FOUNDATIONS OF OPTIMIZATION: IE6001

LP Duality

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Semester I, AY2022/2023

Primal LP

Consider an LP in standard form (primal):

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0, \end{array}$$

with an optimal solution $x^\star$.

Relax the constraints $Ax = b$ by introducing a penalty $p \in \mathbb{R}^m$

$$\begin{array}{rll} g(p) = & \text{minimize} & c^T x + p^T(b - Ax) \\ & \text{subject to} & x \geq 0 \end{array}$$

Lower bound property:

$$g(p) \leq c^T x^\star + p^T(b - Ax^\star) = c^T x^\star$$

Dual LP

Searching for the best possible lower bound:

$$
\begin{aligned}
\max_p \; g(p) &= \max_p \min_{x \geq 0} p^T b + (c - A^T p)^T x \\
&= \max_p p^T b + \min_{x \geq 0} (c - A^T p)^T x \\
&= \max_p p^T b + \begin{cases} 0 & \text{if } c - A^T p \geq 0 \\ -\infty & \text{otherwise} \end{cases}
\end{aligned}
$$

is equivalent to solving another linear program (dual):

$$
\begin{array}{ll}
\text{maximize} & b^T p \\
\text{subject to} & A^T p \leq c.
\end{array}
$$

Primal-Dual Pair

Primal LP:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual LP:

$$\begin{aligned} \text{maximize} \quad & b^T p \\ \text{subject to} \quad & A^T p \leq c \end{aligned}$$

Main result in duality theory:

- The dual of dual coincides with the primal.
- The optimal objective value of the dual problem is equal to that of the primal problem, i.e. $c^T x^\star$.
- When p is chosen optimally, then the option of violating the primal constraints $Ax = b$ is of no value.

Dual of Dual

Standardizing the dual LP

$$\begin{array}{rl} -\text{ minimize} & (-b)^T(p^+ - p^-) \\ \text{subject to} & A^T(p^+ - p^-) + s = c \\ & p^+ \geq 0, p^- \geq 0, s \geq 0, \end{array}$$

allows us to derive the dual of the dual

$$\begin{array}{rl} -\text{ maximize} & c^T y \\ \text{subject to} & Ay \leq -b,\ -Ay \leq b \\ & y \leq 0, \end{array} \Longrightarrow \begin{array}{rl} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0, \end{array}$$

which is the primal.

Shortcut for Deriving the Dual

Primal LP:
($a_{i\cdot} :=$ i^{th} row of A)

$$
\begin{array}{lll}
\min & c^T x & \\
\text{s.t.} & a_{i\cdot} x \geq b_i & i \in \mathcal{M}_1 \\
 & a_{i\cdot} x \leq b_i & i \in \mathcal{M}_2 \\
 & a_{i\cdot} x = b_i & i \in \mathcal{M}_3 \\
 & x_j \geq 0 & j \in \mathcal{N}_1 \\
 & x_j \leq 0 & j \in \mathcal{N}_2 \\
 & x_j \text{ free} & j \in \mathcal{N}_3
\end{array}
$$

Dual LP:
($a_j :=$ j^{th} column of A)

$$
\begin{array}{lll}
\max & b^T p & \\
\text{s.t.} & a_j^T p \leq c_j & j \in \mathcal{N}_1 \\
 & a_j^T p \geq c_j & j \in \mathcal{N}_2 \\
 & a_j^T p = c_j & j \in \mathcal{N}_3 \\
 & p_i \geq 0 & i \in \mathcal{M}_1 \\
 & p_i \leq 0 & i \in \mathcal{M}_2 \\
 & p_i \text{ free} & i \in \mathcal{M}_3
\end{array}
$$

Shortcut for Deriving the Dual

PRIMAL	minimize	maximize	DUAL
constraints	$\geq b_i$	≥ 0	variables
	$\leq b_i$	≤ 0	
	$= b_i$	free	
variables	≥ 0	$\leq c_j$	constraints
	≤ 0	$\geq c_j$	
	free	$= c_j$	

Example

Primal LP:

$$
\begin{array}{rcccccccl}
\underset{x_1 \geq 0, x_2 \leq 0, x_3}{\text{minimize}} & x_1 & + & 2x_2 & + & 3x_3 & & & \\
\text{subject to} & -x_1 & + & 3x_2 & & & = & 5 & \quad (p_1 \text{ free}) \\
 & 2x_1 & - & x_2 & + & 3x_3 & \geq & 6 & \quad (p_2 \geq 0) \\
 & & & & & x_3 & \leq & 4 & \quad (p_3 \leq 0)
\end{array}
$$

Dual LP:

$$
\begin{array}{rcccccccl}
\underset{p_2 \geq 0, p_3 \leq 0, p_1}{\text{maximize}} & 5p_1 & + & 6p_2 & + & 4p_3 & & \\
\text{subject to} & -p_1 & + & 2p_2 & & & \leq & 1 \\
 & 3p_1 & - & p_2 & & & \geq & 2 \\
 & & & 3p_2 & + & p_3 & = & 3
\end{array}
$$

