Foundations of Inventory Management **Fall 2022, NUS**

Lecture 7: Dynamic Stochastic Inventory Model I

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7.1 Basic Model

Similar to the Wagner-Whitin Model, consider a planning horizon consisting of $t = 0, 1, ..., T-1$ periods. Demand in each period t, however, is now random, denoted as D_t, which is assumed to be independent and identically distributed according to cdf $F(\cdot)$ with pdf $f(\cdot)$ on $\mathbb{R}_+$. Here, we assume that there is no fixed ordering cost, and each order only incurs a variable ordering cost c per unit ordered. On-hand inventory incurs holding cost h and backlogged demand incurs backlogging cost p. We consider a discounting rate of $\alpha \in [0, 1]$, and assume that $p > (1-\alpha)c$, i.e., it is not optimal to not ordering anything forever ($\sum_{t=0}^{\infty} \alpha^t p > c$).

Let q_t be the amount ordered in period t and x_t be the amount of the inventory or backlogged demand *at the beginning* of period t. It is often more convenient to use the inventory level immediately after ordering (also known as order-up-to level) at period t, denoted as, y_t as the decision variable. Note that $y_t = x_t + q_t$.

As in the newsvendor model, given a current inventory level y, we let

$$L(y) = h\mathbb{E}[(y - D)^+] + p\mathbb{E}[(D - y)^+]$$

denote the expected overage and underage cost in any period. The end-of-hoirizon cost is given by a general convex function $v_T(x)$.

The problem is then to find a policy $\pi = \{y_0(\cdot), ..., y_T(\cdot)\}$ that minimizes

$$\mathbb{E}\left[\sum_{t=0}^{T-1} \alpha^t \left(c(y_t(x_t) - x_t) + L(y_t(x_t))\right) + \alpha^T v_T(x_T)\right]$$

subject to the constraint that $y_t(x_t) \geq x_t$ and the state transition $x_{t+1} = y_t - D_t$.

Let $v_t(x)$ be the cost-to-go function from period t to $T-1$. Then the Bellman equation to this problem can be written as

$$\begin{aligned} v_t(x) &= \min_{y \geq x}\{c(y - x) + L(y) + \alpha\mathbb{E}[v_{t+1}(y - D)]\} \\ &= \min_{y \geq x}\{\underbrace{cy + L(y) + \alpha\mathbb{E}[v_{t+1}(y - D)]}_{G_t(y)}\} - cx \end{aligned}$$

for $t = 0, ..., T-1$ with the terminal condition $v_T(x)$. Note that when $T = 1, v_T(x) = 0, x_0 = 0$ (or when $\alpha = 0$), the problem reduces to the newsvendor problem with the optimal solution $y^* = F^{-1}((p-c)/(p+h))$.

When $T = 1, v_T(x) = 0$ but $x_0 \in \mathbb{R}$, we then need to solved a constrained-version of the newsvendor problem:

$$v_0(x) = \min_{y \geq x}\{cy + L(y)\}.$$

Due to convexity, the optimal solution in this case is then

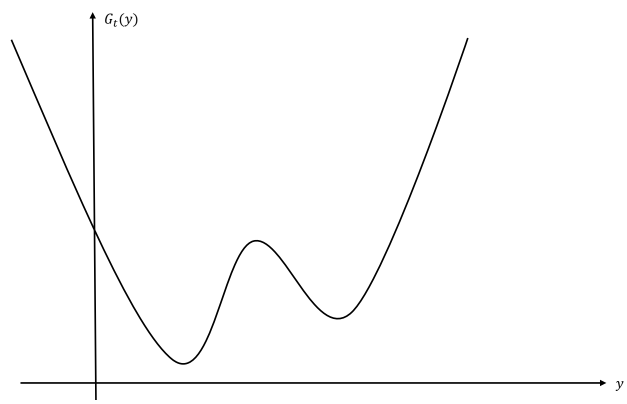
$$y^*(x) = \begin{cases} F^{-1}((p - c)/(p + h)) & \text{if } x \leq F^{-1}((p - c)/(p + h)) \\ x & \text{otherwise.} \end{cases}$$

The policy characterized here is known as the *base-stock policy*, where one only needs to monitor a single parameter—the base-stock level: $s = F^{-1}((p-c)/(p+h))$. If the current inventory level is above the base-stock level, then no order takes place; otherwise one orders to the base-stock level.

