Foundations of Inventory Management **Fall 2022, NUS**

Lecture 8: Dynamic Stochastic Inventory Model II

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8.1 Model with Fixed-Ordering Cost

We consider here the same basic model as in Lecture 7 with the addition of a fixed ordering cost K. The Bellman equation with this change becomes

$$v_t(x) = \min_{y \geq x}\{c(y-x) + K \cdot 1_{\{y>x\}} + L(y) + \alpha\mathbb{E}[v_{t+1}(y-D)]\},$$

where recall that

$$L(y) = h\mathbb{E}[(y-D)^+] + p\mathbb{E}[(D-y)^+].$$

Again, let us start from a single-period problem with $T = 1, v_T(x) = 0$. Let $G(y) = cy + L(y)$ and $S = \arg\min_y G(y)$. Then we have the following two cases:

- If $x \geq S$, then $G(y)$ is increasing in y for $y \geq x$. In addition, for any $y > x$, we have

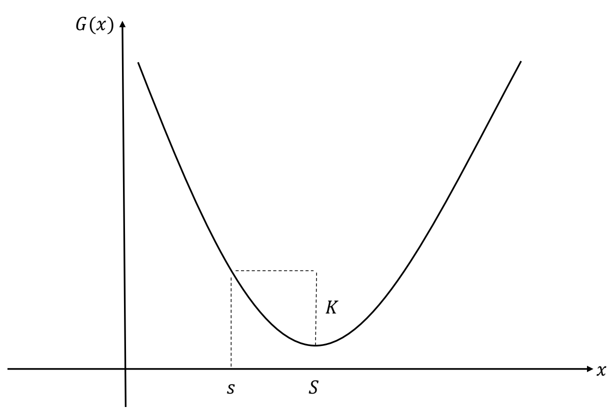
  $$K + cy + L(y) > cy + L(y) \geq cx + L(x).$$

  Hence, we must have $y^* = x$.

- If $x < S$, then given that an order is placed, the optimal total cost is $K + G(S)$. If no order is placed, then the corresponding cost is $G(x)$. Hence, order up to S is optimal if and only if

  $$K + G(S) \leq G(x).$$

  Since $G(x)$ is decreasing for $x < S$, there exists s such that $K + G(S) \leq G(x)$ for $x \leq s$ and $K + G(S) > G(x)$ for $s < x < S$. The thresholds s and S are illustrated in the figure below.



In summary, we have

$$y^*(x) = \begin{cases} S & \text{if } x \le s \\ x & \text{otherwise.} \end{cases}$$

The policy characterized here is known as the *s-S policy*, where one monitors two parameters parameter: the reordering point s and the order-up-to level S. If the current inventory level is above the reordering point s, then no order takes place; otherwise one orders up to S.

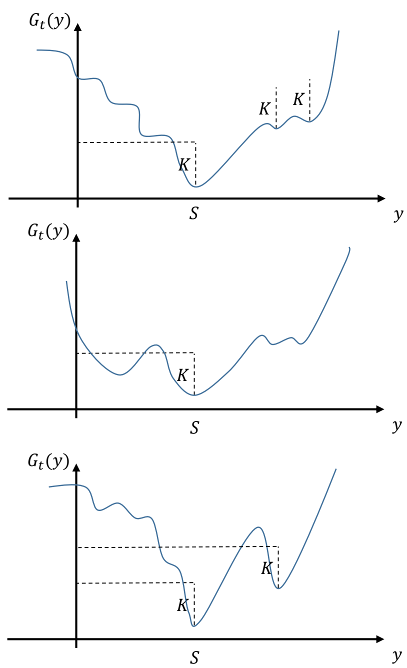
For the general problem, let $G_t(y) = cy + L(y) + \alpha\mathbb{E}[v_{t+1}(y - D)]$. Then the Bellman equation can be rewritten as

$$\begin{aligned} v_t(x) &= \min_{y \ge x}\{K \cdot 1_{\{y>x\}} + G_t(y)\} - cx, \\ &= -cx + \min\{G_t(x), \min_{y \ge x}\{K + G_t(y)\}\}. \end{aligned}$$

Our analysis for the single problem solves the last-period problem here with terminal condition $v_T(x) = 0$. Correspondingly, we have

$$v_{T-1}(x) = \begin{cases} K + G_{T-1}(S) - cx & \text{if } x \le s \\ G_{T-1}(x) - cx & \text{otherwise.} \end{cases}$$

Note here that $v_{T-1}(x)$ is no longer convex. This raises the question of what properties we need for $G_t(y)$ can ensure the optimality of s-S policy in period t. Consider G_t functions of the form in the figure below. In which case do we have s-S policy being optimal?



8.2 K-Convexity

For any $K \geq 0$, a function $f : \mathbb{R} \to \mathbb{R}$ is said to be K-convex if for any $x \leq y$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)(K + f(y)).$$

By definition, a convex function is K-convex for any $K \geq 0$. The following lemma summarizes some of the commonly used properties for K-convex functions.

