Foundations of Inventory Management **Fall 2022, NUS**

Lecture 4: Single Warehouse Multi-Retailer Model

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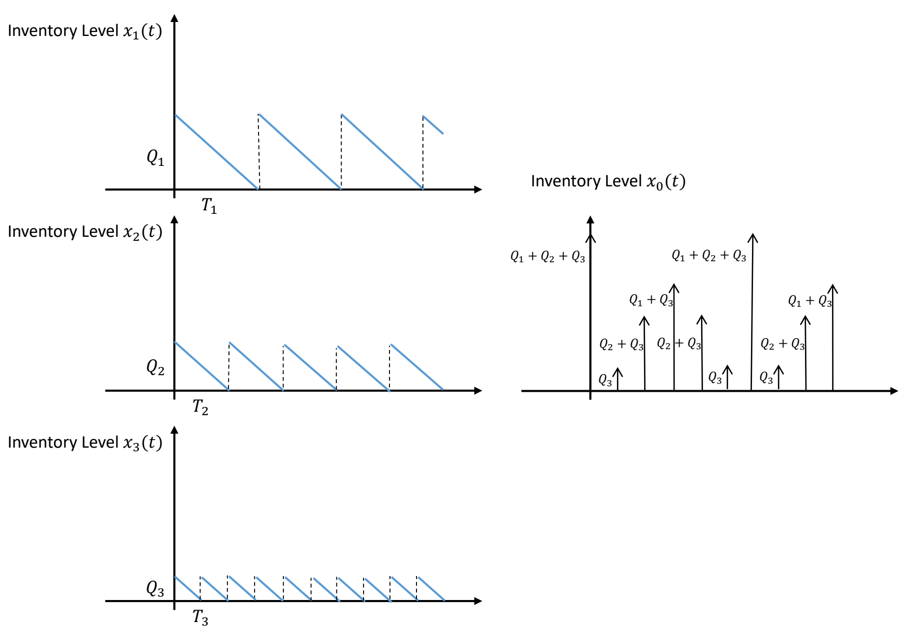
4.1 The Model

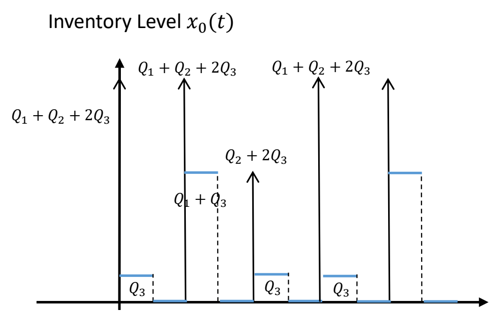
Consider a single warehouse indexed by 0, whose inventory is used to fulfill the demands at n retailers, indexed by $i = 1, 2, ..., n$. Like the EOQ model, each retailer i faces a constant demand rate λ_i. Retailer i incurs a fixed ordering K_i whenever an order is placed from the warehouse and pays a holding cost h_i per unit time per unit inventory.

The warehouse also needs to replenish inventory from the outside supplier. A fixed ordering cost K_0 is incurred whenever the warehouse places order, and the holding cost h_0 is assumed to be smaller compared to that of each retailer, i.e., $h_0 \le h_i, i = 1, ..., n$. No shortage is allowed and we assume that there is no shipping cost from the warehouse to the retailer and the shipping time is zero.

