RADIOSITY: DAY 4 NOTES

1. Last Time

We were considering the integral equation:

$$y(x) = f(x) + \int_0^1 k(x,t)y(t)\ \mathrm{d}t \tag{1}$$

Where:

$$f(x) = \frac{x^3}{6} + \frac{5x}{6}$$

$$k(x,t) = \begin{cases} t(1-x) & t < x \\ x(1-t) & t > x \end{cases}$$

Given that y is a solution to (1), we could then multiply by an arbitrary function $v(x)$ and integrate to get:

$$\int_0^1 y(x)v(x)\ \mathrm{d}x = \int_0^1 f(x)v(x)\ \mathrm{d}x + \int_0^1 v(x) \int_0^1 k(x,t)y(t)\ \mathrm{d}t\ \mathrm{d}x$$

It turns out that we can also go the other way; that is, if y solves the second equation for all v, it also solves the original equation. To approximate y, we can search through a simpler space of functions. Choose a small h, and let interval I_j be $((j-1)h, jh)$. Then $I_1, I_2, \ldots, I_n$ covers the range from 0 to 1, and we can define $P_0((0,1))$ to be the space of functions v where v is constant on each I_j. This is also a vector space, with basis vectors φ_i, where:

$$\varphi_i(x) = \begin{cases} 1 & x \in I_i \\ 0 & \text{otherwise} \end{cases}$$

We can then look for an approximate solution $\tilde{y} \in P_0((0,1))$. In terms of the basis vectors, we can write $\tilde{y}(x) = \sum_{i=1}^{n} \tilde{y}_i \varphi_i(x)$. We can simplify the approximation further by restricting the arbitrary function v to also be in $P_0((0,1))$. Our problem is now to find $\tilde{y} \in P_0((0,1))$, where, for every $v \in P_0((0,1))$:

$$\int_0^1 \tilde{y}(x)v(x)\ \mathrm{d}x = \int_0^1 f(x)v(x)\ \mathrm{d}x + \int_0^1 v(x) \int_0^1 k(x,t)\tilde{y}(t)\ \mathrm{d}t\ \mathrm{d}x \tag{2}$$

2. Solving the Approximate Equation

If $\tilde{y}$ solves (2), then it must hold when v is any of the φ_j. Plugging in any particular φ_j gives:

$$\begin{aligned}
\int_0^1 \tilde{y}(x)\varphi_j(x)\ \mathrm{d}x &= \int_0^1 f(x)\varphi_j(x)\ \mathrm{d}x + \int_0^1 \varphi_j(x) \int_0^1 k(x,t)\tilde{y}(t)\ \mathrm{d}t\ \mathrm{d}x \\
\int_{I_j} \tilde{y}(x)\ \mathrm{d}x &= \int_{I_j} f(x)\ \mathrm{d}x + \int_{I_j}\int_0^1 k(x,t)\tilde{y}(t)\ \mathrm{d}t\ \mathrm{d}x \\
\int_{I_j} \sum_{i=1}^n \tilde{y}_i\varphi_i(x)\ \mathrm{d}x &= \int_{I_j} f(x)\ \mathrm{d}x + \int_{I_j}\int_0^1 k(x,t)\sum_{i=1}^n \tilde{y}_i\varphi_i(t)\ \mathrm{d}t\ \mathrm{d}x \\
\sum_{i=1}^n \tilde{y}_i \int_{I_j} \varphi_i(x)\ \mathrm{d}x &= \int_{I_j} f(x)\ \mathrm{d}x + \sum_{i=1}^n \tilde{y}_i \int_{I_j}\int_0^1 k(x,t)\varphi_i(t)\ \mathrm{d}t\ \mathrm{d}x \\
\tilde{y}_j \int_{I_j} \varphi_j(x)\ \mathrm{d}x &= \int_{I_j} f(x)\ \mathrm{d}x + \sum_{i=1}^n \tilde{y}_i \int_{I_j}\int_{I_i} k(x,t)\ \mathrm{d}t\ \mathrm{d}x \\
h\tilde{y}_j &= \int_{I_j} f(x)\ \mathrm{d}x + \sum_{i=1}^n \tilde{y}_i \int_{I_j}\int_{I_i} k(x,t)\ \mathrm{d}t\ \mathrm{d}x
\end{aligned}$$

