FOUNDATIONS OF OPTIMIZATION: IE6001

Simplex Algorithm

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Simplex Tableaux

- Consider a basic representation of the form

$$\left.\begin{array}{rcrcl} x_0 & - & r^T x_N & = & c_B^T B^{-1} b \\ x_B & + & B^{-1} N x_N & = & B^{-1} b \end{array}\right\} \qquad (*)$$

  where $r = c_N - N^T B^{-T} c_B$ is the reduced cost vector.

- To simplify notation we represent $(*)$ as a tableau

BV	x_0	x_B^T	x_N^T	RHS
x_0	1	0^T	$-r^T$	$c_B^T B^{-1} b$
x_B	0	I	$B^{-1} N$	$B^{-1} b$

  where I is the $m \times m$ identity matrix.

Example: Simplex Tableaux

Basic representation (4) from Example 1 ($I = \{1, 2, 5\}$).

Explicit formulation:

$$
\begin{array}{ccccccccccccc}
x_0 & & & & - & \frac{2}{5}x_3 & - & \frac{1}{5}x_4 & & & = & -8 \\
& & & x_2 & - & \frac{1}{5}x_3 & + & \frac{2}{5}x_4 & & & = & 5 \\
& & & & - & \frac{3}{5}x_3 & + & \frac{1}{5}x_4 & + & x_5 & = & 1 \\
& x_1 & & & + & \frac{3}{5}x_3 & - & \frac{1}{5}x_4 & & & = & 3
\end{array}
$$

Basic representation (4) from Example 1 ($I = \{1, 2, 5\}$).

Removing variables and mathematical operators:

$$
\begin{array}{cccccc|c}
1 & & & -\frac{2}{5} & -\frac{1}{5} & & -8 \\
 & & 1 & -\frac{1}{5} & \frac{2}{5} & & 5 \\
 & & & -\frac{3}{5} & \frac{1}{5} & 1 & 1 \\
 & 1 & & \frac{3}{5} & -\frac{1}{5} & & 3
\end{array}
$$

Example: Simplex Tableaux

Basic representation (4) from Example 1 ($I = \{1, 2, 5\}$).

Appending zeroes:

$$
\begin{array}{cccccc|c}
1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -8 \\
0 & 0 & 1 & -\frac{1}{5} & \frac{2}{5} & 0 & 5 \\
0 & 0 & 0 & -\frac{3}{5} & \frac{1}{5} & 1 & 1 \\
0 & 1 & 0 & \frac{3}{5} & -\frac{1}{5} & 0 & 3
\end{array}
$$

Example: Simplex Tableaux

Basic representation (4) from Example 1 ($I = \{1, 2, 5\}$).

Introducing labels:

BV	x_0	x_1	x_2	x_3	x_4	x_5	RHS
x_0	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	0	-8
x_2	0	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	0	5
x_5	0	0	0	$-\frac{3}{5}$	$\frac{1}{5}$	1	1
x_1	0	1	0	$\frac{3}{5}$	$-\frac{1}{5}$	0	3

Rows are indexed by the respective basic variables.

Properties of the Simplex Tableau

- x_0 appears only in objective row and has coefficient 1;
- each basic variable appears in only one other row and has coefficient 1;
- only one basic variable can occur in each row;
- the RHS of the objective row is the objective value of the current BS
- the RHS's of the other rows are the values of the basic variables at the current BS;
- the coefficients of the nonbasic variables in the objective row are the negative reduced costs.

Note: The current BS is feasible iff the RHS's are nonnegative in all rows (except objective row).

Notational Conventions

Tableau for an admissible index set I (with $p \in I$, $q \notin I$):

BV	x_0	x_1	$\cdots$	x_p	$\cdots$	x_q	$\cdots$	x_n	RHS
x_0	1	β_1	$\cdots$	β_p	$\cdots$	β_q	$\cdots$	β_n	β_0
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_p	0	y_{p1}	$\cdots$	y_{pp}	$\cdots$	y_{pq}	$\cdots$	y_{pn}	y_{p0}
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	$\vdots$

- $y_{ii} = 1 \ \forall i \in I$ and $y_{i'i} = 0 \ \forall i \in I, \ i' \in I \backslash \{i\}$
- $\beta_i = -r_i \ \forall i \notin I$, $i \neq 0$ (negative reduced cost)
- $\beta_i = 0 \ \forall i \in I$

Sequences of BFSs

Simplex Algorithm: if BFS x is not optimal, then there is some neighbouring BFS x' with better objective value.

