RADIOSITY: DAY 3 NOTES

1. Radiative Transport Equation

On day 2, we derived the radiative transport equation:

$$\vec{\omega} \cdot \nabla u(\vec{x}, \vec{\omega}) + \sigma_t(\vec{x}) u(\vec{x}, \vec{\omega}) = \sigma_s \int_{S^2} k(\vec{x}, \vec{\omega}', \vec{\omega}) u(\vec{x}, \vec{\omega}') \, \mathrm{d}\vec{\omega}' + q(\vec{x}, \vec{\omega})$$

Where $\vec{x}$ is position, $\vec{\omega}$ is direction, u is the particle flux (light intensity), σ_t is the absorption probability, σ_s and k measure scattering probability, and q is the particle emission rate.

This is an integro-differential equation, as it contains integrals and derivatives of the unknown function u. Like integral or differential equations, it will have infinitely many solutions unless we specify some sort of initial and boundary conditions. Because the equation is based on a steady-state assumption, initial conditions don't make sense to apply. Instead, we'll specify boundary conditions.

2. Boundary Conditions

Let $H^+(\vec{x})$ be the set of directions that point into the scene for any given point $\vec{x} \in \partial X$, as opposed to those that point outside ($H^-(\vec{x})$). It turns out we don't care what happens in the directions in $H^-(\vec{x})$, since those particles never leave the scene anyway.

The values of $u(\vec{x}, \vec{\omega})$ for $\vec{\omega} \in H^+(\vec{x})$ will be determined by two effects: emission and scattering. Emission covers the cases of having a light source on the boundary (like a ceiling light) and having light enter the scene (like from the sky). Scattering is light from within the scene that scatters off the boundary. These two terms can be put together to give:

$$u(\vec{x}, \vec{\omega}) = q_b(\vec{x}, \vec{\omega}) + \int_{H^-(\vec{x})} k_b(\vec{x}, \vec{\omega}', \vec{\omega}) u(\vec{x}, \vec{\omega}') \, \mathrm{d}\vec{\omega}'$$

Where q_b is the surface emission rate and k_b is the probability of scattering from direction $\vec{\omega}'$ to $\vec{\omega}$.

3. Simplifying Assumptions

The radiative transport equation and the boundary conditions are extremely difficult to solve, so we'll make two more simplifying assumptions. The first is that the interior of the scene doesn't interact with the light. This is reasonable in most rendering scenarios, as air doesn't interact with light in any meaningful amount. In the equation, this sets $\sigma_t = \sigma_s = 0$. Furthermore, we assume the only sources of light are on the boundary of the scene. This further sets $q = 0$. With these assumptions, we can set up the differential equation:

$$\vec{\omega} \cdot \nabla u(\vec{x}, \vec{\omega}) = 0 \qquad \text{For } \vec{x} \in X \backslash \partial X$$
$$u(\vec{x}, \vec{\omega}) = q_b(\vec{x}, \vec{\omega}) + \int_{H^-(\vec{x})} k_b(\vec{x}, \vec{\omega}', \vec{\omega}) u(\vec{x}, \vec{\omega}') \, \mathrm{d}\vec{\omega}' \qquad \text{For } \vec{x} \in \partial X$$

This is a much simpler equation to make progress on. In particular, the first one states that the directional derivative of u in the direction of $\vec{\omega}$ is zero, which means that $u(\vec{x}, \vec{\omega})$ is constant along a line in direction $\vec{\omega}$.

Let's define a function $\gamma(\vec{x}, \vec{\omega})$ to be the first point on ∂X in the direction $\vec{\omega}$ from $\vec{x}$. Using the previous observation, we can conclude $u(\vec{x}, \vec{\omega}) = u(\gamma(\vec{x}, -\vec{\omega}), \vec{\omega})$; that is, the light from the boundary passes unimpeded to each point $\vec{x}$. We can plug this idea into the boundary condition to get:

$$u(\vec{x}, \vec{\omega}) = q_b(\vec{x}, \vec{\omega}) + \int_{H^-(\vec{x})} k_b(\vec{x}, \vec{\omega}', \vec{\omega}) u(\gamma(\vec{x}, -\vec{\omega}'), \vec{\omega}') \, \mathrm{d}\vec{\omega}'$$

Next, we'll assume that u is independent of $\vec{\omega}$. That is, the intensity of light from any given point on an object does not depend on which angle you observe it from. This is definitely not the case with reflective surfaces, but it is generally accurate for dull (*Lambertian*) surfaces like walls. Under this assumption, we can write:

$$k_b(\vec{x}, \vec{\omega}', \vec{\omega}) = \rho(\vec{x}) \vec{\omega} \cdot \vec{n}(\vec{x})$$

Where ρ is the probability a given photon gets scattered rather than absorbed, and $\vec{n}$ is the normal vector to ∂X at $\vec{x}$. The dot product accounts for the fact that more photons can hit a surface from more direct angles. Using the same assumption, we can remove our $\vec{\omega}$ dependence to get:

$$u(\vec{x}) = q_b(\vec{x}) + \int_{H^-(\vec{x})} \rho(\vec{x}) \vec{\omega}' \cdot \vec{n}(\vec{x}) u(\gamma(\vec{x}, -\vec{\omega}')) \, \mathrm{d}\vec{\omega}'$$

