FOUNDATIONS OF OPTIMIZATION: IE6001

Introduction

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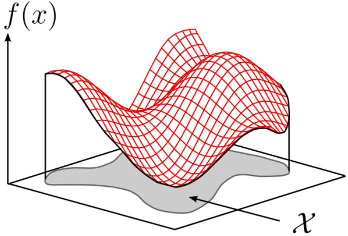
Course Information

- Lecturer: Napat Rujeerapaiboon (napat.rujeerapaiboon@nus.edu.sg)
- Teaching assistant: Xue Yilin and Zhu Yufei
- Lecture slides are available on Canvas.
- In addition to mid-term (25%) and final (50%) exams, there will be several assessed courseworks.
- Active participation is strongly encouraged!

Mathematical Programming

OR solves mathematical programming models:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{X}, \end{array}$$



where

- $x \in \mathbb{R}^n$ are the decision variables
- $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function (e.g., cost)
- $\mathcal{X} \subseteq \mathbb{R}^n$ is the feasible set (set of admissible decisions)

Optimal Decision Tool

- Linear objective function
- Linear constraints (equalities and inequalities)

Amongst the most popular mathematical models:
85% of Fortune 500 firms said they use LP

Example 1

Manufacturer produces: A (acid) and C (caustic).
Ingredients used for producing A and C are: X and Y.

- Each ton of A requires: 2lb of X; 1lb of Y
- Each ton of C requires: 1lb of X ; 3lb of Y
- Supply of X limited to: 11lb/week
- Supply of Y limited to: 18lb/week
- A sells for: £1000/ton
- C sells for: £1000/ton

Market research: max 4 tons of A/week can be sold.

$\boxed{\text{Maximize weekly value of sales of A and C.}}$

Example 1 (MP Model)

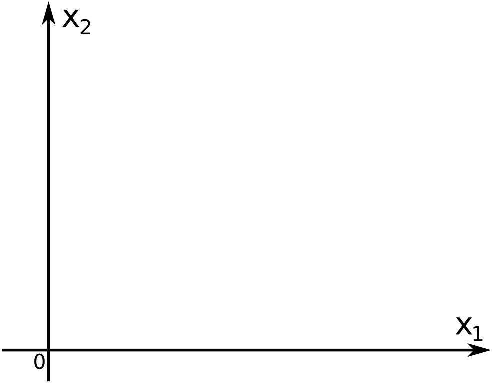
How much A and C to produce?

$\Rightarrow$ Formulate a mathematical programming model!

- Decision variables
  - $x_1 =$ weekly production of A (in tons)
  - $x_2 =$ weekly production of C (in tons)
- Objective function
  - $f(x_1, x_2) =$ weekly profit (in 1000 £)
- Feasible set
  - $\mathcal{X} =$ set of all implementable/admissible production plans $x = (x_1, x_2)$
  - e.g., $x = (27, 2)$ is not possible (not enough supply!)

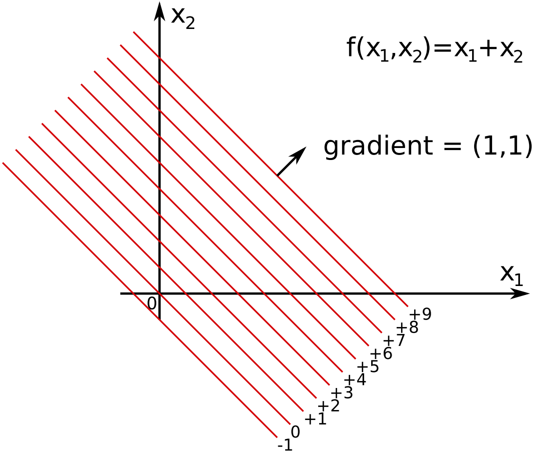
Example 1 (Decision Variables)

A production plan is representable as $x = (x_1, x_2)$



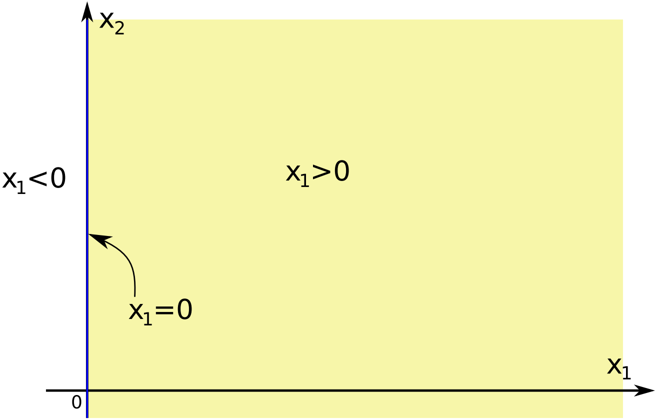
Example 1 (Objective Function)

Profit: $f(x_1, x_2) = x_1 + x_2$ (in 1000 £)