Consider the sequence of BFS's when solving Example 1. We found the following sequence of index sets I_t of BV's:

$$\begin{array}{ccccccc} \{3,4,5\} & \to & \{3,4,1\} & \to & \{2,4,1\} & \to & \{2,5,1\} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ I_1 & & I_2 & & I_3 & & I_4 \end{array}$$

Neighbouring BFS's

I_{t+1} is obtained from I_t by removing one element and replacing it by a new element:

$$|I_t \backslash I_{t+1}| = |I_{t+1} \backslash I_t| = 1.$$

Definition: Two index sets I and I' with $B(I)$ and $B(I')$ non-singular, are said to be neighbours if

$$|I \backslash I'| = |I' \backslash I| = 1.$$

Pivoting

Given a basic representation, suppose a basic variable is to be made nonbasic and a nonbasic variable is to be made basic. What is the new basic representation?

Given: $*$ Two neighbouring index sets I and I'
$*$ Basic representation for I with tableau (y_{ij}, β_j)

Sought: $*$ Basic representation for I' with tableau (y'_{ij}, β'_j)

The basic representation for I' is constructed via pivoting.

Pivoting (cont)

Suppose that $I' = (I \backslash \{p\}) \cup \{q\}$, i.e.,

- the basic variable x_p becomes nonbasic;
- the nonbasic variable x_q becomes basic.

We say that

- x_p leaves the basis;
- x_q enters the basis.

Pivoting (cont)

To swap x_p and x_q we

1. divide row p by the pivot element y_{pq}:

$$y'_{qj} = \frac{y_{pj}}{y_{pq}} \quad \forall j = 0, \ldots, n \tag{1}$$

2. subtract the $\frac{y_{iq}}{y_{pq}}$-fold multiple of row p from row $i \in I \setminus \{p\}$:

$$y'_{ij} = y_{ij} - \frac{y_{iq}}{y_{pq}} y_{pj} \quad \forall j = 0, \ldots, n \tag{2}$$

3. subtract the $\frac{\beta_q}{y_{pq}}$-fold multiple of row p from objective row:

$$\beta'_j = \beta_j - \frac{\beta_q}{y_{pq}} y_{pj} \quad \forall j = 0, \ldots, n \tag{3}$$

Pivoting (Remarks)

- The formulas (1)–(3) are termed pivot equations.
- The pivot equations ensure that:
  - the qth column of the new tableau contains only 0's except for its qth entry, which is 1;
  - the other basic columns remain unchanged.
- Row p is relabelled as row q after pivoting.
- Pivoting is possible iff $y_{pq} \neq 0$.

Pivoting (Example)

Basic representation for $I = \{1, 2, 3\}$:

BV	x_0	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0	1	0	0	0	1	2	1	1
x_1	0	1	0	0	1	1	-1	5
x_2	0	0	1	0	2	-3	1	3
x_3	0	0	0	1	-1	2	-1	-1

Find the neighbouring basic representation for $I' = \{4, 2, 3\}$.

Pivoting (Example)

Exchange: x_1 becomes nonbasic, x_4 becomes basic.

BV	x_0	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0	1	0	0	0	1	2	1	1
x_1	0	1	0	0	$\boxed{1}$	1	-1	5
x_2	0	0	1	0	2	-3	1	3
x_3	0	0	0	1	-1	2	-1	-1

Pivot Element: $y_{14} = 1$

Pivoting (Example)

Exchange: x_1 becomes nonbasic, x_4 becomes basic.

BV	x_0	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0	1	0	0	0	1	2	1	1
x_1	0	1	0	0	$\boxed{1}$	1	-1	5
x_2	0	0	1	0	2	-3	1	3
x_3	0	0	0	1	-1	2	-1	-1
x_0								
x_4	0	1	0	0	1	1	-1	5
x_2								
x_3								

Equation (1) $\Rightarrow$ row for x_1 remains unchanged,
but now relabelled with x_4.

Pivoting (Example)

Exchange: x_1 becomes nonbasic, x_4 becomes basic.

BV	x_0	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0	1	0	0	0	1	2	1	1
x_1	0	1	0	0	$\boxed{1}$	1	-1	5
x_2	0	0	1	0	2	-3	1	3
x_3	0	0	0	1	-1	2	-1	-1
x_0								
x_4	0	1	0	0	1	1	-1	5
x_2	0	-2	1	0	0	-5	3	-7
x_3								

Equation (2) $\Rightarrow$ subtract row for x_1 twice from row for x_2,
no relabelling necessary.

