Classical Electrodynamics

Gauss's theorem:

$$\oint \vec{E} \cdot \vec{S} = 4\pi k Q = \frac{Q}{\epsilon_0}$$

$$\nabla \cdot \vec{E} = -\nabla^2 \phi = \frac{\rho}{\epsilon_0}$$

$$\oint \vec{E} \cdot d\vec{l} = 0$$

Uniqueness of solution to Poisson's equation:

If we impose Dirichlet or Neumann boundary conditions, the solution is unique.

- Dirichlet boundary condition: the value of ϕ is specified on the boundary
- Neumann boundary condition: $\nabla\phi \cdot \hat{n}$ is specified on the boundary

Electrostatic Energy:

$$W = \frac{1}{2}\int d^3x\ \rho(\vec{x})\phi(\vec{x}) = \frac{1}{2}\epsilon_0 \int d^3x\ |\vec{E}|^2$$

$\frac{1}{2}\epsilon_0|\vec{E}|^2$ can be identified as the energy density stored in electrostatic field.

Force on a conducting surface:

- Energy charge from a virtual displacement, the force per unit area is

$$f = \frac{F}{\Delta A} = -\frac{\Delta W/\Delta x}{\Delta A} = \frac{\alpha^2}{2\epsilon_0}$$

where α is the charge area density on the surface.

- Directly from electronic field, the force per unit area is

$$f = \frac{F}{\Delta A} = \frac{(\alpha\Delta A)\vec{E}_{external}}{\Delta A} = \frac{\alpha^2}{2\epsilon_0}$$

Usage of Green function:

We need to generalize the Green function method to address more general boundary conditions, where either ϕ or $\vec{E} \cdot \hat{n}$ are specified on the boundaries.

* If $\phi(\vec{x})$ satisfies Dirichlet boundary conditions, the value of ϕ is specified at boundary, **choose** $G_D(\vec{x}', \vec{x}) = 0$ **for all** $\vec{x}'$ **on the surface** S to get

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G_D(\vec{x}', \vec{x})\ d^3\vec{x}' - \frac{1}{4\pi}\oint_S \phi(\vec{x}')(\nabla' G_D(\vec{x}', \vec{x})\ \cdot \hat{n}')\ ds'$$

* For Neumann boundary conditions, $\nabla\phi \cdot \hat{n}$ is specified on the boundary. The situation is more complicated, because we cannot choose $\nabla' G_N(\vec{x}', \vec{x})\ \cdot \hat{n}' = 0$ for all $\vec{x}'$ on the surface S.

This choice is inconsistent because

$$\oint_S \nabla' G_N(\vec{x}', \vec{x}) \cdot \hat{n}' \, ds' = \int_V \nabla'^2 G_N(\vec{x}', \vec{x}) \, d^3\vec{x} = \int_V -4\pi\delta^3(\vec{x}' - \vec{x}) \, d^3\vec{x} = -4\pi$$

The simplest consistent choice is

$$\nabla' G_N(\vec{x}', \vec{x}) \cdot \hat{n}' = -\frac{4\pi}{S}, \text{ where S is the area of the surface}$$

for all $\vec{x}'$ on the surface S. The solution is

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{x}') G(\vec{x}', \vec{x}) \, d^3\vec{x}' + \frac{1}{4\pi} \oint_S [G(\vec{x}', \vec{x})(\nabla'\phi(\vec{x}') \cdot \hat{n}')] \, ds' + \langle\phi\rangle_S$$

where $\langle\phi\rangle_S$ is the average value of ϕ over whole surface. If one of the surface is at infinity, $\langle\phi\rangle_S$ is typically vanishes. Under the Neumann boundary condition, we have $G_N(\vec{x}', \vec{x}) = G_N(\vec{x}, \vec{x}')$.

