FOUNDATIONS OF OPTIMIZATION: IE6001

Shadow Prices

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Example 1 (perturbed)

Assume that p_1, the availability of X, is not precisely known.

$$\begin{array}{rll}
\max & x_1 + x_2 & \text{objective function} \\
\text{s.t.} & 2x_1 + x_2 \leq p_1 & \text{constraint on availability of X} \\
 & x_1 + 3x_2 \leq 18 & \text{constraint on availability of Y} \\
 & x_1 \leq 4 & \text{constraint on demand of A} \\
 & x_1, x_2 \geq 0 & \text{non-negativity constraints}
\end{array}$$

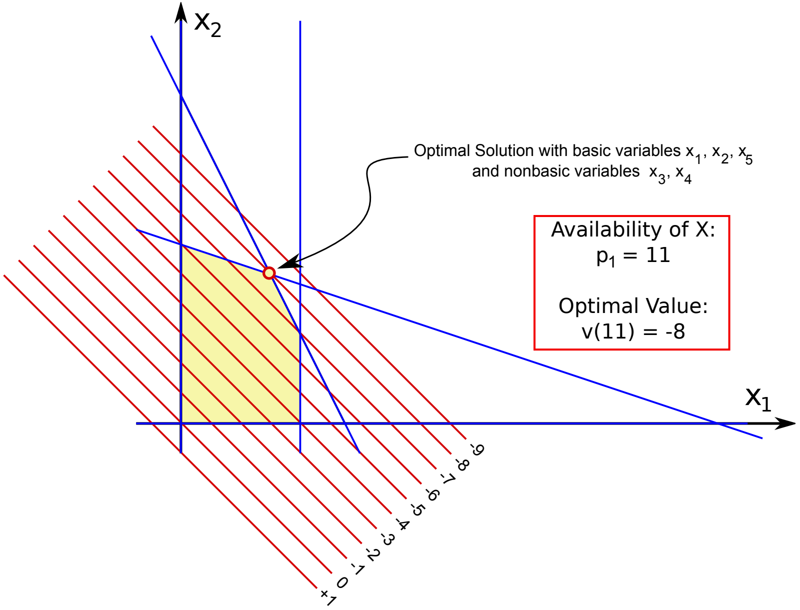
$$\begin{array}{rl}
-\min & -x_1 - x_2 \\
\text{s.t.} & 2x_1 + x_2 + x_3 = p_1 \\
 & x_1 + 3x_2 + x_4 = 18 \\
 & x_1 + x_5 = 4 \\
 & x_1, x_2, x_3, x_4, x_5 \geq 0
\end{array}$$

Example 1 (perturbed)

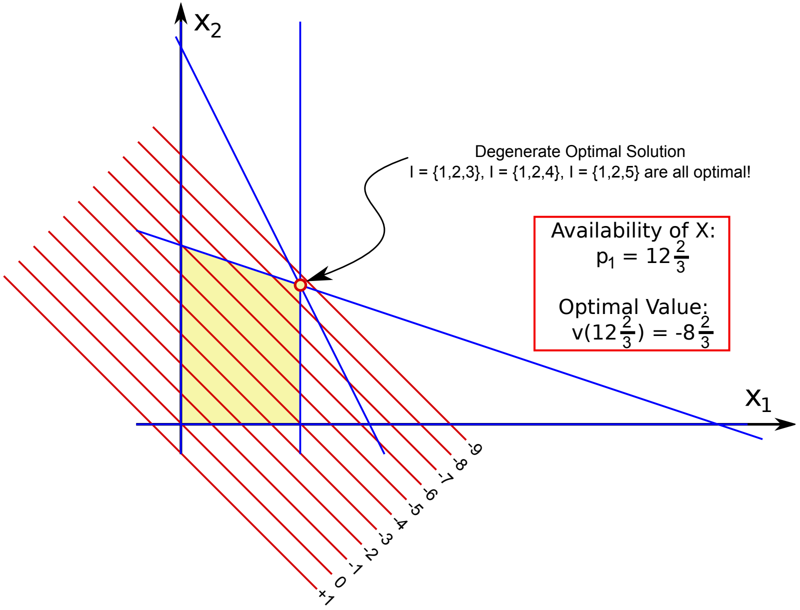
The value function $v(p_1)$ expresses the optimal value of the LP as a function of the unknown availability parameter p_1.

$$
\begin{aligned}
v(p_1) \quad = \quad \min \quad & -x_1 - x_2 \\
\text{s.t.} \quad & 2x_1 + x_2 + x_3 = p_1 \\
& x_1 + 3x_2 + x_4 = 18 \\
& x_1 + x_5 = 4 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

