a.

(i) $\{ x \in \mathbb{R}^n : Ax \leq b \} = P$

let $x_1, x_2 \in P$

then $Ax_1 \leq b$, $Ax_2 \leq b$

let $t \in [0,1]$

$tAx_1 + (1-t)Ax_2 \leq tb + (1-t)b = b$

$A\underbrace{[tx_1 + (1-t)x_2]} \leq b$

So a polyhedron is convex

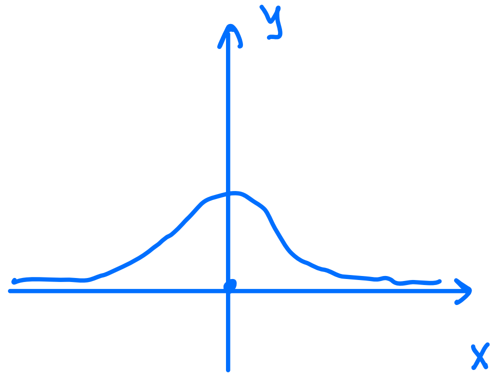
(ii) let $x, y \in C$, then $x \in C_i$ and $y \in C_i$, $i = 1 \cdots n$

for any $t \in [0,1]$, $tx + (1-t)y \in C_i$

So $tx + (1-t)y \in \bigcap_{i=1}^{n} C_i = C$

then C is convex.

(iii) convex hull of a closed set need not be closed:



$C = \{(x,y) : y \geq \frac{1}{1+x^2}\}$

$\mathrm{conv}(C) = \{(x,y) : y > 0\}$

(iv) $A^{-1}(S) = \{ x \in \mathbb{R}^n : Ax \in S \}$

let $x, y \in A^{-1}(S)$, then $Ax, Ay \in S$

$\because$ S is convex

$\therefore$ $tAx + (1-t)Ay \in S$

$$A\underline{\left(tx + (1-t)y\right)} \in S$$

So $tx + (1-t)y \in A^{-1}(S)$

then $A^{-1}(S)$ is also convex

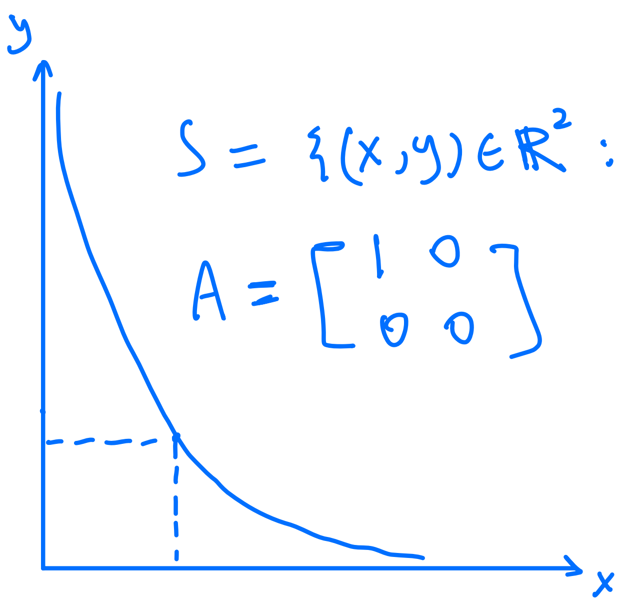
(v) $A(S) = \{ Ax : x \in S \}$

let $x, y \in S$

$tAx + (1-t)Ay = A\left(tx + (1-t)y\right) \in A(S)$

So $A(S)$ is convex.

(vi) S is convex and closed but $A(S)$ need not be closed.



$S = \{ (x,y) \in \mathbb{R}^2 : x > 0, y \geq \frac{1}{x} \}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\qquad A(S) = \{ (x, 0) : x > 0 \}$

b.

(i) $P = \{x \in \mathbb{R}^n : Cx \le d\}$

$A(P) = \{Ax \in \mathbb{R}^m : Cx \le d\}$

$= \{y \in \mathbb{R}^m : y = Ax,\ Cx \le d\}$

because $\{(x, y) \in \mathbb{R}^{m+n} : y = Ax,\ Cx \le d\}$

is a polyhedron,

according to the property (1),

$A(P)$ is a polyhedron.

