Lagrangian Duality Theory

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Chapters 11.7-8, 14.1-2

Recall Primal and Dual of Conic LP

$$
\begin{array}{lll}
(CLP) & \text{minimize} & \mathbf{c} \bullet \mathbf{x} \\
 & \text{subject to} & \mathbf{a}_i \bullet \mathbf{x} = b_i, i = 1, 2, ..., m, \ \mathbf{x} \in K;
\end{array}
$$

and it dual problem

$$
\begin{array}{lll}
(CLD) & \text{maximize} & \mathbf{b}^T \mathbf{y} \\
 & \text{subject to} & \sum_i^m y_i \mathbf{a}_i + \mathbf{s} = \mathbf{c}, \ \mathbf{s} \in K^*,
\end{array}
$$

where $\mathbf{y} \in \mathcal{R}^m$, $\mathbf{s}$ is called the dual slack vector/matrix, and K^* is the dual cone of K.

In general, K can be decomposed to $K = K_1 \oplus K_2 \oplus ... \oplus K_p$, that is,

$$\mathbf{x} = (\mathbf{x}_1; \mathbf{x}_2; ...; \mathbf{x}_p), \ \mathbf{x}_i \in K_i, \ \forall i.$$

Note that $K^* = K_1^* \oplus K_2^* \oplus ... \oplus K_p^*$, or

$$\mathbf{s} = (\mathbf{s}_1; \mathbf{s}_2; ...; \mathbf{s}_p), \ \mathbf{s}_i \in K_i^*, \ \forall i.$$

This is a powerful but very structured duality form.

Lagrangian Function Again

We now consider the general constrained optimization:

$$
\text{(GCO)} \qquad \begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & c_i\mathbf{x}) \quad (\leq, =, \geq) \quad 0, \; i = 1, ..., m, \end{array}
$$

For Lagrange Multipliers.

$$
Y := \{y_i \quad (\leq, '\text{free}', \geq) \quad 0, \; i = 1, ..., m\},
$$

the Lagrangian Function is again given by

$$
L(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) - \mathbf{y}^T\mathbf{c}(\mathbf{x}) = f(\mathbf{x}) - \sum_{i=1}^{m} y_i c_i(\mathbf{x}), \; \mathbf{y} \in Y.
$$

We now develop the Lagrangian Duality theory as an alternative to Conic Duality theory. For general nonlinear constraints, the Lagrangian Duality theory is more applicable.

Toy Example Again

$$\begin{array}{ll} \text{minimize} & (x_1-1)^2+(x_2-1)^2 \\ \text{subject to} & x_1+2x_2-1 \le 0, \\ & 2x_1+x_2-1 \le 0. \end{array}$$

$$L(\mathbf{x},\mathbf{y}) = f(\mathbf{x}) - \mathbf{y}^T\mathbf{c}(\mathbf{x}) = f(\mathbf{x}) - \sum_{i=1}^{2} y_i c_i(\mathbf{x}) =$$

$$= (x_1-1)^2+(x_2-1)^2 - y_1(x_1+2x_2-1) - y_2(2x_1+x_2-1), \ (y_1;y_2) \le \mathbf{0}$$

where

$$\nabla L_x(\mathbf{x},\mathbf{y}) = \begin{pmatrix} 2(x_1-1)-y_1-2y_2 \\ 2(x_2-1)-2y_1-y_2 \end{pmatrix}$$

Lagrangian Relaxation Problem

For given multipliers $\mathbf{y} \in Y$, consider problem

$$\begin{array}{lll} (LRP) & \inf & L(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) - \mathbf{y}^T \mathbf{c}(\mathbf{x}) \\ & \text{s.t.} & \mathbf{x} \in R^n. \end{array}$$

Again, $\mathbf{y}_i$ can be viewed as a penalty parameter to penalize constraint violation $c_i(\mathbf{x}), i = 1, ..., m$.

