Classical Mechanics

Calculus of variations

$$S = \int_{x_1}^{x_2} dx\ f[y(x), y'(x), x],\ y(x_1) = y_1,\ y(x_2) = y_2$$

find the $y_0(x)$ to minimize S.

Set $y(x) = y_0 + \epsilon\eta(x),\ \epsilon \to 0$, to make $dS = 0$, we can get the E-L equation.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

E-L equation for unconstrained motion

We have the action $S = \int dt\ L$ and $L = T - V$.

Generalized force: $\frac{\partial L}{\partial q_i} = F_i$

Generalized momentum: $\frac{\partial L}{\partial \dot{q}_i} = P_i$

E-L equation: $\frac{d}{dt}P_i = F_i$

E-L equation for constrained motion directly

$L = T - U$ where U contains only non-constraint forces, is stationary at true path with respect to all path that satisfy the constrain.

Oct. 11: three examples

Noether's Theorem

Consider a system of particles with pair-like interactions, the Lagrangian is

$$L = \frac{1}{2}\sum_i m_i \dot{x}_i^2 + \sum_{i>j} V(x_i - x_j)$$

Let $x_i' = x_i + a$, where $a \to 0$ is a constant, then we can rewrite L with x_i' and $\dot{x}_i' = \dot{x}_i$.

$$L' = \frac{1}{2}\sum_i m_i (\dot{x}_i')^2 + \sum_{i>j} V(x_i' - x_j')$$

So, L' has the same form as L. The Lagrangian is "form invariant" (covariant) under the transformation.

Conservation of Angular Momentum

The angular momentum $\vec{A} = \vec{r} \times \dot{\vec{r}}$ is conserved, the plane of $\vec{r}$ and $\dot{\vec{r}}$ must be the same plane.

For ϕ, we will get the conservation of angular momentum. For r, replace $\dot{\phi}$ with A, we will get

$$\mu\ddot{r} = -\frac{d}{dr}U_{eff}(r)$$
$$\mu\dot{r}\cdot\ddot{r} = -\frac{d}{dr}U_{eff}(r)\cdot\dot{r}$$
$$\mu\frac{d}{dt}\left(\frac{\dot{r}^2}{2}\right) = -\frac{d}{dt}U_{eff}(r)$$
$$\mu\frac{\dot{r}^2}{2} + U_{eff}(r) = E = const$$
$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U_{eff})}$$

Hamiltonian Dynamics

We define the Hamiltonian: $H = \sum_i p_i\dot{q}_i - L$, then we have

$$\frac{\partial H}{\partial p_i} = \dot{q}_i\ ;\ \frac{\partial H}{\partial q_i} = -\dot{p}_i$$

Liouville's Theorem

Consider a region in phase space and follow its evolution over time. In general, the shape of the region will change, but the volume remains the same.

Liouville's Equation

With $\frac{d\rho}{dt} = 0$, we have

$$\frac{\partial\rho}{\partial t} + \{\rho, H\} = 0$$

Poincare Recurance Theorem

Poisson Bracket

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

properties:

- $\{f, g\} = -\{g, f\}$
- $\{af + bg, h\} = a\{f, h\} + b\{g, h\}$
- $\{fg, h\} = f\{g, h\} + \{f, h\}g$
- Jacobi Identity $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$

- $\dfrac{df}{dt} = \{f, H\} + \dfrac{\partial f}{\partial t}$
- $\{q_i, q_j\} = \{p_i, p_j\} = 0$ and $\{q_i, p_j\} = \delta_{ij}$

Canonical Transformation

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \ ; \ \frac{\partial H}{\partial q_i} = -\dot{p}_i$$
$$q_i \to Q_i(q, p, t)$$
$$p_i \to P_i(q, p, t)$$
$$H(p, q, t) \to K(P, Q, t)$$

If the transformation preserve

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i \ ; \ \frac{\partial K}{\partial Q_i} = -\dot{P}_i$$

we call it canonical transformation.

Restricted Canonical Transformation

$$H(p, q) = K(P, Q)$$
$$\{Q_i, Q_j\} = \{P_i, P_j\} = 0 \, and \, \{Q_i, P_i\} = \delta_{ij}$$

Theorem: The Poisson Bracket is invariant under canonical transform.

Action-Angle variables

Canonical transform: $(p, q) \to (I, \theta)$ [respectively]

So that to make $H = H(I)$, then we have

$$\dot{\theta} = \frac{\partial H}{\partial I} \text{ and } I \text{ are constants}$$

This choice is always possible for any 1-D system, I is called the "action variable", θ is "angle variable".

Claim: The correct choice of I is $I = \dfrac{1}{2\pi}\oint pdq$.