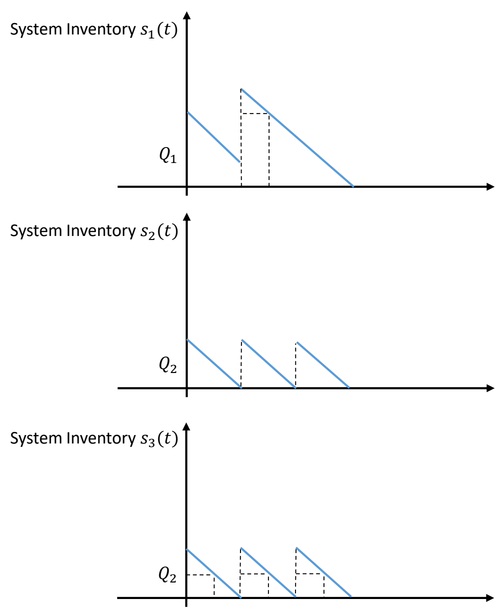
Note that if $K_0 = 0$, the warehouse can place an order whenever one of the retailer needs to replenish its inventory. Hence, the problem can be decoupled into n separate EOQ problems with retailer i placing an order $Q_i^* = \sqrt{2\lambda_i K_i / h_i}$ with a time $T_i^* = \sqrt{2K_i/(\lambda_i h_i)}$ between each order. The figure below illustrates the patterns of inventory levels at each facility for the case of three retailers. Here, for simplicity, we let $\lambda_i = 1$ for $i = 1, 2, 3$, and consider $T_1 = 3, T_2 = 2, T_3 = 1$. The warehouse orders in every period and orders exactly the amounts that are required by those retailers who are placing the order in this particular period. As a result, all orders are shipped out immediately and the warehouse does not hold any inventory.



When K_0 is large, such policy clearly is not optimal since the warehouse orders too frequently and would incur a very large fixed ordering cost. Observe that in general given a fixed policy, even evaluating the total system cost could be non-trivial. Consider, for example, an alternative policy where the warehouse places an order every two period (the retailers still order as in the figure above). In this case, the warehouse has to hold inventory and the resulting pattern of inventory level is illustrated in the figure below, which has a very different pattern compared to the inventory held at the retailer level.



To facilitate the computation of holding cost, it is convenient to introduce the notion of system inventory. Retailer i's system inventory level, denoted as $s_i(t)$, is defined as the inventory held at retailer i plus the part of inventory held at the warehouse that is to-be-shipped to retailer i. The system inventory of each retailer for the above three-retailer-system is illustrated below. The inventory pattern then resembles that of the EOQ model.



The holding cost for the system inventory at retailer i can then be computed as

$$h_i \int_0^t x_i(\tau)d\tau + h_0 \int_0^t s_i(\tau)d\tau - h_0 \int_0^t x_i(\tau)d\tau = (h_i - h_0) \int_0^t x_i(\tau)d\tau + h_0 \int_0^t s_i(\tau)d\tau.$$

One can see that the system's total holding cost is then the sum of the holding cost for the system inventory at each retailer.

4.2 Power-of-Two Policies

Let us now consider the heuristic that the warehouse and all retailers adopt power-of-two policies. In particular, let T_i be the time between each order for facility i, $i = 0, 1, ..., n$ and we have

$$T_i = T_B \cdot 2^{k_i}, k_i \in \mathbb{Z}, i = 0, 1, ..., n.$$

Note that with the power-of-two policies, we have:

- If $T_0 \leq T_i$, i.e., retailer i has a longer replenishment cycle, then whenever the retailer i is ordering, the warehouse is ordering as well. In this case, the warehouse does not need to hold inventory retailer i and the system inventory for retailer i coincides with its inventory level, i.e., $s_i(t) = x_i(t)$. The average holding cost for the system inventory is then

$$H_i(T_i, T_0) = \frac{1}{T_i} h_i \int_0^{T_i} x_i(\tau) d\tau = \frac{h_i \lambda_i T_i}{2}.$$

- If $T_0 > T_i$, i.e., retailer i replenishes more frequently than the warehouse, then whenever the warehouse is ordering, the retailer orders as well. In this case, the warehouse needs to hold additional inventory for retailer's future ordering. The system inventory in this case behaves in the same way as the EOQ model with the time between replenishment being T_0. The average holding cost for the system inventory can be computed as

$$H_i(T_i, T_0) = \frac{1}{T_i}(h_i - h_0) \int_0^{T_i} x_i(\tau) d\tau + \frac{1}{T_0} h_0 \int_0^{T_0} s_i(\tau) d\tau = \frac{(h_i - h_0)\lambda_i T_i}{2} + \frac{h_0 \lambda_i T_0}{2}.$$