Laplace operator:

- In rectangular coordinate

$$\nabla^2 V = \frac{\partial^2}{\partial x^2} V + \frac{\partial^2}{\partial y^2} V + \frac{\partial^2}{\partial z^2} V = 0$$

general solution:

$$V(x, y, z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_n y) \sinh(\gamma_{nm} z)$$

where $\alpha_n = \frac{n\pi}{a}$, $\beta_m = \frac{m\pi}{b}$ and $\gamma_{nm} = \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$.

- In spherical coordinate

$$\nabla^2 V = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)\right] + \frac{1}{r^2 \sin\theta} \cdot \frac{\partial}{\partial \theta}\left(\sin\theta \cdot \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

<u>general solution:</u>

$$V(r, \theta, \phi) = \sum_{l=0} \sum_{m=-l}^{+l} \left(A_{lm} r^l + B_{lm} r^{-(l+1)}\right) Y_{lm}(\theta, \phi)$$

general solution with azimuthal symmetry $m = 0$:

$$V(r, \theta) = \sum_{l=0} \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} \left(x^2 - 1\right)^l$$

$$\frac{d}{dx}(P_{l+1} - P_{l-1}) = 2(l+1) P_l$$

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}$$

- In cylindrical coordinates

$$\nabla^2 V = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)\right] + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial\phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0$$

- In 2-D

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} = 0$$

general solution

$$V(\rho,\phi) = R(\rho)\Psi(\phi)$$
$$R(\rho) = a\rho^{\nu} + b\rho^{-\nu}$$
$$\Psi(\phi) = A\cos(\nu\phi) + B\sin(\nu\phi)$$

general solution for full azimuthal range:

$$V(\rho,\phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n\rho^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} b_n\rho^{-n}\sin(n\phi + \beta_n)$$

general solution for $\phi \in [0,\beta]$ with $V(\phi = 0) = V(\phi = \beta)$:

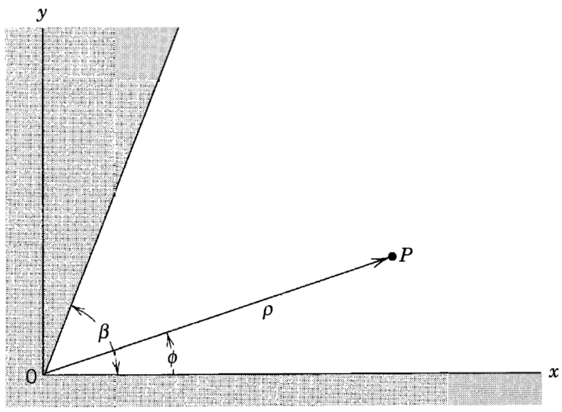

Figure 2.12 Intersection of two conducting planes defining a corner in two dimensions with opening angle β.

$$V(\rho,\phi) = V(\phi = 0) + \sum_{m=1} \left(a_m\rho^{m\pi/\beta} + b_m\rho^{-m\pi/\beta}\right)\sin(m\pi\phi/\beta)$$

Orthogonality relation:

$$\int_0^{2\pi} \sin(n\phi)\cdot\sin(m\phi)\,d\phi = \int_0^{2\pi} \cos(n\phi)\cdot\cos(m\phi)\,d\phi = \pi\delta_{m,n}$$
$$\int_0^{2\pi} \sin(n\phi)\cdot\cos(m\phi)\,d\phi = 0$$
$$\int_{-1}^{1} dx\, P_l(x)P_m(x) = \frac{2}{2l+1}\delta_{lm}$$

Useful expansion:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

where $r_<$ and $r_>$ are smaller one and larger one between $|\vec{x}|$ and $|\vec{x}'|$, γ is the angle between $\vec{x}$ and $\vec{x}'$.

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi)$$

$$\boxed{\left.\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \tfrac{1}{2}(3x^2 - 1) \\ P_3(x) &= \tfrac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \tfrac{1}{8}(35x^4 - 30x^2 + 3) \end{aligned}\right\}}$$