# Classical Harmonic oscillator

## 1-D oscillator

The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

The Hamiltonian equations are

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m},\ \dot{p} = -\frac{\partial H}{\partial x} = -kx$$
$$m\ddot{x} = -kx$$

set $\omega^2 = \dfrac{k}{m}$, we got the equation of motion

$$\ddot{x} = -\omega^2 x$$
$$x = A\sin(\omega t)$$
$$p = m\dot{x} = m\omega A\cos(\omega t)$$

The energy is

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2 = \frac{m}{2}\omega^2 A^2\cos^2(\omega t) + \frac{k}{2}A^2\sin^2(\omega t) = \frac{k}{2}A^2$$

# Quantum Harmonic oscillator

## 1-D oscillator

The Hamiltonian operator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2 = \frac{\hat{p}^2}{2m} + \frac{m}{2}\omega^2\hat{x}^2$$

The time-independent Schrodinger equation is

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

Method 1:

$$\left\langle x\left|\frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2\right|\psi\right\rangle = E\langle x|\psi\rangle$$
$$\frac{1}{2m}\partial_x^2\psi(x) + \frac{k}{2}x^2\psi(x) = E\psi(x)$$

Solve the equation above by assuming $\psi(x) = e^{-x^2/2b^2}$.

Method 2:
Change variables via

$$\tilde{x} = \sqrt{m\omega} \cdot x$$
$$\tilde{p} = \frac{p}{\sqrt{m\omega}}$$
$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m}{2}\omega^2\hat{x}^2 = \frac{\omega}{2}\left(\tilde{p}^2 + \tilde{x}^2\right)$$

Define ladder operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\tilde{x} + i\tilde{p}\right)$$
$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\tilde{x} - i\tilde{p}\right)$$

Then we can replace the original operators by

$$\tilde{x} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right)$$
$$\tilde{p} = \frac{1}{\sqrt{2}i}\left(\hat{a} - \hat{a}^\dagger\right)$$
$$\hat{H} = \frac{\omega}{2}\left(\tilde{p}^2 + \tilde{x}^2\right) = \frac{\omega}{4}\left(\left(\hat{a} + \hat{a}^\dagger\right)^2 - \left(\hat{a} - \hat{a}^\dagger\right)^2\right) = \frac{\omega}{2}\left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right)$$

So,

$$\frac{\omega}{2}\left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right)|\psi\rangle = E|\psi\rangle$$

Considering $\left[\hat{a}, \hat{a}^\dagger\right] = 1$, we can simplify the equation above as

$$\frac{\omega}{2}\left(2\hat{a}^\dagger\hat{a} + 1\right)|\psi\rangle = \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|\psi\rangle = E|\psi\rangle$$

Set $\hat{N} = \hat{a}^\dagger\hat{a}$, we got

$$\omega\left(\frac{1}{2} + \hat{N}\right)|\psi\rangle = E|\psi\rangle$$

The eigenstate wavefunctions of quantum harmonic oscillator are kind of complicated.