Pivoting (Example)

Exchange: x_1 becomes nonbasic, x_4 becomes basic.

BV	x_0	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0	1	0	0	0	1	2	1	1
x_1	0	1	0	0	$\boxed{1}$	1	-1	5
x_2	0	0	1	0	2	-3	1	3
x_3	0	0	0	1	-1	2	-1	-1
x_0								
x_4	0	1	0	0	1	1	-1	5
x_2	0	-2	1	0	0	-5	3	-7
x_3	0	1	0	1	0	3	-2	4

Equation (2) $\Rightarrow$ add row for x_1 to row for x_3,
no relabelling necessary.

Pivoting (Example)

Exchange: x_1 becomes nonbasic, x_4 becomes basic.

BV	x_0	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0	1	0	0	0	1	2	1	1
x_1	0	1	0	0	$\boxed{1}$	1	-1	5
x_2	0	0	1	0	2	-3	1	3
x_3	0	0	0	1	-1	2	-1	-1
x_0	1	-1	0	0	0	1	2	-4
x_4	0	1	0	0	1	1	-1	5
x_2	0	-2	1	0	0	-5	3	-7
x_3	0	1	0	1	0	3	-2	4

Equation (3) $\Rightarrow$ subtract row for x_1 from objective row, no relabelling necessary.

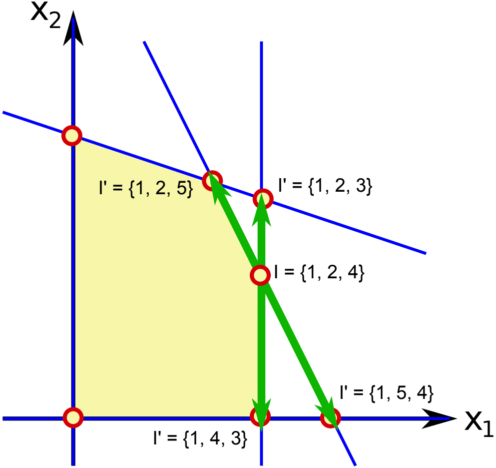
Pivoting (Example)

Exchange: x_1 becomes nonbasic, x_4 becomes basic.

BV	x_0	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0	1	0	0	0	1	2	1	1
x_1	0	1	0	0	$\boxed{1}$	1	-1	5
x_2	0	0	1	0	2	-3	1	3
x_3	0	0	0	1	-1	2	-1	-1
x_0	1	-1	0	0	0	1	2	-4
x_4	0	1	0	0	1	1	-1	5
x_2	0	-2	1	0	0	-5	3	-7
x_3	0	1	0	1	0	3	-2	4

Pivot equations ensure that the other basic columns and the objective column remain unchanged!

Consider the BS for $I = \{1, 2, 4\}$ in Example 1.



We can pivot on:

- $(x_p, x_q) = (x_4, x_5)$
  $\Rightarrow I' = \{1, 2, 5\}$
- $(x_p, x_q) = (x_4, x_3)$
  $\Rightarrow I' = \{1, 2, 3\}$
- $(x_p, x_q) = (x_2, x_3)$
  $\Rightarrow I' = \{1, 4, 3\}$
- $(x_p, x_q) = (x_2, x_5)$
  $\Rightarrow I' = \{1, 5, 4\}$

Pivot Selection (cont)

When choosing a pivot, we impose the following rules:

1. Non-Inferiority: The new BS must have a better objective value than the current BS, i.e., $\beta_0' \leq \beta_0$.
2. Feasibility: The new BS must be feasible, i.e., $y_{i0}' \geq 0$ for all $i \in I'$.

- Rule 1 is used to choose the nonbasic variable x_q, $q \notin I$, which enters the basis.
- Rule 2 is used to choose the basic variable x_p, $p \in I$, which leaves the basis.

Choosing x_q

The objective row of the simplex tableau is equivalent to:

$$x_0 + \sum_{i=1}^{n} \beta_i x_i = \beta_0 \qquad \Longleftrightarrow \qquad x_0 = \beta_0 - \sum_{i \notin I} \beta_i x_i$$

By definition, we have

- $\beta_i = 0$ for all $i \in I$ (basic variables);
- $\beta_i = -r_i$ for all $i \notin I$ (nonbasic variables).

