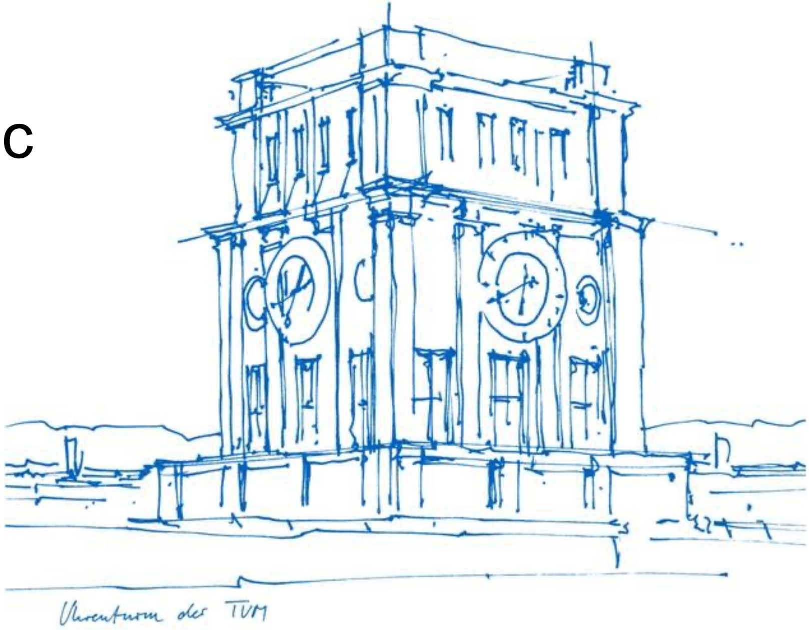
Seminar: Computational Methods for Image Reconstruction

Chapter 3: Analytical Tomographic Image Reconstruction Methods:



Wise 2023/24

Presenter: Boey Kai Zhe

Topics

X-ray CT Geometries

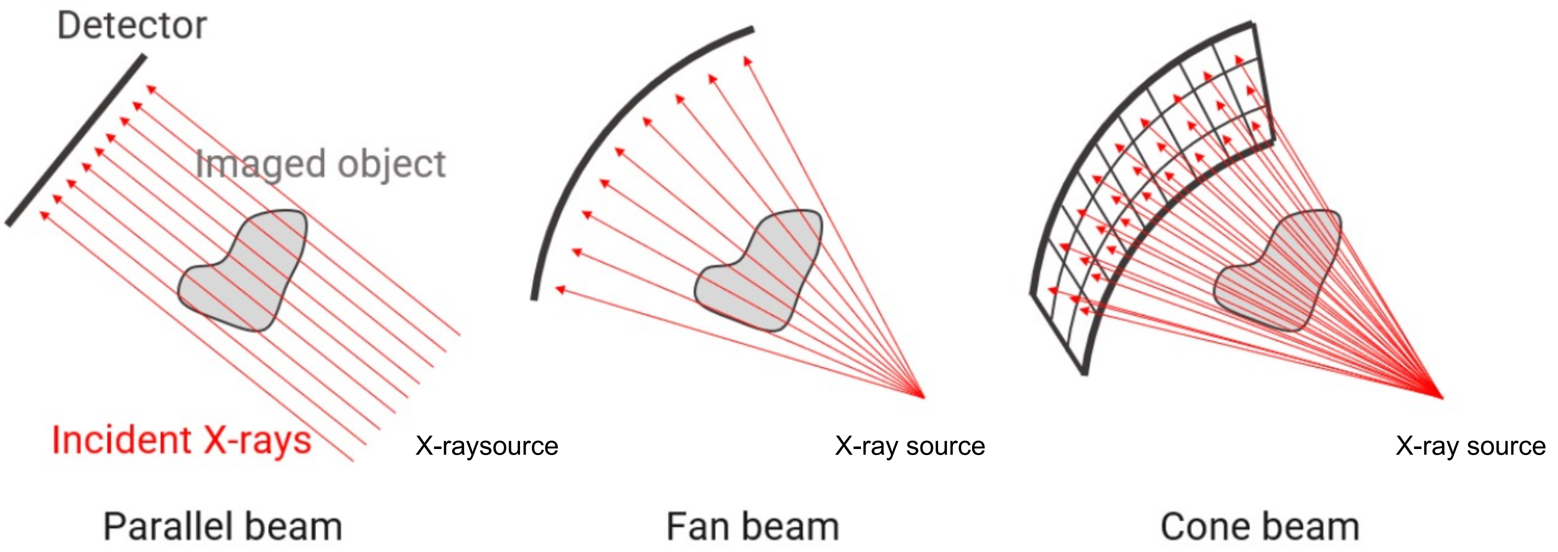
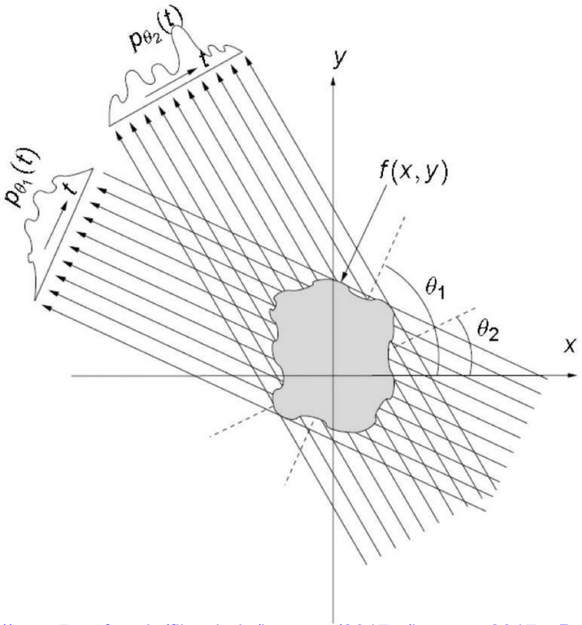