Now, let's turn to the general problem:

$$v_t(x) = \min_{y \geq x}\{G_t(y)\} - cx.$$

What is the property required for $G_t(y)$ in order for the base-stock policy to be optimal? For example, if $G_t(y)$ has the shape in the figure below, then a base-stock policy will no longer be optimal. This will never happen, if, for instance, one can show that $G_t(y)$ is convex in y.



Lemma 7.1 *If v_{t+1} is convex, then*

(a) G_t is convex;

(b) A base-stock policy is optimal in period t, and the base-stock level s_t is the global minimizer[1] *of G_t;*

(c) v_t is convex.

Proof: For Part (a), by definition

$$G_t(y) = cy + L(y) + \alpha\mathbb{E}[v_{t+1}(y-D)].$$

Since $L(y)$ is convex, and the convexity of v_{t+1} implies that $\mathbb{E}[v_{t+1}(y-D)]$ is also convex, we know that $G_t(y)$ is convex in y.

Part (b) is then a direct consequence of the convexity of G_t.

For Part (c), since $\{(x,y)|y \geq x\}$ is convex, and $G_t(y)$ is convex in (x,y), by preservation of convexity under minimization, we then know that $\min_{y\geq x}\{G_t(y)\}$ is convex in x and hence $v_t(x)$ is also convex. ∎

Lemma 7.1 then immediately provides a characterization for the optimal policy of the dynamic program.

Theorem 7.2 *A base stock policy is optimal in each period for our finite-horizon problem.*

[1] We adopt the convention that the smallest solution is selected if multiple solutions exist.

Proof: With Lemma 7.1 and the assumption that the terminal condition $v_T(x)$ is convex, then we can prove that $v_t(x)$ is convex for all $t = 0, ..., T$. As a result, by Lemma 7.1 (b), a base stock policy is then optimal in every period. ■

Note that the base-stock level s_t in each period could be different, and one still need to rely on computational methods to compute them. With the base-stock level, we can further write the value functions more explicitly as

$$v_t(x) = \begin{cases} G_t(s_t) - cx & \text{if } x < s_t \\ G_t(x) - cx & \text{otherwise.} \end{cases}$$

Now consider a special case when the terminal condition is $v_T(x) = -cx$. That is, if $x > 0$, the firm can salvage the leftover inventory to the manufacturer at the purchase price; if $x < 0$, the firm must place another order to satisfy all backlogged demand. When $t = T - 1$, we have

$$\begin{aligned} v_{T-1}(x) &= \min_{y \geq x} \{cy + L(y) + \alpha \mathbb{E}[-c(y - D)]\} - cx \\ &= \min_{y \geq x} \{c(1-\alpha)y + L(y)\} + \alpha c\mu - cx. \end{aligned}$$

The base-stock level at period $T - 1$ is then given by the solution to the newsvendor problem with ordering cost $c(1 - \alpha)$:

$$s_{T-1} = F^{-1}\left(\frac{p - (1-\alpha)c}{p + h}\right).$$

By

$$v_{T-1}(x) = \begin{cases} G_{T-1}(s_{T-1}) - cx & \text{if } x < s_{T-1} \\ G_{T-1}(x) - cx & \text{otherwise,} \end{cases}$$

we have

$$v'_{T-1}(x) = \begin{cases} -c & \text{if } x < s_{T-1} \\ G'_{T-1}(x) - c & \text{if } x > s_{T-1}. \end{cases}$$

By definition $G'_{T-2}(y) = c + L'(y) + \alpha \mathbb{E}[v'_{T-1}(y - D)]$. Since $s_{T-1} - D < s_{T-1}$ almost surely, we have

$$G'_{T-2}(s_{T-1}) = c + L'(s_{T-1}) - \alpha c = (1-\alpha)c + L'(s_{T-1}) = 0.$$

That is, $s_{T-2} = s_{T-1}$. The argument here generalizes to any period as we show next.

Theorem 7.3 *If $v_T(x) = -cx$, then the base-stock policy with a stationary base stock level:*

$$s = F^{-1}\left(\frac{p - (1-\alpha)c}{p + h}\right)$$

is optimal in every period.

