.

2.4.4 Flavon vacuum alignment

In this subsection, we show how the flavons can obtain the vevs of equation (2.79). Note that the ordinary Higgs fields are singlets under the family symmetries and that the flavons are singlets under the electroweak gauge group. The potentials of the Higgs fields and the flavons therefore decouple and should be studied separately. For the Higgses, this is just the superpotential of the MSSM.

In the flavon sector, we would like to build a potential for the fields φ_S, φ_T and ξ such that their minima are as in (2.79). This turns out to be possible only if new fields are introduced again, the so-called driving fields. They do not develop vevs themselves but merely help the 'normal' flavons to do so[7]. The driving fields, that we write as φ_0^S, φ_0^T and ξ_0 have the same A_4 and Z_3 charges as the fields they correspond to. To be able to have a non-trivial minimum, a second copy of the field ξ is needed as well; it is called $\tilde{\xi}$ and has exactly the same charges as ξ.

At this stage also a new symmetry is introduced. It is a continuous R-symmetry $U(1)_R$ that has the usual R-parity as a subgroup. The working of R-symmetry on superfields and supercoordinates is such that consistent terms in a superpotential should always have total R-charge 2, see e.g. [14] for more details. The R-charges in the model of Altarelli and Feruglio are given in table 2.8. It can be seen that all terms in the flavour symmetry breaking superpotential have to be linear in one of the driving fields. The Altarelli–Feruglio model does not discuss quarks, but extensions of it that do, will have the same R-charges for quarks as for leptons. In that way, baryon number violating operators that mediate proton decay are forbidden in the same way as in supersymmetric models with R-parity conservation.

Field	L	e^c	μ^c	τ^c	$H_{u,d}$	φ_T	φ_S	ξ	$\tilde{\xi}$	φ_0^T	φ_0^S	ξ_0
A_4	3	1	$1'$	$1''$	1	3	3	1	1	3	3	1
Z_3	ω	ω^2	ω^2	ω^2	1	1	ω	ω	ω	1	ω	ω
$U(1)_R$	1	1	1	1	0	0	0	0	0	2	2	2

Table 2.8: The representations of the fields in the Altarelli–Feruglio model after inclusion of the driving fields.

The requirement of invariance under A_4, Z_3 and $U(1)_R$ allows us to write down the so-called driving superpotential.

$$\begin{aligned}\mathcal{W}_d =& M(\varphi_0^T\varphi_T)_1 + g_0(\varphi_0^T\varphi_T\varphi_T)_1 + \\ & g_1(\varphi_0^S\varphi_S\varphi_S)_1 + g_2\tilde{\xi}(\varphi_0^S\varphi_S)_1 + g_3\xi_0(\varphi_S\varphi_S)_1 + g_4\xi_0\xi^2 + g_5\xi_0\xi\tilde{\xi} + g_6\xi_0\tilde{\xi}^2 + \mathcal{O}(1/\Lambda)\ .\end{aligned} \tag{2.87}$$

Note that the field that appears in the term with g_2 is $\tilde{\xi}$ and not ξ. So far, there was no difference between these two fields and we are free to define $\tilde{\xi}$ as the combination that couples to $(\varphi_0^S\varphi_S)_1$ in the superpotential.

The scalar potential follows from the superpotential, soft supersymmetry breaking terms and D-terms. Given the large scale of Λ found in equation (2.86), we assume that the masses of soft breaking are much smaller than the mass scales in w_d and we minimize the scalar potential in the supersymmetric limit, taking soft susy breaking terms into account later, for instance to select the minimum described below over the trivial minimum. In the supersymmetric limit the scalar potential reads

$$V = \sum_i \left|\frac{\partial \mathcal{W}}{\partial \phi_i}\right|^2 + \cdots \tag{2.88}$$

Requiring supersymmetry to be conserved and thus $V = 0$ requires the derivatives of the superpo-

[7]The fact that the driving superfields do not develop vevs strictly holds only in the exact supersymmetric phase, while in the broken phase they develop a vev proportional to the common soft breaking scale [50–52], usually denoted as m_{SUSY}. This could have a relevant impact on flavour violating processes, as studied in a series of papers [53–57]

tential to all components of the driving fields to be zero. This leads to a set of seven equations.