In the toy example, for given $(y_1; y_2) \leq \mathbf{0}$, the LRP is:

$$\begin{array}{ll} \inf & (x_1 - 1)^2 + (x_2 - 1)^2 - y_1(x_1 + 2x_2 - 1) - y_2(2x_1 + x_2 - 1) \\ \text{s.t.} & (x_1; x_2) \in R^2, \end{array}$$

and it has a close form solution $\mathbf{x}$ for any given $\mathbf{y}$:

$$x_1 = \frac{y_1 + 2y_2}{2} + 1 \quad \text{and} \quad x_2 = \frac{2y_1 + y_2}{2} + 1$$

with the minimal or infimum value function $= -1.25y_1^2 - 1.25y_2^2 - 2y_1y_2 - 2y_1 - 2y_2$.

Inf-Value Function as the Dual Objective

For any $\mathbf{y} \in Y$, the minimal value function (including unbounded from below or infeasible cases) and the Lagrangian Dual Problem (LDP) are given by:

$$\begin{array}{llll} \phi(\mathbf{y}) := & \inf_{\mathbf{x}} & L(\mathbf{x}, \mathbf{y}), & \text{s.t.} \quad \mathbf{x} \in R^n. \\ (LDP) & \sup_{\mathbf{y}} & \phi(\mathbf{y}), & \text{s.t.} \quad \mathbf{y} \in Y. \end{array}$$

Theorem 1 *The Lagrangian dual objective $\phi(\mathbf{y})$ is a concave function.*

Proof: For any given two multiply vectors $\mathbf{y}^1 \in Y$ and $\mathbf{y}^2 \in Y$,

$$\begin{aligned} \phi(\alpha \mathbf{y}^1 + (1-\alpha)\mathbf{y}^2) &= \inf_{\mathbf{x}} L(\mathbf{x}, \alpha \mathbf{y}^1 + (1-\alpha)\mathbf{y}^2) \\ &= \inf_{\mathbf{x}} [f(\mathbf{x}) - (\alpha \mathbf{y}^1 + (1-\alpha)\mathbf{y}^2)^T \mathbf{c}(\mathbf{x})] \\ &= \inf_{\mathbf{x}} [\alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{x}) - \alpha (\mathbf{y}^1)^T \mathbf{c}(\mathbf{x}) - (1-\alpha)(\mathbf{y}^2)^T \mathbf{c}(\mathbf{x})] \\ &= \inf_{\mathbf{x}} [\alpha L(\mathbf{x}, \mathbf{y}^1) + (1-\alpha) L(\mathbf{x}, \mathbf{y}^2)] \\ &\geq \alpha [\inf_{\mathbf{x}} L(\mathbf{x}, \mathbf{y}^1)] + (1-\alpha)[\inf_{\mathbf{x}} L(\mathbf{x}, \mathbf{y}^2)] \\ &= \alpha \phi(\mathbf{y}^1) + (1-\alpha)\phi(\mathbf{y}^2), \end{aligned}$$

Dual Objective Establishes a Lower Bound

Theorem 2 *(Weak duality theorem) For every $\mathbf{y} \in Y$, the Lagrangian dual function $\phi(\mathbf{y})$ is less or equal to the infimum value of the original GCO problem.*

Proof:

$$\begin{aligned}
\phi(\mathbf{y}) \quad &= \inf_{\mathbf{x}} \{f(\mathbf{x}) - \mathbf{y}^T\mathbf{c}(\mathbf{x})\} \\
&\leq \inf_{\mathbf{x}} \{f(\mathbf{x}) - \mathbf{y}^T\mathbf{c}(\mathbf{x}) \text{ s.t. } \mathbf{c}(\mathbf{x})(\leq, =, \geq)\mathbf{0}\ \} \\
&\leq \inf_{\mathbf{x}} \{f(\mathbf{x}) : \text{ s.t. } \mathbf{c}(\mathbf{x})(\leq, =, \geq)\mathbf{0}\ \}.
\end{aligned}$$

The first inequality is from the fact that the unconstrained inf-value is no greater than the constrained one.