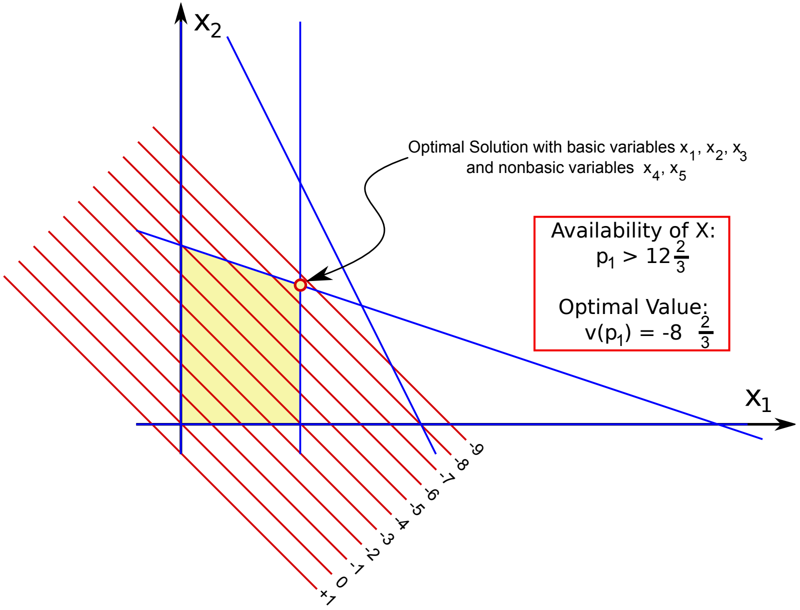
Example 1 (perturbed)



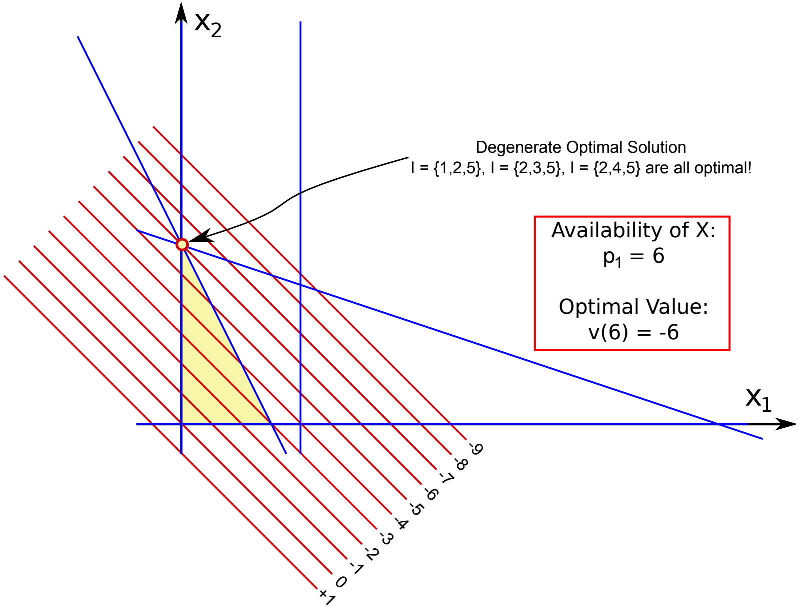
Example 1 (perturbed)



Example 1 (perturbed)



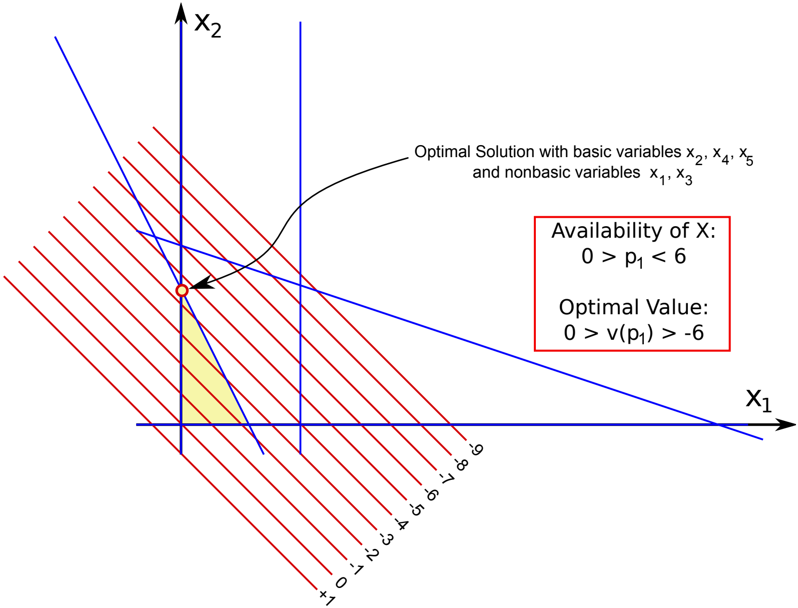
Example 1 (perturbed)