Where various simplifications are made because φ_j is only nonzero on I_j. The integrals of k and f can be computed directly for any given i and j, so this is just a linear equation in the $\tilde{y}_i$. The latter sum can be interpreted as a dot product and written as:

$$\tilde{y}_j \cdot h = \begin{bmatrix} \int_{I_j}\int_{I_1} k(x,t)\ \mathrm{d}t\ \mathrm{d}x & \int_{I_j}\int_{I_2} k(x,t)\ \mathrm{d}t\ \mathrm{d}x & \cdots & \int_{I_j}\int_{I_n} k(x,t)\ \mathrm{d}t\ \mathrm{d}x \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_n \end{bmatrix} + \int_{I_j} f(x)\ \mathrm{d}x$$

Then all of the equations for various j can be written as a single matrix equation:

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_n \end{bmatrix} h = A \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_n \end{bmatrix} + \begin{bmatrix} \int_{I_1} f(x)\ \mathrm{d}x \\ \int_{I_2} f(x)\ \mathrm{d}x \\ \vdots \\ \int_{I_n} f(x)\ \mathrm{d}x \end{bmatrix}$$

Where, for this particular k, the entries of matrix A are:

$$A_{ij} = \int_{I_j}\int_{I_i} k(x,t)\ \mathrm{d}t\ \mathrm{d}x = \begin{cases} -h^3(2jh-h-2)(2i-1)/4 & i<j \\ -h^3(2ih-h-2)(2j-1)/4 & i>j \\ (4j-2)\frac{h^3}{4} - (2j-1)^2\frac{h^4}{4} & i=j \end{cases}$$

Importantly, they are all calculable, so A is just a matrix of known numbers. Similarly, f can be integrated to give:

$$\int_{I_i} f(x)\ \mathrm{d}x = h^2\left(i-\frac{1}{2}\right)\left(5+\left(i^2-i+\frac{1}{2}\right)\frac{h^2}{6}\right)$$

Let $\vec{f}$ be the vector of these numbers. Then the solution to (2) is the solution to the linear equation:

$$\tilde{y}h = A\tilde{y} + \vec{f}$$

Which can be rearranged to give $(hI - A)\tilde{y} = \vec{f}$, where I is the $n\times n$ identity matrix. This is a task that computers can do easily.

3. Back to the Rendering Equation

We want a function u that solves the integral equation:

$$u(\vec{x}) = q_b(\vec{x}) + \rho(\vec{x})\int_{\partial X}\frac{((\vec{z}-\vec{x})\cdot\vec{n}(\vec{x}))\,((\vec{x}-\vec{z})\cdot\vec{n}(\vec{z}))}{\|\vec{x}-\vec{z}\|^4}u(\vec{z})G(\vec{x},\vec{z})\ \mathrm{d}\vec{z} \tag{3}$$

Where q_b is the light emission from point $\vec{x}$, $\vec{n}$ is the normal vector to a point on ∂X, ρ is the probability the light reflects from the surface, and G is a function which is 1 if the line between $\vec{x}$ and $\vec{z}$ is clear of objects and 0 otherwise. This looks fairly similar to the equation we were solving earlier!

In computer graphics, the boundary ∂X is made up of a set of triangles τ. Analogously to the intervals, let $P_0(\tau)$ be the set of functions which are constant on each of the n faces F_i in τ. Like before, $P_0(\tau)$ is a vector space, and it has basis $\{\varphi_i\}_{i=1}^n$ where:

$$\varphi_i(\vec{x}) = \begin{cases} 1 & \vec{x}\in F_i \\ 0 & \text{otherwise} \end{cases}$$

Let's also rewrite (3) using the idea of multiplying by an arbitrary function v. For brevity, let:

$$H(\vec{x},\vec{z}) = \frac{((\vec{z}-\vec{x})\cdot\vec{n}(\vec{x}))\,((\vec{x}-\vec{z})\cdot\vec{n}(\vec{z}))}{\|\vec{x}-\vec{z}\|^4}G(\vec{x},\vec{z})$$