The second inequality is from $\mathbf{c}(\mathbf{x})(\leq, =, \geq)\mathbf{0}$ and $\mathbf{y}(\leq, '\text{free}', \geq)\mathbf{0}$ imply $-\mathbf{y}^T\mathbf{c}(\mathbf{x}) \leq 0$.

The Lagrangian Dual Problem for the Toy Example

$$
\begin{array}{ll}
\text{minimize} & (x_1 - 1)^2 + (x_2 - 1)^2 \\
\text{subject to} & x_1 + 2x_2 - 1 \le 0, \\
 & 2x_1 + x_2 - 1 \le 0;
\end{array}
$$

where $\mathbf{x}^* = \left(\frac{1}{3}; \ \frac{1}{3}\right)$.

$$\phi(\mathbf{y}) = -1.25y_1^2 - 1.25y_2^2 - 2y_1y_2 - 2y_1 - 2y_2, \ \mathbf{y} \le \mathbf{0}.$$

$$
\begin{array}{ll}
\max & -1.25y_1^2 - 1.25y_2^2 - 2y_1y_2 - 2y_1 - 2y_2 \\
\text{s.t.} & (y_1; y_2) \le \mathbf{0}.
\end{array}
$$

where $\mathbf{y}^* = \left(\frac{-4}{9}; \ \frac{-4}{9}\right)$.

The Lagrangian Dual of LP I

Consider LP problem

$$(LP) \quad \text{minimize} \quad \mathbf{c}^T\mathbf{x}$$
$$\text{subject to} \quad A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0};$$

and its conic dual problem is given by

$$(LD) \quad \text{maximize} \quad \mathbf{b}^T\mathbf{y}$$
$$\text{subject to} \quad A^T\mathbf{y} + \mathbf{s} = \mathbf{c},\ \mathbf{s} \geq \mathbf{0}.$$

We now derive the Lagrangian Dual of (LP). Let the Lagrangian multipliers be $\mathbf{y}('\text{free}')$ for equalities and $\mathbf{s} \geq \mathbf{0}$ for constraints $\mathbf{x} \geq \mathbf{0}$. Then the Lagrangian function would be

$$L(\mathbf{x}, \mathbf{y}, \mathbf{s}) = \mathbf{c}^T\mathbf{x} - \mathbf{y}^T(A\mathbf{x} - \mathbf{b}) - \mathbf{s}^T\mathbf{x} = (\mathbf{c} - A^T\mathbf{y} - \mathbf{s})^T\mathbf{x} + \mathbf{b}^T\mathbf{y};$$

where $\mathbf{x}$ is “free”.

The Lagrangian Dual of LP II

Now consider the Lagrangian dual objective

$$\phi(\mathbf{y},\mathbf{s}) = \inf_{\mathbf{x}\in R^n} L(\mathbf{x},\mathbf{y},\mathbf{s}) = \inf_{\mathbf{x}\in R^n} \left[(\mathbf{c} - A^T\mathbf{y} - \mathbf{s})^T\mathbf{x} + \mathbf{b}^T\mathbf{y}\right].$$

If $(\mathbf{c} - A^T\mathbf{y} - \mathbf{s}) \neq \mathbf{0}$, then $\phi(\mathbf{y},\mathbf{s}) = -\infty$. Thus, in order to maximize $\phi(\mathbf{y},\mathbf{s})$, the dual must choose its variables $(\mathbf{y},\mathbf{s} \geq \mathbf{0})$ such that $(\mathbf{c} - A^T\mathbf{y} - \mathbf{s}) = \mathbf{0}$.

This constraint, together with the sign constraint $\mathbf{s} \geq \mathbf{0}$, establish the Lagrangian dual problem:

$$\begin{array}{lll} (LDP) & \text{maximize} & \mathbf{b}^T\mathbf{y} \\ & \text{subject to} & A^T\mathbf{y} + \mathbf{s} = \mathbf{c},\ \mathbf{s} \geq \mathbf{0}. \end{array}$$

which is consistent with the conic dual of LP.