(ii) See part (a.vi)

c. $\because \{x : x \ge 0\}$ is a polyhedron

$\therefore C = \{Ax : x \ge 0\}$ is a polyhedron (b.i)

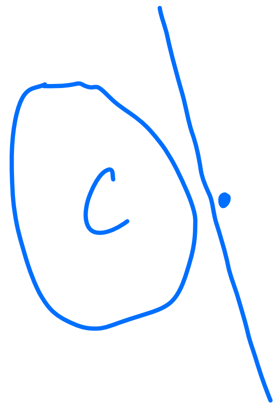
So, C is convex and closed. (a.i)

let $D = \{b\}$, a singleton set.

1. $b \in C$, then (1)

2. $b \notin C$, there exists a hyperplane the separates b from C.

$\{x : y^T x = z\}$.



① $y^T b < z$

② $y^T A x > z$, for $\forall x \geqslant 0$.

②: $(A^T y)^T x > z$, for $\forall x \geqslant 0$, then $\boxed{A^T y \geqslant 0}$

①: if $y^T b \geqslant 0$

$y^T A x > z > y^T b \geqslant 0$ for $\forall x \geqslant 0$

this is not ~~true~~ (when $x = 0$, $y^T A x = 0$)

so $\boxed{y^T b < 0}$

a. when $xy \geq 0$, $f(x,y) = xy + a(x+y)$

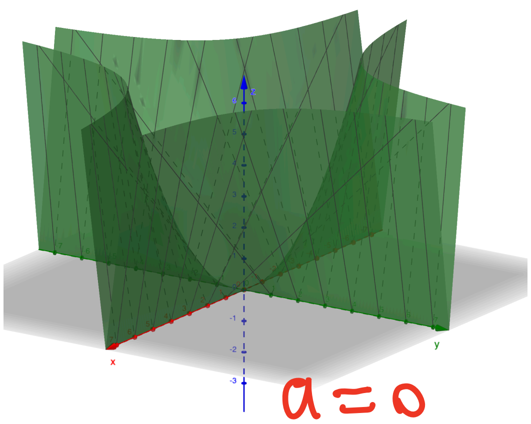
f is convex iff $\nabla^2 f(x,y) = \begin{bmatrix} 2a & 1 \\ 1 & 2a \end{bmatrix} \succeq 0$

$$\begin{cases} \lambda_1 + \lambda_2 = 4a \geq 0 \\ \lambda_1 \lambda_2 = 4a^2 - 1 \geq 0 \end{cases} \Leftrightarrow a \geq \frac{1}{2}$$

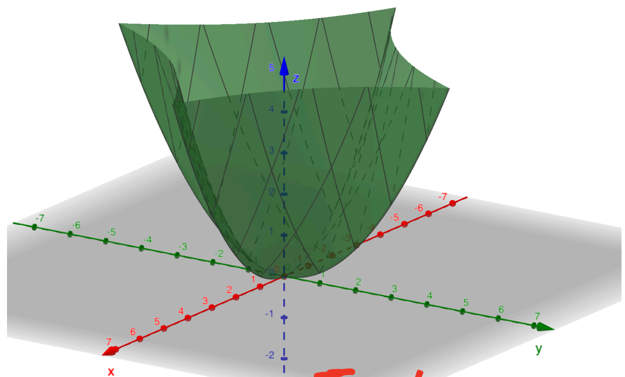
when $xy < 0$, $\nabla^2 f(x,y) = \begin{bmatrix} 2a & -1 \\ -1 & 2a \end{bmatrix} \succeq 0 \Leftrightarrow a \geq \frac{1}{2}$