Then we have

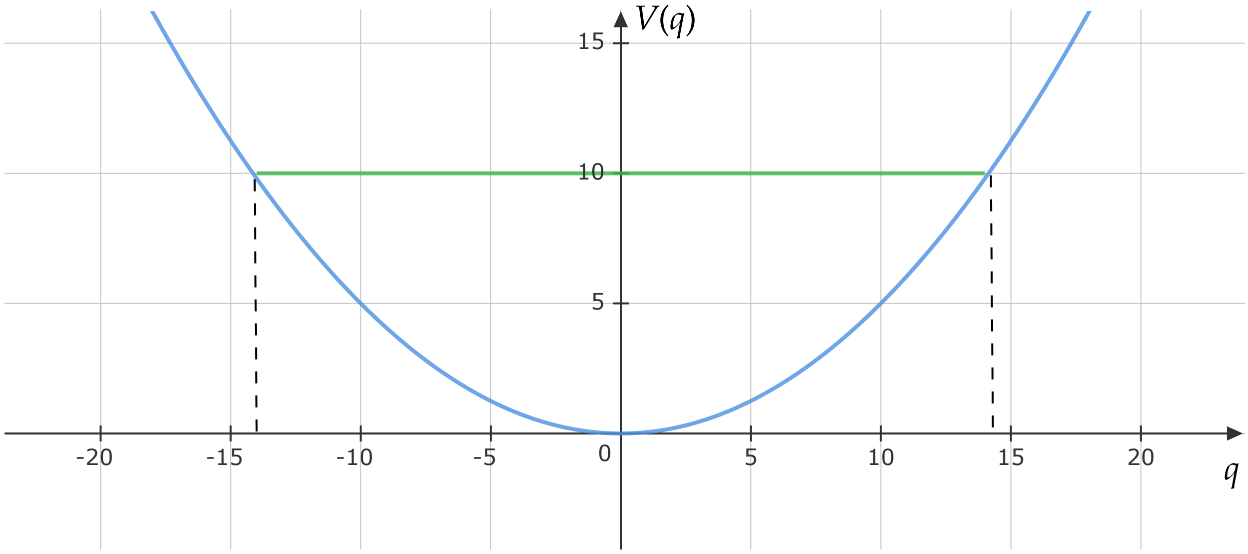
- $H = H(I) = E$
- $\dot{\theta} = \dfrac{\partial H}{\partial I} = \dfrac{dE}{dI} = \omega$
- $t = \dfrac{d}{dE}\displaystyle\int_{q(0)}^{q(t)} pdq$
- $\theta = \omega t = \dfrac{d}{dI}\displaystyle\int_{q(0)}^{q(t)} pdq$

Adiabatic Invariants

consider only 1 d.o.f.

$$H = \frac{p^2}{2m} + V(q)$$

Assume motion is bounded, so motion is periodic.



Let the potential depend on some parameter λ, so that $V = V(q, \lambda)$. We wish to explore what happens if λ changes slowly ("adiabatically") with time. For example, we may change the length of a pendulum.

Since H is time-dependent, energy is no longer conserved.

$$\frac{dE}{dt} = \left(\frac{\partial H}{\partial t}\right)_{p,q} = \left(\frac{\partial H}{\partial \lambda}\right)_{p,q} \cdot \frac{d\lambda}{dt}$$

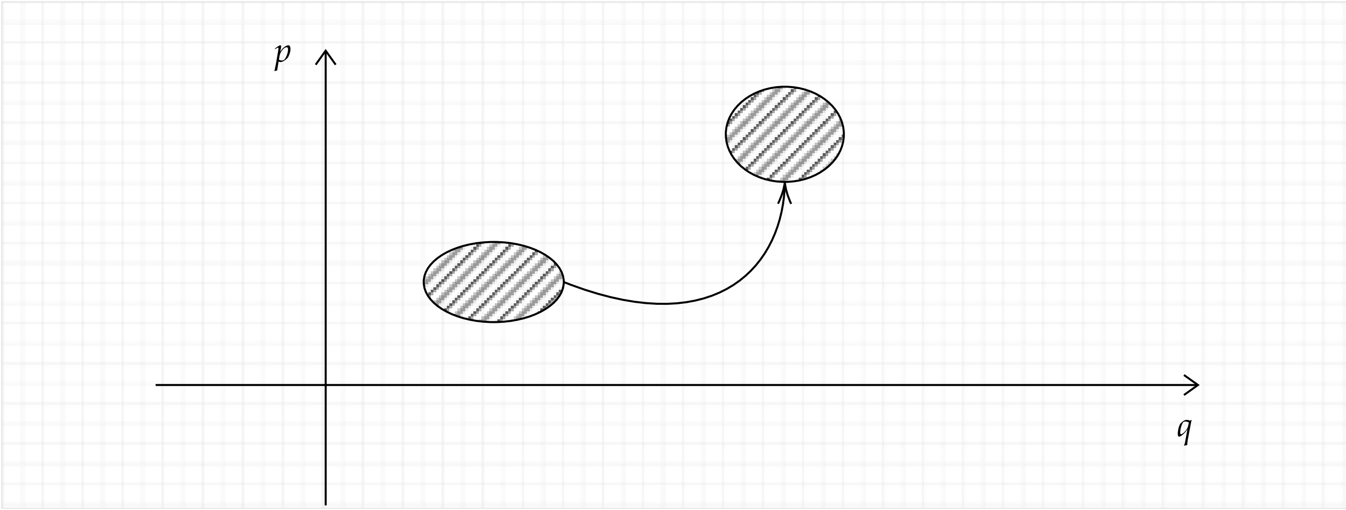
There are <u>specific combinations of E and λ which remain constant as λ is slowly changed</u>. These combinations are called "adiabatic invariants".

<u>Claim</u>: The adiabatic invariant for this system is

$$I = \frac{1}{2\pi}\oint p dq$$

where p is now $p = \sqrt{2m(\,E(t) \,-\, V(q, \lambda(t))\,)}$.

With the adiabatic assumption, the orbit in the phase sapce below can be closed.



e.g.

$$H = \frac{p^2}{2m} + \frac{1}{2}k(t)q^2,\ k\ varying\ slowly$$

We have $I = \frac{E}{\omega} = E\sqrt{\frac{m}{k}}$, with the adiabatic invariant claim, we can get

$$\frac{E(t_1)}{\sqrt{k(t_1)}} = \frac{E(t_2)}{\sqrt{k(t_2)}}$$