FOUNDATIONS OF OPTIMIZATION: IE6001

Basic Solutions

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Assumptions

- From now on we focus on LPs in standard form

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \qquad (\mathcal{LP})$$

  with data $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ (where $b \geq 0$).
- We assume:
  - # of variables $= n \geq m =$ # of equations (otherwise, the system $Ax = b$ is overdetermined);
  - rows of A are linearly independent (otherwise, the constraints are redundant or inconsistent).
  - $\Rightarrow \text{rank}(A) = m$

Linear Dependency

Linear dependency among rows of A implies either:

- contradictory constraints
  i.e., no solution to $A\,x = b$, e.g.

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_2 &= 2 \end{aligned}$$

- redundancy, e.g.

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + 2x_2 &= 2 \end{aligned}$$

Index Sets

- Consider only the linear equations in problem $\mathcal{LP}$.

$$\boxed{A\,x = b}$$

- Let $A = [a_1, \ldots, a_n]$, where $a_i \in \mathbb{R}^m$ is the ith column vector of A.
- Select an index set $I \subseteq \{1, \ldots, n\}$ of cardinality m such that the vectors $\{a_i\}_{i \in I}$ are linearly independent.
- At least one such index set exist since $n \geq m = \text{rank}(A)$.

Definition: The matrix $B = B(I) \in \mathbb{R}^{m \times m}$ consisting of the columns $\{a_i\}_{i \in I}$ is termed the basis corresponding to I.

Basic Solutions

Definition: A solution x to $Ax = b$ with $x_i = 0$ for all $i \notin I$ is a basic solution (BS) to $Ax = b$ w.r.t. the index set I.

Definition: A solution x satisfying both $Ax = b$ and $x \geq 0$ is a feasible solution (FS).

Definition: A feasible solution which is also basic is a basic feasible solution (BFS).

Basic Solutions (cont)

Observation: The basic solution corresponding to I is unique.

As the vectors $\{a_i\}_{i \in I}$ are linearly independent, the basis B is invertible. Thus, the system

$$\boxed{B\, x_B = b}$$

has a unique solution $x_B = B^{-1} b \in \mathbb{R}^m$.

Define $x = (x_1, \ldots, x_n)$ through

$$(x_i)_{i \in I} = x_B \quad \text{and} \quad (x_i)_{i \notin I} = 0.$$

This x is the unique basic solution to $Ax = b$ w.r.t. I.

Basic Solutions (cont)

Assume for example that $I = \{1, \ldots, m\}$.

$$\begin{array}{rcl}
a_{11}x_1 + \ldots + a_{1m}x_m + a_{1,m+1}x_{m+1} + \ldots + a_{1n}x_n & = & b_1 \\
a_{21}x_1 + \ldots + a_{2m}x_m + a_{2,m+1}x_{m+1} + \ldots + a_{2n}x_n & = & b_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots & & \vdots \\
a_{m1}x_1 + \ldots + a_{mm}x_m + a_{m,m+1}x_{m+1} + \ldots + a_{mn}x_n & = & b_m
\end{array}$$

To find basic solution, set $x_i = 0$ for all $i \notin I$.

$$\begin{array}{rcrcl}
a_{11}x_1 + \ldots + a_{1m}x_m & + & a_{1,m+1}0 + \ldots + a_{1n}0 & = & b_1 \\
a_{21}x_1 + \ldots + a_{2m}x_m & + & a_{2,m+1}0 + \ldots + a_{2n}0 & = & b_2 \\
\vdots \qquad\qquad \vdots & & \vdots \qquad\qquad \vdots & & \vdots \\
a_{m1}x_1 + \ldots + a_{mm}x_m & + & a_{m,m+1}0 + \ldots + a_{mn}0 & = & b_m
\end{array}$$

Assume for example that $I = \{1, \ldots, m\}$.

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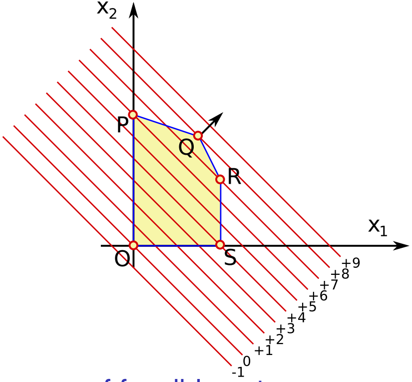
\vdots \qquad\qquad \vdots \qquad\qquad \vdots & & \vdots \\
a_{m1}x_1 + \ldots + a_{mm}x_m + a_{m,m+1}x_{m+1} + \ldots + a_{mn}x_n & = & b_m
\end{array}$$

The following system is equivalent to $Bx_B = b$.

