Task 1

Determine whether the following sets are linearly dependent or linearly independent.

1. $\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \right\}$ in $M_{2\times2}(R)$

$$a_1 \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & -1 \\ 3 & 1 \end{bmatrix} + a_3 \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{cases} a_1 + 0a_2 + a_3 = 0 \\ 2a_1 - a_2 - a_3 = 0 \\ 0a_1 + 3a_2 + 2a_3 = 0 \\ 0a_1 + a_2 - a_3 = 0 \end{cases}$$

$2a_1 = 0 \Rightarrow a_1 = 0$ $\quad a_3 = 0$ $\quad a_2 = 0$

$a_1 = a_2 = a_3 = 0$

$\Rightarrow$ The set is linearly independent.

2. $\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} -2 & 3 \\ 2 & 6 \end{bmatrix} \right\}$ in $M_{2\times2}(R)$

$$a_1 \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix} + a_3 \begin{bmatrix} -2 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{cases} a_1 + 0a_2 - 2a_3 = 0 \\ 0a_1 + 2a_2 + 3a_3 = 0 \\ -a_1 + 0a_2 + 2a_3 = 0 \\ 0a_1 + 4a_2 + 6a_3 = 0 \end{cases} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & \underset{2}{\cancel{4}} & \underset{3}{\cancel{6}} & 0 \end{bmatrix} \times(-1) \Rightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times 1 \Rightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} a_1 - 2a_3 = 0 \\ 2a_2 + 3a_3 = 0 \end{cases} \Rightarrow \begin{matrix} a_1 = 2a_3 \\ a_2 = -\frac{3}{2}a_3 \end{matrix} \Rightarrow$$

$a_3 \neq 0$ is a non-trivial solution

$\therefore$ The set is linearly dependent.

3. $\{(1,0,-2,1), (0,-1,1,1), (-1,2,1,0), (2,1,-4,4)\}$ in R^4

$$a_1(1,0,-2,1) + a_2(0,-1,1,1) + a_3(-1,2,1,0) + a_4(2,1,-4,4) = (0,0,0,0)$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ -2 & 1 & 1 & -4 & 0 \\ 1 & 1 & 0 & 4 & 0 \end{bmatrix} \times 2 \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 4 & 0 \end{bmatrix} \times(-1) \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \times(-1) \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & \underset{1}{\cancel{2}} & \underset{1}{\cancel{2}} & 0 \end{bmatrix} \times 1 \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \times(-1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \times 1 \\ \times(-1) \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{cases} a_1 + 0a_2 + 0a_3 + 3a_4 = 0 \\ -a_2 + a_3 + 0a_4 = 0 \\ a_3 + a_4 = 0 \end{cases} \Rightarrow \begin{matrix} a_3 = -a_4 \\ a_2 = a_3 = -a_4 \\ a_1 = -3a_4 \end{matrix}$$

$\Rightarrow a_4 \neq 0$ is a non-trivial solution

$\therefore$ The set is linearly dependent.

4. $\{(1,0,-2,1), (0,-1,1,1), (-1,2,1,0), (2,1,2,-2)\}$ in R^4.

$$a_1(1,0,-2,1) + a_2(0,-1,1,1) + a_3(-1,2,1,0) + a_4(2,1,2,-2) = (0,0,0,0)$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ -2 & 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & -2 & 0 \end{bmatrix} \times(-1) \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ -2 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -4 & 0 \end{bmatrix} \times 2 \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 6 & 0 \\ 0 & 1 & 1 & -4 & 0 \end{bmatrix} \times 1 \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 6 & 0 \\ 0 & 0 & \underset{1}{\cancel{2}} & \underset{-1}{-\cancel{2}} & 0 \end{bmatrix} \times 1 \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \times(-1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & \underset{1}{\cancel{-8}} & 0 \end{bmatrix} \begin{cases} a_4 = 0 \\ a_3 + 7a_4 = 0 \Rightarrow a_3 = 0 \\ -a_2 + 2a_3 + a_4 = 0 \Rightarrow a_2 = 0 \\ a_1 + 2a_3 - a_2 = 0 \Rightarrow a_1 = 0 \end{cases} \Rightarrow a_1 = a_2 = a_3 = a_4 = 0$$

$\therefore$ The set is linearly independent.

Task 2

Give an example of three linearly dependent vectors in $\mathbb{R}^3$ such that none of the three is a multiple of another. Describe how you get this example?

* Linearly dependent in $\mathbb{R}^3$: $c_1v_1 + c_2v_2 + c_3v_3 = 0$
where
c_1, c_2, c_3 aren't all 0.
$\Rightarrow v_1, v_2, v_3$ lie in the same plane.

Let v_1, v_2, v_3 lie in $z=0$.