So f is convex iff $a \geq \frac{1}{2}$

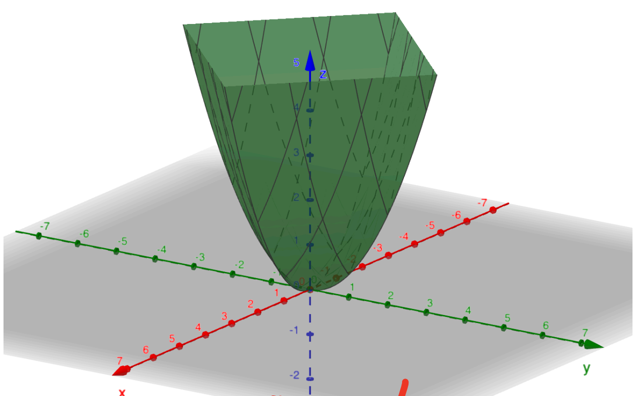
Strong convex $\Leftrightarrow \nabla^2 f \succ 0 \Leftrightarrow a > \frac{1}{2}$



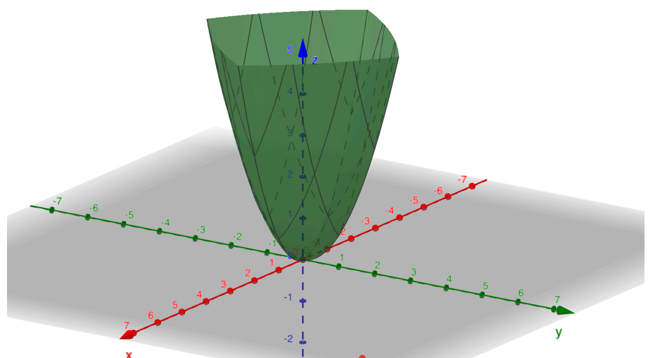
$a = 0$



$a = 1/4$



$a = 1/2$



$a = 3/4$

b.

i. $f(x) = -\sum_{i=1}^{n} \log(x_i)$

strictly convex ($\nabla^2 f(x) \succ 0$)

not strongly convex ($-\log x_i - \frac{m}{2} x_i^2$ is not convex)

ii. $f(x) = \begin{cases} -\sum_{i=1}^{n} x_i \log(x_i) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$ $\quad x \in \mathbb{R}_+^n, \sum_{i=1}^{n} x_i = 1$

when $x \in \mathbb{R}_{++}^n$, $\nabla^2 f(x) = \operatorname{diag}(-\frac{1}{x_1}, -\frac{1}{x_2}, \cdots, -\frac{1}{x_n}) \not\succeq 0$

So $f(x)$ is concave.

c. let $x, y \in \operatorname{dom}(f)$

f is convex iff $f(y) \geq f(x) + \nabla f(x)^T (y-x)$

in the same way $f(x) \geq f(y) + \nabla f(y)^T (x-y)$

then $f(x) + f(y) \geq f(x) + f(y) + (\nabla f(x) - \nabla f(y))^T (y-x)$

So $(\nabla f(x) - \nabla f(y))^T (x-y) \geq 0$

a.

$i \Rightarrow ii$

$\langle \nabla f(x) - \nabla f(y), x-y \rangle \leq \|\nabla f(x) - \nabla f(y)\| \|x-y\| \leq L\|x-y\|^2$

$ii \Leftrightarrow g(x) = \frac{L}{2}\|x\|_2^2 - f(x)$ is convex $\Leftrightarrow iii$

$\downarrow$ monotonicity of g

$\downarrow$ second-order convexity

$ii \Leftrightarrow iv$:

$ii \Leftrightarrow g$ is convex

$g(y) \geq g(x) + \nabla g(x)^T (y-x) \Leftrightarrow iv$

$iv \Rightarrow i$

For any y and z we have

$$f(z) \le f(y) + \langle \nabla f(y), z-y \rangle + \frac{L}{2}\|z-y\|_2^2 \qquad (*)$$

Since f is convex we have

$$f(x) + \langle \nabla f(x), z-x \rangle \le f(z)$$

$$f(x) - \langle \nabla f(x), x \rangle \le f(z) - \langle \nabla f(x), z \rangle$$

$$\le f(y) - \langle \nabla f(x), y \rangle + \langle \nabla f(y) - \nabla f(x), z-y \rangle + \frac{L}{2}\|z-y\|_2^2$$