$$\begin{array}{ccccccc}
a_{11}x_1 & + & \ldots & + & a_{1m}x_m & = & b_1 \\
a_{21}x_1 & + & \ldots & + & a_{2m}x_m & = & b_2 \\
\vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & \ldots & + & a_{mm}x_m & = & b_m
\end{array}$$

Algebra vs. Geometry

- Geometric intuition: LP solution at corner of feasible set



- Algebra: Corners of feasible set correspond to basic feasible solutions

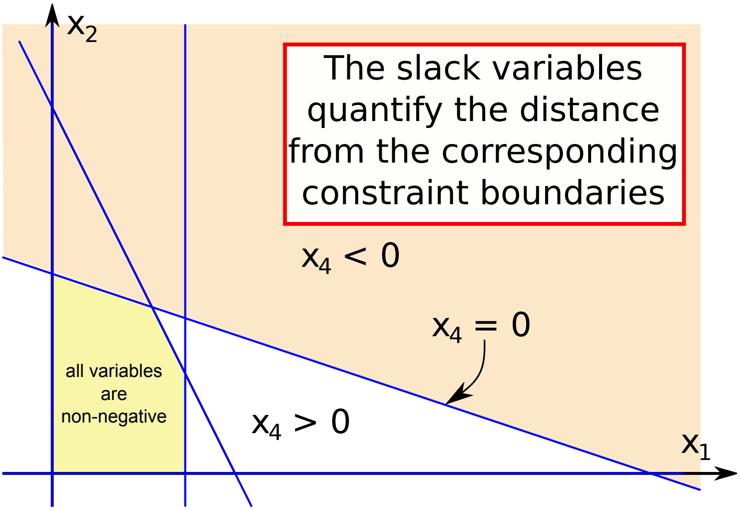
Example 1 (revisited)

$$\begin{array}{lll} \max & x_1 + x_2 & \text{objective function} \\ \text{s.t.} & 2x_1 + x_2 \leq 11 & \text{constraint on availability of X} \\ & x_1 + 3x_2 \leq 18 & \text{constraint on availability of Y} \\ & x_1 \leq 4 & \text{constraint on demand of A} \\ & x_1, x_2 \geq 0 & \text{non-negativity constraints} \end{array}$$

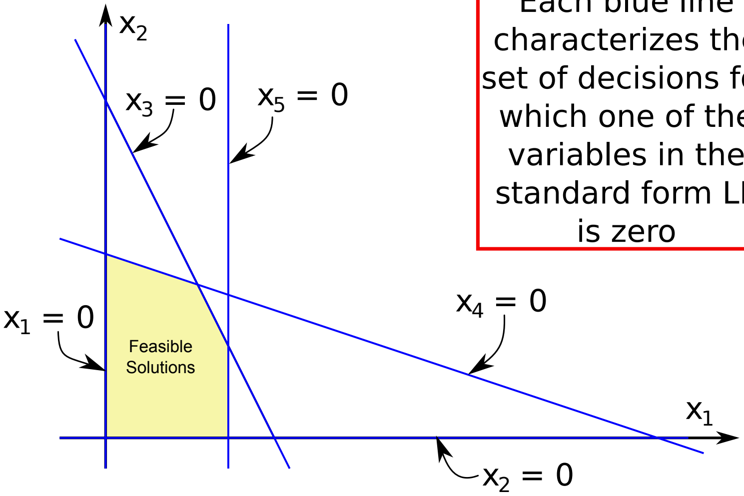
In standard form: $n = 5$ variables & $m = 3$ constraints

$$\begin{array}{ll} \min & -x_1 - x_2 \\ \text{s.t.} & 2x_1 + x_2 + x_3 = 11 \\ & x_1 + 3x_2 + x_4 = 18 \\ & x_1 + x_5 = 4 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

Algebra vs. Geometry (cont)

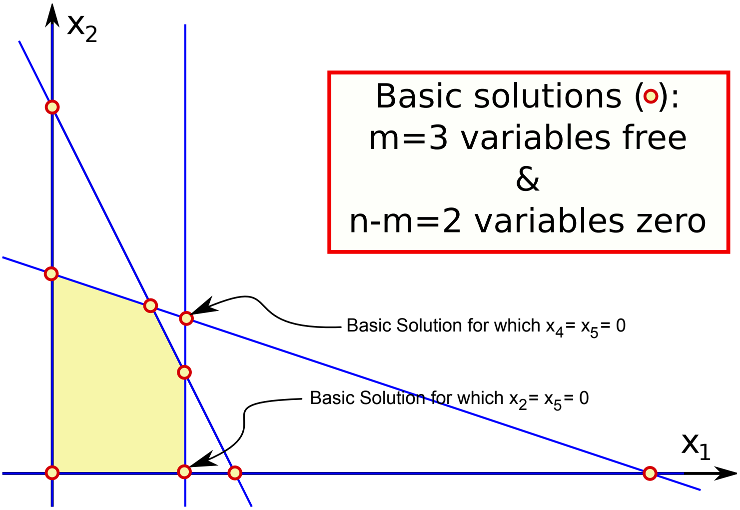


Algebra vs. Geometry (cont)