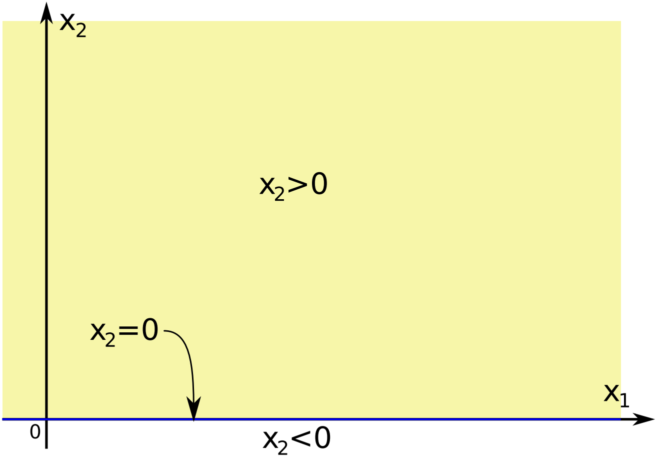
Example 1 (Feasible Set)

Amount of A produced is non-negative: $x_1 \geq 0$



Example 1 (Feasible Set)

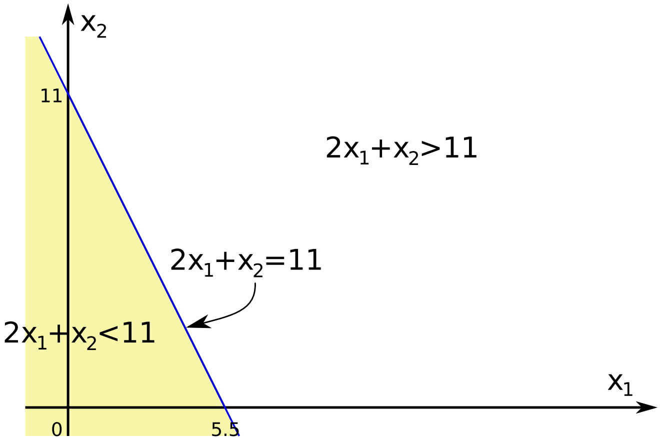
Amount of C produced is non-negative: $x_2 \geq 0$



Example 1 (Feasible Set)

x_1 tons of A & x_2 tons of C require $2x_1 + x_2$ lb of X

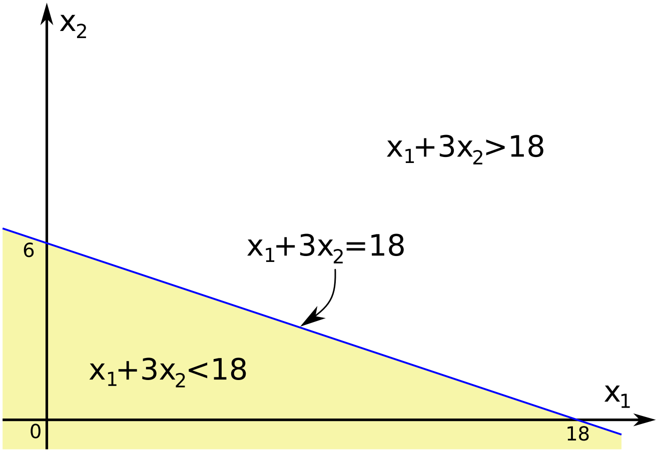
X is limited to 11lb/week: $2x_1 + x_2 \leq 11$



Example 1 (Feasible Set)

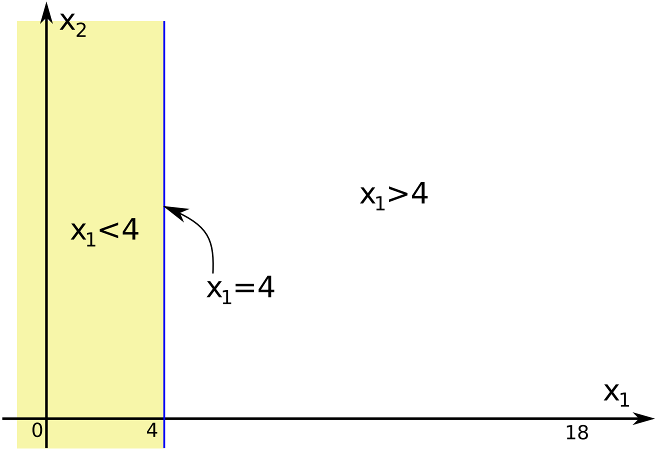
x_1 tons of A & x_2 tons of C require $x_1 + 3x_2$ lb of Y