So

$$f(x) - f(y) - \langle \nabla f(x), x-y \rangle \le \underbrace{\langle \nabla f(y) - \nabla f(x), z-y \rangle + \frac{L}{2}\|z-y\|_2^2}_{\psi(z)}$$

this is true for any z. Since $\psi(z) \to \infty$ as $|z| \to \infty$

the function ψ attains minimum at $\nabla \psi(z) = 0$

$$0 = \nabla \psi(z) = \nabla f(y) - \nabla f(x) + L(z-y)$$

$$z - y = -\frac{1}{L}(\nabla f(y) - \nabla f(x))$$

With the value of z we have

$$\psi(z) = -\frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|_2^2$$

So

$$f(x) - f(y) + \langle \nabla f(x), y-x \rangle \le -\frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|_2^2$$

$$f(y) - f(x) + \langle \nabla f(y), x-y \rangle \le -\frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|_2^2$$

$$\langle \nabla f(x) - \nabla f(y), x-y \rangle \ge \frac{1}{L}\|\nabla f(y) - \nabla f(x)\|_2^2$$

$$\frac{1}{L}\|\nabla f(y) - \nabla f(x)\|_2^2 \le \|\nabla f(x) - \nabla f(y)\| \, \|x-y\|$$

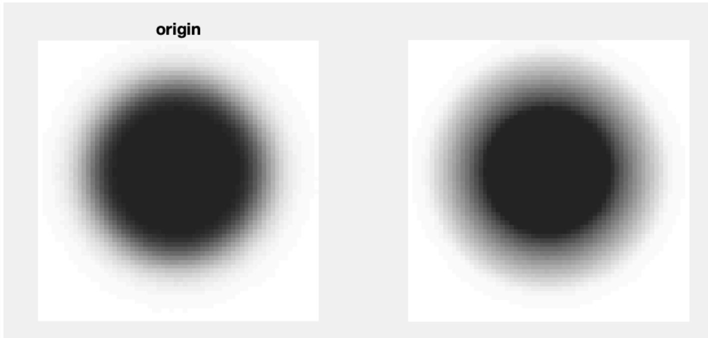
$$\|\nabla f(y) - \nabla f(x)\| \le L\|x-y\|$$

$$\begin{tikzcd}
\text{i} \arrow[d] & \\
\text{ii} \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & g \text{ is convex } \& \text{ iii} \\
\text{iv} \arrow[uu, bend left] &
\end{tikzcd}$$

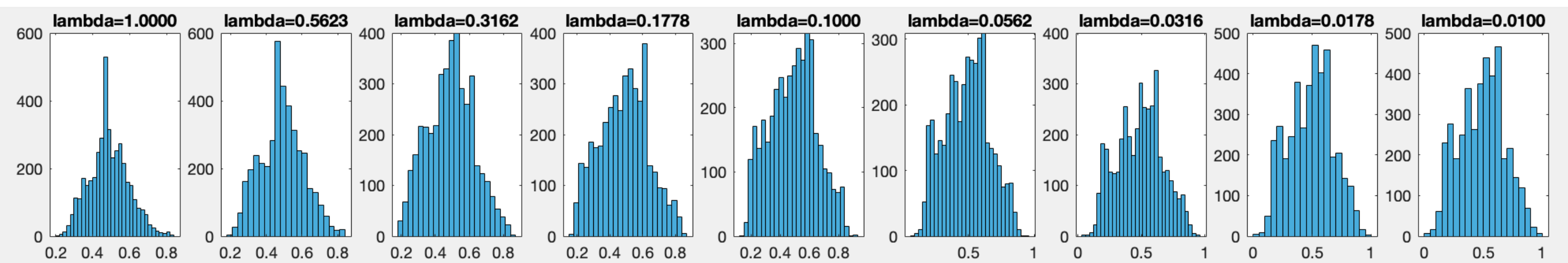
b. similar to a.

4. Solving Optimization Problem with CVX

a.1.



a.2.



b.1. ?