Combining the two cases, we can write the average holding cost for the system inventory as

$$H_i(T_i, T_0) = \frac{(h_i - h_0)\lambda_i}{2} T_i + \frac{h_0 \lambda_i}{2} \max\{T_i, T_0\}.$$

The total average cost is then

$$C(T_0, T_1, ..., T_n) = \sum_{i=0}^{n} \frac{K_i}{T_i} + \sum_{i=1}^{n} H_i(T_i, T_0),$$

and the optimal power-of-two policies can be found via

$$\min_{k_i \in \mathbb{Z}, i=0,\ldots,n} C(T_B \cdot 2^{k_0}, T_B \cdot 2^{k_1}, ..., T_B \cdot 2^{k_n}), \tag{4.1}$$

which is still a nonlinear integer program.

Instead of solving for the optimal k_i^*, we consider the following relaxed problem

$$\min_{T_i \in \mathbb{R}_+, i=0,\ldots,n} C(T_0, T_1, ..., T_n).$$

We remark that this is NOT the average cost of the policy where the time between replenishment at facility i is T_i since the formula for $H_i(T_i, T_0)$ represents the average holding cost only when T_i is a power of two.

For a fixed value of T_0, the problem decouples by T_i:

$$\bar{C}(T_0) = \min_{T_i \geq 0, i \geq 1} C(T_0, T_1, ..., T_n) = \frac{K_0}{T_0} + \sum_{i=1}^{n} f_i(T_0),$$

where

$$f_i(T_0) = \min_{T_i \geq 0} \left\{ \frac{K_i}{T_i} + H_i(T_i, T_0) \right\}.$$

For convenience, we denote $m_i = \frac{(h_i - h_0)\lambda_i}{2}$ and $w_i = \frac{h_0 \lambda_i}{2}$. Then,

$$f_i(T_0) = \min_{T_i \geq 0} \left\{ \frac{K_i}{T_i} + m_i T_i + w_i \max\{T_0, T_i\} \right\},$$

whose objective is convex in T_i. We can solve the problem by separately imposing the constraints $T_i \geq T_0$ and $T_i \leq T_0$.[1]

- If we restrict $T_i \geq T_0$, the objective becomes $\frac{K_i}{T_i} + (m_i + w_i)T_i$, whose unconstrained minimizer is
$$\sqrt{\frac{K_i}{m_i + w_i}}.$$
It follows that if $\sqrt{\frac{K_i}{m_i+w_i}} \geq T_0$, then we must have $T_i^* = \sqrt{\frac{K_i}{m_i+w_i}}$, and $T_i^* = T_0$ otherwise.
- If we restrict $T_i \leq T_0$, the objective becomes $\frac{K_i}{T_i} + m_i T_i + w_i T_0$, whose unconstrained minimizer is
$$\sqrt{\frac{K_i}{m_i}}.$$
It follows that if $\sqrt{\frac{K_i}{m_i}} \geq T_0$, then we must have $T_i^* = \sqrt{\frac{K_i}{m_i}}$, and $T_i^* = T_0$ otherwise.

Combining the two cases, we then arrive at the solution to the original problem:

$$T_i^* = \begin{cases} \sqrt{\frac{K_i}{m_i+w_i}} & \text{if } T_0 < \sqrt{\frac{K_i}{m_i+w_i}} \\ T_0 & \text{if } \sqrt{\frac{K_i}{m_i+w_i}} \leq T_0 < \sqrt{\frac{K_i}{m_i}} \\ \sqrt{\frac{K_i}{m_i}} & \text{if } \sqrt{\frac{K_i}{m_i}} \leq T_0 \end{cases}$$

with the corresponding optimal value

$$f_i(T_0) = \begin{cases} 2\sqrt{(m_i + w_i)K_i} & \text{if } T_0 < \sqrt{\frac{K_i}{m_i+w_i}} \\ \frac{K_i}{T_0} + (m_i + w_i)T_0 & \text{if } \sqrt{\frac{K_i}{m_i+w_i}} \leq T_0 < \sqrt{\frac{K_i}{m_i}} \\ 2\sqrt{m_i K_i} + w_i T_0 & \text{if } \sqrt{\frac{K_i}{m_i}} \leq T_0. \end{cases}$$

