Foundations of Inventory Management **Fall 2022, NUS**

Lecture 3: EOQ Model

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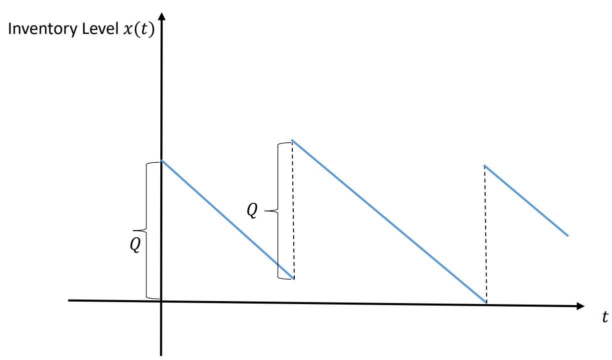
3.1 The EOQ Model

Consider a retailer selling a single product over an infinite horizon in continuous time. Demand comes continuously at a fixed rate λ per unit time and must be satisfied with the available inventory. The retailer replenishes the inventory by ordering from the supplier. There is no lead time in delivering the product and each order incurs a fixed ordering cost K and a variable ordering cost c. On-hand inventory incurs holding cost h per-unit held, per-unit time.

Consider an ordering policy that places a fixed Q units of product at discrete time epochs. The resulting pattern of inventory level is illustrated in the figure below. Whenever the order is placed, an ordering cost $K + cQ$ is incurred. The resulting inventory level incurs a holding cost of $h \cdot \int x(t)dt$.



An observation here is that the optimal ordering policy must satisfy the "zero inventory property": an order is placed only when inventory level drops to zero. As a result, the time T between each order is a constant which is given by

$$T = Q/\lambda.$$

The objective is then to find the optimal Q (or equivalently T) so that the total cost per unit of time is minimized. In partciular, the ordering cost per ordering cycle is simply $K + cQ$. The holding cost per cycle is

$$\begin{aligned} h\int_0^T x(t)dt &= h\int_0^T (Q - \lambda t)dt \\ &= h(QT - \lambda T^2/2) \\ &= h(Q^2/\lambda - \lambda(Q/\lambda)^2/2) \\ &= h\frac{Q^2}{2\lambda} = h\frac{QT}{2}. \end{aligned}$$

The total cost per unit of time is then

$$\frac{K + cQ + hQT/2}{T} = \frac{K}{T} + \frac{hQ}{2} + c\lambda = \frac{\lambda K}{Q} + \frac{hQ}{2} + c\lambda.$$

Let $C(Q) = \frac{\lambda K}{Q} + \frac{hQ}{2}$. It is easy to check that $C(Q)$ is convex in Q, and we have the first-order condition

$$C'(Q) = -\frac{\lambda K}{Q^2} + \frac{h}{2} = 0.$$

The optimal order quantity is then

$$Q^* = \sqrt{\frac{2\lambda K}{h}},$$

with the corresponding optimal cycle length being

$$T^* = \sqrt{\frac{2K}{\lambda h}},$$

and the optimal cost being

$$C(Q^*) = \lambda K \sqrt{\frac{h}{2\lambda K}} + \frac{h}{2}\sqrt{\frac{2\lambda K}{h}} = \sqrt{\frac{\lambda h K}{2}} + \sqrt{\frac{\lambda h K}{2}} = \sqrt{2\lambda h K}.$$

Besides its simplicity, the EOQ formula here also enjoys some nice robustness properties. Suppose one mistakenly orders αQ^* instead of Q^* for some $\alpha \geq 0$. Then the corresponding cost is

$$C(\alpha Q^*) = \frac{1}{\alpha}\frac{\lambda K}{Q^*} + \alpha \frac{hQ^*}{2} = \frac{1}{\alpha}\sqrt{\frac{\lambda h K}{2}} + \alpha\sqrt{\frac{\lambda h K}{2}} = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right) C(Q^*).$$

For example, if one mistakenly doubles the order quantity, i.e., $\alpha = 2$, the resulting cost $C(2Q^*) = \frac{5}{4}C(Q^*)$, only increased 25%.

