FOUNDATIONS OF OPTIMIZATION: IE6001

Two Phase Simplex Algorithm

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Initial Basic Feasible Solution

- In STEP 0 the simplex algorithm requires an initial BFS and the corresponding basic representation.

- One can show that finding a feasible solution is in general as hard as finding an optimal solution!

$\Rightarrow$ How to construct an initial BFS?

  - In general, an initial BFS can be found by using a variant of the simplex algorithm.
  - In some special cases, an initial BFS can be constructed "manually".

Problems with an "All Slack Basis"

minimize $\quad x_0 = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to:

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & \leq & b_m
\end{array}$$

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

Problems with an "All Slack Basis"

$$\text{minimize} \quad x_0 = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

subject to:

$$\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & + & x_{n+1} & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & + & x_{n+2} & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & + & x_{n+m} & = & b_m
\end{array}$$

$$x_1 \geq 0, \ldots, x_n \geq 0, \quad x_{n+1} \geq 0, \ldots, x_{n+m} \geq 0$$

$\Rightarrow$ This is a basic representation for $I = \{n+1, \ldots, n+m\}$.
The corresponding BS is feasible if $b_i \geq 0$, $i = 1, \ldots, m$.

Example w/o an Obvious Initial BFS

Consider a system with

- equalities,
- "$\geq$" inequalities and
- "$\leq$" inequalities,

and assume that all variables and RHS's are nonnegative.

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 10 \\ 2x_1 & - & x_2 & & & \geq & 2 \\ x_1 & - & 2x_2 & + & x_3 & \leq & 6 \end{array}$$

$$x_i \geq 0 \ \forall i = 1, \ldots, 3$$

Example w/o an Obvious Initial BFS

Standardise the system by

- adding slack variables and
- subtracting surplus variables.

$$
\begin{array}{rcrcrcrcrcr}
x_1 & + & x_2 & + & x_3 & & & & & = & 10 \\
2x_1 & - & x_2 & & & - & x_4 & & & = & 2 \\
x_1 & - & 2x_2 & + & x_3 & & & + & x_5 & = & 6
\end{array}
$$

$$x_i \geq 0 \ \forall i = 1, \ldots, 5$$

$\Rightarrow$ No basic representation!
Only slack variables behave like basic variables!

Example w/o an Obvious Initial BFS

Idea: Add new artificial variables to those constraints that were originally equalities and "$\geq$" inequalities.

$$\begin{array}{rcrcrcrcrcrcrcr}
x_1 & + & x_2 & + & x_3 & & & + & \xi_1 & & & & & = & 10 \\
2x_1 & - & x_2 & & & - & x_4 & & & + & \xi_2 & & & = & 2 \\
x_1 & - & 2x_2 & + & x_3 & & & & & & & + & x_5 & = & 6
\end{array}$$

$$x_i \geq 0 \ \forall i = 1, \ldots, 5, \ \xi_1 \geq 0, \ \xi_2 \geq 0.$$

The artificial variables behave like basic variables.
$\Rightarrow$ We have found a basic representation!

But this system is not equivalent to the original one!

Example w/o an Obvious Initial BFS

Important Observation:

Any nonnegative FS $(x_1, \ldots, x_5, \xi_1, \xi_2)$ for

$$\begin{array}{rcrcrcrcrcrcrcr}
x_1 & + & x_2 & + & x_3 & & & + & \xi_1 & & & & & = & 10 \\
2x_1 & - & x_2 & & & - & x_4 & & & + & \xi_2 & & & = & 2 \\
x_1 & - & 2x_2 & + & x_3 & & & & & & & + & x_5 & = & 6
\end{array}$$

with $\xi_1 = \xi_2 = 0$ provides a nonnegative FS $(x_1, \ldots, x_5)$ for

$$\begin{array}{rcrcrcrcrcr}
x_1 & + & x_2 & + & x_3 & & & & & = & 10 \\
2x_1 & - & x_2 & & & - & x_4 & & & = & 2 \\
x_1 & - & 2x_2 & + & x_3 & & & + & x_5 & = & 6
\end{array}$$

