Foundations of Inventory Management **Fall 2022, NUS**

Lecture 6: Newsvendor Model

Lecturer: Zhenyu Hu

6.1 Basic Model

Consider a retailer selling a single product at price p over a single period. The retailer faces random demand D with pdf $f(\cdot)$, cdf $F(\cdot)$, and complementary cdf $\bar{F} = 1 - F$. We denote the mean and variance of D as μ and σ^2 respectively. The retailer needs to order the product from the supplier at a unit cost c with $c < p$ before the demand realizes.

Let q denote the retailer's ordering quantity. Then, the retailer's problem of maximizing the expected profit can written as

$$\max_q \left\{ p\mathbb{E}[q \wedge D] - cq \right\}.$$

By using the relationship that

$$\begin{aligned} q \wedge D &= D - (D-q)^+ \\ &= q - (q-D)^+, \end{aligned}$$

we can rewrite the newsvendor objective: $v(q) := p\mathbb{E}[q \wedge D] - cq$ in different forms and provide different interpretations:

1. $$\begin{aligned} v(q) &= p\mathbb{E}[D - (D-q)^+] - cq \\ &= p\mu - cy - p\mathbb{E}[(D-q)^+]. \end{aligned}$$

   Here, $p\mu - cy$ can be interpreted as the profit in the hypothetical scenario where all demand could be satisfied, and $p\mathbb{E}[(D-q)^+]$ is then the part that should be deducted due to shortages (underage).

2. $$\begin{aligned} v(q) &= p\mathbb{E}[q - (q-D)^+] - cq \\ &= (p-c)q - p\mathbb{E}[(q-D)^+]. \end{aligned}$$

   Here, $(p-c)q$ can be interpreted as the profit in the hypothetical scenario where all inventory could be sold, and $p\mathbb{E}[(q-D)^+]$ is then the part that should be deducted due to oversupply (overage).

3. By further noting that $q - D = (q-D)^+ - (D-q)^+$, we have

   $$\begin{aligned} v(q) &= (p-c)\mathbb{E}[D + (q-D)^+ - (D-q)^+] - p\mathbb{E}[(q-D)^+] \\ &= (p-c)\mu - (p-c)\mathbb{E}[(D-q)^+] - c\mathbb{E}[(q-D)^+]. \end{aligned}$$

   Here, $(p-c)\mu$ can be interpreted as the risk-less profit in the hypothetical scenario where demand is deterministic. The terms $(p-c)\mathbb{E}[(D-q)^+]$ and $c\mathbb{E}[(q-D)^+]$ are interpreted as underage and overage cost due to randomness in the demand respectively.

One can incorporate more complicating features such as (i) left-over inventory would incur a per-unit holding cost h; (ii) unsatisfied demand incurs a shortage penalty cost s; (iii) left-over inventory can be further salvaged at price r. The expected profit after incorporating all three features can be written as

$$\begin{aligned} v(q) &= p\mathbb{E}[q \wedge D] - cq + (r-h)\mathbb{E}[(q-D)^+] - s\mathbb{E}[(D-q)^+] \\ &= (p-c)\mu - (p+s-c)\mathbb{E}[(D-q)^+] - (h+c-r)\mathbb{E}[(q-D)^+], \end{aligned}$$

which retains the essential feature of balancing the overage cost and underage cost. By letting $\bar{h} = h+c-r$, $\bar{s} = p+s-c$ and $L(q) = \bar{h}\mathbb{E}[(q-D)^+] + \bar{s}\mathbb{E}[(D-q)^+]$, it is then equivalent to solve

$$\min_q L(q) = \min_q \bar{h}(q-\mu) + (\bar{h}+\bar{s})\mathbb{E}[(D-q)^+] = \min_q \bar{h}(q-\mu) + (\bar{h}+\bar{s}) \int_q^{+\infty} (x-q) f(x)dx.$$

Since $L(q)$ is convex in q, we can use first order condition:

$$\frac{dL(q)}{dq} = \bar{h} - (\bar{h}+\bar{s}) \int_q^{+\infty} f(x)dx = \bar{h} - (\bar{h}+\bar{s})\bar{F}(q) = -\bar{s} + (\bar{h}+\bar{s})F(q) = 0.^1$$

It follows that the optimal ordering quantity is $q^* = F^{-1}(\bar{s}/(\bar{h}+\bar{s}))$. In particular, in the basic case when $h = s = r = 0$, we have $q^* = F^{-1}((p-c)/p)$.