The optimal cost function also enjoys certain "robustness" property concerning the change of parameters. In particular, given a differentiable function $f(x)$, its elasticity is defined as

$$\mathcal{E}(f(x)) = \frac{xf'(x)}{f(x)},$$

which can be interpreted as measuring the percentage change in $f(x)$ given a percentage change in x:

$$\mathcal{E}(f(x)) \approx \frac{x}{f(x)} \cdot \frac{f(x+\delta) - f(x)}{\delta} = \frac{(f(x+\delta) - f(x))/f(x)}{\delta/x}.$$

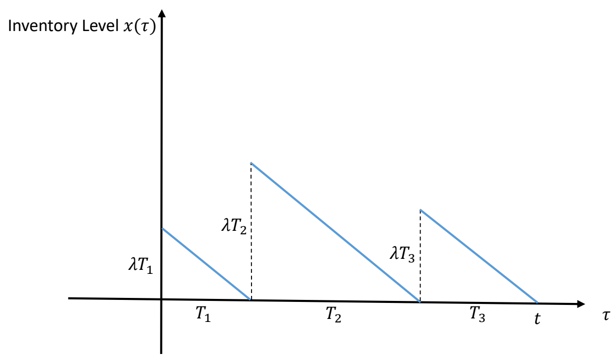
The elasticity of the cost with respect to change in demand λ, for example, can be computed as

$$\frac{\lambda \partial C(Q^*)/\partial \lambda}{C(Q^*)} = \frac{\lambda \frac{hK}{\sqrt{2\lambda h K}}}{\sqrt{2\lambda h K}} = \frac{1}{2}.$$

That is, a 2% change in demand only results in 1% change in cost.

3.2 Finite Horizon Model

Suppose now that we have planning horizon from 0 to t, and we consider a general ordering policy that allows the ordering quantity to be different across each order. In particular, consider an ordering policy that places $m \geq 1$ orders in the interval $[0, t]$. Let T_i be the time between the i-th and $i+1$-th order, $i = 1, ..., m-1$, and T_m be the time between the last order and t. Clearly, $t = \sum_{i=1}^{m} = T_i$ and zero inventory property still holds in this setting. The figure below illustrates the pattern of inventory level $x(\tau)$ under a particular ordering policy that places three orders during $[0, t]$.



The total cost per unit of time can then be computed as

$$\frac{1}{t}\left[Km + h\int_0^t x(\tau)d\tau\right],$$

where

$$\int_0^t x(\tau)d\tau = \sum_{i=1}^{m} \frac{\lambda T_i^2}{2}.$$

For a fixed m, the problem of finding the optimal order quantity in each order can then be solved via

$$\begin{aligned}\min \ & \sum_{i=1}^{m} T_i^2 \\ \text{s.t.} \ & \sum_{i=1}^{m} T_i = t, \\ & T_i \geq 0, i = 1, ..., m.\end{aligned}$$

The Lagrangian of the problem is $\mathcal{L}(T_1, ..., T_m, \theta) = \sum_{i=1}^{m} T_i^2 + \theta(t - \sum_{i=1}^{m} T_i)$. From the first order condition $\frac{\partial \mathcal{L}}{\partial T_i} = 0$, we have

$$T_i = \frac{\theta}{2},$$

and from the primal feasibility, we require

$$\sum_{i=1}^{m} T_i = m\frac{\theta}{2} = t.$$

As a result, we have $\theta = 2t/m$, and $T_i^* = t/m$. The corresponding total cost per unit of time is

$$\frac{Km}{t} + h\frac{\lambda t}{2m}.$$

Optimizing over m, we obtain

$$m^* = t\sqrt{\frac{\lambda h}{2K}}.$$

Correspondingly,

$$T^* = \sqrt{\frac{2K}{\lambda h}},\ Q^* = \sqrt{\frac{2\lambda K}{h}}$$

recovering the solution to the infinite horizon problem.