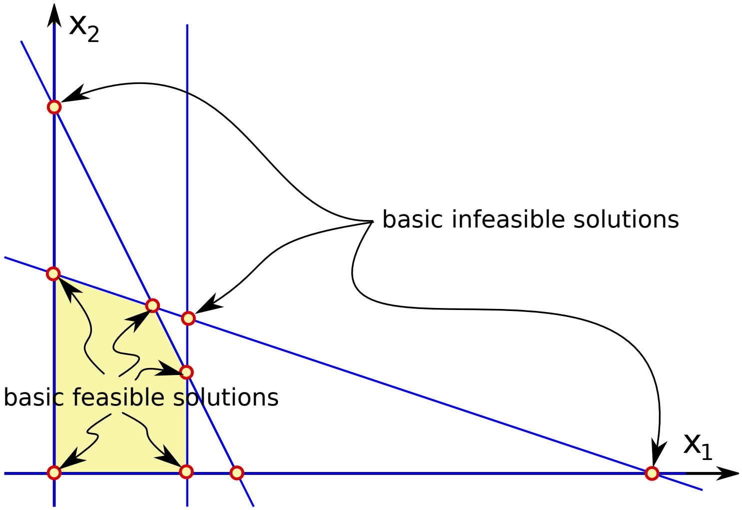


Each blue line characterizes the set of decisions for which one of the variables in the standard form LP is zero

Algebra vs. Geometry (cont)



Algebra vs. Geometry (cont)



Importance of BFS

$\boxed{\text{Vertices of the feasible set} = \text{basic feasible solutions!}}$

- Geometry: optimum always achieved at a vertex
- Algebra: optimum always achieved at a BFS

Definition: Given an LP in standard form, a feasible solution to the constraints $\{A\,x = b;\, x \geq 0\}$ that achieves the optimal value of the objective function is called an optimal feasible solution. If the solution is basic then it is an optimal BFS.

Fundamental Theorem of LP

Theorem 1: For an LP in standard form with $\text{rank}(A) = m \leq n$:

1. $\exists$ a feasible solution $\Rightarrow \exists$ a BFS.
2. $\exists$ an optimal solution $\Rightarrow \exists$ an optimal BFS.

Fundamental Theorem of LP (cont)

The naïve statement "an LP has an optimal BFS" is in general not true as the LP can be:

INFEASIBLE

$$\min\{2x_1 + x_2 \mid -x_1 - x_2 = 1;\ x_1, x_2 \geq 0\}$$

(no (x_1, x_2) satisfies the constraints)

or UNBOUNDED

$$\max\{x_1 \mid x_1 - x_2 = 1;\ x_1, x_2 \geq 0\}$$

(x_1 can grow arbitrarily).

Searching for Optima

- Theorem 1 reduces solving an LP to searching over BFS's.
- For an LP in standard form with n variables and m constraints, there are

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

possibilities of selecting m columns in the A matrix.

$\Rightarrow$ There are at most $\binom{n}{m}$ basic solutions: a finite number of possibilities!

$\Rightarrow$ Theorem 1 offers an obvious but terribly inefficient way of computing the optimum through a finite search.

Number of BFS

Note: There are $\binom{n}{m}$ choices of $I \subseteq \{1, \ldots, n\}$ with $|I| = m$.

$\Rightarrow$ The number of distinct BFS is finite and usually $< \binom{n}{m}$ for the following reasons:

1. $B(I)$ may be singular (i.e., I fails to be an index set),
2. the BS corresponding to I may not be feasible.

A “Small” Problem

Let $m = 30$, and $n = 100$.

$$\binom{100}{30} = \frac{100!}{30!\,70!} \approx 2.9 \times 10^{25}.$$

It takes approximately 10^{12} years if we check 10^6 sets/sec.

(The age of the universe is $\approx 14 \times 10^9$ years!)