Example 6.1 *Suppose D is normally distributed with mean μ and variance σ^2. Let $\phi(\cdot)$ and $\Phi(\cdot)$ be the pdf and cdf for standard normal distribution. Then,*

$$\begin{aligned} & \mathbb{P}(D \le q^*) = \frac{\bar{s}}{\bar{h}+\bar{s}} \\ \iff & \mathbb{P}\left(\frac{D-\mu}{\sigma} \le \frac{q^*-\mu}{\sigma}\right) = \frac{\bar{s}}{\bar{h}+\bar{s}} \\ \iff & \frac{q^*-\mu}{\sigma} = \Phi^{-1}\left(\frac{\bar{s}}{\bar{h}+\bar{s}}\right) \\ \iff & q^* = \mu + \sigma\Phi^{-1}\left(\frac{\bar{s}}{\bar{h}+\bar{s}}\right). \end{aligned}$$

Note here that when $\bar{s}/(\bar{h}+\bar{s}) > 1/2$, we have $q^* > \mu$ and q^* is increasing in σ. That is, there is a positive safety stock, and the larger the variance, the more safety stock one prepares.

By letting $z^* = \Phi^{-1}\left(\frac{\bar{s}}{\bar{h}+\bar{s}}\right)$, we further have

$$\mathbb{E}[(D-q^*)^+] = \sigma\mathbb{E}\left[\left(\frac{D-\mu}{\sigma} - \frac{q^*-\mu}{\sigma}\right)^+\right] = \sigma\mathbb{E}[(Z-z^*)^+],$$

where Z denotes a standard normal random variable. It follows that the optimal cost

$$L(q^*) = \sigma\left[\bar{h}z^* + (\bar{h}+\bar{s})\mathbb{E}[(Z-z^*)^+]\right],$$

which is independent of μ and increasing in σ.

[1] One can alternatively exchange the order of integration and taking right/left derivative:

$$\partial_+ L(q) = \bar{h} - (\bar{h}+\bar{s})\mathbb{E}[1_{\{q<D\}}] = \bar{h} - (\bar{h}+\bar{s})\bar{F}(q).$$

6.1.1 Inventory Pooling (Eppen, 1979)

Consider n newsvendors operating at different locations, each facing a random demand D_i, $i = 1, ..., n$. Suppose that D_i are independent and identically distributed according normal distribution with mean μ and variance σ^2. The overage and underage costs $\bar{h}$ and $\bar{s}$ are identical at all locations.

In a system where each newsvendor orders independently and uses its own inventory to satisfy demand, we can compute the total cost as

$$C(n) = n\sigma\left[\bar{h}z^* + (\bar{h}+\bar{s})\mathbb{E}[(Z - z^*)^+]\right].$$

Consider now a centralized system where all inventory of the newsvendor are pooled in a single warehouse and demands from all newsvendors are directly satisfied from the warehouse. Alternatively, one can also consider a centralized system where the inventories are still stored physically at each newsvendor, but one can costlessly transship inventory from one location to another to satisfy demand.[2] In such system, one is deciding a total ordering quantity (either at the warehouse or sum of all inventories at the newsvendors) in order to satisfy the total demand $D = \sum_{i=1}^n D_i \sim \mathcal{N}(n\mu, n\sigma^2)$. It follows that the total cost in such system is

$$\bar{C}(n) = \sqrt{n}\sigma\left[\bar{h}z^* + (\bar{h}+\bar{s})\mathbb{E}[(Z - z^*)^+]\right].$$

Hence, $\bar{C}(n)/C(n) = 1/\sqrt{n}$, i.e., the benefit from inventory pooling scales in square-root order when demands are independent.

In the other extreme case when demands are perfectly correlated, we have $D = nD_i \sim \mathcal{N}(n\mu, n^2\sigma^2)$. In this case, $\bar{C}(n) = C(n)$ and there is no benefit from inventory pooling at all.

6.1.2 Sample Average Approximation (SAA)

The newsvendor problem can also be viewed as a two-stage stochastic program where the first-stage (here-and-now) decision is the ordering quantity q and the second-stage decision is how many demands to fulfill, denoted as w. Given a first-stage decision q and a demand realization d, we can write the second-stage problem as the following simple linear program:

$$\begin{aligned} g(q,d) = \max_{w}\ & pw \\ s.t.\ & w \le q, \\ & w \le d. \end{aligned}$$

Clearly, $w^* = q \wedge d$, and $g(q,d) = p(q \wedge d)$. The first-stage problem is then given by

$$\max_q \mathbb{E}[g(q,D)] - cq.$$

When the integration is hard to compute or when one only has data $d_1, d_2, ..., d_m$ sampled from the distribution F, a common approach to solve the problem is via Sample Average Approximation (SAA), i.e., one replaces $\mathbb{E}[g(q,D)]$ with its sample estimate: $\frac{1}{m}\sum_{j=1}^m g(q, d_j)$. The two-stage stochastic program can then

[2] Centralized system here refers to the assumption that there is one decision maker who is responsible for all ordering decisions at all the newsvendors. In a decentralized system where each newsvendor is still responsible for its own ordering quantity, then one needs specify how the additional revenue earned from transshipment is allocated among newsvendors so that each newsvendor's objective is properly specified. One is referred to Rudi et al. (2001) and the follow-up works for this line of research.

be solved via the following linear program

$$\begin{aligned}
\max_{q,w_j} \quad & \frac{1}{m}\sum_{j=1}^{m} pw_j - cq \\
s.t. \quad & w_j \le q, j = 1, ..., m \\
& w_j \le d_j, j = 1, ..., m.
\end{aligned}$$

6.2 Robust Newsvendor (Scarf, 1958; Gallego and Moon, 1993)

In the above development, we have assumed that one has perfect knowledge of the demand distribution F, or in case of the SAA, one believes that the empirical distribution approximates the true distribution F well. Here, we consider the case where both assumptions fail and one only has knowledge (or good estimate) about the mean μ and the variance σ^2. The robust newsvendor problem seeks to find the optimal ordering quantity q that maximizes the worst case expected profit over all possible distributions that has mean μ and variance σ^2:

$$\max_q \min_{F\in\mathcal{F}} \{p\mathbb{E}[q \wedge D] - cq\} = \max_q \min_{F\in\mathcal{F}} \left\{p\mu - (cq + p\mathbb{E}[(D-q)^+])\right\},$$

where

$$\mathcal{F} = \left\{ F \in \mathcal{P} \,\middle|\, \int x dF(x) = \mu, \int (x-\mu)^2 dF(x) = \sigma^2 \right\}$$

with $\mathcal{P}$ being the set of all probability distributions on $\mathbb{R}$.

To evaluate the worst-case profit, it is then equivalent to evaluate

$$\max_{F\in\mathcal{F}} \mathbb{E}[(D-q)^+].$$

Observe that

$$\mathbb{E}[(D-q)^+] = \frac{1}{2}\left(\mathbb{E}[(D-q) + |D-q|]\right) = \frac{1}{2}\left(\mu - q + \mathbb{E}[|D-q|]\right) \le \frac{1}{2}\left(\mu - q + \sqrt{\mathbb{E}[(D-q)^2]}\right),$$

where the inequality is due to Jensen's inequality. Since

$$\mathbb{E}[(D-q)^2] = \mathbb{E}[(D-\mu+\mu-q)^2] = (\mu-q)^2 + 2\mathbb{E}[(\mu-q)(D-\mu)] + \mathbb{E}[(D-\mu)^2] = (\mu-q)^2 + \sigma^2,$$

we have

$$\mathbb{E}[(D-q)^+] \le \frac{1}{2}\left(\mu - q + \sqrt{(\mu-q)^2+\sigma^2}\right).$$

To show that the above bound is tight, consider a two-point distribution[3]

$$\mathbb{P}(D=a) = \theta,\ \mathbb{P}(D=b) = 1-\theta.$$

In this case, we have

$$\mathbb{E}[|D-q|] = \theta\sqrt{(a-q)^2} + (1-\theta)\sqrt{(b-q)^2},\ \sqrt{\mathbb{E}[(D-q)^2]} = \sqrt{\theta(a-q)^2 + (1-\theta)(b-q)^2}.$$

[3] In general, for a moment problem of the form:

$$\begin{aligned}
\sup_{F\in\mathcal{P}} \quad & \mathbb{E}_F[\phi(D)] \\
s.t. \quad & \int f_i(x)dF(x) = \mu_i, i = 1, ..., m,
\end{aligned}$$

it is sufficient to focus on the extreme points of the feasible set, which are probability distributions supported on at most $m+1$ points (see Smith, 1995).

Here, Jensen's inequality holds with equality if and only if $(a-q)^2 = (b-q)^2$. We can then let $a = q - \delta$ and $b = q + \delta$. To satisfy the moment constraints, we then require

$$\theta(q-\delta) + (1-\theta)(q+\delta) = q + (1-2\theta)\delta = \mu$$
$$\theta(q-\delta)^2 + (1-\theta)(q+\delta)^2 = q^2 + 2(1-2\theta)q\delta + \delta^2 = \mu^2 + \sigma^2.$$

From the first equation, we have $1 - 2\theta = (\mu - q)/\delta$, and combined with the second equation we can solve

$$\delta = \sqrt{\sigma^2 + (\mu-q)^2},\ \theta = \frac{1}{2} - \frac{\mu - q}{2\delta}.$$