Choosing x_q (cont)

Two cases:

- if $\beta_i \leq 0 \; \forall i \notin I$, then the current BFS is optimal.

$$x_0 = \beta_0 - \sum_{i \notin I} \beta_i x_i \geq \beta_0 \quad \forall x_i \geq 0$$

$\Rightarrow$ No other FS achieves a lower objective value.

- if $\exists q \notin I$ with $\beta_q > 0$, then increasing the nonbasic variable x_q decreases the current objective value.

$\Rightarrow$ $\boxed{\text{Any nonbasic } x_q \text{ with } \beta_q > 0 \text{ can enter the basis.}}$

Note: Decreasing nonbasic variables is not allowed.

Choosing x_p

- Increase some nonbasic x_q with $\beta_q > 0$ while fixing all other nonbasic variables at zero, i.e., $x_i = 0 \; \forall i \notin I \cup \{q\}$.
- The rows for the basic variables x_i, $i \in I$, in the simplex tableau then imply

$$x_i + \sum_{j \notin I} y_{ij} x_j = y_{i0} \quad \text{and} \qquad x_i = y_{i0} - y_{iq} x_q \quad \forall i \in I.$$

- Aim: increase x_q as much as possible while ensuring that all basic variables x_i, $i \in I$, remain nonnegative.

Choosing x_p (cont)

- We must choose the value of $x_q > 0$ such that

$$x_i = y_{i0} - y_{iq} x_q \geq 0 \quad \forall i \in I. \tag{$*$}$$

  Otherwise, some basic variable(s) become infeasible.

- Each $i \in I$ gives a constraint on x_q:

$$x_i = y_{i0} - y_{iq} x_q \geq 0 \iff \begin{cases} x_q \leq \overline{x}_{iq} \triangleq \dfrac{y_{i0}}{y_{iq}} & \text{if } y_{iq} > 0, \\ x_q \leq \overline{x}_{iq} \triangleq \infty & \text{if } y_{iq} \leq 0. \end{cases}$$

$\Rightarrow$ The feasibility requirement $(*)$ is equivalent to

$$x_q \leq \min_{i \in I} \overline{x}_{iq}. \tag{$**$}$$

Choosing x_p (cont)

Two cases:

- Suppose $\min_{i \in I} \overline{x}_{iq} = \infty$ ($\forall i \in I: \; y_{iq} \leq 0$), then
  - x_q can grow indefinitely.
  - As $\beta_q > 0$, the objective value $x_0 = \beta_0 - \beta_q x_q$ can drop indefinitely. $\Rightarrow$ The LP is unbounded below.
- Suppose $\min_{i \in I} \overline{x}_{iq} < \infty$ ($\exists i \in I: \; y_{iq} > 0$), then
  - The best solution is obtained by making x_q as large as possible, i.e., we set $x_q = \min_{i \in I} \overline{x}_{iq}$.
  - $\exists p \in \arg\min_{i \in I} \overline{x}_{iq} \Rightarrow x_q = \overline{x}_{pq} = \frac{y_{p0}}{y_{pq}}$
  - Thus, $x_p = y_{p0} - y_{pq} x_q = 0$ becomes nonbasic!

Pivot Selection (Summary)

- $\boxed{\text{If } \beta_i \leq 0 \text{ for all } i \notin I\text{, then the current BFS is optimal.}}$

  $\Rightarrow$ We are done! Otherwise...

- $\boxed{\text{Any } x_q \text{ with } q \notin I \text{ and } \beta_q > 0 \text{ can enter the basis.}}$

  $\Rightarrow$ Choose a suitable x_q...

- $\boxed{\text{If } y_{iq} \leq 0 \text{ for all } i \in I\text{, then the LP is unbounded.}}$

  $\Rightarrow$ We are done! Otherwise...

- $\boxed{\text{Any } x_p \text{ with } p \in \arg\min_{i \in I} \overline{x}_{iq} \text{ can leave the basis.}}$

  $\Rightarrow$ Choose a suitable x_p...

Recall: $$\overline{x}_{iq} \triangleq \begin{cases} \dfrac{y_{i0}}{y_{iq}} & \text{if } y_{iq} > 0, \\ \infty & \text{if } y_{iq} \leq 0. \end{cases}$$

Pivoting Conventions

- If there are several $q \notin I$ with $\beta_q > 0$, then choose q with

$$\beta_q = \max_{j \notin I} \left\{ \beta_j \right\} . \qquad (*)$$

  Such a q produces the maximum decrease in x_0 per unit of increase in x_q.
- If several $q \notin I$ satisfy $(*)$, choose the one with the smallest index.
- If there are several $p \in \arg\min_{i \in I} \overline{x}_{iq}$, choose the one with the smallest index.