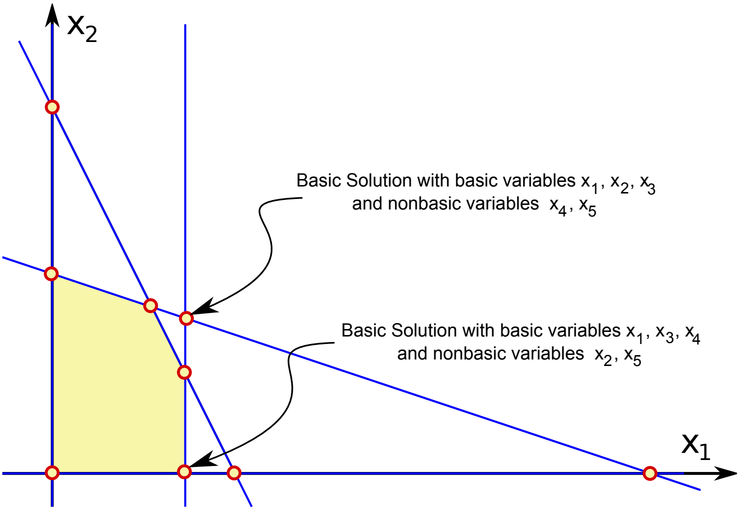
Basic Variables

Fix an index set I with $|I| = m$ and $B(I)$ invertible.

Definition The variables $\{x_i\}_{i \in I}$ are referred to as the basic variables (BV), while the variables $\{x_i\}_{i \notin I}$ are called the nonbasic variables (NBV) corresponding to I.

Note: By construction, the nonbasic variables are always zero, but the basic variables can be zero or non-zero.

Example: Basic vs Nonbasic Variables



Basic Representation

Fix an index set I with $|I| = m$ and $B(I)$ invertible.

Definition: The basic representation corresponding to I is the (unique) reformulation of the system $(x_0 = c^T x, Ax = b)$ which expresses the objective function value x_0 and each BV as a linear function of the NBV's:

$$\begin{bmatrix} x_0 \\ x_B \end{bmatrix} = f(x_N),$$

where

- $x_B = [x_i | i \in I]$ (BV's),
- $x_N = [x_i | i \notin I]$ (NBV's) and
- $f : \mathbb{R}^{n-m} \to \mathbb{R}^{m+1}$ is linear.

Matrix Partition

Let $A = [a_1, \ldots a_n]$, where $a_i \in \mathbb{R}^m$ is the ith column of A. For any index set $I \subseteq \{1, \ldots, n\}$ with $|I| = m$. Define

- $B = B(I) = [a_i | i \in I]$;
- $N = N(I) = [a_i | i \notin I]$;
- $c_B = c_B(I) = [c_i | i \in I]$;
- $c_N = c_N(I) = [c_i | i \notin I]$;
- $x_B = x_B(I) = [x_i | i \in I]$;
- $x_N = x_N(I) = [x_i | i \notin I]$.

This implies

$$Ax = Bx_B + Nx_N \quad \text{and} \quad c^T x = c_B^T x_B + c_N^T x_N \,.$$

Example: Partition of A

$$A = \begin{bmatrix} 2 & 4 & 3 & 3 & 1 & 0 \\ 3 & -3 & 4 & 2 & 0 & 1 \\ -1 & 2 & 1 & 2 & 0 & 0 \end{bmatrix}$$

Choose $I = \{1, 5, 2\}$

$$\Rightarrow B(I) = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 0 & -3 \\ -1 & 0 & 2 \end{bmatrix} \quad \text{and} \quad N(I) = \begin{bmatrix} 3 & 3 & 0 \\ 4 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Example: Partition of c and x

Let $n = 6$ and $I = \{5, 3, 2\}$

$$
\begin{aligned}
c_B &= [c_i \mid i \in I] &= [c_5, c_3, c_2]^T \\
c_N &= [c_j \mid j \notin I] &= [c_1, c_4, c_6]^T
\end{aligned}
$$

$$
\begin{aligned}
x_B &= [x_i \mid i \in I] &= [x_5, x_3, x_2]^T \\
x_N &= [x_j \mid j \notin I] &= [x_1, x_4, x_6]^T
\end{aligned}
$$

Basic Represenation (cont)

Given this partition, we have:

$$\left.\begin{array}{rcl} x_0 - c^T x & = & 0 \\ A\,x & = & b \end{array}\right\} \quad\Longleftrightarrow\quad \left\{\begin{array}{rcl} x_0 - c_B^T x_B - c_N^T x_N & = & 0 \\ B\,x_B + N\,x_N & = & b \end{array}\right.$$

Since B is invertible by construction, this implies that

$$x_B = B^{-1} B x_B = B^{-1}(b - N x_N) = B^{-1} b - B^{-1} N x_N \,.$$

Substituting this formula into the expression for x_0 we find

$$x_0 = c_B^T x_B + c_N^T x_N = c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N \,.$$

Basic Representation (cont)

Thus, the original system $x_0 = c^T x$, $Ax = b$ is equivalent to the basic representation

$$\left.\begin{array}{rcl} x_0 & = & c_B^T B^{-1} b + (c_N - N^T B^{-T} c_B)^T x_N \\ x_B & = & B^{-1} b - B^{-1} N x_N, \end{array}\right\} \qquad (*)$$

which expresses x_0 and x_B as linear functions of x_N.