The Lagrangian Dual of LP with the Log-Barrier I

For a fixed $\mu > 0$, consider the problem

$$\begin{array}{ll} \min & \mathbf{c}^T\mathbf{x} - \mu\sum_{j=1}^n \log(x_j) \\ \text{s.t.} & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

Again, the non-negativity constraints can be “ignored” if the feasible region has an ”interior”, that is, any minimizer must have $\mathbf{x}(\mu) > \mathbf{0}$. Thus, the Lagrangian function would be simply given by

$$L(\mathbf{x},\mathbf{y}) = \mathbf{c}^T\mathbf{x} - \mu\sum_{j=1}^n \log(x_j) - \mathbf{y}^T(A\mathbf{x}-\mathbf{b}) = (\mathbf{c}-A^T\mathbf{y})^T\mathbf{x} - \mu\sum_{j=1}^n \log(x_j) + \mathbf{b}^T\mathbf{y}.$$

Then, the Lagrangian dual objective (we implicitly need $\mathbf{x} > \mathbf{0}$ for the function to be defined)

$$\phi(\mathbf{y}) := \inf_{\mathbf{x}} L(\mathbf{x},\mathbf{y}) = \inf_{\mathbf{x}} \left[(\mathbf{c}-A^T\mathbf{y})^T\mathbf{x} - \mu\sum_{j=1}^n \log(x_j) + \mathbf{b}^T\mathbf{y} \right].$$

The Lagrangian Dual of LP with the Log-Barrier II

First, from the view point of the dual, the dual needs to choose $\mathbf{y}$ such that $\mathbf{c} - A^T\mathbf{y} > \mathbf{0}$, since otherwise the primal can choose $\mathbf{x} > \mathbf{0}$ to make $\phi(\mathbf{y})$ go to $-\infty$.

Now for any given $\mathbf{y}$ such that $\mathbf{c} - A^T\mathbf{y} > \mathbf{0}$, the $\inf$ problem has a unique finite close-form minimizer $\mathbf{x}$

$$x_j = \frac{\mu}{(\mathbf{c} - A^T\mathbf{y})_j}, \ \forall j = 1, ..., n.$$

Thus,

$$\phi(\mathbf{y}) = \mathbf{b}^T\mathbf{y} + \mu \sum_{j=1}^{n} \log(\mathbf{c} - A^T\mathbf{y})_j + n\mu(1 - \log(\mu)).$$

Therefore, the dual problem, for any fixed μ, can be written as

$$\max_{\mathbf{y}} \ \phi(\mathbf{y}) = n\mu(1 - \log(\mu)) + \max_{\mathbf{y}} \,[\mathbf{b}^T\mathbf{y} + \mu \sum_{j=1}^{n} \log(\mathbf{c} - A^T\mathbf{y})_j].$$

This is actually the LP dual with the Log-Barrier on dual inequality constraints $\mathbf{c} - A^T\mathbf{y} \geq \mathbf{0}$.

Lagrangian Strong Duality Theorem

Theorem 3 *Let (GCO) be a convex minimization problem and the infimum* f^* *of (GCO) be finite, and the suprermum of (LDP) be* ϕ^*. *In addition, let (GCO) have an interior-point feasible solution with respect to inequality constraints, that is, there is* $\hat{\mathbf{x}}$ *such that all inequality constraints are strictly held. Then,* $f^* = \phi^*$, *and (LDP) admits a maximizer* $\mathbf{y}^*$ *such that*

$$\phi(\mathbf{y}^*) = f^*.$$

Furthermore, if (GCO) admits a minimizer $\mathbf{x}^*$, *then*

$$y_i^* c_i(\mathbf{x}^*) = 0,\ \forall i = 1, ..., m.$$

The assumption of "interior-point feasible solution" is called Constraint Qualification condition, which was also needed as a condition to prove the strong duality theorem for general Conic Linear Optimization.