Image: https://imaging.rigaku.com/blog/how-does-ct-reconstruction-work

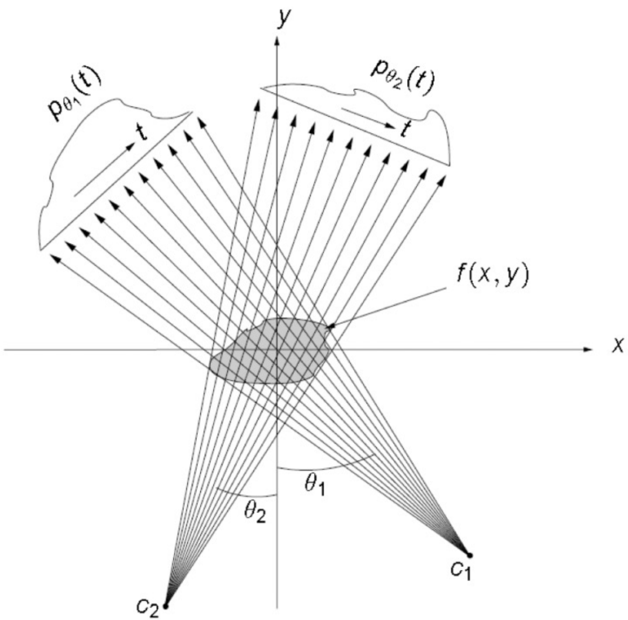
Parallel Beam Geometry



- 1^{st} Generation CT
- Year developed: 1971
- Anatomy: Head only
- Principle: Translate – Rotate (Pencil beams were translated to cover the object from each view and then rotated)
- Time to acquire 1 image: ~5 min
- Slow, not practical!!

Image: https://www5.cs.fau.de/fileadmin/lectures/2017w/Lecture.2017w.ProjFCR/fanBeamRecon.pdf

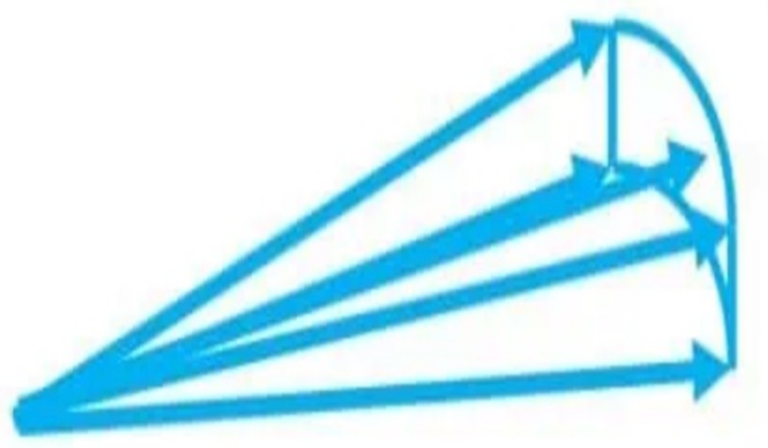
Fan Beam Geometry (Topic of presentation)



- 3rd Generation CT
- Year developed: 1975
- Anatomy: All anatomy
- Principle: Rotate – Rotate (source and detector rotated together)
- Time to acquire 1 image: 1 sec
- Fast, widely use today.