Note: By setting $x_N = 0$ in $(*)$ we obtain the basic solution $x = (x_B, x_N) = (B^{-1}b, 0)$ with objective value $x_0 = c_B^T B^{-1} b$.

Definition: We call $r = c_N - N^T B^{-T} c_B$ the reduced cost vector. It characterises the sensitivity of the objective function value x_0 w.r.t. the nonbasic variables x_N.

Example: Basic Representation

Consider the following LP:

$$\min x_0 = 6x_1 + 3x_2 + 4x_3 + 2x_4 - 3x_5 + 4x_6$$

subject to:

$$\begin{array}{lllllllll} 2x_1 & -1x_2 & +3x_3 & +2x_4 & +3x_5 & +2x_6 & +x_7 & & = 4 \\ 3x_1 & +4x_2 & +2x_3 & +2x_4 & +3x_5 & & & +x_8 & = 2 \end{array}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0$$

Example: Basic Representation (cont)

Thus, we are given the following problem data:

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 & 3 & 2 & 1 & 0 \\ 3 & 4 & 2 & 2 & 3 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad c = \begin{bmatrix} 6 \\ 3 \\ 4 \\ 2 \\ -3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

Example: Basic Representation (cont)

Choose $I = \{4, 3\}$. Then, we have

$$B = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix} \quad \Rightarrow \quad B^{-1} = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix},$$

$$N = \begin{bmatrix} 2 & -1 & 3 & 2 & 1 & 0 \\ 3 & 4 & 3 & 0 & 0 & 1 \end{bmatrix},$$

$$c_B^T = \begin{bmatrix} 2 & 4 \end{bmatrix}, \quad c_N^T = \begin{bmatrix} 6 & 3 & -3 & 4 & 0 & 0 \end{bmatrix}.$$

Example: Basic Representation (cont)

$$x_B + B^{-1}\, N\, x_N = B^{-1}\, b$$

$$\iff \quad x_B + B^{-1}\, N\, x_N = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\iff \quad x_B + \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 & 2 & 1 & 0 \\ 3 & 4 & 3 & 0 & 0 & 1 \end{bmatrix} x_N = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$x_B + \begin{bmatrix} \frac{5}{2} & 7 & \frac{3}{2} & -2 & -1 & \frac{3}{2} \\ -1 & -5 & 0 & 2 & 1 & -1 \end{bmatrix} x_N = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Example: Basic Representation (cont)

$$
\begin{aligned}
x_0 &= c_B^T B^{-1} b + r^T x_N = c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N)\, x_N \\
&= \begin{bmatrix} 2 \\ 4 \end{bmatrix}^T \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} + (c_N^T - c_B^T B^{-1} N)\, x_N \\
&= 6 + \left(c_N^T - \begin{bmatrix} 2 \\ 4 \end{bmatrix}^T \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 & 2 & 1 & 0 \\ 3 & 4 & 3 & 0 & 0 & 1 \end{bmatrix} \right) x_N \\
&= 6 + \left(c_N^T - \begin{bmatrix} 1 & -6 & 3 & 4 & 2 & -1 \end{bmatrix} \right) x_N \\
&= 6 + \left(\begin{bmatrix} 6 & 3 & -3 & 4 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -6 & 3 & 4 & 2 & -1 \end{bmatrix} \right) x_N \\
&= 6 + \begin{bmatrix} 5 & 9 & -6 & 0 & -2 & 1 \end{bmatrix} x_N
\end{aligned}
$$

Example: Basic Representation (cont)

Thus, the original system

$$\begin{array}{rrrrrrrrrl}
x_0 & -6x_1 & -3x_2 & -4x_3 & -2x_4 & +3x_5 & -4x_6 & & & = 0 \\
 & 2x_1 & -1x_2 & +3x_3 & +2x_4 & +3x_5 & +2x_6 & +x_7 & & = 4 \\
 & 3x_1 & +4x_2 & +2x_3 & +2x_4 & +3x_5 & & & +x_8 & = 2
\end{array}$$

is equivalent to the basic representation w.r.t. $I = \{4, 3\}$

$$\begin{array}{rrrrrrrrrl}
x_0 & -5x_1 & -9x_2 & & & +6x_5 & & +2x_7 & -x_8 & = 6 \\
 & \frac{5}{2}x_1 & +7x_2 & & +x_4 & +\frac{3}{2}x_5 & -2x_6 & -x_7 & +\frac{3}{2}x_8 & = -1 \\
 & -x_1 & -5x_2 & +x_3 & & & +2x_6 & +x_7 & -x_8 & = 2
\end{array}$$

The corresponding BS is not feasible:

$$(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (6, 0, 0, 2, -1, 0, 0, 0, 0)$$

Importance of Basic Representations

Fix an index set I with $|I| = m$ and $B = B(I)$ invertible.