Note that the problem would be a convex minimization problem if all equality constraints are hyperplane or affine functions $c_i(\mathbf{x}) = \mathbf{a}_i\mathbf{x} - b_i$, all other level sets are convex.

Same Example when the Constraint Qualification Failed

Consider the problem ($\mathbf{x}^* = (0;\ 0)$):

$$\begin{array}{lll} \min & & x_1 \\ \text{s.t.} & x_1^2 + (x_2 - 1)^2 - 1 \le 0, & (y_1 \le 0) \\ & x_1^2 + (x_2 + 1)^2 - 1 \le 0, & (y_2 \le 0) \end{array}$$

$$L(\mathbf{x}, \mathbf{y}) = x_1 - y_1(x_1^2 + (x_2 - 1)^2 - 1) - y_2(x_1^2 + (x_2 + 1)^2 - 1).$$

$$\phi(\mathbf{y}) = \frac{1 + (y_1 - y_2)^2}{y_1 + y_2}, \quad (y_1, y_2) \le 0$$

Although there is no duality gap, but the dual does not admit a (finite) maximizer...

In summary: The primal and Lagrangian dual may 1) have a duality gap; 2) have zero duality gap but the optimal solutions are not attainable; or 3) have zero duality gap and the optimal solutions are attainable.

Proof of Lagrangian Strong Duality Theorem where the Constraints $\mathbf{c}(\mathbf{x}) \geq \mathbf{0}$

Consider the convex set

$$C := \{(\kappa; \mathbf{s}) : \ \exists \mathbf{x} \text{ s.t. } f(\mathbf{x}) \leq \kappa, \ -\mathbf{c}(\mathbf{x}) \leq \mathbf{s}\}.$$

Then, $(f^*; \mathbf{0})$ is on the closure of C. From the supporting hyperplane theorem, there exists $(y_0^*; \mathbf{y}^*) \neq \mathbf{0}$ such that

$$y_0^* f^* \leq \inf_{(\kappa;\mathbf{s}) \in C} (y_0^* \kappa + (\mathbf{y}^*)^T \mathbf{s}).$$

First, we show $\mathbf{y}^* \geq \mathbf{0}$, since otherwise one can choose some $(0; \mathbf{s} \geq \mathbf{0})$ such that the inequality is violated. Secondly, we show $y_0^* > 0$, since otherwise one can choose $(\kappa \to \infty; \mathbf{0})$ if $y^* < 0$, or $(0; \mathbf{s} = -\mathbf{c}(\hat{\mathbf{x}}) < \mathbf{0})$ if $y^* = 0$ (then $\mathbf{y}^* \neq \mathbf{0}$), such that the above inequality is violated.

Now let us divide both sides by y_0^* and let $\mathbf{y}^* := \mathbf{y}^* / y_0^*$, we have

$$f^* \leq \inf_{(\kappa;\mathbf{s}) \in C} (\kappa + (\mathbf{y}^*)^T \mathbf{s}) = \inf_{\mathbf{x}} (f(\mathbf{x}) - (\mathbf{y}^*)^T \mathbf{c}(\mathbf{x})) = \phi(\mathbf{y}^*) \leq \phi^*.$$

Then, from the weak duality theorem, we must have $f^* = \phi^*$.

If (GCO) admits a minimizer $\mathbf{x}^*$, then $f(\mathbf{x}^*) = f^*$ so that

$$f(\mathbf{x}^*) \leq \inf_{\mathbf{x}} \left[f(\mathbf{x}) - (\mathbf{y}^*)^T \mathbf{c}(\mathbf{x}) \right] \leq f(\mathbf{x}^*) - (\mathbf{y}^*)^T \mathbf{c}(\mathbf{x}^*) = f(\mathbf{x}^*) - \sum_i^m y_i^* c_i(\mathbf{x}^*),$$

which implies that

$$\sum_i^m y_i^* c_i(\mathbf{x}^*) \leq 0.$$

Since $y_i^* \geq 0$ and $c_i(\mathbf{x}^*) \geq 0$ for all i, it must be true $y_i^* c_i(\mathbf{x}^*) = 0$ for all i.