$$\begin{aligned}
\frac{\partial \mathcal{W}_d}{\partial \varphi_{01}^T} &= M\varphi_{T1} + \frac{2g_0}{3}\Big(\varphi_{T1}^2 - \varphi_{T2}\varphi_{T3}\Big), \\
\frac{\partial \mathcal{W}_d}{\partial \varphi_{02}^T} &= M\varphi_{T3} + \frac{2g_0}{3}\Big(\varphi_{T2}^2 - \varphi_{T1}\varphi_{T3}\Big), \\
\frac{\partial \mathcal{W}_d}{\partial \varphi_{03}^T} &= M\varphi_{T2} + \frac{2g_0}{3}\Big(\varphi_{T3}^2 - \varphi_{T1}\varphi_{T2}\Big), \\
\frac{\partial \mathcal{W}_d}{\partial \varphi_{01}^S} &= g_2\tilde{\xi}\varphi_{S1} + \frac{2g_1}{3}\Big(\varphi_{S1}^2 - \varphi_{S2}\varphi_{S3}\Big), \\
\frac{\partial \mathcal{W}_d}{\partial \varphi_{02}^S} &= g_2\tilde{\xi}\varphi_{S3} + \frac{2g_1}{3}\Big(\varphi_{S2}^2 - \varphi_{S1}\varphi_{S3}\Big), \\
\frac{\partial \mathcal{W}_d}{\partial \varphi_{03}^S} &= g_2\tilde{\xi}\varphi_{S2} + \frac{2g_1}{3}\Big(\varphi_{S3}^2 - \varphi_{S1}\varphi_{S2}\Big), \\
\frac{\partial \mathcal{W}_d}{\partial \xi_0} &= g_4\xi^2 + g_5\xi\tilde{\xi} + g_6\tilde{\xi}^2 + g_3\Big(\varphi_{S1}^2 + 2\varphi_{S2}\varphi_{S3}\Big).
\end{aligned} \tag{2.89}$$

The first three and the last four equations of (2.89) can be separately solved and a non-trivial solution is given by

$$\begin{aligned}
\varphi_T &= (v_T, 0, 0), \qquad v_T = -\frac{3M}{2g}, \\
\xi &= v_\xi, \\
\tilde{\xi} &= 0, \\
\varphi_S &= (v_S, v_S, v_S), \qquad v_S^2 = -\frac{g_4}{3g_3}v_\xi^2.
\end{aligned} \tag{2.90}$$

This is indeed of the form (2.79) and breaks the A_4 flavour symmetry to Z_3 and Z_2 respectively in the charged lepton and neutrino sectors.

2.4.5 Higher order corrections and quark masses

So far we have proven that it is possible to obtain the tribimaximal lepton mixing in leading order (LO) in a $1/\Lambda$ expansion. The lepton superpotential also contains next-to-leading order (NLO) terms, that is terms with one additional flavon. These slightly break the TBM prediction. The same holds for NLO terms in the driving superpotential, as these drive the vacuum expectation values of the flavons away from the values in (2.90). A third source of deviations from the predicted limit comes from the fact that the mass matrices (and thus the mixing matrices) are calculated at a very high scale, up to 1.8×10^{15} GeV according to equation (2.86). Renormalization effects between this scale and the scale where they are observed should be taken into account, although they are generally relatively small; see chapter 4 for a discussion of Yukawa coupling renormalization in a similar model.

These corrections are very welcome, as the observed neutrino mass mixing matrix is exactly of the tribimaximal type. However, as the tribimaximal pattern already fits the data rather well (certainly before the recent signal of non-zero θ_{13}^l), the deviations should be small – in particular the correction to θ_{12}^l should be tiny. We do not discuss the full NLO computation of masses and mixings in the Altarelli–Feruglio model here, but instead present the main conclusion. In order for the NLO corrections not to spoil the LO conclusions, the ratio between the vev of any of the flavons to the cut-off scale Λ should be smaller than a few percent. The upper limit of v_X/Λ in equation (2.85) becomes 0.05 instead of 1.

The aim of the Altarelli–Feruglio model is the reproduction of the lepton mixing matrix. Quarks are thus not considered in the most basic version of the model. Addition of quarks to the model is of course possible and it can be done in several ways. In the most simple extension, the quark doublet transforms in the same way as the lepton doublet, while the anti-quarks of both types transform in the same way as the charged anti-leptons. In this case, at the leading order, the quark mixing matrix is the unit matrix. This is a reasonable approximation, given that the quark mixing angles are fairly small, but the required NLO corrections seem to be larger than those allowed in the lepton sector, which is not entirely satisfactory. We find that while the (extended) model fits the data quite well at leading order, it is hard to improve this at higher orders.

2.4.6 A balance

The original model of Altarelli and Feruglio was published in 2005. At that time, the tribimaximal mixing pattern was well-known in the literature (it was published in 2002 by Harrison, Perkins and Scott). The TBM mixing pattern was in accordance with the experimental data, although the experimental errors were still large. Neutrino oscillations have given new results in the last years. In particular, there now seems to be a signal for a θ_{13}^l angle that is relatively large. This would require a rather large correction to this angle, while such corrections are not allowed for the other two angles. Interestingly enough, a comparably large correction is needed to introduce the Cabibbo angle in the CKM matrix that is the unity at leading order. Alternative models that require the same (larger) corrections in both sectors may address this point; in chapter 4 we provide a model with this mechanism. In chapter 3, we investigate mixing patterns that give non-zero mixing patterns at the leading order.