[1] Alternatively, if we let $g_i(T_i) = \frac{K_i}{T_i} + m_i T_i + w_i \max\{T_0, T_i\}$, one can use the right derivative:

$$\partial_+ g_i(T_i) = \begin{cases} -\frac{K_i}{T_i^2} + m_i & \text{if } T_i < T_0 \\ -\frac{K_i}{T_i^2} + m_i + w_i & \text{if } T_i \geq T_0 \end{cases}$$

and the fact that $T_i^* = \inf\{T_i | \partial_+ g_i(T_i) \geq 0\}$.

For simplicity, we denote $\underline{\tau}_i = \sqrt{\frac{K_i}{m_i+w_i}}$ and $\overline{\tau}_i = \sqrt{\frac{K_i}{m_i}}$, and note that $f_i(T_0)$ is convex and differentiable in T_0. Now we consider the problem of

$$\min_{T_0 \geq 0} \bar{C}(T_0) = \min_{T_0 \geq 0} \frac{K_0}{T_0} + \sum_{i=1}^{n} f_i(T_0).$$

The points $\{(\underline{\tau}_1, \overline{\tau}_1), (\underline{\tau}_2, \overline{\tau}_2), ..., (\underline{\tau}_n, \overline{\tau}_n)\}$ divide $\mathbb{R}_+$ into $2n+1$ intervals (we allow some interval to be empty). On interval I_j, $j = 1, ..., 2n+1$, $\bar{C}(T_0)$ must take the form:

$$\frac{A_j}{T_0} + B_j + C_j T_0,$$

for some constants A_j, B_j and C_j. To compute A_j, B_j, C_j, let $I_j = [\ell_j, u_j]$, we further define the following index sets:

$$\begin{aligned} \mathcal{G}_j &= \{i \mid u_j \leq \underline{\tau}_i\} \\ \mathcal{E}_j &= \{i \mid \underline{\tau}_i \leq \ell_j, u_j \leq \overline{\tau}_i\} \\ \mathcal{L}_j &= \{i \mid \ell_j \geq \overline{\tau}_i\}. \end{aligned}$$

It then follows that on I_j, we have

$$\begin{aligned} A_j &= K_0 + \sum_{i \in \mathcal{E}_j} K_i; \\ B_j &= \sum_{i \in \mathcal{G}_j} 2\sqrt{(m_i + w_i)K_i} + \sum_{i \in \mathcal{L}_j} 2\sqrt{m_i K_i}; \\ C_j &= \sum_{i \in \mathcal{E}_j} (m_i + w_i) + \sum_{i \in \mathcal{L}_j} w_i. \end{aligned}$$

Example 4.1 *Consider the case when $n = 2$ and $\underline{\tau}_1 < \underline{\tau}_2 < \overline{\tau}_1 < \overline{\tau}_2$. The four positive numbers divide $\mathbb{R}_+$ into five intervals. On I_1, for example, we have $\mathcal{G}_1 = \{1, 2\}, \mathcal{E}_1 = \emptyset, \mathcal{L}_1 = \emptyset$. It follows that $A_1 = K_0, B_1 = 2\sqrt{(m_1 + w_1)K_1} + 2\sqrt{(m_2 + w_2)K_2}, C_1 = 0$ and*

$$\bar{C}(T_0) = \frac{K_0}{T_0} + 2\sqrt{(m_1 + w_1)K_1} + 2\sqrt{(m_2 + w_2)K_2}$$

on I_1. As another example, on I_3, $\mathcal{G}_3 = \emptyset, \mathcal{E}_3 = \{1, 2\}, \mathcal{L}_3 = \emptyset$. It follows that $A_3 = K_0 + K_1 + K_2, B_3 = 0, C_3 = m_1 + w_1 + m_2 + w_2$ and

$$\bar{C}(T_0) = \frac{K_0 + K_1 + K_2}{T_0} + (m_1 + w_1 + m_2 + w_2)T_0$$

on I_1.