More on Lagrangian Duality

Consider the constrained problem with additional constraints

$$
\begin{array}{lll}
(GCO) & \inf & f(\mathbf{x}) \\
& \text{s.t.} & \mathbf{c}_i(\mathbf{x}) \ (\leq, =, \geq) \ 0, \ i = 1, ..., m, \\
& & \mathbf{x} \in \Omega \subset R^n.
\end{array}
$$

Typically, Ω has a simple form such as the cone

$$\Omega = R^n_+ = \{\mathbf{x} : \ \mathbf{x} \geq \mathbf{0}\}$$

or the box

$$\Omega := \{\mathbf{x} : \ -\mathbf{e} \leq \mathbf{x} \leq \mathbf{e}.\}$$

Then, when derive the Lagrangian dual, there is not need to introduce multipliers for Ω constraints.

Lagrangian Relaxation Problem

Consider again the (partial) Lagrangian Function:

$$L(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) - \mathbf{y}^T \mathbf{c}(\mathbf{x}), \ \mathbf{y} \in Y;$$

and define the dual objective function of $\mathbf{y}$ be

$$\begin{array}{rll} \phi(\mathbf{y}) := & \inf_{\mathbf{x}} & L(\mathbf{x}, \mathbf{y}) \\ & \text{s.t.} & \mathbf{x} \in \Omega. \end{array}$$

Theorem 4 *The Lagrangian dual function $\phi(\mathbf{y})$ is a concave function.*

Theorem 5 *(Weak duality theorem) For every $\mathbf{y} \in Y$, the Lagrangian dual function value $\phi(\mathbf{y})$ is less or equal to the infimum value of the original GCO problem.*

The Lagrangian Dual Problem

$$
\begin{array}{lll}
(LDP) & \sup & \phi(\mathbf{y}) \\
& \text{s.t.} & \mathbf{y} \in Y.
\end{array}
$$

would called the Lagrangian dual of the original GCO problem:

Theorem 6 *(Strong duality theorem) Let (GCO) be a convex minimization problem, the infimum f^* of (GCO) be finite, and the suprermum of (LDP) be ϕ^*. In addition, let (GCO) have an interior-point feasible solution with respect to inequality constraints, that is, there is $\hat{\mathbf{x}}$ such that all inequality constraints are strictly held. Then, $f^* = \phi^*$, and (LDP) admits a maximizer $\mathbf{y}^*$ such that*

$$\phi(\mathbf{y}^*) = f^*.$$

Furthermore, if (GCO) admits a minimizer $\mathbf{x}^*$, then

$$y_i^* c_i(\mathbf{x}^*) = 0, \ \forall i = 1, ..., m.$$

Rules to Construct the Lagrangian Dual

$$\text{(GCO)} \qquad \begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & c_i\mathbf{x}) \quad (\leq, =, \geq) \quad 0, \; i = 1, ..., m, \end{array}$$

- All multipliers are dual variables.
- Derive the LDC

$$\nabla f(\mathbf{x}) = \mathbf{y}^T \nabla \mathbf{c}(\mathbf{x})$$

If no $\mathbf{x}$ appeared in an equation, set it as an equality constraint for the dual; otherwise, express $\mathbf{x}$ in terms of $\mathbf{y}$ and replace $\mathbf{x}$ in the Lagrange function, which becomes the Dual objective. (This may be very difficult ...)

- Add the MSC as dual constraints.