Proof: We prove via induction. Suppose $v'_{t+1}(x) = -c$ for $x < s$, then the base-stock policy with a base stock level s is optimal in period t and $v'_t(x) = -c$ for $x < s$. The boundary case when $t = T - 1$ is already proved above. For general t, since $s - D < s$ almost surely, we again have

$$G'_t(s) = (1-\alpha)c + L'(s) = 0.$$

Hence, the base-stock policy with a base stock level s is optimal in period t. As a result,

$$v_t(x) = \begin{cases} G_t(s) - cx & \text{if } x < s \\ G_t(x) - cx & \text{otherwise,} \end{cases}$$

which implies $v'_t(x) = -c$ for $x < s$. ■

7.2 Lost Sales Model

Instead of assuming that unsatisfied demand can be backlogged, consider the case when the unsatisfied demand is lost. This then changes the state transition to $x_{t+1} = (y_t - D_t)^+$ and the corresponding Bellman equation becomes

$$v_t(x) = \min_{y \geq x}\{cy + L(y) + \alpha\mathbb{E}[v_{t+1}((y-D)^+)]\} - cx.$$

If we want to repeat the argument used in the backlogged model, then we need to inductively show

$$G_t(y) = cy + L(y) + \alpha\mathbb{E}[v_{t+1}((y-D)^+)]$$

is convex in y for all $t = 0, .., T-1$. Now, if v_{t+1} is convex, is G_t convex? Unfortunately not necessarily since $v_{t+1}((y-D)^+)$ is now a composition of two convex functions. The composition of the two convex functions is still convex, if we further know that v_{t+1} is increasing convex. If v_{t+1} is increasing convex, then $G_t(y)$ would be convex and the base-stock policy would be optimal in period t. However, to complete the induction argument, we need to further show that v_t is also increasing convex. While convexity of v_t follows from the preservation of convexity under minimization, the increasing property is not preserved as evident from the expression

$$v_t(x) = \begin{cases} G_t(s_t) - cx & \text{if } x < s_t \\ G_t(x) - cx & \text{otherwise.} \end{cases}$$

This suggests that convexity could be too strong to pursue here. A weaker concept is quasi-convexity. A function f is said to be quasi-convex on a convex set $\mathcal{X}$ if for all $\lambda \in [0,1]$ and $x, y \in \mathcal{X}$, we have

$$f(\lambda x + (1-\lambda)y) \leq f(x) \vee f(y).$$

For one dimensional function, if we let x^* be the global minimizer of f, then given any $x_1 \leq x_2 \leq x^*$, we can always find λ such that $x_2 = \lambda x_1 + (1-\lambda)x^*$. By definition, we then have

$$f(x_2) \leq f(x_1) \vee f(x^*) = f(x_1).$$

That is, $f(x)$ is decreasing when $x \leq x^*$ and one can similarly show that $f(x)$ is increasing when $x \geq x^*$. Unlike convexity, quasi-convexity, however, is not preserved under addition. For example, $f_1(x) = \log(x)$ and $f_2(x) = -x$ are both quasi-convex functions on $\mathbb{R}_+$. The function $\log(x) - x$, however, is concave.

Now consider the case when $V_T(x) = -cx$. By letting $\tilde{v}_t(x) = v_t(x) + cx$, we can rewrite the Bellman equation in terms of $\tilde{v}_t$ as

$$\begin{aligned} \tilde{v}_t(x) &= \min_{y \geq x}\{cy + L(y) + \alpha\mathbb{E}[\tilde{v}_{t+1}((y-D)^+) - c(y-D)^+]\} \\ &= \min_{y \geq x}\{\tilde{L}(y) + \alpha\mathbb{E}[\tilde{v}_{t+1}((y-D)^+)]\} \end{aligned}$$

with the terminal condition $\tilde{v}_T(x) = 0$. Here, we have let

$$\begin{aligned} \tilde{L}(y) &= cy + L(y) - \alpha c\mathbb{E}[(y-D)^+] \\ &= cy + (h - \alpha c)\mathbb{E}[(y-D)^+] + p\mathbb{E}[(D-y)^+]. \end{aligned}$$

The following result shows that as long as $\tilde{L}(y)$ is well-behaved, one can still establish the optimality of stationary base-stock policy.

Theorem 7.4 *Suppose $\tilde{L}(y)$ is any quasi-convex function and let $s = \arg\min \tilde{L}(y)$. Then,*

$$\tilde{G}_t(y) = \tilde{L}(y) + \alpha\mathbb{E}[\tilde{v}_{t+1}((y-D)^+)]$$

is also quasi-convex for any $0 \leq t \leq T-1$ and a base-stock policy with base-stock level s is optimal.