We then have the following algorithm in finding the optimal T_0^*. Note that the unconstrained minimizer of the function $\frac{A_j}{T_0} + B_j + C_j T_0$ is given by $\sqrt{A_j/C_j}$. Hence, starting from any given interval I_j, if $\sqrt{A_j/C_j} \in I_j$, by convexity, we must have $T_0^* = \sqrt{A_j/C_j}$. If $\sqrt{A_j/C_j} > u_j$, then the objective must be decreasing on I_j, we go to I_{j+1}. If $\sqrt{A_j/C_j} < \ell_j$, then the objective must be increasing on I_j, we go to I_{j-1}.

Let j^* be the index such that $\sqrt{A_{j^*}/C_{j^*}} \in I_{j^*}$. Correspondingly, we let $\mathcal{G}^* = \mathcal{G}_{j^*}, \mathcal{E}^* = \mathcal{E}_{j^*}, \mathcal{L}^* = \mathcal{L}_{j^*}$ and

$A^* = A_{j^*}, B^* = B_{j^*}, C^* = C_{j^*}$. Then,

$$\begin{aligned}\bar{C}(T_0^*) &= \frac{A^*}{T_0^*} + B^* + C^* T_0^* \\ &= A^* \sqrt{\frac{C^*}{A^*}} + B^* + C^* \sqrt{\frac{A^*}{C^*}} \\ &= 2\sqrt{A^* C^*} + B^* \\ &= 2\frac{A^*}{T_0^*} + B^*.\end{aligned}$$

Substituting the expressions for A^*, B^*, we have

$$\begin{aligned}\bar{C}(T_0^*) &= 2\frac{K_0}{T_0^*} + \sum_{i \in \mathcal{E}^*} \frac{2K_i}{T_0^*} + \sum_{i \in \mathcal{G}^*} 2\sqrt{(m_i + w_i)K_i} + \sum_{i \in \mathcal{L}^*} 2\sqrt{m_i K_i} \\ &= 2\sqrt{\frac{K_0}{(T_0^*)^2} K_0} + \sum_{i \in \mathcal{E}^*} 2\sqrt{\frac{K_i}{(T_0^*)^2} K_i} + \sum_{i \in \mathcal{G}^*} 2\sqrt{(m_i + w_i)K_i} + \sum_{i \in \mathcal{L}^*} 2\sqrt{m_i K_i}.\end{aligned}$$

The above expression for the optimal cost has some intuitive interpretations. To this end, we define the "reallocated holding cost" as

$$\bar{H}_i = \begin{cases} m_i + w_i & \text{if } i \in \mathcal{G}^* \\ m_i & \text{if } i \in \mathcal{L}^* \\ \frac{K_i}{(T_0^*)^2} & \text{if } i \in \mathcal{E}^* \cup \{0\}. \end{cases}$$

Note that $\bar{H}_0 + \sum_{i=1}^n \bar{H}_i = \sum_{i=1}^n (m_i + w_i)$. That is, we are reallocation the system holding cost $\sum_{i=1}^n (m_i + w_i)$ to each facility $i, i = 0, 1, ..., n$. Then,

$$\bar{C}(T_0^*) = \sum_{i=0}^n 2\sqrt{\bar{H}_i K_i} = \sum_{i=0}^n \min_{T_i} C_i(T_i),$$

where $C_i(T_i) = \frac{K_i}{T_i} + \bar{H}_i T_i$ for $i = 0, 1, ..., n$. That is, the reformulation here decomposes the problem back into $n+1$ EOQ models:

- For $i \in \mathcal{G}^*$, we have $T_i^* > T_0^*$. Retailers in $\mathcal{G}^*$ are slow-moving retailers and the warehouse never holds inventory for them. The reallocated holding cost is simply

$$m_i + w_i = \frac{(h_i - h_0)\lambda_i}{2} + w_i = \frac{h_0 \lambda_i}{2} = \frac{\lambda_i h_i}{2}.$$

