Task 1:

For each T defined below, (1) prove that T is a linear transformation, and (2) find bases for both $N(T)$ and $R(T)$. Then (3) compute the nullity and rank of T, and (4) verify the dimension theorem. Finally, (5) determine whether T is one-to-one or onto.

- $T:\ R^3 \to R^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

(1) Prove: $T(cu+v) = c\cdot T(u) + T(v)$

Let $u = (a_1, a_2, a_3)$, $v = (b_1, b_2, b_3)$

$$T(cu+v) = T(ca_1+b_1, ca_2+b_2, ca_3+b_3)$$
$$= (ca_1+b_1-ca_2-b_2,\ 2\cdot(ca_3+b_3))$$
$$= (c(a_1-a_2)+(b_1-b_2),\ 2\cdot(ca_3+b_3))$$
$$= c((a_1-a_2), 2a_3) + (b_1-b_2, 2b_3) = cT(u)+T(v)$$ #

(2) Basis for $N(T)$:

$N(T) = \{(a_1, a_2, a_3) \in R^3 : T(a_1, a_2, a_3) = (0,0)\}$

$\begin{cases} a_1 - a_2 = 0 \\ 2a_3 = 0 \end{cases} \Rightarrow \begin{matrix} a_3 = 0 \\ a_1 = a_2 = t \end{matrix} = (t, t, 0) = t(1,1,0)$ $\therefore$ A basis of $N(T) = (1,1,0)$ #

Basis for $R(T)$:

$R(T) = \{T(a_1, a_2, a_3) : (a_1, a_2, a_3) \in R^3\}$, a_1, a_2, a_3 can be any R.

$T(a_1, a_2, a_3) = (x, y)$

Let $a_1 = x$, $a_2 = 0$ s.t. $a_1 - a_2 = x$ $\Rightarrow T(x, 0, \frac{y}{2}) = (x, y)$

$a_3 = \frac{y}{2}$ s.t. $2a_3 = y$

$\therefore R(T) = R^2 \Rightarrow$ A basis of $R(T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ #

(3) Nullity of $T = \dim(N(T)) = 1$ #

Rank of $T = \dim(R(T)) = 2$ #

(4) $\text{Nullity}(T) + \text{Rank}(T) = 1 + 2 = 3 = \dim(R^3)$ #

(5) $N(T) = t(1,1,0) \neq \{\emptyset\}$ $\therefore T$ is not one-to-one #

$R(T) = R^2 \Rightarrow T$ is onto #

- $T: R^6 \to R^4$ defined by $T(a_1, a_2, a_3, a_4, a_5, a_6) = (2a_1 - a_2, a_3 + a_2, 0, 0)$

(1) Let $u = (a_1, a_2, a_3, a_4, a_5, a_6)$, $v = (b_1, b_2, b_3, b_4, b_5, b_6)$

$$T(cu+v) = T(ca_1+b_1, ca_2+b_2, ca_3+b_3, ca_4+b_4, ca_5+b_5, ca_6+b_6)$$
$$= (2(ca_1+b_1)-(ca_2+b_2), (ca_2+b_2+ca_3+b_3), 0, 0)$$
$$= c(2a_1-a_2, a_2+a_3, 0, 0) + (2b_1-b_2, b_2+b_3, 0, 0) = cT(u)+T(v)_{\#}$$

(2) $N(T) = \{(a_1, a_2, a_3, a_4, a_5, a_6) \in R^6 : (2a_1-a_2, a_2+a_3, 0, 0) = (0,0,0,0)\}$

$$\begin{cases} 2a_1 - a_2 = 0 \\ a_2 + a_3 = 0 \end{cases} \Rightarrow 2a_1 + a_3 = 0 \quad \begin{matrix} a_2 = 2a_1 \\ a_3 = -2a_1 \end{matrix}$$

$$N(T) = (a_1, 2a_1, -2a_1, a_4, a_5, a_6)$$
$$= a_1(1,2,-2,0,0,0) + a_4(0,0,0,1,0,0) + a_5(0,0,0,0,1,0) + a_6(0,0,0,0,0,1)$$