Y is limited to 18lb/week: $x_1 + 3x_2 \leq 18$



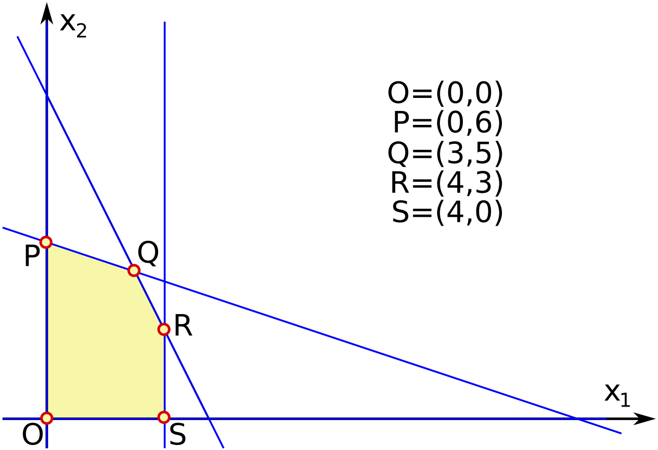
Example 1 (Feasible Set)

Cannot sell more than 4 tons of A/week: $x_1 \leq 4$



Example 1 (Feasible Set)

To obtain the overall feasible set,
intersect the feasible sets of all individual constraints



Example 1 (Feasible Set)

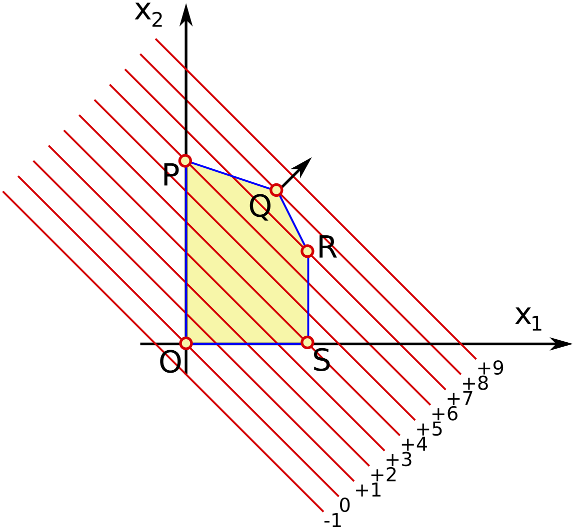
- The feasible set is a convex polygon
- The corners O,P,Q,R,S of the feasible set are termed extreme points or vertices
- Each vertex is given by the intersection of two blue lines; its coordinates can be computed by jointly solving the two linear equations defining the blue lines
- We obtain O=(0,0), P=(0,6), Q=(3,5), R=(4,3), S=(4,0)

Example 1 (Summary)

The best production plan is obtained by solving the following linear program:

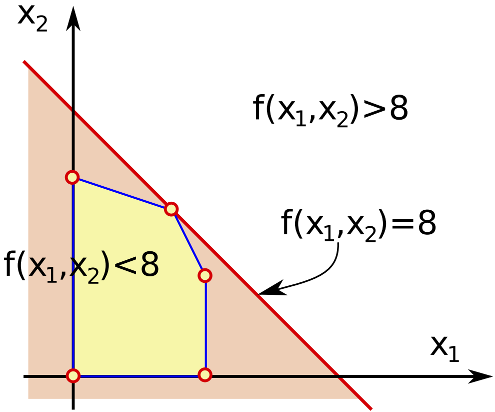
$$
\begin{array}{lll}
\text{maximize} & x_1 + x_2 & : \text{objective function} \\
\text{subject to} & 2x_1 + x_2 \leq 11 & : \text{constraint on availability of X} \\
 & x_1 + 3x_2 \leq 18 & : \text{constraint on availability of Y} \\
 & x_1 \leq 4 & : \text{constraint on demand of A} \\
 & x_1, x_2 \geq 0 & : \text{non-negativity constraints}
\end{array}
$$

Example 1 (Graphical Solution)



Example 1 (Graphical Solution)

All feasible points satisfy $f(x_1, x_2) \leq 8$
Q is the only feasible point (x_1, x_2) with $f(x_1, x_2) = 8$



Linear Programming

- A Linear program (LP) is a mathematical program that
  - optimizes (maximizes or minimizes) a linear objective function
  - over a polyhedral feasible set described by linear equality/inequality constraints.
- The feasible set has finitely many vertices.
- One can prove that every LP (with a bounded nonempty feasible set) has a vertex solution.

$\Rightarrow$ To solve the LP it is sufficient to examine only the vertices of its feasible set!

Variants of Example 1

- minimize $3x_1 - x_2$ over feasible set of Example 1

O=(0,0)	P=(0,6)	Q=(3,5)	R=(4,3)	S=(4,0)
0	-6	4	9	12

$\Rightarrow$ P: $x_1 = 0$, $x_2 = 6$ is optimal.

- maximize $2x_1 + x_2$ over feasible set of Example 1:

  Any point on the line segment QR is optimal.

$\Rightarrow$ points other than vertices can be optimal, but there is always an optimal vertex

- The number of vertices of the feasible set is always finite, but it is typically exponential in the problem dimensions.
- The Simplex Algorithm is an efficient method for finding an optimal vertex without necessarily examining all vertices.