Example w/o an Obvious Initial BFS

To find such a solution, we solve the auxiliary LP:

$$\text{minimize} \quad \zeta = \xi_1 + \xi_2$$

subject to:

$$\begin{array}{rcrcrcrcrcrcrcr}
x_1 & + & x_2 & + & x_3 & & & + & \xi_1 & & & & & = & 10 \\
2x_1 & - & x_2 & & & - & x_4 & & & + & \xi_2 & & & = & 2 \\
x_1 & - & 2x_2 & + & x_3 & & & & & & & + & x_5 & = & 6
\end{array}$$

$$x_1 \geq 0, \ldots, x_5 \geq 0, \ \xi_1 \geq 0, \ \xi_2 \geq 0$$

The initial BFS for this LP is given by $\xi_1 = 10$, $\xi_2 = 2$, $x_5 = 6$ (basic variables) and $x_1 = \cdots = x_4 = 0$ (nonbasic variables).

Example w/o an Obvious Initial BFS

- To solve the auxiliary LP with the simplex algorithm, we need a basic representation for the initial BFS.
- However, the objective function value $\zeta = \xi_1 + \xi_2$ is expressed in terms of the basic variables ξ_1 and ξ_2.
- To express ζ as a function of the nonbasic variables, we add all equations with artificial variables to the objective.

$$
\begin{array}{r|lrrrrrrrrrrrrr}
\text{Obj.} & \zeta & & & & & & & & - & \xi_1 & - & \xi_2 & = & 0 \\
+\ \text{Eq. 1} & & x_1 & + & x_2 & + & x_3 & & & + & \xi_1 & & & = & 10 \\
+\ \text{Eq. 2} & & 2x_1 & - & x_2 & & & - & x_4 & & & + & \xi_2 & = & 2 \\
\hline
= & \zeta + & 3x_1 & & & + & x_3 & - & x_4 & & & & & = & 12
\end{array}
$$

Example w/o an Obvious Initial BFS

- The auxiliary LP is feasible and bounded by construction ($\zeta = \xi_1 + \xi_2 \geq 0$ cannot drop indefinitely!).

$\Rightarrow$ The simplex algorithm must terminate in STEP 1 with an optimal BFS. There are two cases:

  - $\zeta = 0$ at optimality: this implies that $\xi_1 = \xi_2 = 0$, and the optimal BFS of the auxiliary LP provides a BFS for the original system.

  - $\zeta > 0$ at optimality: the auxiliary LP has no feasible solution with $\xi_1 = \xi_2 = 0 \Rightarrow$ the original system has no BFS $\Rightarrow$ it is infeasible!

Example w/o an Obvious Initial BFS

Solve the auxiliary LP with the simplex algorithm.

BV	x_1	x_2	x_3	x_4	x_5	ξ_1	ξ_2	RHS
ζ	3		1	-1				12
ξ_1	1	1	1			1		10
ξ_2	2	-1		-1			1	2
x_5	1	-2	1		1			6

Example w/o an Obvious Initial BFS

Solve the auxiliary LP with the simplex algorithm.

BV	x_1	x_2	x_3	x_4	x_5	ξ_1	ξ_2	RHS
ζ	3		1	-1				12
ξ_1	1	1	1			1		10
ξ_2	2	-1		-1			1	2
x_5	1	-2	1		1			6
ζ		$1\frac{1}{2}$	1	$\frac{1}{2}$			$-1\frac{1}{2}$	9
ξ_1		$1\frac{1}{2}$	1	$\frac{1}{2}$		1	$-\frac{1}{2}$	9
x_1	1	$-\frac{1}{2}$		$-\frac{1}{2}$			$\frac{1}{2}$	1
x_5		$-1\frac{1}{2}$	1	$\frac{1}{2}$	1		$-\frac{1}{2}$	5

Example w/o an Obvious Initial BFS

Solve the auxiliary LP with the simplex algorithm.