Availability of X:
$p_1 = 6$

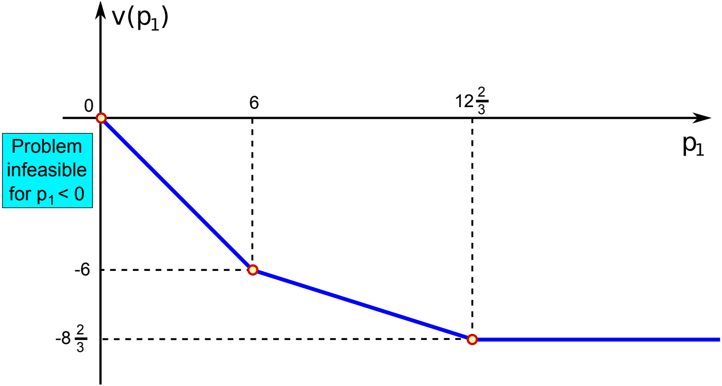
Optimal Value:
$v(6) = -6$

Example 1 (perturbed)



Example 1 (perturbed)

Note: $v(p_1)$ is non-increasing, convex and piecewise linear.



Perturbation

Let $p \in \mathbb{R}^m$ denote a general RHS and define the value function $v(p) : \mathbb{R}^m \to \mathbb{R}$ by:

$$v(p) = \min \left\{ c^T x \mid A\, x = p; x \geq 0 \right\}$$

Solving the original LP (the reference problem)

$$\min \left\{ x_0 = c^T x \mid A\, x = b, x \geq 0 \right\}$$

thus computes $v(b)$.

Shadow Prices

Suppose we have solved the reference problem

$$\min\left\{x_0 = c^T x \mid Ax = b, x \geq 0\right\}$$

and found an optimal basis matrix B satisfying

$$x_B = B^{-1} b \geq 0 \quad \text{(Feasibility)}$$

and

$$r = c_N - N^T (B^{-1})^T c_B \geq 0 \quad \text{(Optimality)}.$$

Shadow Prices (cont)

Definition: The vector of shadow prices $\Pi \in \mathbb{R}^m$ is defined as

$$\Pi = (B^{-1})^T c_B,$$

where $B = B(I)$ is an optimal basis.

Note that there can be more than one optimal basis
$\Rightarrow$ The shadow prices need not be unique.

The shadow Prices give information about the sensitivity of the value function $v(p)$ at $p = b$.

Local Behaviour of Value Function

Theorem: $v(p) = v(b) + \Pi^T(p - b)$ for all $p \in \mathbb{R}^m$ with $B^{-1}p \geq 0$.

Proof:

- If $B^{-1}p \geq 0$, then B remains the optimal basis for

$$\min\{c^T x : Ax = p, x \geq 0\}$$

  since r is not affected by changing b to p.

- Thus, we find

$$\begin{aligned} v(p) &= c_B^T B^{-1} p \\ &= c_B^T B^{-1} b + c_B^T B^{-1}(p - b) \\ &= v(b) + \Pi^T(p - b) \qquad \square \end{aligned}$$

Global Behaviour of Value Function

Theorem: $v(p) \geq v(b) + \Pi^T(p-b)$ for all $p \in \mathbb{R}^m$.

Proof:

$$
\begin{aligned}
v(p) &= \min_{x \geq 0; Ax=p} \left\{ c^T x \right\} \\
&= \min_{x \geq 0; Ax=p} \left\{ c^T x - \Pi^T(Ax-p) \right\} \\
&\geq \min_{x \geq 0} \left\{ c^T x - \Pi^T(Ax-p) \right\} \\
&= \min_{x \geq 0} \left\{ (c^T - \Pi^T A)x + \Pi^T p \right\} \\
&= \Pi^T p + \underbrace{\min_{x \geq 0} \left\{ (c^T - \Pi^T A)x \right\}}_{\geq 0 \text{ see next slide!}}
\end{aligned}
$$

