Task 1

Let β and γ be the standard ordered bases for R^n and R^m, respectively. For each linear transformation: $T: R^n \to R^m$, compute $[T]_\beta^\gamma$.

1. $T: R^3 \to R^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

Let $\beta = \{\underset{v_1}{(1,0,0)}, \underset{v_2}{(0,1,0)}, \underset{v_3}{(0,0,1)}\}$, $\gamma = \{\underset{e_1}{(1,0)}, \underset{e_2}{(0,1)}\}$

$T(v_1) = T(1,0,0) = (1-0, 2\cdot 0) = (1,0) = 1\cdot e_1 + 0\cdot e_2$
$T(v_2) = T(0,1,0) = (0-1, 2\cdot 0) = (-1,0) = -1e_1 + 0e_2$
$T(v_3) = T(0,0,1) = (0-0, 2\cdot 1) = (0,2) = 0e_1 + 2e_2$

$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ *

2. $T: R^6 \to R^4$ defined by $T(a_1, a_2, a_3, a_4, a_5, a_6) = (2a_1 - a_2, a_3 + a_2, 0, 0)$

Let $\beta = \{\underset{v_1}{(1,0,0,0,0,0)}, \underset{v_2}{(0,1,0,0,0,0)}, \underset{v_3}{(0,0,1,0,0,0)}, \underset{v_4}{(0,0,0,1,0,0)}, \underset{v_5}{(0,0,0,0,1,0)}, \underset{v_6}{(0,0,0,0,0,1)}\}$,

$\gamma = \{\underset{e_1}{(1,0,0,0)}, \underset{e_2}{(0,1,0,0)}, \underset{e_3}{(0,0,1,0)}, \underset{e_4}{(0,0,0,1)}\}$

$T(v_1) = (2,0,0,0)$, $T(v_4) = (0,0,0,0)$
$T(v_2) = (-1,1,0,0)$, $T(v_5) = (0,0,0,0)$
$T(v_3) = (0,1,0,0)$, $T(v_6) = (0,0,0,0)$

$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ *

3. $T: R^2 \to R^3$ defined by $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$

Let $\beta = \{\underset{v_1}{(1,0)}, \underset{v_2}{(0,1)}\}$, $\gamma = \{\underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)}\}$

$T(v_1) = (2,3,1) = 2e_1 + 3e_2 + e_3$
$T(v_2) = (-1,4,0) = -e_1 + 4e_2 + 0e_3$

$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$ *

4. $T: R^n \to R^n$ defined by $T(a_1, a_2, \dots, a_n) = (a_n, a_{n-1}, \cdots, a_1)$

Let $\beta = \{e_1, e_2 \dots e_n\} = \gamma$

$T(e_1) = e_n$
$T(e_2) = e_{n-1}$
$\vdots$
$T(e_{n-1}) = e_2$
$T(e_n) = e_1$

$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & & 0 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n \times n}$ *

Task 2

Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard ordered bases for $\mathbb{R}^2$ and $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$. Compute $[T]_\beta^\gamma$. If $\alpha = \{(1,2), (2,3)\}$, compute $[T]_\alpha^\gamma$.

① $\beta = \{\underset{v_1}{(1,0)}, \underset{v_2}{(0,1)}\}$, $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$, standard $\mathbb{R}^3 = \{\underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)}\}$

$T(v_1) = (1,1,2) = 1e_1 + 1e_2 + 2e_3$
$T(v_2) = (-1,0,1) = -1e_1 + 0e_2 + 1e_3$ $\Rightarrow [T]_\beta = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}$

* For γ-basis:

$(1,1,2) = a_1(1,1,0) + a_2(0,1,1) + a_3(2,2,3)$

$\begin{cases} a_1 + 2a_3 = 1 \\ a_1 + a_2 + 2a_3 = 1 \\ a_2 + 3a_3 = 2 \end{cases}$ $>$ $a_2 = 0$, $a_3 = \frac{2}{3}$, $a_1 = -\frac{1}{3}$ $\Rightarrow (1,1,2) = -\frac{1}{3}(1,1,0) + 0(0,1,1) + \frac{2}{3}(2,2,3)$

$(-1,0,1) = b_1(1,1,0) + b_2(0,1,1) + b_3(2,2,3)$

$\begin{cases} b_1 + 2b_3 = -1 \\ b_1 + b_2 + 2b_3 = 0 \\ b_2 + 3b_3 = 1 \end{cases}$ $b_2 = 1$, $b_3 = 0$, $b_1 = -1$ $\Rightarrow (-1,0,1) = -1\cdot(1,1,0) + 1(0,1,1) + 0(2,2,3)$