- Assume that
  - the corresponding BS with $x_B = B^{-1}b$ and $x_N = 0$ is feasible, i.e., $B^{-1}b \geq 0$.
- The objective value of this BFS is $x_0 = c_B^T B^{-1} b$.
- Any other feasible solution satisfies $x_N \geq 0$.
- The basic representation

$$x_0 = c_B^T B^{-1} b + r^T x_N \quad \text{and} \quad x_B = B^{-1} b - B^{-1} N x_N$$

  tells us how x_0 and x_B change when the nonbasic variables increase.

Importance of Basic Representations

In particular, the reduced cost vector r enables us to:

- recognise whether the current BFS is optimal (this is the case iff $r \geq 0$; then, no other feasible solution can have a lower objective value than the current BFS);
- find a new BFS with a lower objective value if the current BFS is not optimal (by increasing a nonbasic variable with a negative reduced cost).

Idea of the Simplex Algorithm

$$\begin{array}{ll} \text{min} & x_0 = c^T x \\ \text{s.t.} & Ax = b,\ x \geq 0 \end{array} \qquad (\mathcal{LP})$$

1. Among the FS to $\mathcal{LP}$, an important finite subset are the BFS. We know that (at least) one BFS is optimal.
2. Each BFS is associated with a basic representation, i.e., a set of equations equivalent to $x_0 = c^T x$, $A\,x = b$, that expresses the BV's in terms of the NBV's.
3. The basic representation tells us if increasing any NBV will improve the objective. If there is one, increase it until a new, better, BFS is reached. If no such a NBV exists, we have an optimal solution.

Example 1 (revisited)

After standardising Example 1 (max $\rightarrow$ min & adding slack variables), the equations $x_0 = c^T x$ and $Ax = b$ become:

$$
\begin{array}{ccccccccccccc}
x_0 & + & x_1 & + & x_2 & & & & & & & = & 0 \\
 & & 2x_1 & + & x_2 & + & x_3 & & & & & = & 11 \\
 & & x_1 & + & 3x_2 & & & + & x_4 & & & = & 18 \\
 & & x_1 & & & & & & & + & x_5 & = & 4
\end{array} \qquad (1)
$$

This happens to be a basic representation where the slack variables play the role of the basic variables ($I = \{3, 4, 5\}$).

$\boxed{\text{BV: } \{x_3, x_4, x_5\}}$ $\boxed{\text{NBV: } \{x_1, x_2\}}$

Example 1 (revisited)

In equations (1) set NBV's to zero:

$$x_1 = x_2 = 0$$

"Solve" for the remaining BV's

$$x_0 = 0,\ x_3 = 11,\ x_4 = 18,\ x_5 = 4$$

The corresponding BS is:

$$\boxed{(x_0, \underline{x_1}, \underline{x_2}, x_3, x_4, x_5) = (0, 0, 0, 11, 18, 4)}$$

It is also a BFS!

Example 1 (revisited)

For this BFS the objective function is $x_0 = 0$.

In order to find a better BFS, search for a nonbasic variable x_j such that increasing x_j improves x_0.

Looking at the objective function

$$x_0 = -x_1 - x_2$$

we see that we can decrease x_0 either by increasing x_1 or x_2 (increasing both simultaneously is too complicated).

Example 1 (revisited)

E.g. consider increasing x_1 to λ and leaving $x_2 = 0$.

$\Rightarrow$ The decrease for x_0 will be: $x_0 = -x_1 - x_2 = -\lambda$

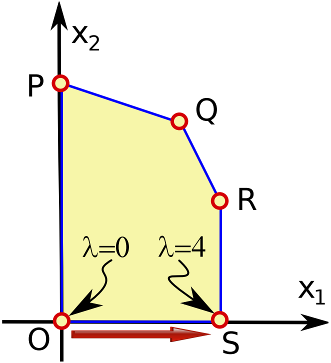
We cannot increase λ indefinitely. The basic variables must remain feasible ($x_i \geq 0 \; \forall i \in I$):

$$\begin{array}{lllll} x_3 & = & 11 - 2\lambda & \geq 0 & \Rightarrow & \lambda \leq \frac{11}{2} \\ x_4 & = & 18 - \lambda & \geq 0 & \Rightarrow & \lambda \leq 18 \\ x_5 & = & 4 - \lambda & \geq 0 & \Rightarrow & \lambda \leq 4 \end{array} \qquad (1')$$

We want the best (largest) λ satisfying (1') $\Rightarrow \boxed{\lambda = 4}$

Example 1 (revisited)

From (1') we see that λ takes values between 0 and 4.
Any solution defined by (1'), i.e., $x_1 = \lambda, x_2 = 0$,
corresponds to a point along 0S.