Image: https://www5.cs.fau.de/fileadmin/lectures/2017w/Lecture.2017w.ProjFCR/fanBeamRecon.pdf

Cone Beam Geometry (Topic of presentation)



- Very similar in geometry to fan beam CT.
- Difference in the detector where it has more rows.
- Even fewer rotations needed than fan-beam

Image: https://www5.cs.fau.de/fileadmin/lectures/2017w/Lecture.2017w.ProjFCR/fanBeamRecon.pdf

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Topics

Properties of Fan Beam Geometry

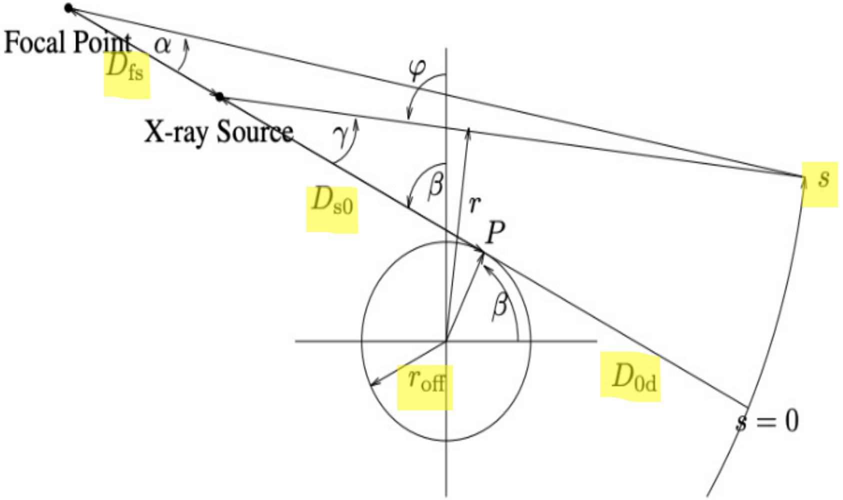


Image: J. Fessler. Image reconstruction: Algorithms and analysis. [5]

Distance parameters

- Dfs - the distance from the focal point f to the X-ray source s;
- Dso - distance from the X-ray source s to P
- Dod - distance from p to the center of detector d
- Dsd = Dod + Dso (total distance from the X-ray source to the center of the detector)
- s∈ [−smax, smax] - arc length along the detector
- roff – mismatch distance between isocenter and Dsd

2 Cases of Fan Beam Geometry

Image: https://www5.cs.fau.de/fileadmin/lectures/2017w/Lecture.2017w.ProjFCR/fanBeamRecon.pdf

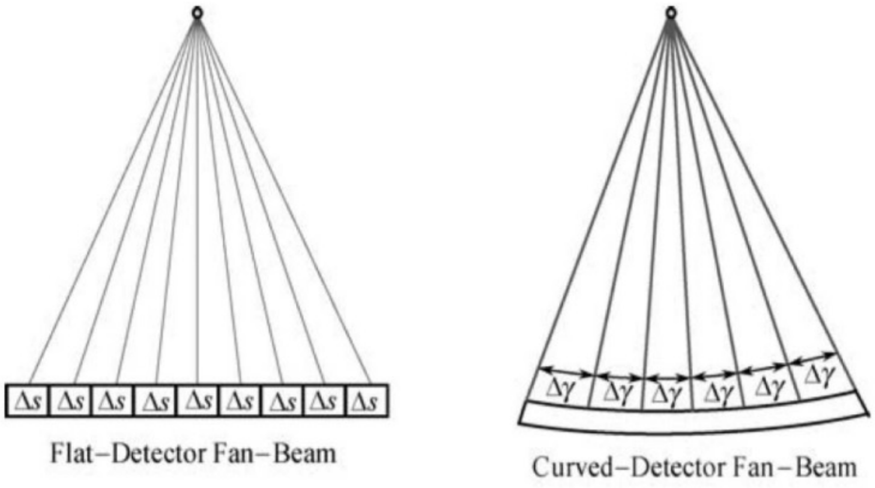


Image: Zeng, 2009 [4]

Equidistant case

- Flat detector
- Dfs = ∞

Equiangular case

- Curve detector
- Dfs = 0

Properties of Fan Beam Geometry (Continue)

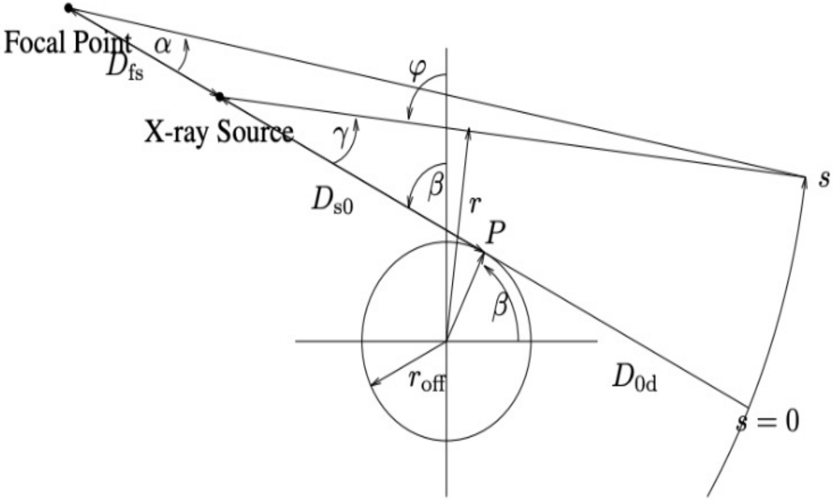


Image: J. Fessler. Image reconstruction: Algorithms and analysis. [5]

Angular parameters

- β - angle that the line segment between the X-ray source and the detector center makes with the y axis.

$$\gamma(s) = \begin{cases} \frac{s}{D_{sd}}, & \text{if } D_{fs} = 0 \text{ (equiangular)} \\ \arctan(\frac{s}{D_{sd}}), & \text{if } D_{fs} = \infty \text{ (equidistant)} \end{cases} \qquad (1)$$

$$s(\gamma) = \gamma^{-1}(s) = \begin{cases} D_{sd}\gamma, & \text{if } D_{fs} = 0 \text{ (equiangular)} \\ D_{fd}[\gamma - \arcsin(\frac{D_{fs}}{D_{fd}} \sin\gamma)], & \text{if } 0 \leqslant D_{fs} \leqslant \infty \\ D_{sd} \tan\gamma, & \text{if } D_{fs} = \infty \text{ (equidistant)} \end{cases} \qquad (2)$$

Forward projection

<u>More definations:</u>

- L(s, β) = {(x, y) : x cos ϕ(s, β) + y sin ϕ(s, β) = r(s)} - the ray corresponding to angle β and detector element s;
- γ_{max} = γ(s_{max}) and s_{max} is half of the total arc length of the detector.
- r_{max} - field of view
- $2\gamma_{max}$ - fan angle
- |r(s)| ⩽ r_{max} = Dso sin γ_{max}

With some derivations, we obtain the forward projection:

$$p(s, \beta) = \int_{L(s,\beta)} f(x, y)\, d\ell = \iint f(x, y)\delta(x \cos \phi(s.\beta) + y \sin \phi(s.\beta) - r(s))\, dx\, dy \quad (3)$$

Goal: reconstruct f(x,y) by estimating its absorption coefficient

Filter-backproject (FBP) for 360° scan

Simple derivation:

- Begin with the definition of the Parallel-beam fbp:

$$f(x,y) = \frac{1}{2}\int_0^{2\pi}\int p_\phi(r)\, h_*(x\cos\phi(s.\beta) + y\sin\phi(s.\beta) - r)\, dr\, d\phi \qquad (5)$$

,where h∗(.) denotes the ramp filter

- Change of variable r=r(s) & φ = φ(s, β), jacobian of the determinant: $J(s) \triangleq |D_{s0}\cos\gamma(s) - r_{\text{off}}\sin\gamma(s)|\,|\dot{\gamma}(s)|\,.$

- Applying change of variable to (5), we get:

$$f(x,y) = \frac{1}{2}\int_0^{2\pi}\int w_{2\pi}(x,y;\phi(s,\beta))\, p(s,\beta)\, h_*(x\cos\phi(s.\beta) + y\sin\phi(s.\beta) - r(s)) J(s)\, ds\, d\beta \qquad (6)$$

- Given x_β = x cos β + y sin β and y_β = −x sin β + y cos β, using trigonometric identities, the following relationship are obtained:

Continuation:

$$L_\beta(x,y) = \sqrt{(D_{s0} - y_\beta)^2 + (x_\beta - r_{off})^2} \quad (7)$$

$$\gamma_\beta(x,y) = \arctan\left(\frac{x_\beta - r_{off}}{D_{s0} - y_\beta}\right) \quad (8)$$

$$s_\beta(x,y) = \begin{cases} D_{sd}\gamma_\beta(x,y), & \text{if } D_{fs} = 0 \text{ (equiangular)} \\ D_{sd}\tan\gamma_\beta(x,y), & \text{if } D_{fs} = \infty \text{ (equidistant)} \end{cases} \quad (9)$$

- Substituting equation (7), (8) and (9) into (6) and applying some scaling property and algebraic manipulation, the formula for fan beam reconstruction can be simplified as:

$$f(x,y) = \int_0^{2\pi} \frac{1}{W_2^2(x,y,\beta)L_\beta^2(x,y)} \left[\int p(s,\beta)\frac{w_{2\pi(s,\beta)J()s}}{W_1^2(s)} g_*(s_\beta(x,y) - s)ds\right] d\beta \quad (10)$$

★

- W1 and W2 are some the weight function and g∗(s) is the ramp filter, which are different for both equiangular and equidistant case.

Outline of the FBP formula

- Step 1: Compute weighted projections for each β

$$\overline{p}(s,\beta) = p(s,\beta)\frac{w_{2\pi}(s,\beta)J(s)}{W_1^2(s)} \qquad (11)$$

- Step 2: Filter those weighted projections (along s) for each β using the modified ramp filter

$$\overline{\overline{p}}(s,\beta) = \overline{p}(s,\beta) * g_s(s) \qquad (12)$$

- *Step 3:* Perform a weighted backprojection of those filtered projections

$$f(x,y) = \int_0^{2\pi} \frac{1}{W_2^2(x,y,\beta)L_\beta^2(x,y)}\overline{\overline{p}}(s_\beta(x,y),\beta)d\beta \qquad (13)$$

Equiangular case/ curved detector

- Projection weighting ratio in equation (11)

$$\frac{J(s)}{W_1^2(s)} = J(s) = \frac{1}{D_{sd}}|D_{so}\cos\frac{s}{D_{sd}} - r_{off}\sin\frac{s}{D_{sd}} \approx \frac{D_{so}}{D_{sd}}\cos\frac{s}{D_{sd}} \qquad (15)$$

- Modified ramp filter in equation (12), where $\triangle s$ denotes the detector element spacing:

$$h[n] = \begin{cases} \frac{1}{4\triangle_s^2}, & n = 0 \\ 0, & n \text{ even} \\ \frac{-1}{[\pi D_{sd}\sin(n\triangle_s/D_{sd})]^2}, & n \text{ odd} \end{cases} \qquad (16)$$

- Backprojection weighting ratio in equation (13)

$$\frac{1}{W_2^2 L_\beta^2(x,y)} = \frac{D_{sd}^2}{L_\beta^2(x,y)} = \frac{D_{sd}^2}{(D_{s0} - y_\beta)^2 + (x_\beta - r_{off})^2} \qquad (17)$$

Equidistant case/ flat detector

- Projection weighting ratio in equation (11), reminder γ(s) = arctan(s/ Dsd)

$$w_{2\pi}(s,\beta)\frac{J(s)}{W_1^2(s)} = \frac{w_{2\pi}(s,\beta)}{D_{sd}}|D_{so}\cos\gamma(s) - r_{off}\sin\gamma(s) \approx w_{2\pi}(s,\beta)\frac{D_{so}}{\sqrt{D_{sd}^2+s^2}} \tag{19}$$

- Ordinary ramp filter in equation (12)：

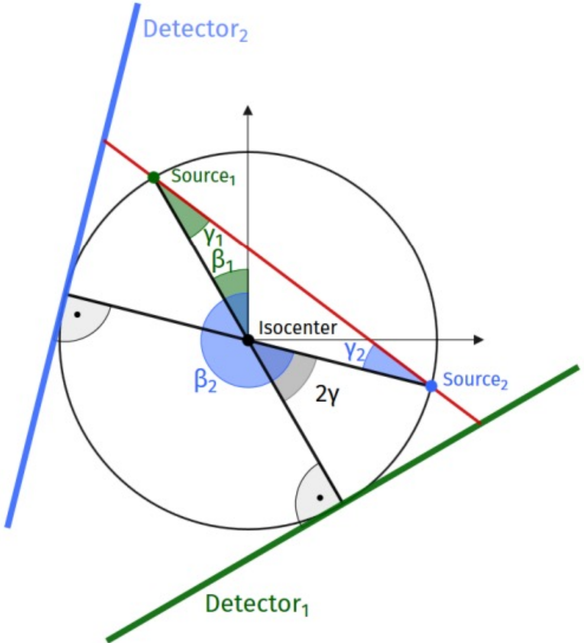
$$h[n] = \begin{cases} 1, & n = 0 \\ 0, & n \text{ even} \\ \frac{-1}{(\pi n/2)^2}, & n \text{ odd} \end{cases} \tag{20}$$

Demo !!

- Backprojection weighting ratio in equation (13)

$$\frac{1}{W_2^2 L_\beta^2(x,y)} = \frac{D_{sd}^2}{(\cos\gamma_\beta(x,y)L_\beta(x,y))^2} = \frac{D_{sd}^2}{(D_{s0}-y_\beta)^2} \tag{21}$$

FBP for short scans (<360°)



- For a 360◦ rotation, every ray is sampled exactly twice.
- Given 2 identical rays (β1, γ1) and (β2, γ2) are identical:

  $\gamma_2 = -\gamma_1$

  $\beta_2 = \beta_1 + 2\gamma_1 + \pi$

- Short scan setting: $\beta_{max} = \pi + 2\gamma_{max}$, every point in the equivalent parallel-beam sinogram is sampled at least once.

Image: https://www5.cs.fau.de/fileadmin/lectures/2017w/Lecture.2017w.ProjFCR/fanBeamRecon.pdf

Solution: Parker's weighting

- Replace Parker's weighting with regular projection weighting in (11)
- Parker's weighting:

$$w_{2\pi}(s,\beta) = \begin{cases} \boxed{q\left(\frac{\beta}{2(\gamma_{max}-\gamma)}\right), \quad 0 \leqslant \beta \leqslant 2(\gamma_{max}-\gamma)} \\ 1, \quad 2(\gamma_{max}-\gamma) < \beta < \pi - 2\gamma \\ \boxed{q\left(\frac{\pi+2\gamma_{max}-\beta}{2(\gamma_{max}+\gamma)}\right), \quad \pi - 2\gamma \leqslant \beta \leqslant \pi + 2\gamma_{max}} \end{cases} \quad (22)$$

Identical rays

- q(x) = $sin^2(\frac{\pi}{2}x)$

Demo !!

Backproject-filter (BPF) approach

- Given fan-beam projection p(s, β), β∈[0, β_{max}], weighting function $w_{bpf}(s,\beta)$, the weighted backprojection

$$b(x,y) == \int_{-s_{max}}^{s_{max}} \int_{0}^{\beta_{max}} \delta(x\cos\phi(s.\beta) + y\sin\phi(s.\beta) - r(s)) w_{BPF}(s,\beta)\, ds\, d\beta \quad (23)$$

- Algorithm
  - Step 1. Backproject the projection measurements
  - Step 2. Take the two-dimensional FFT of the backprojection image.
  - Step 3. Apply the ramp filter to the Fourier transformed image.
  - Step 4. Take the inverse two-dimensional FFT of the filtered image to obtain the reconstruction

Demo !!

Fan-parallel rebinning

- Transform the fan-beam projections into parallel-beam projections by finding equal rays in both geometries
- Idea, express p(s, β) in terms of parallel beam projections.
- How? Use change of variable (i) $\phi = \beta + \gamma$ and (ii) r = Dso sinγ, we can write:

$$p_\phi(r) = p(s,\beta)|_{s=s(r),\beta=\beta(r,\phi)} = p(\gamma^{-1}(\arcsin r/D_{so}, \phi - \arcsin r/D_{so})) \quad (4)$$

- Perform 2 interpolation steps:
  (i) angular interpolation - 1D interpolation along the source position using the relationship $\phi = \beta + \gamma$
  (ii) radial interpolation - 1D interpolation along the detector using the relationship
  r = Dso sin γ

- Inaccurate due to to interpolation, hence, choose FBP if possible

Demo !!

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Topics

Cone Beam Geometry

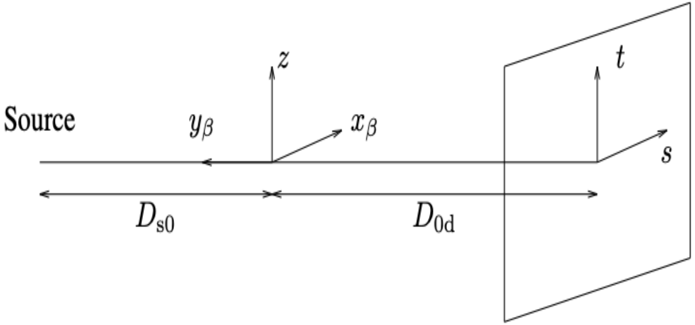

Image: J. Fessler. Image reconstruction: Algorithms and analysis. [5]

- β - angle of the source point counter-clockwise from the y axis
- (x_β, y_β) – rotated coordinates.
- Extra height parameter **t** in the detector.
- Other parameters are similar to fan-beam geometry described previously.
- Here, we considers only circular X-ray source trajectory (axial scan)
- Reconstruction algorithm used: Feldkamp conebeam algorithm or FDK approach

Short digression on Axial scan

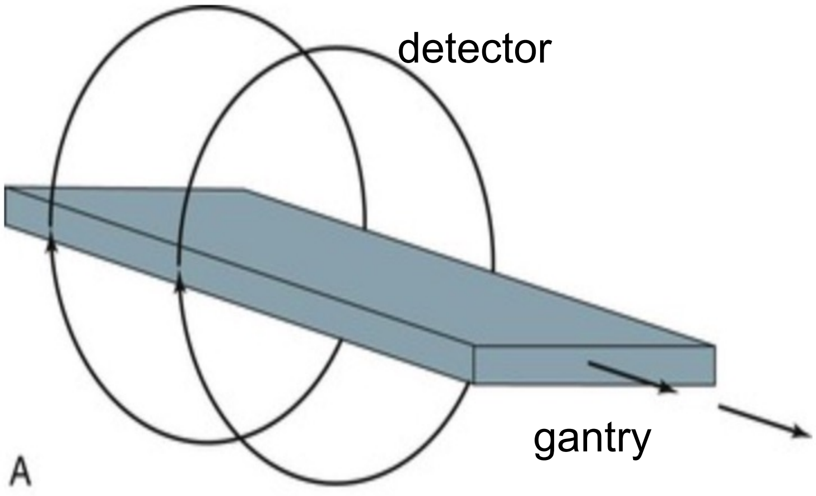

Image: https://clinicalgate.com/computed-tomography-3/

- X-ray tube and detector are stationary
- Patient moved through the gantry for cross sectional image to be taken at that specific location.
- Multiple slices are stacked to create the image.

Equidistant case (flat detector)

- Forward projection: $p(s,t;\beta) = \int_{L(s,t;\beta)} f(x,y,z)\, dl \qquad (24)$
- FDK algorithm:
  - Step 1: Compute weighted projections using the projection weighting

$$\overline{p}(s,t;\beta) = w_1(s,t)p(s,t;\beta), \qquad w_1(s,t) = \frac{D_{so}}{\sqrt{D_{sd}^2 + s^2 + t^2}} \qquad (25)$$

  - Step 2: Filter those projections along each row of the detector as if it were part of a 2D fan-beam acquisition using the ordinary ramp filter h∗(s) from equation (20)

$$\overline{p}(s,t;\beta) = \overline{p}(s,t;\beta) * h_*(s) \qquad (26)$$

$$h[n] = \begin{cases} 1, & n = 0 \\ 0, & n \text{ even} \\ \frac{-1}{(\pi n/2)^2}, & n \text{ odd} \end{cases}$$

Reminder of equation 20 !

  - Step 3: Perform weighted cone-beam backprojection of those filtered projections

$$\hat{f}(x,y,z) = \frac{1}{2}\int_0^{2\pi} w_2(x,y,\beta)\overline{p}(s_\beta(x,y), t_\beta(x,y,z);\beta), d\beta \qquad (27)$$

$$w_2(x,y,\beta) = \frac{D_{sd}^2}{(D_{so} - y_\beta)^2} \qquad s_\beta(x,y) = \frac{D_{sd}}{D_{so} - y_\beta} x_\beta \qquad t_\beta(x,y,z) = \frac{D_{sd}}{D_{so} - y_\beta} z \qquad (28)$$

Equiangular case (curved detector)

- Procedure is similar to the FDK algorithm
- Differences:
  - Replace the 1D weighting equation (25) by: $$w_1(s,t) = \frac{D_{so}}{\sqrt{D_{sd}^2 + t^2}} \cos\left(\frac{s}{D_{sd}}\right) \quad (29)$$
  - Replace the filter in equation (26) with the modified ramp filter in equation (16)

$$h[n] = \begin{cases} \frac{1}{4\Delta_s^2}, & n = 0 \\ 0, & n \text{ even} \\ \frac{-1}{[\pi D_{sd} \sin(n\Delta_s / D_{sd})]^2}, & n \text{ odd} \end{cases} \quad (16)$$

Reminder!

  - Replace weighting ratio w2 in (28) with (17), s_β with (9) and t_β unchanged.

$$\frac{1}{W_2^2 L_\beta^2(x,y)} = \frac{D_{sd}^2}{L_\beta^2(x,y)} = \frac{D_{sd}^2}{(D_{so} - y_\beta)^2 + (x_\beta - r_{off})^2} \quad (17)$$

$$s_\beta(x,y) = \begin{cases} D_{sd}\gamma_\beta(x,y), & \text{if } D_{fs} = 0 \text{ (equiangular)} \\ D_{sd}\tan\gamma_\beta(x,y), & \text{if } D_{fs} = \infty \text{ (equidistant)} \end{cases} \quad (9)$$

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Topics

Helical (Spiral) scan

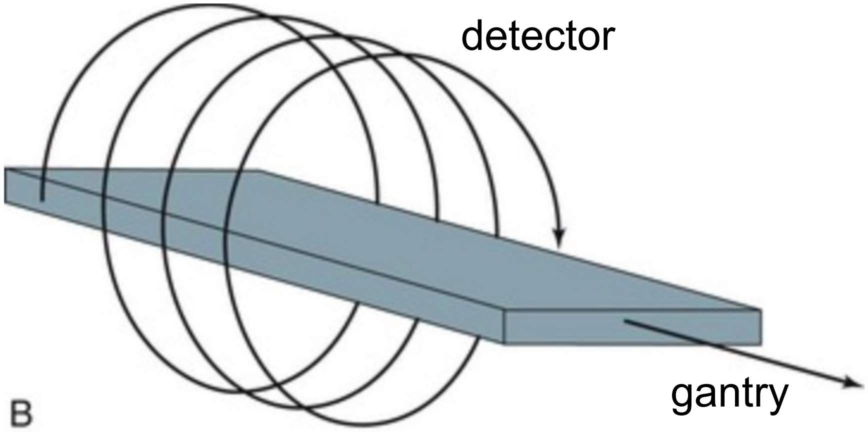

Image: https://clinicalgate.com/computed-tomography-3/

- Continuous rotation of the X-ray tube (emitter) and detector around the patient.
- Continuous data acquisition.
- More efficient than axial scanning, faster acquisition.
- More commonly in use today.
- Importantly, FBP type reconstruction exists.

Cone-parallel rebinning

- Simplified computation and reduced noise for both axial and helical scans.
- converts the fans for each detector row into cone-parallel rays.

$$p_{\mathrm{CP}}(r,\varphi;t) \triangleq p(s,t;\beta)\Big|_{s=s(r),\ \beta=\beta(r,\phi)} = p\left(\gamma^{-1}\left(\arcsin\left(\frac{r}{D_{\mathrm{s0}}}\right)\right),\, t;\, \phi - \arcsin\left(\frac{r}{D_{\mathrm{s0}}}\right)\right)$$

where,

$$s(\gamma) = \gamma^{-1}(s) = \begin{cases} D_{sd}\gamma, & \text{if } D_{fs} = 0 \text{ (equiangular)} \\ D_{fd}[\gamma - \arcsin(\frac{D_{fs}}{D_{fd}}\sin\gamma)], & \text{if } 0 \leqslant D_{fs} \leqslant \infty \\ D_{sd}\tan\gamma, & \text{if } D_{fs} = \infty \text{ (equidistant)} \end{cases} \quad (2)$$

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Topics

Comparison

Geometries	Parallel-beam	Fan-beam	Cone-beam
Dimension	2D		3D
projection	$p(r,\phi)$	$p(s,\beta)$	$p(s,t,\beta)$
Reconstruction algorithm	FBP , BPF, CBP	“weighted” FBP, fan parallel-rebinning, BPF	FDK (axial)
Generation	1st	3rd	3rd
Acquisition time	Slow	fast	faster

References

1. https://imaging.rigaku.com/blog/how-does-ct-reconstruction-work
2. https://howradiologyworks.com/ctgenerations/
3. https://www5.cs.fau.de/fileadmin/lectures/2017w/Lecture.2017w.ProjFCR/fanBeamRecon.pdf
4. Gengsheng Lawrence “Larry” Zeng.“Medical Image Reconstruction – A Conceptual Tutorial”. Springer 2009
5. J. Fessler. Image reconstruction: Algorithms and analysis. Book draft, 20:3.28, 2023
6. . Kak and M Slaney. Principles of computerized tomographic imaging. Book, pages 49–107, 198
7. G. L. Zeng G. T. Gullberg. Backprojection filtering for variable orbit fan-beam tomography. IEEE TRANSACTIONS ON NUCLEAR SCIENCE, 42, 1995
8. https://clinicalgate.com/computed-tomography-3/

The End

(and Thank you!!)