3.3 Power-of-Two Policies

Let $\bar{h} = \lambda h/2$ denote the "demand rate adjusted" holding cost. With a slight abuse of notation, we let

$$C(T) = \frac{K}{T} + \frac{h\lambda}{2}T = \frac{K}{T} + \bar{h}T$$

Correspondingly, $T^* = \sqrt{K/\bar{h}}$. The motivation of the power-of-two policies stems from the difficulty in implementing the EOQ in practice—T^* may well be some irrational number. As a result, power-of-two policies restricts

$$T = T_B \cdot 2^k, k \in \mathbb{Z},$$

where T_B is some base planning period, e.g., a week or a month.[1] The question is how good is such simple heuristics?

By convexity of $C(T)$, the optimal k^* can be found via

$$k^* = \min\{k \in \mathbb{Z} | C(T_B \cdot 2^k) \le C(T_B \cdot 2^{k+1})\}.$$

It then follows that

$$\begin{aligned}
& C(T_B \cdot 2^{k^*}) \le C(T_B \cdot 2^{k^*+1}) \\
\iff & \frac{K}{T_B \cdot 2^{k^*}} + \bar{h}T_B \cdot 2^{k^*} \le \frac{K}{T_B \cdot 2^{k^*+1}} + \bar{h}T_B \cdot 2^{k^*+1} \\
\iff & \frac{K}{T_B \cdot 2^{k^*+1}} \le \bar{h}T_B \cdot 2^{k^*} \\
\iff & \frac{K}{2\bar{h}} \le \left(T_B \cdot 2^{k^*}\right)^2 \\
\iff & \frac{1}{\sqrt{2}}T^* \le T_B \cdot 2^{k^*}
\end{aligned}$$

Similarly, by $C(T_B \cdot 2^{k^*-1}) > C(T_B \cdot 2^{k^*})$, we can arrive at $T_B \cdot 2^{k^*} < \sqrt{2}T^*$. That is, regardless of the chosen base planning period T_B, the optimal power-of-two policy must lie in $[T^*/\sqrt{2}, \sqrt{2}T^*)$ (and the only one that lies in the interval).

We can then bound the performance of power-of-two policy by $C(T^*/\sqrt{2})$ or $C(\sqrt{2}T^*)$, whichever is higher. Similar to the derivation for $C(\alpha Q^*)$, we can obtain

$$C(\alpha T^*) = \frac{1}{\alpha}\sqrt{\bar{h}K} + \alpha\sqrt{\bar{h}K} = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right)C(T^*).$$

[1] See Muckstadt and Roundy (1993) for some practices observed in a major U.S. automotive manufacturer. The restriction also helps planner in scheduling detailed operations, e.g., the same operation takes place at the same time on the same machine every, say, fourth week.

Hence,

$$C(T^*/\sqrt{2}) = C(\sqrt{2}T^*) = \frac{1}{2}\left(\sqrt{2} + \frac{1}{\sqrt{2}}\right) C(T^*).$$

It follows that

$$\frac{C(T_B \cdot 2^{k^*})}{C(T^*)} \leq \frac{\max\{C(T^*/\sqrt{2}), C(\sqrt{2}T^*)\}}{C(T^*)} = \frac{1}{2}\left(\sqrt{2} + \frac{1}{\sqrt{2}}\right) \approx 1.06.$$

How about power-of-three? One can consider the following more general class of policies $T = T_B \cdot a^k$ for $a \in \{2, 3, ...\}$. Again, by using the relationship that

$$C(T_B \cdot a^{k^*}) \leq C(T_B \cdot a^{k^*+1})$$
$$C(T_B \cdot a^{k^*}) > C(T_B \cdot a^{k^*-1})$$

one can derive that $T^*/\sqrt{a} \leq T_B \cdot a^{k^*} < \sqrt{a}T^*$. It then follows that

$$\frac{C(T_B \cdot a^{k^*})}{C(T^*)} \leq \frac{1}{2}\left(\sqrt{a} + \frac{1}{\sqrt{a}}\right).$$

Note that the function $f(x) = x + 1/x$ is increasing in x for $x \geq 1$. As a result, the bound is the tightest when $a = 2$.

References

Muckstadt, J. A. and R. O. Roundy (1993). Analysis of multistage production systems. *Handbooks in operations research and management science 4*, 59–131.