A basis of $N(T) = \{(1,2,-2,0,0,0), (0,0,0,1,0,0), (0,0,0,0,1,0), (0,0,0,0,0,1)\}_{\#}$

$R(T) = (x, y, 0, 0)$ $\therefore$ A basis of $R(T)$ is $\{(1,0,0,0), (0,1,0,0)\}_{\#}$

(3) Nullity of $T = \dim(N(T)) = 4_{\#}$

Rank of $T = \dim(R(T)) = 2_{\#}$

(4) $\text{Nullity}(T) + \text{Rank}(T) = 4+2 = 6 = \dim(R^6)_{\#}$

(5) $N(T) \neq \{\phi\} \Rightarrow T$ is not one-to-one.$_{\#}$

$R(T) = (x, y, 0, 0)$ doesn't span $R^4 \Rightarrow T$ is not onto.$_{\#}$

- $T: M_{1x2}(R) \to M_{1x3}(R)$ defined by $T([a_1, a_2]) = T([a_1 + a_2, 0, 2a_1 - a_2])$

(1) Let $u = [a_1 \ a_2]$, $v = [b_1 \ b_2]$

$T(cu+v) = T([ca_1+b_1 \quad ca_2+b_2])$

$= [ca_1+b_1+ca_2+b_2 \quad 0 \quad 2ca_1+2b_1-ca_2-b_2]$

$= c[a_1+a_2 \quad 0 \quad 2a_1-a_2] + [b_1+b_2 \quad 0 \quad 2b_1-b_2]$

$= c \cdot T(u) + T(v)$ ※

(2) $N(T) = \{[a_1 \ a_2] \in M_{1x2}(R) \mid T([a_1 \ a_2]) = [0 \ 0 \ 0]\}$

$\Rightarrow (a_1+a_2, 0, 2a_1-a_2) = (0, 0, 0)$

$\begin{cases} a_1+a_2=0 \\ 2a_1-a_2=0 \end{cases} \Rightarrow \begin{matrix} 3a_1=0 \\ a_1=0, \ a_2=0 \end{matrix}$ $\therefore N(T) = \{(0,0)\} \Rightarrow$ Basis of $N(T) = \{\phi\}$ ※

$R(T) = \{T([a_1 \ a_2]) \in M_{1x3}(R) : [a_1 \ a_2] \in M_{1x2}(R)\}$, a_1, a_2 can be any R.

Let $(a_1, a_2) = (1, 0) \Rightarrow T([1 \ 0]) = (1, 0, 2)$

$(a_1, a_2) = (0, 1) \Rightarrow T([0 \ 1]) = (1, 0, -1)$

$\Rightarrow R(T) = \text{span}\{(1,0,2), (1,0,-1)\}$

$\therefore$ A basis of $R(T)$ is $\{(1,0,2), (1,0,-1)\}$ ※

(3) Nullity of $T = \dim(N(T)) = 0$ ※

Rank of $T = \dim(R(T)) = 2$ ※

(4) Nullity(T) + Rank$(T) = 0 + 2 = 2 = \dim(M_{1x2}(R))$ ※

(5) $N(T) = \{\phi\} \Rightarrow T$ is one-to-one ※

$R(T)$ doesn't span $M_{1x3}(R) \Rightarrow T$ is not onto ※.

Task 2:

We define T: $R^2 \rightarrow R^2$ is a function. For each of the following parts, state why T is not linear.

- 1. $T(a_1, a_2) = (1, a_2)$
- 2. $T(a_1, a_2) = (a_1, a_1^2)$
- 3. $T(a_1, a_2) = (|a_1|, a_2)$

linear transformation:

(1) $\exists\, T(0) = 0$

(2) $T(cu+v) = cT(u) + T(v)$

1. (1) $T(0,0) = (1,0) \neq (0,0)$ $\therefore T$ is not linear.