The Dual of SVM

$$
\begin{array}{ll}
\text{minimize}_{\mathbf{x},x_0,\beta} & \beta + \mu\|\mathbf{x}\|^2 \\
\text{subject to} & \mathbf{a}_i^T\mathbf{x} + x_0 + \beta \geq 1,\ \forall i,\ (\mathbf{y}_a \geq \mathbf{0}) \\
& -\mathbf{b}_j^T\mathbf{x} - x_0 + \beta \geq 1,\ \forall j,\ (\mathbf{y}_b \geq \mathbf{0}) \\
& \beta \geq 0.\ (\alpha \geq 0)
\end{array}
$$

$$L(\mathbf{x}, x_0, \beta, \mathbf{y}_a, \mathbf{y}_b, \alpha) = \beta + \mu\|\mathbf{x}\|^2 - \mathbf{y}_a^T(A^T\mathbf{x} + x_0\mathbf{e} + \beta\mathbf{e} - \mathbf{e}) - \mathbf{y}_b^T(-B^T\mathbf{x} - x_0\mathbf{e} + \beta\mathbf{e} - \mathbf{e}) - \alpha\beta.$$

$$\nabla_{\mathbf{x}} L(\cdot) = 2\mu\mathbf{x} - A\mathbf{y}_a + B\mathbf{y}_b = \mathbf{0},\ \text{(replace } \mathbf{x}\text{)}$$

$$\nabla_{x_0} L(\cdot) = -\mathbf{e}^T\mathbf{y}_a + \mathbf{e}^T\mathbf{y}_b = 0,\ \text{(dual constraint)}$$

$$\nabla_{\beta} L(\cdot) = 1 - \mathbf{e}^T\mathbf{y}_a - \mathbf{e}^T\mathbf{y}_b - \alpha = 0.\ \text{(dual constraint)}$$

Then the dual objective is

$$\frac{-1}{4\mu}\|A\mathbf{y}_a - B\mathbf{y}_b\|^2 + \mathbf{e}^T\mathbf{y}_a + \mathbf{e}^T\mathbf{y}_b.$$

The Lagrangian Dual of LP with Bound Constraints

Sometimes the dual can be constructed by simple reasoning: consider

$$\begin{array}{lll} (LP) & \text{minimize} & \mathbf{c}^T\mathbf{x} \\ & \text{subject to} & A\mathbf{x} = \mathbf{b},\ -\mathbf{e} \le \mathbf{x} \le \mathbf{e}\ (\|\mathbf{x}\|_\infty \le 1); \end{array}$$

Let the Lagrangian multipliers be $\mathbf{y}$ for equality constraints. Then the Lagrangian dual objective would be

$$\phi(\mathbf{y}) = \inf_{-\mathbf{e}\le\mathbf{x}\le\mathbf{e}} L(\mathbf{x},\mathbf{y}) = \inf_{-\mathbf{e}\le\mathbf{x}\le\mathbf{e}} \left[(\mathbf{c} - A^T\mathbf{y})^T\mathbf{x} + \mathbf{b}^T\mathbf{y}\right];$$

where if $(\mathbf{c} - A^T\mathbf{y})_j \le 0$, $x_j = 1$; and otherwise, $x_j = -1$.

Therefore, the Lagrangian dual is

$$\begin{array}{lll} (LDP) & \text{maximize} & \mathbf{b}^T\mathbf{y} - \|\mathbf{c} - A^T\mathbf{y}\|_1 \\ & \text{subject to} & \mathbf{y} \in R^m. \end{array}$$

Farkas Lemma for Nonlinear Constraints I

Consider the convex constrained system:

$$(\text{CCS}) \qquad \begin{array}{lll} \min & \mathbf{0}^T\mathbf{x} & \\ \text{s.t.} & c_i\mathbf{x}) & \geq \quad 0,\ i = 1, ..., m, \end{array}$$

where $c_i(.)$ are concave functions and the Lagrangian Function is given by

$$L(\mathbf{x}, \mathbf{y}) = -\mathbf{y}^T\mathbf{c}(\mathbf{x}) = -\sum_{i=1}^{m} y_i c_i(\mathbf{x}),\ \mathbf{y} \geq \mathbf{0}.$$

Again, let

$$\phi(\mathbf{y}) := \inf_{\mathbf{x}} L(\mathbf{x}, \mathbf{y}).$$

Theorem 7 *If there exists $\mathbf{y} \geq \mathbf{0}$ such that $\phi(\mathbf{y}) > 0$, then (CSS) is infeasible.*

The proof is directly from the dual objective function $\phi(\mathbf{y})$ is a homogeneous function and the dual has its objective value unbounded from above.