Proof: We prove by induction. When $t = T-1$, the conclusion clearly holds. Suppose now that $\tilde{L}(y) + \alpha\mathbb{E}[\tilde{v}_{t+2}((y-D)^+)]$ is quasi-convex with the global minimizer s. We next show that $\tilde{L}(y)+\alpha\mathbb{E}[\tilde{v}_{t+1}((y-D)^+)]$ is also quasi-convex with the global minimizer s.

Suppose first that $s \le y_t \le y_t'$. We want to establish that

$$\tilde{L}(y_t) + \alpha\mathbb{E}[\tilde{v}_{t+1}((y_t - D_t)^+)] \le \tilde{L}(y_t') + \alpha\mathbb{E}[\tilde{v}_{t+1}((y_t' - D_t)^+)].$$

By quasi-convexity of $\tilde{L}(y)$, we already have $\tilde{L}(y_t) \le \tilde{L}(y_t')$. For any realization d_t, we let $x_{t+1} = (y_t - d_t)^+$ and $x_{t+1}' = (y_t' - d_t)^+$, and note that we must have $x_{t+1} \le x_{t+1}'$ for any realization d_t. By definition, we have

$$\tilde{v}_{t+1}(x_{t+1}) = \min_{y \ge x_{t+1}} \{\tilde{L}(y) + \alpha\mathbb{E}[\tilde{v}_{t+2}((y-D)^+)]\}$$
$$\tilde{v}_{t+1}(x_{t+1}') = \min_{y \ge x_{t+1}'} \{\tilde{L}(y) + \alpha\mathbb{E}[\tilde{v}_{t+2}((y-D)^+)]\}.$$

By $x_{t+1} \le x_{t+1}'$, we then have $\tilde{v}_{t+1}(x_{t+1}) \le \tilde{v}_{t+1}(x_{t+1}')$ for any realization d_t, and hence $\tilde{G}_t(y_t) \le \tilde{G}_t(y_t')$.

Suppose now that $y_t \le y_t' \le s$. Again, we have $\tilde{L}(y_t) \ge \tilde{L}(y_t')$, and for any realization d_t

$$x_{t+1} = (y_t - d_t)^+ \le (y_t' - d_t)^+ = x_{t+1}' \le s.$$

By our induction hypothesis, the global minimizer s of $\tilde{G}_{t+1}(y)$ is then feasible for both problems at states x_{t+1} and x_{t+1}' respectively. Hence,

$$\tilde{v}_{t+1}(x_{t+1}) = \tilde{v}_{t+1}(x_{t+1}') = \tilde{G}_{t+1}(s).$$

As a result, we have $\tilde{G}_t(y_t) \ge \tilde{G}_t(y_t')$. ■

Recall that $s_b = F^{-1}\left(\frac{p-(1-\alpha)c}{p+h}\right)$ is the base-stock level for the back-order model, and let $s_l = F^{-1}\left(\frac{p-c}{p+h-(1-\alpha)c}\right)$ be the base-stock level for the lost-sales model and $s_m = F^{-1}\left(\frac{p-c}{p+h}\right)$ be the newsvendor solution. It can be shown that $s_b \ge s_l \ge s_m$.

7.3 Model with Lead Time (Karlin and Scarf, 1958)

Here we consider another variation based on the backlogging model: when an order is placed at time t, it can only arrive at peirod $t+L$, where L is referred to as the lead time. In this case, one not only needs to monitor the current inventory level x_0 but also all the orders in the pipeline: q_0—order placed L periods ago and arriving today; q_1—order placed $L-1$ periods ago and arriving tomorrow; all the way to q_{L-1}—order placed yesterday and arriving $L-1$ periods later. Suppose we order q at period t and demand is realized to be d. Then the state transition is

$$(x_0, q_0, ..., q_{L-2}, q_{L-1}) \Longrightarrow (x_0 + q_0 - d, q_1, ..., q_{L-1}, q).$$

Let $v_t(x_0, q_0, ..., q_{L-1})$ be the cost-to-go from period t to $T-1$ when the current state is $(x_0, q_0, ..., q_{L-1})$. The Bellman equation for v_t can be written as

$$v_t(x_0, q_0, ..., q_{L-1}) = \min_{q \ge 0}\{cq + L(x_0 + q_0) + \alpha\mathbb{E}\left[v_{t+1}(x_0 + q_0 - D, q_1, ..., q)\right]\},$$

and we let the terminal condition be $v_T(\cdot) \equiv 0$.