With the worst-case bound, the problem of deciding the optimal ordering quantity becomes

$$\begin{aligned}&\max_q \left\{p\mu - cq - \frac{p}{2}\left(\mu - q + \sqrt{(\mu-q)^2 + \sigma^2}\right)\right\}\\ =&\max_q \left\{(p-c)\mu + c(\mu - q) - \frac{p}{2}\left(\mu - q + \sqrt{(\mu-q)^2 + \sigma^2}\right)\right\}\end{aligned}$$

By letting $x = \mu - q$, it is equivalent to maximize

$$h(x) = cx - \frac{p}{2}\left(x + \sqrt{x^2 + \sigma^2}\right),$$

which is concave in x. From the first-order-condition:

$$h'(x) = c - \frac{p}{2}\left(1 + \frac{x}{\sqrt{x^2+\sigma^2}}\right) = 0 \iff \frac{x}{\sqrt{x^2+\sigma^2}} = \frac{2c-p}{p}$$

we have $4c(p-c)x^2 = (2c-p)^2\sigma^2$. It follows that

$$x^* = \frac{\sigma}{2}\frac{2c-p}{\sqrt{c(p-c)}},$$

and correspondingly

$$q^* = \mu - x^* = \mu + \frac{\sigma}{2}\frac{p-c-c}{\sqrt{c(p-c)}} = \mu + \frac{\sigma}{2}\left(\sqrt{\frac{p-c}{c}} - \sqrt{\frac{c}{p-c}}\right).$$

Recall that the newsvendor ordering quantity under the normal demand is $\mu + \sigma\Phi^{-1}((p-c)/p)$. Consistently, when $p \geq 2c$ (i.e., $(p-c)/p > 1/2$), we have $q^* > \mu$ and q^* is increasing in σ.

Finally, with

$$h(x^*) = cx^* - \frac{p}{2}\left(x^* + \frac{p}{2c-p}x^*\right) = \frac{2c(c-p)}{2c-p}x^* = -\sigma\sqrt{c(p-c)}$$

the worst-case profit under the optimal ordering quantity is

$$(p-c)\mu - \sigma\sqrt{c(p-c)}.$$

6.3 Data-Driven Setting

Given that we have data $d_{[m]} = (d_1, d_2, ..., d_m)$ sampled from the distribution F, both the SAA method and the robust newsvendor provides a data-driven policy that maps the data to an ordering quantity. In particular, we can view the SAA method as

$$\pi^{SAA}(d_{[m]}) = \hat{F}^{-1}((p-c)/p),$$

where $\hat{F}$ is the empirical distribution of $d_{[m]}$, and the robust newsvendor solution as

$$\pi^{RO}(d_{[m]}) = \hat{\mu} + \frac{\hat{\sigma}}{2}\left(\sqrt{\frac{p-c}{c}} - \sqrt{\frac{c}{p-c}}\right),$$

where $\hat{\mu}, \hat{\sigma}$ are the sample mean and sample variance of the data $d_{[m]}$.

More generally, we can think of a data-driven policy as any function $\pi : \mathbb{R}^m \to \mathbb{R}$ that maps the data $d_{[m]}$ to an ordering quantity. Correspondingly, the out-of-sample expected profit under a policy π given the historical data $d_{[m]}$ and the true underlying distribution F can be defined as

$$v(\pi, F, d_{[m]}) = p\mathbb{E}_{D\sim F}[\pi(d_{[m]}) \wedge D] - c\pi(d_{[m]}).$$

One can imagine that given any two policies, the comparison of the out-of-sample expected profit would then be heavily depending on the true underlying distribution F and the exact values of historical data $d_{[m]}$ (which itself is generated from F). To eliminate the dependence on the actual data generated, one can further define an ex-ante performance metric by averaging over all possible data that can be generated:

$$\mathcal{V}(\pi, F, m) = \mathbb{E}_{D_{[m]} \overset{\text{i.i.d.}}{\sim} F}[v(\pi, F, D_{[m]})].$$

Two policies can then be compared, for example, via the worst-case expected out-of-sample performance $\inf_{F\in\mathcal{F}} \mathcal{V}(\pi, F, m)$.

Besbes and Mouchtaki (2022) consider instead the following worst-case relative regret as the performance metric: $\sup_{F\in\mathcal{F}} \mathcal{R}(\pi, F, m)$, where

$$\mathcal{R}(\pi, F, m) = \frac{\text{opt}(F) - \mathcal{V}(\pi, F, m)}{\text{opt}(F)},$$

with

$$\text{opt}(F) = p\mathbb{E}[F^{-1}((p-c)/p) \wedge D] - cF^{-1}((p-c)/p),$$

being the optimal profit attained in the case when distribution is known. Besbes and Mouchtaki (2022) further analytically characterize $\sup_{F\in\mathcal{F}} \mathcal{R}(\pi, F, m)$ for SAA policy and provide an algorithm to compute the optimal data-drive policy.

References

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