BV	x_1	x_2	x_3	x_4	x_5	ξ_1	ξ_2	RHS
ζ		$1\frac{1}{2}$	1	$\frac{1}{2}$			$-1\frac{1}{2}$	9
ξ_1		$1\frac{1}{2}$	1	$\frac{1}{2}$		1	$-\frac{1}{2}$	9
x_1	1	$-\frac{1}{2}$		$-\frac{1}{2}$			$\frac{1}{2}$	1
x_5		$-1\frac{1}{2}$	1	$\frac{1}{2}$	1		$-\frac{1}{2}$	5
ζ						-1	-1	0
x_2		1	$\frac{2}{3}$	$\frac{1}{3}$			$-\frac{1}{3}$	6
x_1	1		$\frac{1}{3}$	$-\frac{1}{3}$			$\frac{1}{3}$	4
x_5			2	1	1		-1	14

This is a BFS for the original system!

Two Phase Simplex: Phase 1

Step 1: Modify the constraints so that all RHS's are nonnegative (constraints with negative RHS $\times -1$).

Step 2: Identify now all equality and $\geq$ constraints. In Step 4 we will add artificial variables to these constraints.

Step 3: Standardise inequalities: for $\leq$ constraints, add slacks; for $\geq$ constraints, subtract excesses.

Step 4: Add now artificial variables ξ_i to all $\geq$ or equality constraints identified in Step 2.

Step 5: Let ζ be the sum of all artificial variables and derive the basic representation of ζ.

Step 6: Find minimum value of ζ using the simplex algorithm.

Two Phase Simplex: Phase 2

Case 1: $\zeta^* > 0$

$\Rightarrow$ The original LP is infeasible.

Case 2: $\zeta^* = 0$ and all ξ_i are nonbasic at optimality.

$\Rightarrow$ Remove all artificial columns from the optimal Phase 1 tableau.

$\Rightarrow$ Derive the basic representation of x_0 (original objective) w.r.t. optimal index set of Phase 1.

$\Rightarrow$ Solve the original LP with the simplex algorithm (Phase 2). The final basis of Phase 1 is the initial basis of Phase 2. The optimal solution to Phase 2 is the optimal solution to the original LP.

Two Phase Simplex: Phase 2

Case 3: $\zeta^* = 0$ and at least one ξ_i is basic at optimality.

$\Rightarrow$ As $\zeta^* = 0$ we conclude that all $\xi_i = 0$.

$\Rightarrow$ We have found a degenerate BFS for the original problem and a basic representation for the auxiliary problem.

$\Rightarrow$ As the BFS is degenerate, we can pivot on any $y_{pq} \neq 0$ corresponding to an artificial ξ_p and an original variable x_q without changing the BFS!

Example (Case 2)

$$\min \; x_0 = 2x_1 + 3x_2$$

subject to

$$\begin{array}{rcrcr} \frac{1}{2}x_1 & + & \frac{1}{4}x_2 & \leq & 4 \\ x_1 & + & 3x_2 & \geq & 20 \\ x_1 & + & x_2 & = & 10 \end{array}$$

and

$$x_1, x_2 \geq 0$$

Example (Case 2)

Steps 1–4 of Phase 1 transform the equality constraints to:

$$\begin{array}{rcrcrcrcrcrcr}
\frac{1}{2}x_1 & + & \frac{1}{4}x_2 & + & x_3 & & & & & & & = & 4 \\
x_1 & + & 3x_2 & & & - & x_4 & + & \xi_2 & & & = & 20 \\
x_1 & + & x_2 & & & & & & & + & \xi_3 & = & 10
\end{array}$$

Initial BFS for Phase 1:

Basic variables: $x_3 = 4,\ \xi_2 = 20,\ \xi_3 = 10$

Nonbasic variables: $x_1 = x_2 = x_4 = 0$

Example (Case 2)

In Step 5 of Phase 1 define $\zeta = \xi_2 + \xi_3$ and derive the basic representation for ζ w.r.t. the basic variables x_3, ξ_2 and ξ_3.