Lemma 8.1 **(a)** *If f is K-convex and a is a positive scalar, then af is k-convex for all $k \geq aK$.*

(b) *The sum of a K-convex function and a k-convex function is $(K + k)$-convex.*

(c) *If v is K-convex, ϕ is the probability density of a positive random variable, and $G(y) := E[v(y - D)] = \int_0^\infty v(y-\xi)\phi(\xi)d\xi$, then G is K-convex.*

(d) *If f is K-convex, $x < y$ and $f(x) = K + f(y)$, then $f(z) \leq K + f(y)$ for all $z \in [x, y]$ and $f(z) \geq K + f(y)$ for all $z < x$.*

The proof is left as homework. The next lemma shows that K-convexity is indeed sufficient to guarantee the optimality of s-S policy.

Lemma 8.2 *Let $S_t = \arg\min_y G_t(y)$. If $G_t(y) = cy + L(y) + \alpha\mathbb{E}[v_{t+1}(y - D)]$ is continuous and K-convex with $\lim_{y\to-\infty} G_t(y) > K + G_t(S_t)$, then s-S policy is optimal in period t.*

Proof: Let

$$s_t = \inf\{y | G_t(y) = K + G_t(S_t)\}.$$

By continuity and $\lim_{y\to-\infty} G_t(y) > K + G_t(S_t)$, s_t exists and since $G(S_t) \leq K + G_t(S_t)$, we further have $s_t \leq S_t$.

For any $x < s_t$, by Lemma 8.1 (d), we must have

$$G_t(x) \geq K + G_t(S_t),$$

which implies $y^*(x) = S_t$. For any $s_t \leq x \leq S_t$, again by Lemma 8.1 (d), we must have

$$G_t(x) \leq K + G_t(S_t),$$

which implies $y^*(x) = x$. For $x > S_t$, suppose there exists $y > x$ such that $G_t(y) + K < G_t(x)$. Then by continuity of G_t and the fact that $G_t(S_t) \leq G_t(y) + K$, there must exist some $w \in [S_t, x)$ such that

$$G_t(w) = G_t(y) + K.$$

Lemma 8.1 (d) then implies that $G_t(x) \leq K + G_t(y)$, a contradiction. Hence, $y^*(x) = x$ in this case as well, and in summary, s-S policy is optimal. ∎

To show that s-S policy is optimal in every period, it is then sufficient to show that the K-convexity (and other conditions used in Lemma 8.2) is preserved in the dynamic programming iteration. The condition that $\lim_{y\to-\infty} G_t(y) > K + G_t(S_t)$ can be verified separately by noting that $\lim_{y\to-\infty} G_t(y) = +\infty$ for $t = 0, 1..., T-1$. The following lemma establishes the preservation of K-convexity.

Lemma 8.3 *If v_{t+1} is continuous and K-convex, then both G_t and v_t are continuous and K-convex.*

Proof: The continuity part is straightforward: one can either invoke the general maximum theorem (see, for instance, Ok, 2011) or verify directly.

For K-convexity, by Lemma 8.1, we know that $\mathbb{E}[v_{t+1}(y-D)]$ and hence $G_t(y) = cy + L(y) + \alpha\mathbb{E}[v_{t+1}(y-D)]$ are both K-convex. By Lemma 8.2, s-S policy is then optimal in period t, and we have

$$v_t(x) = \begin{cases} K + G_t(S_t) - cx & \text{if } x \le s_t \\ G_t(x) - cx & \text{otherwise.} \end{cases}$$

By letting

$$G_t^*(x) = \begin{cases} K + G_t(S_t) & \text{if } x \le s_t \\ G_t(x) & \text{otherwise.} \end{cases}$$

It is then sufficient to show that $G_t^*(x)$ is K-convex. Clearly, $G_t^*(x)$ is K-convex on the intervals $[s_t, +\infty)$ and $(-\infty, s_t]$ separately. It then remains to show that for any $x < s_t < y$ and $\lambda \in [0,1]$, we also have

$$G_t^*(\lambda x + (1-\lambda)y) \le \lambda G_t^*(x) + (1-\lambda)(K + G_t^*(y)).$$

Let $w = \lambda x + (1-\lambda)y$ and note that $G_t^*(x) = K + G_t(S_t), G_t^*(y) = G_t(y)$. If $w \le s_t$, then

$$G_t^*(w) = K + G_t(S_t) \le K + G_t(y) = K + G_t^*(y).$$

It follows that $G_t^*(\lambda x + (1-\lambda)y) \le \lambda G_t^*(x) + (1-\lambda)(K + G_t^*(y))$.

If $s_t < w \le y$, then $G_t^*(w) = G_t(w)$ and we need to show that

$$G_t(w) \le \lambda(K + G_t(S_t)) + (1-\lambda)(K + G_t(y)) = \lambda G_t(s_t) + (1-\lambda)(K + G_t(y)).$$

Since $x < s_t$, there exists $\lambda < \tilde{\lambda} \le 1$ such that $w = \tilde{\lambda} s_t + (1-\tilde{\lambda})y$. By K-convexity of G_t, we have

$$G_t(w) \le \tilde{\lambda} G_t(s_t) + (1-\tilde{\lambda})(K + G_t(y)) \le \lambda G_t(s_t) + (1-\lambda)(K + G_t(y)),$$

where in the last inequality we used the fact that $\tilde{\lambda} > \lambda$ and $K + G_t(y) \ge G_t(s_t)$. ∎

By Lemmas 8.2 and 8.3, we then know that as long as the terminal condition v_T is K-convex, then v_t is K-convex in every period, and an (s_t, S_t)-policy is optimal in period t.

References

Ok, E. A. (2011). *Real analysis with economic applications.* Princeton University Press.