When $L = 1$, the model reduces to

$$v_t(x_0, q_0) = \min_{q\geq 0}\{cq + L(x_0 + q_0) + \alpha\mathbb{E}\left[v_{t+1}(x_0 + q_0 - D, q)\right]\}.$$

An observation here is that the optimal q^* depends on the state (x_0, q_0) only through $x_0 + q_0$. Hence, we can define a new state $x = x_0 + q_0$ which can be interpreted as the inventory level after all outstanding orders have arrived. Correspondingly, let $\tilde{v}_t(x)$ be the cost-to-go from period t to $T-1$ when the current inventory level after all outstanding orders have arrived is x. The Bellman equation for $\tilde{v}_t$ is then

$$\begin{aligned}\tilde{v}_t(x) &= \min_{q\geq 0}\{cq + L(x) + \alpha\mathbb{E}\left[\tilde{v}_{t+1}(x - D + q)\right]\},\\ &= \min_{y\geq x}\{cy + \alpha\mathbb{E}\left[\tilde{v}_{t+1}(y - D)\right]\} + L(x) - cx,\end{aligned}$$

with $\tilde{v}_T(x) = 0$. It is straightforward to show via induction that $v_t(x_0, q_0) = \tilde{v}(x_0 + q_0)$ for all $t = 0, 1..., T$, and $\tilde{v}_t(x)$ is convex in x. By letting s_t be the global minimizer of $cy + \alpha\mathbb{E}\left[\tilde{v}_{t+1}(y - D)\right]$, a base-stock policy with base-stock level s_t is then optimal. Note that the base-stock level at the last period: $s_{T-1} = -\infty$, i.e., one never orders in the last period since it will not arrive in time anyway.

When $L = 2$, the model reduces to

$$v_t(x_0, q_0, q_1) = \min_{q\geq 0}\{cq + L(x_0 + q_0) + \alpha\mathbb{E}\left[v_{t+1}(x_0 + q_0 - D, q_1, q)\right]\}.$$

Similar to the case when $L = 1$, the optimal q^* depends on the state (x_0, q_0) only through $x_0 + q_0$, and we can rewrite the Bellman equation as

$$\begin{aligned}\tilde{v}_t(x, q_1) &= \min_{q\geq 0}\{cq + L(x) + \alpha\mathbb{E}\left[\tilde{v}_{t+1}(x - D + q_1, q)\right]\}\\ &= L(x) + \min_{q\geq 0}\{cq + \alpha\mathbb{E}\left[\tilde{v}_{t+1}(x + q_1 - D, q)\right]\}.\end{aligned}$$

One can now again see that q^* only depends on the state (x, q_1) through $x + q_1$. We let

$$g_t(x + q_1) = \min_{q\geq 0}\{cq + \alpha\mathbb{E}\left[\tilde{v}_{t+1}(x + q_1 - D, q)\right]\}.$$

Then $\tilde{v}_t(x, q_1) = L(x) + g_t(x + q_1)$, and $\tilde{v}_{t+1}(x + q_1 - d, q) = L(x + q_1 - d) + g_{t+1}(x + q_1 - d + q)$. By letting $\tilde{x} = x + q_1$, g_t can be recursively computed via

$$\begin{aligned}g_t(\tilde{x}) &= \min_{q\geq 0}\{cq + \alpha\mathbb{E}\left[L(\tilde{x} - D)\right] + \alpha\mathbb{E}\left[g_{t+1}(\tilde{x} - D + q)\right]\}\\ &= \min_{y\geq \tilde{x}}\{cy + \alpha\mathbb{E}\left[g_{t+1}(y - D)\right]\} + \alpha\mathbb{E}\left[L(\tilde{x} - D)\right] - c\tilde{x}\end{aligned}$$

with the boundary condition $g_{T-1}(\tilde{x}) = 0$. Again one can show that $g_t(\tilde{x})$ is convex via induction and hence a base-stock policy is optimal. The state we are monitoring in the base-stock policy is now $\tilde{x} = x_0 + q_0 + q_1$ which represents the inventory level plus all orders in transit. The same argument can be repeatedly used to establish the optimality of base-stock policy for the model with any lead time L.

The above argument fails if one considers a lost-sales model instead. The optimal policy in this case could be very complicated and the structural results are very limited. One is referred to Zipkin (2008) for some characterizations and Xin and Goldberg (2016) for the analysis of a simple heuristic for this model.

References

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