Task 1:

For each of the following linear transformations T, determine whether T is invertible and justify your answer.

① one-to-one
② onto

- 1. $T\colon R^2 \to R^3$ defined by $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$
- 2. $T\colon R^3 \to R^3$ defined by $T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$
- 3. $T\colon M_{2x2}(R) \to P_2(R)$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + 2bx + (c+d)x^2$

1. $T(a_1, a_2) = A\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $T(1,0) = (2,3,1)$, $T(0,1) = (-1,4,0)$, $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$

① one-to-one : $N(T) = \{0\}$

$\begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2a_1 - a_2 = 0 \\ 3a_1 + 4a_2 = 0 \\ a_1 = 0 \end{cases}$ $a_1 = 0$, $a_2 = 0$ $\therefore N(T) = \{0\}$ $\therefore T$ is one-to-one

② onto : $Rank(T) = dim(codomain)$

$dim(codomain) = 3$ $\because$ codomain : R^3

$Rank(T) = 2$ $\because A$ is linearly independent

$\Rightarrow Rank(T) \neq dim(codomain)$ $\therefore T$ isn't onto $\Rightarrow$ Not invertible

2. $T(a_1, a_2, a_3) = A\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $T(1,0,0) = (3,0,3)$, $T(0,1,0) = (0,1,4)$, $T(0,0,1) = (-2,0,0)$

$A = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix}$

① one-to-one :

$A\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 3a_1 - 2a_3 = 0 \\ a_2 = 0 \\ 3a_1 + 4a_2 = 0 \end{cases}$ $a_1 = 0$, $a_3 = 0$ $\Rightarrow N(T) = \{0\}$ $\therefore T$ is one-to-one

② onto :

$A = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix} \times(-1) = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 4 & 2 \end{pmatrix} \times(-4) = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow Rank(A) = Rank(T) = 3$

$dim(codomain) = 3$ $\therefore T$ is onto

$\because T$ is both one-to-one & onto $\Rightarrow$ Invertible

3. ① one-to-one :

$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0 \Rightarrow a + 2bx + (c+d)x^2 = 0 \Rightarrow \begin{cases} a = 0 \\ 2b = 0 \Rightarrow b = 0 \\ c + d = 0 \Rightarrow d = -c \end{cases}$

$N(T) = \left\{\begin{bmatrix} 0 & 0 \\ c & -c \end{bmatrix}, \text{where } c \in R\right\} = 1 \neq \{0\} \Rightarrow T$ is not one-to-one

$\Rightarrow$ Not invertible

Task 2:

Which of the following pairs of vector spaces are isomorphic? Justify your answers.

- 1. R^3 *and* $P_3(R)$
- 2. R^4 *and* $P_3(R)$
- 3. $V = \{A \in M_{2x2}(R)\ where\ tr(A) = 0\}$ *and* R^3

⇒ ① dim(domain) = dim(codomain)
② one-to-one
③ onto

1. $\dim(R^3) = 3$
$\dim(P_3(R)) = 4$
$\because \dim(R^3) \neq \dim(P_3(R)) \Rightarrow$ Not isomorphic

2. ① $\dim(R^4) = \dim(P_3(R)) = 4$
② one-to-one:
Let $v = (a,b,c,d) \in R^4$, $T(a,b,c,d) = a + bx + cx^2 + dx^3$
If $T(a,b,c,d) = 0 \Rightarrow a + bx + cx^2 + dx^3 = 0 \Rightarrow a = b = c = d = 0 \Rightarrow N(T) = \{0\}$
$\therefore T$ is one-to-one
③ Any $p(x) \in P_3(R)$ can be the output of $T \Rightarrow T$ is onto

⇒ Isomorphic

3. ① Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \because tr(A) = 0$, $a + d = 0$, $\Rightarrow d = -a$
A basis of $A = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ $\therefore \dim(A) = 3$
$\dim(R^3) = 3 \Rightarrow \dim(V) = \dim(R^3) = 3$
② one-to-one: $T: V \to R^3$
Let $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, $T(A) = (a,b,c)$
If $T(a,b,c) = (0,0,0) \Rightarrow a = b = c = 0 \Rightarrow A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow N(T) = \{0\}$
$\therefore T$ is one-to-one.
③ Any $(a,b,c) \in R^3$ can construct $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \Rightarrow T$ is onto.

⇒ Isomorphic

Task 3:

Let A be an $n\,x\,n$ matrix and 0 means zero matrix.

- 1. Suppose that $A^2 = 0$. Prove that A is not invertible.
- 2. Suppose that $AB = 0$ for some nonzero $n\,x\,n$ matrix B. Could A be invertible? Explain.

1. A is invertible if A^{-1} exists such that $A \cdot A^{-1} = I$

$A^2 = 0 \Rightarrow A \cdot A = 0$

$A^{-1} \cdot A \cdot A = A^{-1} \cdot 0$

$I \cdot A = 0 \Rightarrow$ A is a zero matrix.

If A is a zero matrix:

$A \cdot A^{-1} = 0 \cdot A^{-1} = 0 \neq I$. $\Rightarrow$ Zero matrix is not invertible.

$\Rightarrow$ A is not invertible.

2. $AB = 0$

$A^{-1} A \cdot B = A^{-1} \cdot 0$

$I \cdot B = 0 \Rightarrow$ B isn't a nonzero matrix.

$\therefore$ A couldn't be invertible.

Task 4:

Let V and W be n-dimensional vector spaces, and let $T: V \to W$ be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.

Target: Prove T is an isomorphism $\Longleftrightarrow$ $T(\beta)$ is a basis of W.

($\Rightarrow$) ① Prove $T(\beta)$ is linearly independent.

$\because \beta = \{v_1, v_2, \cdots v_n\}$ is a basis of V, β is linearly independent.

$\Rightarrow c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$

T is one-to-one $\Rightarrow T(v) = 0 \Rightarrow v = 0$

Let $c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n) = 0 \Rightarrow T(c_1 v_1 + c_2 v_2 + \cdots c_n v_n) = 0$

$\because T$ is isomorphic, $c_1 v_1 + c_2 v_2 + \cdots c_n v_n = 0$

and $\because \beta$ is linearly independent, $c_1 = c_2 = \cdots = c_n = 0$

$\therefore T(\beta)$ is linearly independent.

② Prove $T(\beta)$ spans W.

$\because T$ is onto, range$(T) = W$

$T(\beta)$ has n linearly independent vectors that span W. $\Rightarrow$ span$(T(\beta)) = W$

$\therefore T(\beta)$ is a basis for W.

($\Leftarrow$) ① Prove one-to-one.

Suppose $T(v) = T(w)$

$T(v) - T(w) = 0$, $T(v - w) = 0$ $\Rightarrow$ $\because T(\beta)$ is linearly independent, $v - w = 0 \Rightarrow v = w$

$\therefore N(T) = \{0\} \Rightarrow T$ is one-to-one

② Prove onto.

$\because T(\beta)$ is a basis for W, span$(T(\beta)) = W$

$\Rightarrow$ range$(T) = W$

$\therefore T$ is onto.

$\Rightarrow T$ is one-to-one & onto

$\therefore T$ is an isomorphism.