Now we can change our perspective on γ. Rather than looking out from $\vec{x}$ backwards, we can think about looking towards $\vec{x}$ from the rest of the boundary. For a given $\vec{z}$ on the boundary, the vector $\vec{x} - \vec{z}$ is the vector from $\vec{z}$ to $\vec{x}$ (although we should normalize it by dividing by its norm). With this perspective, we can write the equation as:

$$\begin{aligned} u(\vec{x}) &= q_b(\vec{x}) + \int_{\partial X} \rho(\vec{x}) \frac{(\vec{z}-\vec{x}) \cdot \vec{n}(\vec{x})}{\|\vec{z}-\vec{x}\|} u(\vec{z}) \frac{\vec{n}(\vec{z}) \cdot (\vec{x}-\vec{z})}{\|\vec{x}-\vec{z}\|^3} v(\vec{x},\vec{z}) \, \mathrm{d}\vec{z} \\ &= q_b(\vec{x}) + \int_{\partial X} \rho(\vec{x}) \frac{((\vec{z}-\vec{x}) \cdot \vec{n}(\vec{x}))\,((\vec{x}-\vec{z}) \cdot \vec{n}(\vec{z}))}{\|\vec{x}-\vec{z}\|^4} u(\vec{z}) v(\vec{x},\vec{z}) \, \mathrm{d}\vec{z} \end{aligned}$$

Where the term after $u(\vec{z})$ corrects for the fact that the original formulation was with respect to an angle, while this version is over a boundary. The $v(\vec{x},\vec{z})$ is a function defined by:

$$v(\vec{x},\vec{z}) = \begin{cases} 1 & \vec{x} \text{ and } \vec{z} \text{ are connected by a line through } X \\ 0 & \text{otherwise} \end{cases}$$

To ensure the integral is taken over only the parts of the scene that $\vec{x}$ can see.

4. Solving Integral Equations Numerically

This equation is still far too complex to solve analytically. Instead, we'll solve it numerically. To see how, we'll consider a simpler integral equation:

$$y(x) = \frac{x^3}{6} + \frac{5x}{6} + 4 \int_0^1 k(x,t) y(t) \, \mathrm{d}t$$

Where k is defined as:

$$k(x,t) = \begin{cases} t(1-x) & t < x \\ x(1-t) & t > x \end{cases}$$

It turns out this one can be solved analytically, but we won't. Instead, consider multiplying the equation by some function $v(x)$, giving:

$$y(x)v(x) = \left(\frac{x^3}{6} + \frac{5x}{6}\right) v(x) + 4v(x) \int_0^1 k(x,t) y(t) \, \mathrm{d}t$$

If we integrate both sides of this with respect to x, we'll get numbers on both sides. In particular, we get:

$$\int_0^1 y(x)v(x) \, \mathrm{d}x = \int_0^1 \left(\frac{x^3}{6} + \frac{5x}{6}\right) v(x) \, \mathrm{d}x + \int_0^1 4v(x) \int_0^1 k(x,t) y(t) \, \mathrm{d}t \, \mathrm{d}x$$

Such an integral can be approximated by splitting the interval from 0 to 1 into several intervals I_j, where we pick a small h and let $I_j = ((j-1)h, jh)$. We can approximate the functions involved by assuming they are constant over each I_j. Let $P_0((0,1))$ be the set of functions which are piecewise constant on all I_j. We'll try to approximate y by $\tilde{y} \in P_0((0,1))$.

Note that $P_0((0,1))$ is a vector space; it is closed under linear combinations. That means it has a basis. One such basis is the $\varphi_j(x)$, where:

$$\varphi_j(x) = \begin{cases} 1 & x \in I_j \\ \\ 0 & x \notin I_j \end{cases}$$

Then $\{\varphi_j(x)\}_{j=1}^N$, where I_N is the last interval, is a basis for $P_0((0,1))$. Let $\tilde{y}_i$ be the coefficients of $\tilde{y}$ in this basis, that is:

$$\tilde{y}(x) = \sum_{i=1}^{N} \tilde{y}_i \varphi_i(x)$$

To get a good enough approximation, we'll require that:

$$\int_0^1 \tilde{y}(x)v(x)\ \mathrm{d}x = \int_0^1 \left(\frac{x^3}{6} + \frac{5x}{6}\right) v(x)\ \mathrm{d}x + \int_0^1 4v(x) \int_0^1 k(x,t)\tilde{y}(t)\ \mathrm{d}t\ \mathrm{d}x$$

For any $v \in P_0((0,1))$. This is not as good a solution as y is, but it should be close. Expressing $\tilde{y}$ in terms of the $\tilde{y}_i$:

$$\int_0^1 \sum_{i=1}^{N} \tilde{y}_i \varphi_i(x) v(x)\ \mathrm{d}x = \int_0^1 \left(\frac{x^3}{6} + \frac{5x}{6}\right) v(x)\ \mathrm{d}x + \int_0^1 4v(x) \int_0^1 k(x,t) \sum_{i=1}^{N} \tilde{y}_i \varphi_i(t)\ \mathrm{d}t\ \mathrm{d}x$$

If we can find the $\tilde{y}_i$, we can get a good approximation for y. As we'll see next time, this turns out to be a linear system of equations!