Dual of the dual LP:

$$\begin{array}{rrcrcrcl}
\underset{y_1\geq 0, y_2\leq 0, y_3}{\text{minimize}} & y_1 & + & 2y_2 & + & 3y_3 & & \\
\text{subject to} & -y_1 & + & 3y_2 & & & = & 5 \\
 & 2y_1 & - & y_2 & + & 3y_3 & \geq & 6 \\
 & & & & & y_3 & \leq & 4
\end{array}$$

Dual LP:

$$\begin{array}{rrcrcrclc}
\underset{p_2\geq 0, p_3\leq 0, p_1}{\text{maximize}} & 5p_1 & + & 6p_2 & + & 4p_3 & & & \\
\text{subject to} & -p_1 & + & 2p_2 & & & \leq & 1 & (y_1 \geq 0) \\
 & 3p_1 & - & p_2 & & & \geq & 2 & (y_2 \leq 0) \\
 & & & 3p_2 & + & p_3 & = & 3 & (y_3 \text{ free})
\end{array}$$

Weak Duality

Theorem: If x is primal feasible and p is dual feasible, then

$$b^T p \leq c^T x.$$

Proof: WLOG, consider an LP in standard form. It follows that

$$p^T b = p^T(Ax) = (A^T p)^T x \leq c^T x.$$

Corollary: If x is primal feasible, p is dual feasible, $b^T p = c^T x$, then x and p are optimal in their respective problem.

Strong Duality

Theorem: If a (primal) LP has an optimal solution, then so does the dual, and the respective optimal costs are equal.

Proof: WLOG, consider an LP in standard form and suppose that B is an optimal basis discovered by the simplex algorithm.

$$r = c_N - N^T B^{-T} c_B \geq 0$$

Choose $p = B^{-T} c_B$[1], we then have

$$N^T p \leq c_N, \; B^T p = c_B \implies A^T p \leq c, \qquad \text{(dual feas.)}$$

and

$$p^T b \; = \; c_B^T (B^{-1} b) = c_B^T x_B. \qquad \text{(dual obj.)}$$

Hence, by weak duality, p is optimal in the dual.

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[1] shadow prices

Strong Duality

Strong duality may fail if the assumption is violated.

Primal LP:

$$\begin{aligned} \text{minimize} \quad & 1x \\ \text{subject to} \quad & 0x \geq 1 \end{aligned}$$

Dual LP:

$$\begin{aligned} \text{maximize} \quad & 1p \\ \text{subject to} \quad & 0p = 1 \\ & p \geq 0 \end{aligned}$$

Both of the LPs are infeasible.

- The optimal objective of the primal (min) LP is $+\infty$.
- The optimal objective of the dual (max) LP is $-\infty$.

The Different Possibilities

Let $P^\star$ denote the optimal objective value of the primal LP & $D^\star$ denote the optimal objective value of the dual LP.

- $P^\star = D^\star = \begin{cases} \text{finite value} \\ -\infty \\ +\infty \end{cases}$

- $P^\star = +\infty > -\infty = D^\star$ (i.e., both are infeasible)

Ex. $P^\star = D^\star = -\infty$

$$\begin{array}{ll} \text{minimize} & 1x \\ \text{subject to} & 1x \le 0 \end{array}$$

What's the dual?

Ex. $P^\star = D^\star = +\infty$

$$\begin{array}{ll} \text{maximize} & 1x \\ \text{subject to} & 1x \ge 0 \end{array}$$

What's the primal?

Complementary Slackness

Theorem: Let x be primal feasible and p be dual feasible. Then, x and p are optimal in their respective problem iff

$$p_i(a_{i\cdot}x - b_i) = 0 \quad \forall i = 1, \ldots, m$$
$$x_j(c_j - a_j^T p) = 0 \quad \forall j = 1, \ldots, n$$

Proof:

- Define $u_i = p_i(a_{i\cdot}x - b_i)$ and $v_j = x_j(c_j - a_j^T p)$.
- It follows that $u_i \geq 0 \;\; \forall i$ and $v_j \geq 0 \;\; \forall j$.[2]

$$c^T x - b^T p = x^T(c - A^T p) + p^T(Ax - b) = \sum_j v_j + \sum_i u_i$$

- Hence, $c^T x = b^T p$ iff $u_i = 0 \;\; \forall i$ and $v_j = 0 \;\; \forall j$.

[2] see shortcut for deriving the dual

- Similar to the reduced cost vector, the dual LP can provide a certificate of optimality.
- The optimal dual solution is comprised of shadow prices.
- The dual LP may be easier to solve than the primal LP.

Ex. Solve the following LP:

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{n} x_i \\ \text{subject to} & x_i + x_{i+1} \geq 1 \quad i = 1, \ldots, n-1 \end{array}$$