2. (1) $T(0,0) = (0,0)$

(2) Let $u = (a_1, a_2)$, $v = (b_1, b_2)$

$T(cu+v) = T(ca_1+b_1, ca_2+b_2) = (ca_1+b_1, (ca_1+b_1)^2)$

$cT(u) + T(v) = cT(a_1, a_2) + T(b_1, b_2) = c\cdot(a_1, a_1^2) + (b_1, b_1^2)$

$= (ca_1+b_1, ca_1^2+b_1^2)$

$\because T(cu+v) \neq cT(u)+T(v)$

$\therefore T$ is not linear.

3. (1) $T(0,0) = (0,0)$

(2) Let $u = (a_1, a_2)$, $v = (b_1, b_2)$

$T(cu+v) = T(ca_1+b_1, ca_2+b_2) = (|ca_1+b_1|, ca_2+b_2)$

$cT(u) + T(v) = c\cdot T(a_1, a_2) + T(b_1, b_2)$

$= c(|a_1|, a_2) + (|b_1|, b_2)$

$= (c|a_1| + |b_1|, ca_2+b_2)$

$\because |ca_1+b_1| \neq c|a_1|+|b_1| \Rightarrow T(cu+v) \neq cT(u)+T(v)$

$\therefore T$ is not linear.

Task 3:

Suppose that $T: R^2 \rightarrow R^2$ is linear, $T(1,0) = (1,4)$, *and* $T(1,1) = (2,5)$. (1) What is $T(2,3)$? (2) Is T one-to-one? Justify your answer.

(1) $T(1,1) = T((1,0)+(0,1)) = T(1,0) + T(0,1)$

$\Rightarrow T(0,1) = T(1,1) - T(1,0) = (2,5) - (1,4) = (1,1)$

$T(2,3) = 2T(1,0) + 3T(0,1)$

$= 2 \cdot (1,4) + 3(1,1) = (5,11)$ #

(2) One-to-one: $N(T) = \{\phi\}$

$c_1 \cdot (1,4) + c_2 \cdot (1,1) = (0,0)$

$\begin{cases} c_1 + c_2 = 0 \\ 4c_1 + c_2 = 0 \end{cases} \Rightarrow \begin{array}{l} 3c_1 = 0 \\ c_1 = 0 \\ c_2 = 0 \end{array} \Rightarrow N(T) = (0,0) = \{\phi\}$

$\therefore$ T is one-to-one. #

Task 4:

Is there a linear transformation $T\colon R^3 \to R^2$ such that $T(1, 2, 1) = (1, 1)$ *and* $T(3, 6, 3) = (2, 1)$? Justify your answer.

linear transformation: $T(cu) = cT(u)$

$T(3,6,3) = 3T(1,2,1) = 3\cdot(1,1) = (3,3) \neq (2,1)$

$\therefore$ There is no such a linear transformation.

Task 5:

Give an example of a linear transformation T: $R^2 \to R^2$ such that $N(T) = \{(x,\ y) \in R^2\ where\ x = 0\}$. Justify your answer.

$N(T) = \{(x, y) \in R^2$ where $x = 0\}$

$\Rightarrow 1^\circ.$ $x=0$: vectors $(0, y) \to (0,0)$

$2^\circ.$ $x \neq 0$: $(x, y) \not\to (0,0)$

$\Rightarrow T(x, y) = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow T(x, y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$

$1^\circ.$ $x=0$, $T(0, y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} by \\ dy \end{bmatrix}$

$\therefore b = 0,\ d = 0$, $A = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$. $T(x, y) = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ cx \end{bmatrix}$

$2^\circ.$ Ensure when $x \neq 0$, $T(x, y) \neq (0,0)$

$T(x, y) = (ax, cx) \neq (0,0)$

$\Rightarrow a \neq 0$ or $c \neq 0$

$\therefore$ An example of $(a, c) = (1, 2) \Rightarrow A$ can be $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

$3^\circ.$ Justify my answer:

$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $T(x, y) = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix}$

① $x=0$:

$N(T) = (0,0) \Rightarrow \begin{bmatrix} x \\ 2x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = 0.$

② $x \neq 0$:

$T(x, y) = \begin{bmatrix} x \\ 2x \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\hookrightarrow$ x can be any R

$\therefore A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ #