CSCI 4140
Complex Numbers

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Introduction

- The state coefficients in qubits are complex numbers, so we need to know a bit about them
- We won’t be doing any real math with complex numbers, but we do need to know some of their properties
- We will see complex numbers in programming quantum computer, so we need to be familiar with them

Complex Numbers

- A complex number, c, has a real and an imaginary part, written as:

$$c = a + ib$$

- Where both a and b are real numbers and $i = \sqrt{-1}$
- Sometime we will find j used instead of i, we will see this in Qiskit
- We also have the following:

$$Re(c) = a$$

$$Im(c) = b$$

Complex Numbers

- Now to explain something from the overview, we had the following expression:

$$|\alpha|^2 + |\beta|^2 = 1$$

- Why do I need the absolute value if I'm squaring the numbers, shouldn't the result be a positive real number?
- No, it's not, going back to our example number c we have:

$$c^2 = (a+ib)^2 = (a+ib)(a+ib) = a^2-b^2+2iab$$

- In general this is a complex number, not at all what we want

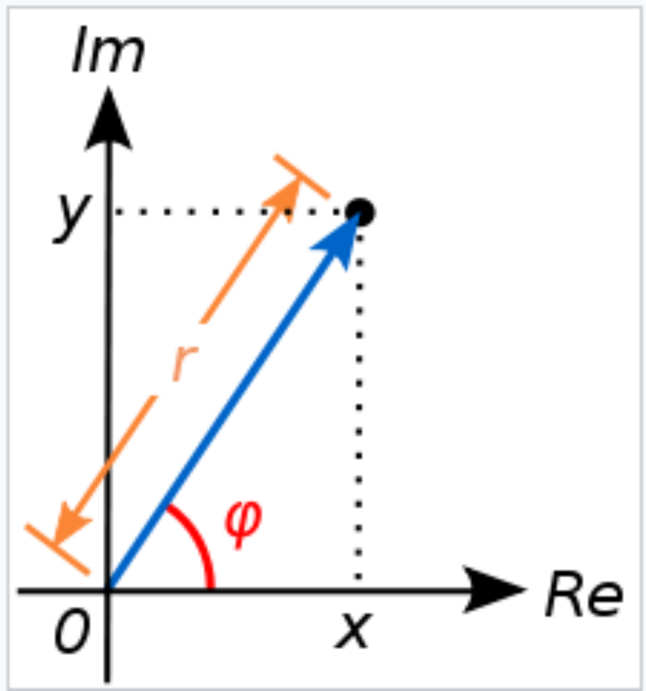
Complex Numbers

- Now lets try the following:

$$(a+ib)(a-ib) = a^2+b^2$$

- The square root of this is called the modulus, or absolute value
- If c =a+ib, we call a-ib the complex conjugate of c and write it as $\bar{c}$ or c^*
- By taking the modulus of a complex number, we end up with a real number
- Thus, our expression for qubit probabilities works out

Polar Representation



- Since a complex number has two real number, we can plot it in 2D space
- X axis for real component and Y axis for imaginary
- The length r is the modulus
- φ is the argument and is given by:

$$\varphi = \mathrm{atan2}(\mathrm{Im}(c), \mathrm{Re}(c))$$

Polar Representation

- Given this we can now write our complex number in the following way:
$$c = r(\cos(\varphi) + isin(\varphi))$$
- Using Euler's formula we can rewrite this as:
$$c = re^{i\varphi}$$
- We can easily go back and forth between the two representations
- We will often use Euler's formula to reduce the amount of writing we need to do
- Note: in our case r=1, making this even more compact

Complex Arithmetic

- Addition, subtraction and multiplication are all easy:

  $(a+ib)+(x+iy) = (a+x) + i(b+y)$

  $(a+ib)-(x+iy) = (a-x) + i(b-y)$

  $(a+ib)*(x+iy) = ax-by+ibx+iay$

- It's division that gets difficult, the easiest way to approach it is using polar representation
- Consider a complex number, x, it must satisfy the following:

  $x*x^{-1} = 1$

Complex Arithmetic

- Let's put this in polar form:

$$re^{i\varphi}\acute{r}e^{i\psi}=1$$

- Now e^0=1, so if $\psi = -\varphi$, we have the following:

$$r\acute{r}=1$$

- Therefore, we have the following:

$$x^{-1} = \frac{1}{r}e^{-\varphi}$$

- For division we just need to multiply by the inverse

Matrix Exponentials

- Occasionally we will come across something that looks like the following, where M is a matrix:

$$U = e^{i\gamma M}$$

- This is a matrix, but we need an expression for this matrix
- Recall the Taylor series for a function f(x) expanded about x_0

$$g(x) = \sum_{n=0}^{\infty} f^{(n)}(x_0) \frac{(x - x_0)^n}{n!}$$

- Where $f^{(n)}()$ is the n^{th} derivative

Matrix Exponentials

- If we take x_0=0 and remember that the derivative of e^x is e^x and e^0=1 we get the following:

$$g(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

- Going back to our original matrix expression we have:

$$e^{i\gamma M} = \sum_{n=0}^{\infty} \frac{(i\gamma M)^n}{n!}$$

Matrix Exponentials

- Most of the entries in M will be less than 1, so raising it to a power will produce even smaller entries
- Also n! grows very quickly
- In our case the first few terms in this sum will be a good approximation, since the terms quickly go to zero

Matrix Exponentials

- One other trick, if B is an involuntary matrix ($B^2 = I$) then we have the following:

$$e^{i\gamma B} = \cos(\gamma)\, I + isin(\gamma)B$$

- It turns out that many of the matrices that we will use are involuntary, so this is a useful relationship