Even if the question of NLO corrections is not answered completely satisfactory, the tribimaximal pattern is still an interesting start for modelbuilding. The Altarelli–Feruglio model can provide this mixing, but, as in the case of the Froggatt–Nielsen mechanism, it requires the introduction of a rather large number of extra fields and symmetries.

The main ingredient of the model is the horizontal symmetry A_4. It has subgroups Z_2 and Z_3 that are at the basis of the tribimaximal mixing. The introduction of the fields φ_S and φ_T seems therefore very reasonable. It is those fields that break in exactly the right directions for the TBM mixing pattern to arrive.

Tribimaximal mixing requires that the Z_2 and Z_3 residual symmetries are not only created by the flavons, but also transmitted in the right way to respectively the neutrinos and the charged leptons. Unfortunately, both φ_S and φ_T are triplets of A_4 (and singlets of the Standard Model gauge group), so a priori both of them would couple to both sectors. To prevent this, a rather *ad hoc* Z_3 symmetry is introduced. This symmetry also requires a new particle: the fourth term in equation (2.78) with two Higgs fields and two lepton doublets is invariant under the Standard Model gauge group and under A_4 with or without ξ and it needs this field only to be safe under Z_3. This chain of action-and-reaction of addition of new fields and symmetries makes the model slightly baroque and unfortunately, this will not be much better (actually quite a bit worse) in more ambitious ones, such as the model of chapter 4 that also aims at reproducing the CKM matrix and some relations between particles of different type.

The version of the Altarelli–Feruglio model described here is supersymmetric. An earlier version of the model [10] has a setting with extra dimensions. We stress that neither are fundamental elements of the model or of non-Abelian flavour models in general. In principle the questions whether flavour symmetries exist in nature and whether supersymmetry or extra dimensions are introduced – for instance to solve the hierarchy problem – should be answered separately, both from a theoretical and from an experimental point of view. The problem is that the theoretical methods developed so far only succeed in giving the right vacuum alignment for two or more flavons in settings with supersymmetry or extra dimensions. We have shown that with supersymmetry the right flavon alignment is possible, but that this is non-trivial. New driving fields had to be introduced as well as a second copy of ξ. We note that it is possible in some other models to find a supersymmetric vacuum without driving fields. The A_5 model of Feruglio and Paris [58] is an elegant example. Alternatively, one can consider models that are non-supersymmetric and do not have extra dimensions. This is discussed in chapter 5 where the role of the flavons is basically given to a multiplet of Higgs bosons.

Concluding we state that the model of Altarelli and Feruglio succeeds in reproducing the tribimaximal mixing pattern in an impressive way. The 'cost' of this is the introduction of quite a number of new symmetries and fields and from an theoretical-economical perspective, one may question whether the gained predictively is worth the price. The last words of this section have the same message as those in section 2.3.1 on the Froggatt–Nielsen mechanism. Ultimately it is up to experiments to confirm or disprove the sometimes very specific predictions of the model.

2.5 Conclusions of the chapter

In this chapter we discussed how fermion masses appear in the Standard Model (extended to include neutrino masses) and in supersymmetric variations. To focus on the different operators that appear, we first studied the case where the Standard Model has only one family. Next we allowed for three family copies of the fermionic content in accordance with observations. We found that with more than one generation, mass eigenstates and flavour eigenstates do not coincide. This gives rise to the CKM and PMNS mixing matrices.

We studied the experimentally found masses and mixing angles of the elementary fermions. The trained eye of the theoretical physicist immediately sees patterns here. We stressed that these patterns do not need to imply new physics. Even if some of the 20 or 22 free parameters of the Standard Model plus neutrinos have some unnatural values, it is perfectly consistent and describes everything that is measured. On the other hand, these patterns are compelling enough to at least justify a thorough study of different possibilities to explain them using symmetry principles.

We studied two simplified models: the Froggatt–Nielsen model, that gives an explanation for the hierarchies of the charged fermion masses and the Altarelli–Feruglio model that explains the emergence of a neutrino mixing close to the tribimaximal mixing pattern. These models are very interesting, but they leave some questions unanswered. For instance, we found no clear way to include quarks and quark mixing in the Altarelli–Feruglio model. Furthermore, we mentioned that the tribimaximal mixing pattern is not be the only phenomenologically interesting mixing pattern. In particular models that accommodate non-zero θ^l_{13} either at first order or naturally via corrections are appealing. In the following chapters we will see examples of both these approaches.