- For $i \in \mathcal{L}^*$, we have $T_i^* < T_0^*$. Retailers in $\mathcal{L}^*$ are fast-moving retailers. Since part of the inventory is held at the warehouse, the reallocated holding cost

$$m_i = \frac{(h_i - h_0)\lambda_i}{2} < \frac{\lambda_i h_i}{2}.$$

The value $\bar{C}(T_0^*)$ provides a lower bound to the optimal cost of (4.1) and now we construct a feasible power-of-two policy to problem (4.1). As in the EOQ model, we can simply let

$$k_i = \min\{k \in \mathbb{Z} | C_i(T_B \cdot 2^k) \le C_i(T_B \cdot 2^{k+1})\} = \min\left\{k \in \mathbb{Z} \,\middle|\, \frac{1}{\sqrt{2}} T_i^* \le T_B \cdot 2^k \right\}$$

and $T_i = T_B 2^{k_i}$ for $i = 0, 1, ..., n$. By definition, we then have $k_i = k_0$ for $i \in \mathcal{E}^*$, $k_i \geq k_0$ for $i \in \mathcal{G}^*$, and $k_i \leq k_0$ for $i \in \mathcal{L}^*$. It then follows that the objective of (4.1) under the constructed power-of-two policy is

$$C(T_B \cdot 2^{k_0}, T_B \cdot 2^{k_1}, ..., T_B \cdot 2^{k_n}) = \sum_{i=0}^{n} \frac{K_i}{T_B 2^{k_i}} + \sum_{i=1}^{n} H_i(T_B 2^{k_i}, T_B 2^{k_0}) = \frac{K_0}{T_0} + \sum_{i=1}^{n} \left(\frac{K_i}{T_i} + m_i T_i + w_i \max\{T_0, T_i\} \right).$$

We claim that

$$C(T_B \cdot 2^{k_0}, T_B \cdot 2^{k_1}, ..., T_B \cdot 2^{k_n}) = \sum_{i=0}^{n} C_i(T_B 2^{k_i}) = \sum_{i=0}^{n} \left(\frac{K_i}{T_B 2^{k_i}} + \bar{H}_i T_B 2^{k_i} \right).$$

Since $\bar{H}_0 = \sum_{i=1}^{n} (m_i + w_i - \bar{H}_i)$, we have

$$\begin{aligned} \sum_{i=0}^{n} C_i(T_B 2^{k_i}) &= \frac{K_0}{T_0} + \bar{H}_0 T_0 + \sum_{i=1}^{n} \left(\frac{K_i}{T_i} + \bar{H}_i T_i \right) \\ &= \frac{K_0}{T_0} + \sum_{i=1}^{n} \left(\frac{K_i}{T_i} + (m_i + w_i - \bar{H}_i) T_0 + \bar{H}_i T_i \right) \end{aligned}$$

It follows that:

- For $i \in \mathcal{G}^*$, we have $\bar{H}_i = m_i + w_i$ and $k_i \geq k_0$. Hence,
$$(m_i + w_i - \bar{H}_i) T_0 + \bar{H}_i T_i = (m_i + w_i) T_i = m_i T_i + w_i \max\{T_0, T_i\}.$$
- For $i \in \mathcal{L}^*$, we have $\bar{H}_i = m_i$ and $k_i \leq k_0$. Hence,
$$(m_i + w_i - \bar{H}_i) T_0 + \bar{H}_i T_i = w_i T_0 + m_i T_i = m_i T_i + w_i \max\{T_0, T_i\}.$$
- For $i \in \mathcal{E}^*$, we have $\bar{H}_i = K_i / (T_0^*)^2$ and $k_i = k_0$ (i.e., $T_i = T_0$). Hence,
$$(m_i + w_i - \bar{H}_i) T_0 + \bar{H}_i T_i = (m_i + w_i) T_i = m_i T_i + w_i \max\{T_0, T_i\}.$$

By using the bound we derived for power-of-two policy for the EOQ model, we then arrive at:

$$C(T_B \cdot 2^{k_0}, T_B \cdot 2^{k_1}, ..., T_B \cdot 2^{k_n}) = \sum_{i=0}^{n} C_i(T_B 2^{k_i}) \leq 1.06 \bar{C}(T_0^*).$$

That is, the power-of-two policy we constructed is within 6% of the cost under the optimal power-of-two policy obtained by solving the integer program (4.1).