Example 1 (revisited)

The original BFS had been:

$$\boxed{(x_0, \underline{x_1}, \underline{x_2}, x_3, x_4, x_5) = (0, 0, 0, 11, 18, 4)}$$

corresponding to point O.

Setting $x_1 = \lambda = 4$ implies also $x_0 = -4$.
The values of x_3, x_4, and x_5 change too.

The new BS is given by:

$$\boxed{(x_0, x_1, \underline{x_2}, x_3, x_4, \underline{x_5}) = (-4, 4, 0, 3, 14, 0)}$$

corresponding to point S.

Example 1 (revisited)

The new BS:

$$\boxed{(x_0, x_1, \underline{x_2}, x_3, x_4, \underline{x_5}) = (-4, 4, 0, 3, 14, 0)}$$

is also a BFS to (1).

The new basic and nonbasic variables are:

$\boxed{\text{BV: } \{x_1, x_3, x_4\}}$ $\boxed{\text{NBV: } \{x_2, x_5\}}$

Task: Obtain a new basic representation: i.e., transform (1) to express x_0, x_1, x_3, and x_4 in terms of x_2, x_5.

Systematic approach: Pivoting (discussed later)

Example 1 (revisited)

Rearranging the equations (1) by using elementary row operations (ERO), we obtain a new basic representation, corresponding to $I = \{1, 3, 4\}$.

$$\begin{array}{ccccccccccccc} x_0 & & & + & x_2 & & & & & - & x_5 & = & -4 \\ & & & & x_2 & + & x_3 & & & - & 2x_5 & = & 3 \\ & & & & 3x_2 & & & + & x_4 & - & x_5 & = & 14 \\ & & x_1 & & & & & & & + & x_5 & = & 4 \end{array} \qquad (2)$$

Any solution to (2) is also a solution to (1) and vice versa.

Example 1 (revisited)

Looking at the new objective function

$$x_0 = -4 - x_2 + x_5$$

we see that we can decrease x_0 by increasing x_2.

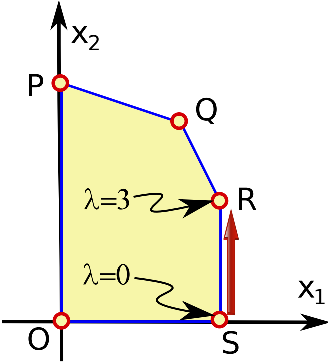
$\Rightarrow$ Increase x_2 to λ and keep $x_5 = 0$, making sure that the basic variables x_3, x_4 and x_1 remain feasible.

$$\begin{array}{rcccccc} x_0 & = & -4 - \lambda & & & & \\ x_3 & = & 3 - \lambda & \geq 0 & \Rightarrow & \lambda \leq 3 & \\ x_4 & = & 14 - 3\lambda & \geq 0 & \Rightarrow & \lambda \leq \frac{14}{3} & \quad (2') \\ x_1 & = & 4 & \geq 0 & \Rightarrow & \lambda \leq \infty & \end{array}$$

We want the best (largest) λ satisfying (2') $\Rightarrow$ $\boxed{\lambda = 3}$

Example 1 (revisited)

From (2') we see that λ takes values between 0 and 3.
Any solution defined by (2'), i.e., $x_1 = 4, x_2 = \lambda$,
corresponds to a point along SR.



Example 1 (revisited)

The previous BFS had been

$$\boxed{(x_0, x_1, \underline{x_2}, x_3, x_4, \underline{x_5}) = (-4, 4, 0, 3, 14, 0)}$$

corresponding to point S.

Setting $x_2 = \lambda = 3$ implies $x_0 = -7$.
The values of x_3, and x_4 change too.

The new BS is given by:

$$\boxed{(x_0, x_1, x_2, \underline{x_3}, x_4, \underline{x_5}) = (-7, 4, 3, 0, 5, 0)}$$

corresponding to point R.

Example 1 (revisited)

The new BS

$$\boxed{(x_0, x_1, x_2, \underline{x_3}, x_4, \underline{x_5}) = (-7, 4, 3, 0, 5, 0)}$$

is also a BFS to (1).

The new basic and nonbasic variables are:

$\boxed{\text{BV: } \{x_1, x_2, x_4\}}$ $\boxed{\text{NBV: } \{x_3, x_5\}}$

Task: Obtain a new basic representation, i.e., transform (2) to express x_0, x_1, x_2, and x_4 in terms of x_3, x_5.

