Task 1:

Determine whether the following sets are subspaces of R^3 under the operations of addition and scalar multiplication defined on R^3. Justify your answers.

- 1、 $W_1 = \{(x, y, z) \in R^3 : x = 2y\}$
- 2、 $W_2 = \{(x, y, z) \in R^3 : y = 0\}$
- 3、 $W_3 = \{(x, y, z) \in R^3 : x = 2y \text{ and } z = 2\}$
- 4、 $W_3 = \{(x, y, z) \in R^3 : x = y^2\}$

* W is a subspace of V iff:
(a) $0 \in W$
(b) $x+y \in W$ whenever $x \in W$, $y \in W$
(c) $cx \in W$ whenever $c \in F$, $x \in W$
(P.1) Theorem 1.3)

1、(a) $0 \in W$: $0 = 2 \cdot 0 \Rightarrow (0,0,0) \in W_1$

(b) $x+y \in W$:

Let $a = (x_1, y_1, 0)$, $b = (x_2, y_2, 0)$

$\begin{cases} x_1 = 2y_1 \\ x_2 = 2y_2 \end{cases} \Rightarrow a+b = (x_1+x_2, y_1+y_2, 0) = (\underbrace{2y_1+2y_2}_{2(y_1+y_2)}, y_1+y_2, 0)$ $\therefore a+b \in W_1$

(c) $cx \in W$:

Let $a = (x, y, 0) \in W$, $c \in R$ $\therefore ca \in W_1$

$ca = (cx, cy, 0)$

$cx = c \cdot 2y = 2(cy)$

Ans: True

2、(a) $0 \in W$: $0 = 0 \Rightarrow (0,0,0) \in W_2$

(b) $x+y \in W$:

Let $a = (x_1, 0, z_1)$, $b = (x_2, 0, z_2)$ $\therefore a+b \in W_2$

$a+b = (x_1+x_2, \underline{\underline{0}}, z_1+z_2)$

(c) $cx \in W$:

Let $a = (x, 0, z) \in W_2$, $c \in R$ $\therefore ca \in W_2$ Ans: True

$c \cdot 0 = 0$

3、(a) $0 \in W$: $\underset{x}{0} = 2 \cdot \underset{y}{0}$, $0 \neq 2 \Rightarrow (0,0,0) \notin W_3$ Ans: False

4、(a) $0 \in W$: $0 = 0^2 \Rightarrow (0,0,0) \in W_4$

(b) $x+y \in W$:

Let $a = (x_1, y_1, 0)$, $b = (x_2, y_2, 0)$ $\therefore a+b \notin W_4$

$\begin{cases} x_1 = y_1^2 \\ x_2 = y_2^2 \end{cases} \Rightarrow a+b = (y_1^2+y_2^2, y_1+y_2, 0) \Rightarrow (y_1+y_2)^2 \neq y_1^2+y_2^2$ Ans: False

Task 2:

Prove that $A + A^t$ is symmetric for any square matrix A.

* A symmetric matrix is a matrix A such that $A^t = A$. (P.18)

$\Rightarrow A + A^t$ is symmetric if $(A+A^t)^t = A+A^t$

$$\boxed{(A+A^t)^t} = A^t + (A^t)^t \rightarrow A = A^t + A = \boxed{A + A^t}$$

$\therefore A + A^t$ is symmetric for any square matrix A.

Task 3:

For each list of polynomials in $P_2(\Re)$, determine whether the first polynomials can be expressed as a linear combination of the other two.

- 1、$-2x^2+3$, x^2+3x, $2x^2+4x-1$
- 2、x^2+2x-3, $-3x^2+2x+1$, $2x^2-x-1$
- 3、$3x^2+4x+1$, x^2-2x+1, $-2x^2-x+1$

1、$-2x^2+0x+3=a(x^2+3x)+b(2x^2+4x-1)$

$$\begin{cases} a+2b=-2 \\ 3a+4b=0 \\ 0a-b=3 \end{cases} \quad \begin{matrix} b=-3 \\ a=4 \end{matrix} \Rightarrow -2x^2+3=4(x^2+3x)-3(2x^2+4x-1)$$

Ans: True

2、$x^2+2x-3=a(-3x^2+2x+1)+b(2x^2-x-1)$

$$\begin{cases} -3a+2b=1 \\ 2a-b=2 \\ a-b=-3 \end{cases} \quad \begin{matrix} a=5 \\ b=8 \end{matrix} \Rightarrow x^2+2x-3=5(-3x^2+2x+1)+8(2x^2-x-1)$$

Ans: True

3、$3x^2+4x+1=a(x^2-2x+1)+b(-2x^2-x+1)$

$-5-2\cdot 6=-17\neq 3$

$$\begin{cases} a-2b=3 \\ -2a-b=4 \\ a+b=1 \end{cases} \Rightarrow a=-5,\ b=6$$

Ans: False

Task 4:

In each part, determine whether the given vector is in the span of S.

- 1、(2, -1, 1), $S = \{(1, 0, 2), (-1, 1, 1)\}$
- 2、(-1, 2, 1), $S = \{(1, 0, 2), (-1, 1, 1)\}$
- 3、(-1, 1, 1, 2), $S = \{(1, 0, 1, -1), (0, 1, 1, 1)\}$

1、 $(2,-1,1) = a(1,0,2) + b(-1,1,1)$

$$\begin{cases} a-b=2 \\ b=-1 \\ 2a+b=1 \end{cases} \Rightarrow a=1,\ b=-1 \qquad \Rightarrow (2,-1,1) = (1,0,2) - (-1,1,1)$$

$2\cdot 1 - 1 = 1$

Ans = True

2、 $(-1,2,1) = a(1,0,2) + b(-1,1,1)$

$$\begin{cases} a-b=-1 \\ b=2 \\ 2a+b=1 \end{cases} \Rightarrow a=1,\ b=2$$

$2\cdot 1 + 2 = 4 \neq 1$

Ans: False

3、 $(-1,1,1,2) = a(1,0,1,-1) + b(0,1,1,1)$

$$\begin{cases} a=-1 \\ b=1 \\ a+b=1 \\ -a+b=2 \end{cases}$$

$-1+1=0\neq 1$

Ans: False

Task 5:

Show that if

$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $M_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric $2x2$ matrices.

* A symmetric matrix is a matrix A such that $A^t = A$. (P.18)

If $A = \begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}$, $A^t = \begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}$ $\Rightarrow$ A is a symmetrix matrix

$\Rightarrow$ The form of a 2×2 symmetric matrix is like $\begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}$

Let $A = \alpha M_1 + \beta M_2 + \gamma M_3$

$$= \alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix} + \begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix} = \begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}$$

$\therefore$ The span of $\{M_1, M_2, M_3\}$ is the set of all symmetrix 2×2 matrices.