4.3 Comparison to the Optimal Policy

Here we show that the power-of-two policy we constructed is not only within 6% of the cost under the optimal power-of-two policy to problem (4.1) but within 6% of the cost under any possible ordering policy for the single-warehouse multi-retailer problem. To establish this, it is sufficient to show that $\bar{C}(T_0^*)$ provides a lower bound to the minimum cost over all feasible policies.

Theorem 4.2 *The value $\bar{C}(T_0^*)$ is also a lower bound on the average cost of all feasible policies over every finite horizon.*

Proof: Consider any policy on $[0,t], t>0$. Let y_i be the corresponding number of orders placed by facility $i, i \geq 0$, and $x_i(t)$ and $s_i(t)$ to be the inventory level and system inventory level respectively at retailer $i, i \geq 1$. The system's total holding cost can be computed as

$$\sum_{i=1}^{n}\left[(h_i-h_0)\int_0^t x_i(\tau)d\tau + h_0\int_0^t s_i(\tau)d\tau\right] = \sum_{i=1}^{n}\int_0^t \left[(h_i-h_0)x_i(\tau)+h_0 s_i(\tau)\right]d\tau.$$

We claim that

$$(h_i-h_0)x_i(\tau)+h_0 s_i(\tau) \geq \frac{2\bar{H}_i}{\lambda_i}x_i(\tau) + \frac{2(m_i+w_i-\bar{H}_i)}{\lambda_i}s_i(\tau).$$

Indeed:

- For $i \in \mathcal{G}^*$, we have $\bar{H}_i = m_i + w_i = \frac{h_i\lambda_i}{2}$. Hence,

  $$\frac{2\bar{H}_i}{\lambda_i}x_i(\tau) + \frac{2(m_i+w_i-\bar{H}_i)}{\lambda_i}s_i(\tau) = h_i x_i(\tau) \leq (h_i-h_0)x_i(\tau)+h_0 s_i(\tau),$$

  where the inequality is due to $s_i(\tau) \geq x_i(\tau)$.

- For $i \in \mathcal{L}^*$, we have $\bar{H}_i = m_i = \frac{h_i-h_0}{2}\lambda_i$ and $m_i + w_i - \bar{H}_i = w_i = \frac{h_0\lambda_i}{2}$. Hence,

  $$\frac{2\bar{H}_i}{\lambda_i}x_i(\tau) + \frac{2(m_i+w_i-\bar{H}_i)}{\lambda_i}s_i(\tau) = (h_i-h_0)x_i(\tau)+h_0 s_i(\tau).$$

- For $i \in \mathcal{E}^*$, we have $\bar{H}_i = K_i/(T_0^*)^2$. By $\underline{\tau_i} \leq T_0^* \leq \overline{\tau_i}$ for all $i \in \mathcal{E}^*$, we have

  $$\frac{K_i}{m_i+w_i} \leq (T_0^*)^2 \leq \frac{K_i}{m_i},$$

  which implies

  $$m_i \leq \frac{K_i}{(T_0^*)^2} \leq m_i + w_i.$$

  Hence, $m_i \leq \bar{H}_i \leq m_i + w_i$. It then follows that

  $$\begin{aligned}\frac{2\bar{H}_i}{\lambda_i}x_i(\tau) + \frac{2(m_i+w_i-\bar{H}_i)}{\lambda_i}s_i(\tau) &\leq \frac{2\bar{H}_i}{\lambda_i}x_i(\tau) + \frac{2(m_i+w_i-\bar{H}_i)}{\lambda_i}s_i(\tau) + \frac{2(\bar{H}_i-m_i)}{\lambda_i}(s_i(\tau)-x_i(\tau)) \\ &= \frac{2m_i}{\lambda_i}x_i(\tau) + \frac{2w_i}{\lambda_i}s_i(\tau) \\ &= (h_i-h_0)x_i(\tau)+h_0 s_i(\tau).\end{aligned}$$