$\therefore [T]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}$ *

② $\alpha = \{\underset{v_1}{(1,2)}, \underset{v_2}{(2,3)}\}$ $\Rightarrow$ $T(v_1) = (-1,1,4)$, $T(v_2) = (-1,2,7)$ $\Rightarrow [T]_\alpha = \begin{pmatrix} -1 & -1 \\ 1 & 2 \\ 4 & 7 \end{pmatrix}$ ← standard basis

* For γ-basis:

$(-1,1,4) = c_1(1,1,0) + c_2(0,1,1) + c_3(2,2,3)$

$\begin{cases} c_1 + 2c_3 = -1 \\ c_1 + c_2 + 2c_3 = 1 \\ c_2 + 3c_3 = 4 \end{cases}$ $c_2 = 2$, $c_3 = \frac{2}{3}$, $c_1 = -\frac{7}{3}$ $\Rightarrow (-1,1,4) = -\frac{7}{3}(1,1,0) + 2(0,1,1) + \frac{2}{3}(2,2,3)$

$(-1,2,7) = d_1(1,1,0) + d_2(0,1,1) + d_3(2,2,3)$

$\begin{cases} d_1 + 2d_3 = -1 \\ d_1 + d_2 + 2d_3 = 2 \\ d_2 + 3d_3 = 7 \end{cases}$ $d_2 = 3$, $d_3 = \frac{4}{3}$, $d_1 = -\frac{11}{3}$ $\Rightarrow (-1,2,7) = -\frac{11}{3}(1,1,0) + 3(0,1,1) + \frac{4}{3}(2,2,3)$

$\therefore [T]_\alpha^\gamma = \begin{pmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}$ *

Task 3

Let A and B be $n \times n$ matrices. Recall that de trace of A is defined by $tr(A) = \sum_{i=1}^{n} A_{ii}$. Prove that $tr(AB) = tr(BA)$.

Let $A = [a_{ij}]$, $B = [b_{ij}]$ where $i, j = 1, 2, \dots, n$

$tr(AB) = \sum_{i=1}^{n} (AB)_{ii}$ By definition (P.88): $(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$\Rightarrow (AB)_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki}$

$\therefore tr(AB) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki}$

$tr(BA) = \sum_{i=1}^{n} (BA)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} a_{ki}$

$\because a_{ik} b_{ki} = b_{ik} a_{ki}$ $\therefore tr(AB) = tr(BA)$ ※

Task 4

Let $\alpha = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, $\beta = \{1, x, x^2\}$, and $\gamma = \{1\}$

- Define $T: M_{2\times2}(R) \to M_{2\times2}(R)$ by $T(A) = A^t$. Compute $[T]_\alpha$ and $[T(A)]_\alpha$, where $A = \begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix}$.

① $\alpha = \{ \underset{E_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{E_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{E_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{E_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \}$

$\because T(A) = A^T$

$\Rightarrow T(E_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = E_1$

$T(E_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = E_3$

$T(E_3) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E_2$

$T(E_4) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = E_4$

$\therefore [T]_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ※

② $T(A) = A^T = \begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix}$, $T(A) = 1 \cdot E_1 + (-1) \cdot E_2 + 4E_3 + 2E_4 \Rightarrow [T(A)]_\alpha = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 2 \end{bmatrix}$ ※

- Define $T: P_2(R) \to R$ by $T(f(x)) = f(2)$. Compute $[T]_\beta^\gamma$ and $[T(f(x))]_\beta^\gamma$, where $f(x) = 4x^2 - 2x + 1$.

① $\beta = \{1, x, x^2\}$, $T(f(x)) = f(2)$

$T(1) = 1 \Rightarrow [T(1)]_\gamma = [1]$

$T(x) = 2 \Rightarrow [T(x)]_\gamma = [2]$

$T(x^2) = 4 \Rightarrow [T(x^2)]_\gamma = [4]$

$\Rightarrow [T]_\beta^\gamma = [1 \ \ 2 \ \ 4]$ ※

② $f(2) = 16 - 4 + 1 = 13$

$\because \gamma = \{1\}$ $\therefore [T(f(x))]_\beta^\gamma = [13]$ ※