Simplex Algorithm (Minimisation)

- Step 0: Find initial BFS and its basic representation.
- Step 1: If $\beta_i \leq 0$ for all $i \notin I$:
  STOP — the current BFS is optimal.
- Step 2: If $\exists j \notin I$ with $\beta_j > 0$ and $y_{ij} \leq 0$ for all $i \in I$:
  STOP — no finite minimum exists.
- Step3: Choose x_q with $\beta_q > 0$
  Entry criterion — x_q enters the basis.
- Step 4: Choose $p \in \arg\min_{i \in I} \overline{x}_{iq}$
  Exit criterion — x_p leaves the basis.
- Step 5: Pivot on y_{pq} and go to STEP 1.

Example (Simplex Algorithm)

minimize:

$$x_0 = -4x_1 - 2x_2 + x_3$$

Subject to:

$$\begin{array}{rrrl} x_1 & +x_2 & +x_3 & \leq 4 \\ x_1 & -x_2 & -2x_3 & \leq 3 \\ 3x_1 & +2x_2 & +x_3 & \leq 12 \end{array}$$

$$x_1, x_2, x_3 \geq 0.$$

Example (Simplex Algorithm)

Adding slack variables x_4, x_5, x_6 we get:

minimize:

$$x_0 = -4x_1 - 2x_2 + x_3$$

Subject to:

$$\begin{array}{rrrrrrl} x_1 & +x_2 & +x_3 & +x_4 & & & = 4 \\ x_1 & -x_2 & -2x_3 & & +x_5 & & = 3 \\ 3x_1 & +2x_2 & +x_3 & & & +x_6 & = 12 \end{array}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

We have thus an initial BFS for $I = \{4, 5, 6\}$.

Example (Simplex Algorithm)

BV	x_0	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0	1	4	2	-1				0
x_4		1	1	1	1			4
x_5		1	-1	-2		1		3
x_6		3	2	1			1	12
x_0	1		6	7		-4		-12
x_4			2	3	1	-1		1
x_1		1	-1	-2		1		3
x_6			5	7		-3	1	3

Example (Simplex Algorithm)

BV	x_0	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0	1		6	7		-4		-12
x_4			2	3	1	-1		1
x_1		1	-1	-2		1		3
x_6			5	7		-3	1	3
x_0	1		$\frac{4}{3}$		$-\frac{7}{3}$	$-\frac{5}{3}$		$-\frac{43}{3}$
x_3			$\frac{2}{3}$	1	$\frac{1}{3}$	$-\frac{1}{3}$		$\frac{1}{3}$
x_1		1	$\frac{1}{3}$		$\frac{2}{3}$	$\frac{1}{3}$		$\frac{11}{3}$
x_6			$\frac{1}{3}$		$-\frac{7}{3}$	$-\frac{2}{3}$	1	$\frac{2}{3}$

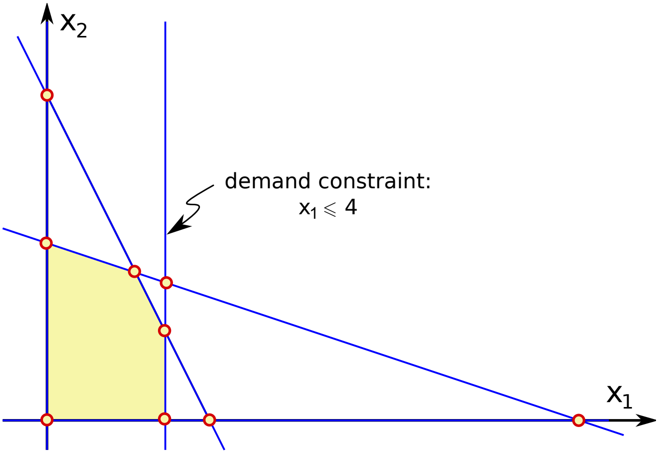
Example (Simplex Algorithm)

BV	x_0	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0	1		$\frac{4}{3}$		$-\frac{7}{3}$	$-\frac{5}{3}$		$-\frac{43}{3}$
x_3			$\frac{2}{3}$	1	$\frac{1}{3}$	$-\frac{1}{3}$		$\frac{1}{3}$
x_1		1	$\frac{1}{3}$		$\frac{2}{3}$	$\frac{1}{3}$		$\frac{11}{3}$
x_6			$\frac{1}{3}$		$-\frac{7}{3}$	$-\frac{2}{3}$	1	$\frac{2}{3}$
x_0	1			-2	-3	-1		-15
x_2			1	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$		$\frac{1}{2}$
x_1		1		$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{7}{2}$
x_6				$-\frac{1}{2}$	$-\frac{5}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$

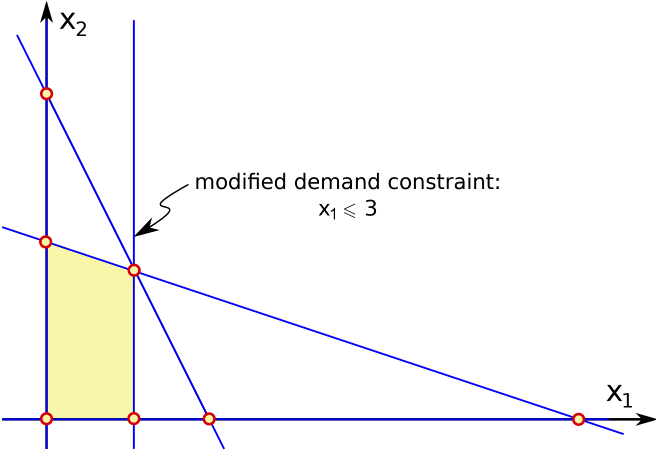
Solution: $x_1 = \frac{7}{2}$, $x_2 = \frac{1}{2}$, and $x_3 = 0$.

simplex algorithm terminate in the finite time

Feasible set of Example 1:

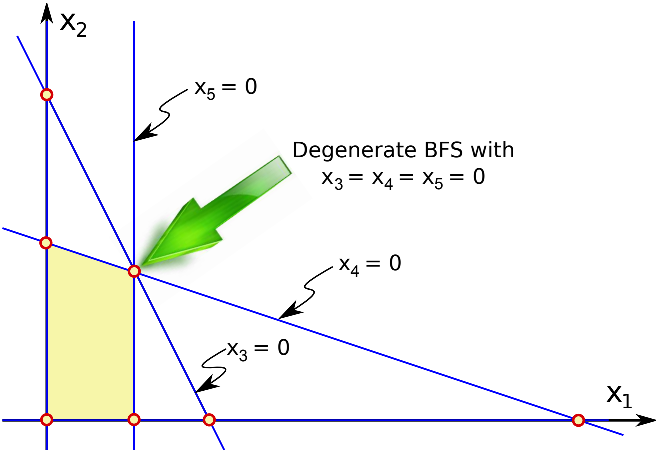


Consider now a variant of Example 1:



Degenerate BS's (cont)

The highlighted BS corresponds to the index sets
$I = \{1, 2, 3\}$, $I = \{1, 2, 4\}$ and $I = \{1, 2, 5\}$.



Degenerate BFS's: Definition

Definition: A BS is called degenerate if more than $n - m$ of its components are zero.

$\Rightarrow$ A degenerate BS has at least one zero-valued basic variable (next to the $n - m$ zero-valued nonbasic variables).

$\Rightarrow$ For a degenerate BS with index set I there exists at least one $i \in I$ with $y_{i0} = 0$.

Definition: A BS is called non-degenerate if all of its basic variables are different from zero.

Finite Termination

Theorem: If all BS's are non-degenerate, then the simplex algorithm must terminate after a finite number of steps with

- either an optimal solution
- or a proof that the problem is unbounded.

Finite Termination (Proof)

- At each step we have $y_{i0} > 0 \; \forall i \in I$ (non-degeneracy).
- Unless optimality or unboundedness is detected in STEP 1 or 2, we find $\beta_0' = \beta_0 - \frac{\beta_q}{y_{pq}} y_{p0} < \beta_0$.
- Thus, the sequence of objective values obtained by the algorithm is strictly decreasing.

$$\beta_0 > \beta_0' > \beta_0'' > \cdots$$

  No basic solution will ever be repeated!

- There are $\leq \binom{n}{m}$ basic solutions. Thus, the process cannot continue indefinitely and must terminate at STEP 1 or 2 after a finite number of iterations.

$\square$