FOUNDATIONS OF OPTIMIZATION: IE6001

Linear Programs in Standard Form

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LPs in Standard Form

- We want to use computers to solve LP problems

$\Rightarrow$ We need a standardised specification of LP problems

Definition: An LP is in standard form if:

- The aim is to minimize a linear objective function;
- All constraints are linear equality constraints;
- All constraint right hand sides are non-negative;
- All decision variables are non-negative.

LPs in Standard Form

An LP in standard form looks as follows:

$$
\begin{array}{lccccccccc}
\text{minimize} & c_1 x_1 & + & c_2 x_2 & + & \dots & + & c_n x_n & & \\
\text{subject to} & a_{11} x_1 & + & a_{12} x_2 & + & \dots & + & a_{1n} x_n & = & b_1 \\
 & a_{21} x_1 & + & a_{22} x_2 & + & \dots & + & a_{2n} x_n & = & b_2 \\
 & \vdots & & \vdots & & & & \vdots & & \vdots \\
 & a_{m1} x_1 & + & a_{m2} x_2 & + & \dots & + & a_{mn} x_n & = & b_m
\end{array}
$$

$$x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$$

The input parameters b_i, c_j, and a_{ij} are fixed real constants that encode the LP problem. We require $b_i \geq 0$ $\forall i = 1, \dots, m$. (The decision variables x_i, $i = 1, \dots, n$, are yet to be found.)

Compact Notation

Collect the input parameters in vectors and matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$c^T = [c_1, c_2, \dots, c_n]$$

LPs in Standard Form (cont)

- With matrix notation, the LP in standard form reduces to

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0, \end{array}$$

where $b \geq 0$.

- Inequalities of the type $x \geq 0$ are understood to hold componentwise.

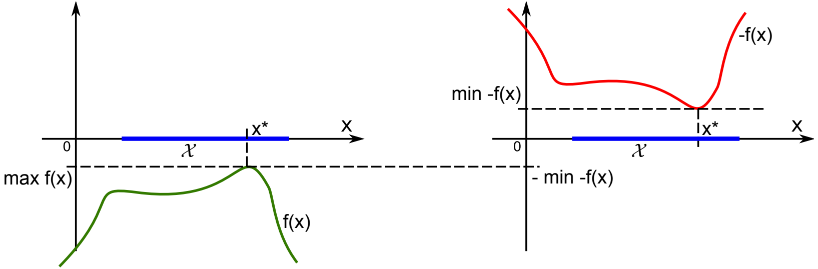
Standardising General LPs

- General LP problems can
  - be maximization (instead of minimization) problems;
  - have inequality (instead of equality) constraints;
  - have equality constrains with negative (instead of non-negative) right hand sides;
  - have free (instead of non-negative) decision variables.
- These general LPs can be transformed to standard LPs in a systematic way.

Maximization $\rightarrow$ Minimization

$$\left.\begin{array}{ll} \max & f(x) \\ \text{s.t.} & x \in \mathcal{X} \end{array}\right\} \quad = \quad \left\{\begin{array}{ll} -\min & -f(x) \\ \text{s.t.} & x \in \mathcal{X} \end{array}\right.$$

Inverting the objective preserves the optimal solution x^*



$\leq$ Inequalities $\rightarrow$ Equalities

$$\text{minimize} \quad c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to:

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & \leq & b_m
\end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

$\leq$ Inequalities $\rightarrow$ Equalities

$$\text{minimize} \quad c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

subject to:

$$\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & + & y_1 & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & + & y_2 & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & + & y_m & = & b_m
\end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, \quad y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0$$

Slack Variables

- To reformulate $\leq$ inequalities as equalities, we introduced *m slack variables*
  - Original variables: $x_1, x_2, \ldots, x_n$
  - Slack variables: $y_1, y_2, \ldots, y_m$

  $\Rightarrow$ After transformation, LP has $n + m$ variables!
- With matrix notation we can write

$$\left.\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array}\right\} \quad = \quad \left\{\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax + y = b \\ & x \geq 0, y \geq 0, \end{array}\right.$$

where $y = (y_1, \ldots, y_m)^T$.

$\geq$ Inequalities $\to$ Equalities

$$\text{minimize} \quad c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

subject to:

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & \geq & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & \geq & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & \geq & b_m
\end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

$\geq$ Inequalities $\rightarrow$ Equalities

$$\text{minimize} \quad c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to:

$$\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & - & y_1 & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & - & y_2 & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & - & y_m & = & b_m
\end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, \quad y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0$$

Surplus Variables

- To reformulate $\geq$ inequalities as equalities, we introduced *m* surplus variables
  - Original variables: $x_1, x_2, \ldots, x_n$
  - Surplus variables: $y_1, y_2, \ldots, y_m$

  $\Rightarrow$ After transformation, LP has $n + m$ variables!
- With matrix notation we can write

$$\left.\begin{array}{ll} \text{min} & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array}\right\} \quad = \quad \left\{\begin{array}{ll} \text{min} & c^T x \\ \text{s.t.} & Ax - y = b \\ & x \geq 0, y \geq 0, \end{array}\right.$$

where $y = (y_1, \ldots, y_m)^T$.

Negative Right Hand Sides

- If the right hand side of the ith constraint is negative, i.e., if $b_i < 0$ in
$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i,$$
then this constraint should be multiplied by -1.
- This yields
$$(-a_{i1})x_1 + (-a_{i2})x_2 + \ldots + (-a_{in})x_n = -b_i.$$
- The new constraint has a non-negative right hand side, i.e., we have $-b_i \geq 0$.

Free Variables (1st Approach)

- Free variables:
  - Suppose there is no constraint $x_j \geq 0$,i.e., x_j can be positive or negative.
  - Substitute $x_j = x_j^+ - x_j^-$ with $x_j^+, x_j^- \geq 0$.
  - The LP has now $(n+1)$ variables:

$$x_1, \ldots, x_{j-1}, x_j^+, x_j^-, x_{j+1}, \ldots, x_n$$

Free Variables (2nd Approach)

- Free variables:
  - Suppose there is no constraint $x_j \geq 0$,i.e., x_j can be positive or negative.
  - Any equality constraint involving x_j can be used to eliminate x_j.
  - Example: x_1 is free

$$\left.\begin{array}{ll} \min & x_1 + 3x_2 + 4x_3 \\ \text{s.t.} & x_1 + 2x_2 + x_3 = 5 \ (*) \\ & 2x_1 + 3x_2 + x_3 = 6 \\ & x_2, x_3 \geq 0 \end{array}\right\} = \left\{\begin{array}{ll} \min & x_2 + 3x_3 + 5 \\ \text{s.t.} & x_2 + x_3 = 4 \\ & x_2, x_3 \geq 0 \end{array}\right.$$

Use $(*)$ to substitute $x_1 = 5 - 2x_2 - x_3$.