Therefore, the total holding cost can be lower bounded by

$$\begin{aligned}\sum_{i=1}^{n}\int_0^t \left[(h_i-h_0)x_i(\tau)+h_0 s_i(\tau)\right]d\tau &\geq \sum_{i=1}^{n}\int_0^t \left[\frac{2\bar{H}_i}{\lambda_i}x_i(\tau) + \frac{2(m_i+w_i-\bar{H}_i)}{\lambda_i}s_i(\tau)\right]d\tau \\ &= \sum_{i=1}^{n}\int_0^t \bar{H}_i\frac{2x_i(\tau)}{\lambda_i}d\tau + \int_0^t \bar{H}_0\frac{2x_0(\tau)}{\lambda_0}d\tau,\end{aligned}$$

where in the equality we defined $x_0(\tau)$ to be the "average" system inventory level

$$x_0(\tau) := \frac{\lambda_0}{2\bar{H}_0} \sum_{i=1}^{n} (m_i + w_i - \bar{H}_i) \frac{2s_i(\tau)}{\lambda_i}$$

and λ_0 is set to be an arbitrary positive number for the ease of notation. An important observation here is that for $i = 1, ..., n$, $2x_i(\tau)/\lambda_i$ can be interpreted as the inventory level in a re-scaled system where the ordering quantity and demand rate are simultaneously scaled by $2/\lambda_i$ (so that the demand rate after scaling is simply 2). For $i = 0$, first note that $s_i(\tau)$ has jumps at the times only when the warehouse makes an order and decreases linearly in τ with slope $-\lambda_i$ otherwise. It follows that $2s_i(\tau)/\lambda_i$ can also be interpreted as the inventory level in a re-scaled system with demand rate 2. Since $\bar{H}_0 = \sum_{i=1}^{n}(m_i + w_i - \bar{H}_i)$, $2x_0(\tau)/\lambda_0$ also has jumps exactly at the times when the warehouse makes an order and decreases linearly in τ with slope -2 otherwise.

Combining with the fixed ordering cost, the total average cost is then lower bounded by

$$\frac{1}{t} \sum_{t=0}^{n} \left(y_i K_i + \int_0^t \bar{H}_i \frac{2x_i(\tau)}{\lambda_i} d\tau \right).$$

Note that for each $i = 0, 1, ..., n$, the cost $\frac{1}{t}(y_i K_i + \int_0^t \bar{H}_i \frac{2x_i(\tau)}{\lambda_i} d\tau)$ exactly represents the average cost in a finite horizon EOQ model with demand rate 2 under a policy that places y_i orders and has inventory level $\frac{2x_i(\tau)}{\lambda_i}$. Recall that optimal number of (equally spaced) order for such system is

$$m^* = t\sqrt{\frac{\lambda h}{2K}}.$$

and the corresponding optimal cost is

$$\frac{Km^*}{t} + h\frac{\lambda t}{2m^*} = \sqrt{2\lambda h K}.$$

Here, we then have

$$\frac{1}{t} \sum_{t=0}^{n} \left(y_i K_i + \int_0^t \bar{H}_i \frac{2x_i(\tau)}{\lambda_i} d\tau \right) \geq \sum_{t=0}^{n} 2\sqrt{\bar{H}_i K_i} = \bar{C}(T_0^*),$$

establishing our claim. ■

The 1.06 bound we derived here is independent of how the base planning period T_B is chosen. In the original paper, Roundy (1985) establish that 1.06 bound holds for a more general class of policy called "integer-ratio" policy and further show that the bound can be tightened to 1.02 if the base planning period in the power-of-two policy can be chosen optimally.

References

Roundy, R. (1985). 98%-effective integer-ratio lot-sizing for one-warehouse multi-retailer systems. *Management science 31*(11), 1416–1430.