Example 1 (revisited)

Rearranging the equations (2) by using EROs, we obtain a new basic representation corresponding to $I = \{1, 2, 4\}$.

$$
\begin{array}{cccccccccccccc}
x_0 & & & & & - & x_3 & & & + & x_5 & = & -7 \\
& & & & x_2 & + & x_3 & & & - & 2x_5 & = & 3 \\
& & & & & - & 3x_3 & + & x_4 & + & 5x_5 & = & 5 \\
& & x_1 & & & & & & & + & x_5 & = & 4
\end{array}
\tag{3}
$$

Any solution to (3) also solves (1) and (2) and vice versa.

Example 1 (revisited)

Looking at the new objective function

$$x_0 = -7 + x_3 - x_5$$

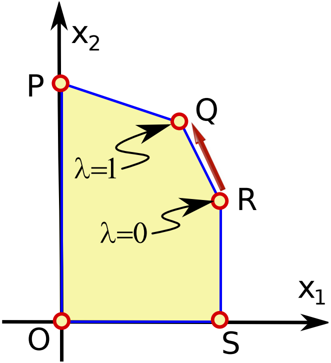
we see that we can decrease x_0 by increasing x_5.

$\Rightarrow$ Increase x_5 to λ and keep $x_3 = 0$, making sure that the basic variables x_2, x_4 and x_1 remain feasible.

$$\begin{array}{lclcccc}
x_0 & = & -7 - \lambda & & & & \\
x_2 & = & 3 + 2\lambda & \geq 0 & \Rightarrow & \lambda \leq \infty & \\
x_4 & = & 5 - 5\lambda & \geq 0 & \Rightarrow & \lambda \leq 1 & \\
x_1 & = & 4 - \lambda & \geq 0 & \Rightarrow & \lambda \leq 4 &
\end{array} \qquad (3')$$

We want the best (largest) λ satisfying (3') $\Rightarrow$ $\boxed{\lambda = 1}$

From (3') we see that λ takes values between 0 and 1.
Any solution defined by (3'), i.e., $x_1 = 4 - \lambda, x_2 = 3 + 2\lambda$,
corresponds to a point along RQ.



Example 1 (revisited)

The previous BFS had been

$$\boxed{(x_0, x_1, x_2, \underline{x_3}, x_4, \underline{x_5}) = (-7, 4, 3, 0, 5, 0)}$$

corresponding to point R.

Setting $x_5 = \lambda = 1$ implies $x_0 = -8$.
The values of x_1, x_2, and x_3 change too.

The new BS is given by

$$\boxed{(x_0, x_1, x_2, \underline{x_3}, \underline{x_4}, x_5) = (-8, 3, 5, 0, 0, 1)}$$

corresponding to point Q.

Example 1 (revisited)

$$\boxed{(x_0, x_1, x_2, \underline{x_3}, \underline{x_4}, x_5) = (-8, 3, 5, 0, 0, 1)}$$

is also a BFS to (1).

The new basic and nonbasic variables are:

$\boxed{\text{BV: } \{x_1, x_2, x_5\}}$ $\boxed{\text{NBV: } \{x_3, x_4\}}$

Task: Obtain a new basic representation, i.e., transform (3) to express x_0, x_1, x_2, and x_5 in terms of x_3, x_4.

Example 1 (revisited)

Rearranging the equations (3) by using EROs, we obtain a new basic representation, corresponding to $I = \{1, 2, 5\}$.

$$
\begin{array}{ccccccccccccc}
x_0 & & & - & \frac{2}{5}x_3 & - & \frac{1}{5}x_4 & & & = & -8 \\
 & & x_2 & - & \frac{1}{5}x_3 & + & \frac{2}{5}x_4 & & & = & 5 \\
 & & & - & \frac{3}{5}x_3 & + & \frac{1}{5}x_4 & + & x_5 & = & 1 \\
 & x_1 & & + & \frac{3}{5}x_3 & - & \frac{1}{5}x_4 & & & = & 3
\end{array} \tag{4}
$$

Any solution to (4) also solves (1), (2), (3) and vice versa.

Example 1 (revisited)

From the first equation in (4) we deduce that any solution to (1), (2), (3), or (4) has to satisfy

$$x_0 = -8 + \frac{2}{5}x_3 + \frac{1}{5}x_4.$$

Any FS further satisfies $x_3, \ x_4 \geq 0$.
$\Rightarrow$ Thus, (4) implies that $x_0 \geq -8$ for any FS.

The BFS corresponding to (5), $(x_0, x_1, x_2, x_3, x_4, x_5) =$
$= (-8, 3, 5, 0, 0, 1)$, has objective value $x_0 = -8$.

$\boxed{\text{This must be a minimal solution!}}$

Note: The optimal value of the original max problem is $+8$!