CSCI 4140
Linear Algebra
Part One

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Introduction

- The overview hints that we will be using vectors and matrices
- There is a certain amount of notation and terminology that we should review
- In particular, the notation for vectors is taken from quantum mechanics, which is different from what you are used to
- I recommend that you quickly go through this material the first time and to familiarize yourself with the ideas
- You can then use it as a reference in the rest of the course

Vectors

- We will use the bra-ket notation for vectors, due to Dirac a prominent theoretical physicist
- We've already seen kets, they are column vectors, bras are row vectors:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

- Now we turn to the inner product, or dot product
- For two vectors a and b we write the inner product as:

$$\langle a|b\rangle$$

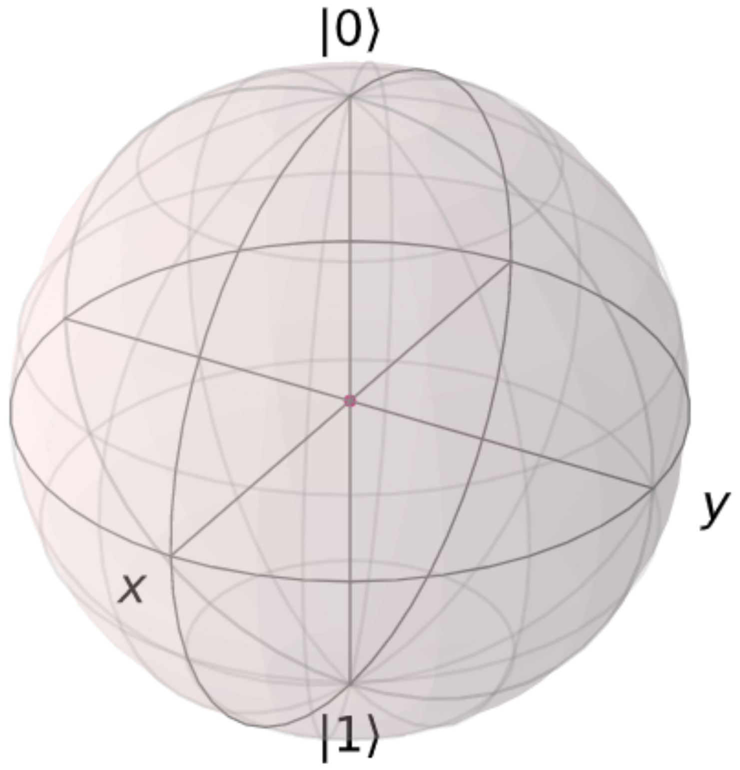
Vectors

- We can view this as a row vector times a column vector, but there is one issue
- For any vector a, we would like $\langle a|a\rangle$ to be a real number, the square of the length of the vector
- But, we are dealing with complex numbers, so we need to be careful, we need to take the complex conjugate of the first vector:
$$\langle a|b\rangle = a_1^* b_1 + a_2^* b_2$$
- This gives us a real result

Vectors

- For a qubit q we have $\langle q|q\rangle$=1, and we know that qubits are 2D vectors
- Since we have two components we have the making of spherical coordinates
- Since $\langle q|q\rangle$=1 the radius of the sphere is 1, the two components given the location on the surface of the sphere
- This is called the Bloch sphere and we will find it handy when describing the operation of 1 qubit gates

Bloch Sphere



Vector Products

- There are two other vector products that are of interest
- The outer product of two vectors is a matrix, in the case of 2D vectors it is a 2x2 matrix
- We construct this matrix by multiplying each element of the first vector by each element of the second vector
- For two vectors a and b we write the outer product as $|a\rangle\langle b|$
- Note that this is just the reverse of the inner product

Vector Products

- For the two vectors:

$$|a\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad |b\rangle = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

- The outer product is given by:

$$|a\rangle\langle b| = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}$$

- This generalizes to nD vectors in the obvious way, and the two vectors don’t need to be the same size

Vector Products

- In terms of more common vector notation we have:

$$\langle a|b\rangle = a^* b$$
$$|a\rangle\langle b| = ab^*$$

- The final product that we need is the tensor product
- So what's a tensor?
- A tensor is a generalization of vectors and matrices to an arbitrary number of dimensions
- An order 0 tensor is a scalar, an order 1 tensor is a vector, an order 2 tensor is a matrix, etc.

Vector Products

- Why do we care?
  - It unifies some of the mathematics, can have 3D tensors, etc.
  - We need them for combining qubits
- We can view the tensor product as a generalization of the outer product in the case of vectors:

$$a \otimes b = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}$$

- This is basically a column vector times a row vector

Vector Products

- Now let's see what happens when we take the tensor product of two column vectors:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}$$

- If I have two qubits and I want to combine them into a single state, I just need to take their tensor product
- This is how we build up multi-qubit systems

Vector Products

- To see how this works consider two qubits, one in state $|0\rangle$ and the other in state $|1\rangle$
- The combined state is:

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

- We can do the same thing for the other three state combinations