$$
\begin{array}{rcrcrcrcrcrcrcr}
\text{Row } 0 & \zeta & & & & & & & & - & \xi_2 & - & \xi_3 & = & 0 \\
+\ \text{Row } 2 & & & x_1 & + & 3x_2 & - & x_4 & + & \xi_2 & & & & = & 20 \\
+\ \text{Row } 3 & & & x_1 & + & x_2 & & & & & & + & \xi_3 & = & 10 \\
\hline
= & \zeta & + & 2x_1 & + & 4x_2 & - & x_4 & & & & & & = & 30
\end{array}
$$

$$\Rightarrow \quad \zeta = \xi_2 + \xi_3 = 30 - 2x_1 - 4x_2 + x_4$$

Example (Case 2)

In Step 6 of Phase 1 we solve the auxiliary LP.

$$\text{minimize } \zeta = 30 - 2x_1 - 4x_2 + s_4$$

subject to:

$$\begin{array}{rcrcrcrcrcrcr}
\frac{1}{2}x_1 & + & \frac{1}{4}x_2 & & & + & x_3 & & & & & = & 4 \\
x_1 & + & 3x_2 & - & x_4 & & & + & \xi_2 & & & = & 20 \\
x_1 & + & x_2 & & & & & & & + & \xi_3 & = & 10
\end{array}$$

$$x_1, x_2, x_3, x_4, \xi_2, \xi_3 \geq 0.$$

Example (Case 2)

BV	x_1	x_2	x_3	x_4	ξ_2	ξ_3	RHS
ζ	2	4		-1			30
x_3	$\frac{1}{2}$	$\frac{1}{4}$	1				4
ξ_2	1	3		-1	1		20
ξ_3	1	1				1	10

Example (Case 2)

BV	x_1	x_2	x_3	x_4	ξ_2	ξ_3	RHS
ζ	2	4		-1			30
x_3	$\frac{1}{2}$	$\frac{1}{4}$	1				4
ξ_2	1	3		-1	1		20
ξ_3	1	1				1	10
ζ	$\frac{2}{3}$			$\frac{1}{3}$	$-\frac{4}{3}$		$\frac{10}{3}$
x_3	$\frac{5}{12}$		1	$\frac{1}{12}$	$-\frac{1}{12}$		$\frac{7}{3}$
x_2	$\frac{1}{3}$	1		$-\frac{1}{3}$	$\frac{1}{3}$		$\frac{20}{3}$
ξ_3	$\frac{2}{3}$			$\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{10}{3}$

Example (Case 2)

BV	x_1	x_2	x_3	x_4	ξ_2	ξ_3	RHS
ζ	$\frac{2}{3}$			$\frac{1}{3}$	$-\frac{4}{3}$		$\frac{10}{3}$
x_3	$\frac{5}{12}$		1	$\frac{1}{12}$	$-\frac{1}{12}$		$\frac{7}{3}$
x_2	$\frac{1}{3}$	1		$-\frac{1}{3}$	$\frac{1}{3}$		$\frac{20}{3}$
ξ_3	$\frac{2}{3}$			$\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{10}{3}$
ζ					-1	-1	0
x_3			1	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{5}{8}$	$\frac{1}{4}$
x_2		1		$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	5
x_1	1			$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	5

$\zeta^* = 0 \Rightarrow$ Phase 1 concluded.

Example (Case 2)

BFS found in Phase 1:

Basic variables: $x_3 = \frac{1}{4},\ x_2 = 5,\ x_1 = 5$

Nonbasic variable: $x_4 = \xi_2 = \xi_3 = 0$

There are no artificial variables in the basis $\Rightarrow$ Case 2

We can drop the columns of all artificial variables:
ξ_2 and ξ_3 are no longer needed!

Example (Case 2)

In Phase 2 we first derive the basic representation of $x_0 = 2x_1 + 3x_2$ w.r.t. the basic variables x_1, x_2 and x_3.