Then, by the same idea of multiplying both sides of (3) by an arbitrary function $v(x)$ and integraing over ∂X, we get that for all $v(x)$, u must satisfy:

$$\int_{\partial X} u(\vec{x})v(\vec{x})\ \mathrm{d}\vec{x} = \int_{\partial X} q_b(\vec{x})v(\vec{x})\ \mathrm{d}x + \int_{\partial X} v(\vec{x})\rho(\vec{x}) \int_{\partial X} H(\vec{x},\vec{z})u(\vec{z})\ \mathrm{d}\vec{z}\ \mathrm{d}\vec{x}$$

Let's look for an approximate solution $\tilde{u} \in P_0(\tau)$, and restrict v to also be in $P_0(\tau)$. These simplifications give us the equation:

$$\int_{\partial X} \tilde{u}(\vec{x})v(\vec{x})\ \mathrm{d}\vec{x} = \int_{\partial X} q_b(\vec{x})v(\vec{x})\ \mathrm{d}x + \int_{\partial X} v(\vec{x})\rho(\vec{x}) \int_{\partial X} H(\vec{x},\vec{z})\tilde{u}(\vec{z})\ \mathrm{d}\vec{z}\ \mathrm{d}\vec{x} \tag{4}$$

Since $\tilde{u} \in P_0(\tau)$, we can represent it as a linear combination of basis vectors $\sum_{i=1}^{n} \tilde{u}_i\varphi_i(\vec{x})$. Consider the case where $v = \varphi_j$. Plugging these simplifications into (4) gives:

$$\int_{\partial X} \sum_{i=1}^{n} \tilde{u}_i\varphi_i(\vec{x})\varphi_j(\vec{x})\ \mathrm{d}\vec{x} = \int_{\partial X} q_b(\vec{x})\varphi_j(\vec{x})\ \mathrm{d}x + \int_{\partial X} \varphi_j(\vec{x})\rho(\vec{x}) \int_{\partial X} H(\vec{x},\vec{z}) \sum_{i=1}^{n} \tilde{u}_i\varphi_i(\vec{z})\ \mathrm{d}\vec{z}\ \mathrm{d}\vec{x}$$

And using the definition of φ_i, we can simplify many of these integrals:

$$\sum_{i=1}^{n} \int_{F_j} \tilde{u}_i\varphi_i(\vec{x})\ \mathrm{d}\vec{x} = \int_{F_j} q_b(\vec{x})\ \mathrm{d}\vec{x} + \int_{F_j} \rho(\vec{x}) \sum_{i=1}^{n} \int_{\partial X} H(\vec{x},\vec{z})\tilde{u}_i\varphi_i(\vec{z})\ \mathrm{d}\vec{z}\ \mathrm{d}\vec{x}$$
$$\tilde{u}_j A_j = \int_{F_j} q_b(\vec{x})\ \mathrm{d}\vec{x} + \sum_{i=1}^{n} \tilde{u}_i \int_{F_j} \rho(\vec{x}) \int_{F_i} H(\vec{x},\vec{z})\ \mathrm{d}\vec{z}\ \mathrm{d}\vec{x}$$

Where A_j is the area of face F_j. We can make some more simplifying assumptions:

- The reflectivity ρ is constant on F_j; let it be ρ_j.
- The emmission q_b is constant on F_j; let it be ε_j.

That simplifies the equation further to:

$$\tilde{u}_j A_j = A_j\varepsilon_j + \sum_{i=1}^{n} \tilde{u}_i\rho_j \int_{F_j} \int_{F_i} H(\vec{x},\vec{z})\ \mathrm{d}\vec{z}\ \mathrm{d}\vec{x}$$

That leaves the integral of H to figure out. We can approximate it by doing a "Riemann sum", calculating H for the vertices of each face and averaging:

$$\int_{F_j} \int_{F_i} H(\vec{x},\vec{z})\ \mathrm{d}\vec{z}\ \mathrm{d}\vec{x} \approx \frac{A_i A_j}{9} \sum_{k=1}^{3} \sum_{\ell=1}^{3} H(F_{ik}, F_{j\ell})$$

Where F_{ik} is the k-th vertex of F_i. Let c_{ji} be the product of this approximation with ρ_j. Then, the equation can be written as:

$$\tilde{u}_j A_j = A_j \varepsilon_j + \sum_{i=1}^{n} \tilde{u}_i c_{ji}$$

We can go a little further by dividing both sides by A_j, as A_j is a common factor to every term. In any case, this is now a system of linear equations whose solution $\tilde{u}$ is a good description of the light in a scene.