Global Behaviour of Value Function

$$
\begin{aligned}
\left[c^T - \Pi^T A\right]\ x &= \left(\left[c_B^T \mid c_N^T\right] - c_B^T B^{-1}\ [B \mid N]\right) \begin{bmatrix} x_B \\ x_N \end{bmatrix} \\
&= \left[c_B^T \mid c_N^T\right] \begin{bmatrix} x_B \\ x_N \end{bmatrix} - c_B^T \left[I \mid B^{-1} N\right] \begin{bmatrix} x_B \\ x_N \end{bmatrix} \\
&= c_B^T\, x_B - c_B^T\, x_B + \left(c_N^T - c_B^T B^{-1} N\right) x_N \\
&= r^T x_N \\
&\geq 0 \qquad\qquad \text{(as } r \geq 0\text{, and } x_N \geq 0\text{)}
\end{aligned}
$$

Global Behaviour of Value Function

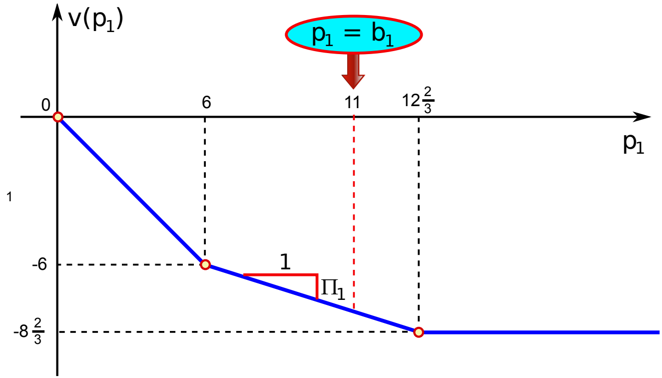
Thus, we find

$$
\begin{aligned}
v(p) &\geq \Pi^T p + \min_{x \geq 0} \left\{ (c^T - \Pi^T A) x \right\} \\
&\geq \Pi^T p \\
&= \Pi^T b + \Pi^T (p - b) \\
&= c_B^T B^{-1} b + \Pi^T (p - b) \\
&= v(b) + \Pi^T (p - b)
\end{aligned}
$$

$\square$

Shadow Prices in Example 1

Note: Π_1 is the shadow price for the budget of ingredient X.



At $p_1 = b_1 = 11$, the optimal costs change by $\Pi_1 = -\frac{2}{5}$ if the availability of X increases by 1.

Interpretation

- Assume the company can buy a ”small” additional amount of ingredient X, at price μ_1 per unit.
- Is it worthwhile to buy additional units of X?
  - Yes if $\mu_1 + \Pi_1 < 0$ (overall cost decreases);
  - No if $\mu_1 + \Pi_1 > 0$ (overall cost increases).

$\Rightarrow$ Therefore, $-\Pi_1$ is the maximum price one should pay for one additional unit of X!

Evaluation of Shadow Prices

Sometimes shadow prices can be read off the final tableau.

Lemma: Suppose row t is initially a "$\leq$-constraint" and a slack variable x_s had been added. Then, $\Pi_t = \beta_s$, where β_s is the objective coefficient of x_s in the final (optimal) tableau.

Proof:

- If x_s is nonbasic in the final tableau, then

$$\begin{aligned} \beta_s &= -r_s = -(c_N - N^T B^{-T} c_B)^T e_s \\ &= -c_s + \Pi^T N e_s = 0 + \Pi^T e_t = \Pi_t. \end{aligned}$$

- If x_s is basic in the final tableau, then

$$\beta_s = 0 = c_s = e_s^T c_B = e_s^T B^T \Pi = e_t^T \Pi = \Pi_t. \qquad \square$$

Evaluation of Shadow Prices (cont)

Sometimes shadow prices can be read off the final tableau.

Lemma: Suppose row t is initially a "$\geq$-constraint" and a surplus variable x_s had been added. Then, $\Pi_t = -\beta_s$, where β_s is the objective coefficient of x_s in the final tableau.

Example 1 (revisited)

The final tableau for Example 1 is:

BV	x_0	x_1	x_2	x_3	x_4	x_5	RHS
x_0	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	0	-8
x_2	0	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	0	5
x_5	0	0	0	$-\frac{3}{5}$	$\frac{1}{5}$	1	1
x_1	0	1	0	$\frac{3}{5}$	$-\frac{1}{5}$	0	3

- The constraint on the availability of X was standardised by introducing the slack variable x_3.
- The shadow price Π_1 for that constraint thus coincides with the coefficient of x_3 in the objective row of the above tableau $\Rightarrow \Pi_1 = -\frac{2}{5}$.