Use Rows 2 and 3 of the optimal Phase 1 tableau to eliminate x_1 and x_2 from Row 0 of Phase 2 (objective x_0).

$$
\begin{array}{rcccccccccc}
\text{Row 0}: & x_0 & - & 2x_1 & - & 3x_2 & & & = & 0 \\
+3 \times (\text{Row 2}): & & & & & 3x_2 & - & \frac{3}{2}x_4 & = & 15 \\
+2 \times (\text{Row 3}): & & & 2x_1 & & & + & x_4 & = & 10 \\
\hline
= \quad : & x_0 & & & & & - & \frac{1}{2}x_4 & = & 25
\end{array}
$$

$$
\Rightarrow \quad x_0 = 2x_1 + 3x_2 = 25 + \frac{1}{2}x_4
$$

Example (Case 2)

We now begin Phase 2 with following basic representation:

$$
\begin{array}{ccccccccc}
x_0 & & & & - & \frac{1}{2}x_4 & = & 25 \\
 & x_3 & & & - & \frac{1}{8}x_4 & = & \frac{1}{4} \\
 & & x_2 & & - & \frac{1}{2}x_4 & = & 5 \\
 & & & x_1 & + & \frac{1}{2}x_4 & = & 5
\end{array}
$$

The corresponding BFS is optimal!

IN THIS CASE: Phase 2 requires no further pivots.
IN GENERAL: Continue with the simplex algorithm.

Example (Case 1)

Increase b_2 from 20 to 36 $\Rightarrow$ LP becomes infeasible.

$$\text{minimize } x_0 = 2x_1 + 3x_2$$

subject to:

$$\begin{array}{ccccc} \frac{1}{2}x_1 & + & \frac{1}{4}x_2 & \leq & 4 \\ x_1 & + & 3x_2 & \geq & 36 \\ x_1 & + & x_2 & = & 10 \end{array}$$

and

$$x_1, x_2 \geq 0$$

Example (Case 1)

After Steps 1-5 we find the auxiliary LP:

$$\text{minimize } \zeta = \xi_2 + \xi_3$$

subject to:

$$\begin{array}{ccccccccccccc}
\frac{1}{2}x_1 & + & \frac{1}{4}x_2 & + & x_3 & & & & & & & = & 4 \\
x_1 & + & 3x_2 & & & - & x_4 & + & \xi_2 & & & = & 36 \\
x_1 & + & x_2 & & & & & & & + & \xi_3 & = & 10
\end{array}$$

$$x_1, x_2, x_3, x_4, \xi_2, \xi_3 \geq 0$$

Initial BFS for Phase 1:

Basic variables: $x_3 = 4,\ \xi_2 = 36,\ \xi_3 = 10$

Nonbasic variables: $x_1 = x_2 = x_4 = 0$

Find the basic representation of ζ w.r.t. this basis.

$$
\begin{array}{lccccccccccccccc}
\text{Row 0} & \zeta & & & & & & & & - & \xi_2 & - & \xi_3 & = & 0 \\
+\ \text{Row 2} & & & x_1 & + & 3x_2 & - & x_4 & + & \xi_2 & & & & = & 36 \\
+\ \text{Row 3} & & & x_1 & + & x_2 & & & & & & + & \xi_3 & = & 10 \\
\hline
= & \zeta & + & 2x_1 & + & 4x_2 & - & x_4 & & & & & & = & 46
\end{array}
$$

$$\Rightarrow \quad \zeta = \xi_2 + \xi_3 = 46 - 2x_1 - 4x_2 + x_4$$

Solve the auxiliary LP with the simplex algorithm.

BV	x_1	x_2	x_3	x_4	ξ_2	ξ_3	RHS
ζ	2	4		-1			46
x_3	$\frac{1}{2}$	$\frac{1}{4}$	1				4
ξ_2	1	3		-1	1		36
ξ_3	1	1				1	10

Example (Case 1)

Solve the auxiliary LP with the simplex algorithm.

BV	x_1	x_2	x_3	x_4	ξ_2	ξ_3	RHS
ζ	2	4		-1			46
x_3	$\frac{1}{2}$	$\frac{1}{4}$	1				4
ξ_2	1	3		-1	1		36
ξ_3	1	1				1	10
ζ	-2			-1		-4	6
x_3	$\frac{1}{4}$		1			$-\frac{1}{4}$	$\frac{3}{2}$
ξ_2	-2			-1	1	-3	6
x_2	1	1				1	10

In Row 0 of Tableau 2 no variable has a positive coefficient.
$\Rightarrow$ Optimal Phase 1 tableau with $\zeta^* = 6 > 0 \Rightarrow$ no FS.