$v_1 = (1, 2, 0)$ $\Rightarrow v_3$ should be a linear combination of v_1 & v_2.
$v_2 = (2, 1, 0)$

$v_3 = sv_1 + tv_2$
$= s(1,2,0) + t(2,1,0)$ where $s \neq 0, t \neq 0$

Let $s = t = 1 \Rightarrow v_3 = (1,2,0) + (2,1,0) = (3,3,0)$

$\therefore \{(1,2,0), (2,1,0), (3,3,0)\}$ is an example of three linearly dependent vectors in $\mathbb{R}^3$.

Task 3

Label the following statements as true or false, give an counterexample if you labeled false.

1. A vector space cannot have more than one basis. False

Counterexample: In $\mathbb{R}^2$, $\{(1,0),(0,1)\}$ is one basis.
$\{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$ is another basis.

2. The dimension of $P_n(F)$ is $n+1$.

True. (P.48 Example 10)

3. If V is a vector space having dimension n, and if S is a subset of V with n vectors, then S is a basis for V. False

Counterexample: If $n=3$, $V = \mathbb{R}^3$, the basis of V should be linearly independent & generates $\mathbb{R}^3$.
Let $S = \{(1,1,1), (2,2,2), (3,3,3)\}$
$\Rightarrow S$ is a subset of V with 3 vectors but isn't linearly independent. $\therefore S$ is not a basis of V.

Task 4

Determine which of the following sets are bases for $P_2(\mathbb{R})$.

1. $\{1-x^2,\ 2+5x+x^2,\ -4x+3x^2\}$ * dimension of $P_2(\mathbb{R}) = 2+1 = 3$
$\Rightarrow$ basis contains 3 polynomials.

$a_1(1,0,-1)+a_2(2,5,1)+a_3(0,-4,3)=(0,0,0)$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 5 & -4 & 0 \\ -1 & 1 & 3 & 0 \end{bmatrix} \times 1 \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 5 & -4 & 0 \\ 0 & \cancel{3} & \cancel{3} & 0 \end{bmatrix} \times(-5) \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & \cancel{-9}\,1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \times(-1) \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$a_2=0,\ a_3=0,\ a_1=0$

$\Rightarrow$ linearly independent & contains 3 vectors

$\therefore$ The set is a basis for $P_2(\mathbb{R})$.

2. $\{2-4x+x^2,\ 3x-x^2,\ 6-x^2\}$

$a_1(2,-4,1)+a_2(0,3,-1)+a_3(6,0,-1)=(0,0,0)$

$$\begin{bmatrix} \cancel{2}\,1 & 0 & \cancel{6}\,3 & 0 \\ -4 & 3 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix} \begin{matrix} \times 4 \\ \times(-1) \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & \cancel{3}\,1 & \cancel{12}\,4 & 0 \\ 0 & -1 & -4 & 0 \end{bmatrix} \times 1 \Rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\begin{cases} a_1+3a_3=0 \\ a_2+4a_3=0 \end{cases} \Rightarrow \begin{matrix} a_1=-3a_3 \\ a_2=-4a_3 \end{matrix} \Rightarrow a_3 \neq 0$ is a non-trivial solution.
$\therefore$ linearly dependent

$\therefore$ The set is not a basis for $P_2(\mathbb{R})$.

3. $\{1+2x-x^2,\ 1+2x^2,\ 2+x+x^2\}$

$a_1(1,2,-1)+a_2(1,0,2)+a_3(2,1,1)=(0,0,0)$

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 \end{bmatrix} \times 1 \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & \cancel{3}\,1 & \cancel{3}\,1 & 0 \end{bmatrix} \times(-2) \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \times 2 \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \times 1$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cancel{-1}\,1 & 0 \end{bmatrix}$$

$a_3=0$
$a_2=0$ $\Rightarrow$ linearly independent &
$a_1=0$ contains 3 vectors

$\therefore$ The set is a basis for $P_2(\mathbb{R})$.

Task 5

Find bases for the following subspaces of R^5.

$W = \{(a_1, a_2, a_3, a_4, a_5) \in R^5 : a_1 - a_3 - a_4 = 0\}$

What are the dimensions of W?

$a_1 - a_3 - a_4 = 0 \Rightarrow a_1 = a_3 + a_4$

$$\Rightarrow W = \{(a_3 + a_4, a_2, a_3, a_4, a_5) : a_2, a_3, a_4, a_5 \in R\}$$

$$= \{a_2(0,1,0,0,0) + a_3(1,0,1,0,0) + a_4(1,0,0,1,0) + a_5(0,0,0,0,1) : a_2, a_3, a_4, a_5 \in R\}$$

$$= \mathrm{span}\{\underbrace{(0,1,0,0,0)}_{v_1}, \underbrace{(1,0,1,0,0)}_{v_2}, \underbrace{(1,0,0,1,0)}_{v_3}, \underbrace{(0,0,0,0,1)}_{v_4}\}$$

$\Downarrow$

basis for W

$\Rightarrow \dim(W) = 4$