Farkas Lemma for Nonlinear Constraints II

Consider the system, for a parameter $b \geq 0$,

$$-x_1^2 - (x_2 - 1)^2 + b \geq 0, \quad (y_1 \geq 0)$$
$$-x_1^2 - (x_2 + 1)^2 + b \geq 0, \quad (y_2 \geq 0)$$

$$L(\mathbf{x}, \mathbf{y}) = y_1(x_1^2 + (x_2 - 1)^2 - b) + y_2(x_1^2 + (x_2 + 1)^2 - b).$$

Then, if $y_1 + y_2 \neq 0$,

$$\phi(\mathbf{y}) = \frac{4y_1y_2 - b(y_1 + y_2)^2}{y_1 + y_2}, \quad (y_1, y_2) \geq 0$$

When $b \geq 1$, $\phi(\mathbf{y}) \leq 0$; and, otherwise, one can choose $y_1 = y_2 = y > 0$ such that

$$\phi(\mathbf{y}) = 2(1 - b)y > 0$$

which implies that the original constrained system is infeasible.

The Conic Duality vs. Lagrangian Duality I

Consider SOCP problem

$$(SOCP) \quad \text{minimize} \quad \mathbf{c}^T\mathbf{x}$$
$$\text{subject to} \quad A\mathbf{x} = \mathbf{b},\ x_1 - \|\mathbf{x}_{-1}\|_2 \geq 0;$$

and it conic dual problem

$$(SOCD) \quad \text{maximize} \quad \mathbf{b}^T\mathbf{y}$$
$$\text{subject to} \quad A^T\mathbf{y} + \mathbf{s} = \mathbf{c},\ s_1 - \|\mathbf{s}_{-1}\|_2 \geq 0.$$

Let the Lagrangian multipliers be $\mathbf{y}$ for equalities and scalar $s \geq 0$ for the single constraint $x_1 \geq \|\mathbf{x}_{-1}\|_2$. Then the Lagrangian function would be

$$L(\mathbf{x}, \mathbf{y}, s) = \mathbf{c}^T\mathbf{x} - \mathbf{y}^T(A\mathbf{x} - \mathbf{b}) - s(x_1 - \|\mathbf{x}_{-1}\|_2) = (\mathbf{c} - A^T\mathbf{y})^T\mathbf{x} - s(x_1 - \|\mathbf{x}_{-1}\|_2) + \mathbf{b}^T\mathbf{y}.$$

The Conic Duality vs. Lagrangian Duality II

Now consider the Lagrangian dual objective

$$\phi(\mathbf{y}, s) = \inf_{\mathbf{x} \in R^n} L(\mathbf{x}, \mathbf{y}, \mathbf{s}) = \inf_{\mathbf{x} \in R^n} \left[(\mathbf{c} - A^T \mathbf{y})^T \mathbf{x} - s(x_1 - \|\mathbf{x}_{-1}\|_2) + \mathbf{b}^T \mathbf{y} \right].$$

The objective function of the problem may not be differentiable so that the classical optimal condition theory do not apply. Consequently, it is difficult to write a clean/explicit form of the Lagrangian dual problem.

On the other hand, many nonlinear optimization problems, even they are convex, are difficult to transform them into structured CLP problems (especially to construct the dual cones). Therefore, each of the duality form, Conic or Lagrangian, has its own pros and cons.