RADIOSITY: DAY 2 NOTES

1. Recap

We want to color a scene, which is composed of a region X (with boundary ∂X). To do that, we want to calculate the intensity of photons at each point in X. Solving this problem for red, green, and blue photons (three separate steps) will give results that can be combined into a full-color image.

2. Density

Suppose a metal rod has density $\delta(x)$ at a point x along its length. We can't say anything about the amount of mass *at* point x, since there's no volume at a point. However, we can talk about the mass in a region of width $\mathrm{d}x$ centered at x. For a sufficiently small $\mathrm{d}x$, we can say the mass is $\delta(x) \cdot \mathrm{d}x$. For a larger interval I, we can calculate the mass with the integral $\displaystyle\int_I \delta(x)\ \mathrm{d}x$.

3. Photon Counts and Phase Space

We can refer to photons as having positions in phase space, which has axes for $\vec{x}$ and $\vec{\omega}$. Denote the density of photons at point $\vec{x}$, in direction $\vec{\omega}$, and at time t by $n(\vec{x}, \vec{\omega}, t)$. This density can be measured by drawing a small region around $\vec{x}$ and considering a small cone of angles around $\vec{\omega}$, counting the number of photons at time t that match both criteria, and then dividing by the sizes of the region and the cone. In a 2D problem, the units of n will be $\frac{\text{number of photons}}{m^2 \cdot rad}$, while in 3D the units will be $\frac{\text{number of photons}}{m^3 \cdot std}$ (where std stands for steradians).

The number of photons in a large phase space region $V \times \Omega$ at time t is then:

$$N(V, \Omega, t) = \int_V \int_\Omega n(\vec{x}, \vec{\omega}, t)\ \mathrm{d}\vec{\omega}\ \mathrm{d}\vec{x}$$

Define the function $u(\vec{x}, \vec{\omega}, t)$ to be the *phase space flux*: the rate of flow of photons in direction $\vec{\omega}$ at point $\vec{x}$ at time t. We can measure u by drawing a small line segment (or surface in 3D) centered at $\vec{x}$ and perpendicular to direction $\vec{\omega}$, and counting the number of photons that pass through that line in a direction close to $\vec{\omega}$ over a certain period of time. In a 2D problem, the units of u will then be $\frac{\text{photons}}{s \cdot m \cdot rad}$; in a 3D problem, it's $\frac{\text{photons}}{s \cdot m^2 \cdot std}$.

Suppose the value of u is known at all positions, angles, and times; in particular, let's say it's always 4. If we observe a line segment of length 0.01 centered at $\vec{x}$ (perpendicular to $\vec{\omega}$, within a range of 0.05 radians around $\vec{\omega}$ over 1 second, we should expect to see $4 \cdot 1 \cdot 0.01 \cdot 0.05 = 0.002$ photons.

Now suppose the line segment is no longer perpendicular to $\vec{\omega}$, but the other measurements are the same. This situation is similar to the previous, but the line segment's effective length is shorter because of its angle. If θ is the angle between the line's normal vector $\vec{n}$ and $\vec{\omega}$, this effective length is $\cos\theta$ times the actual length. This cosine can be computed with a dot product, yielding $4 \cdot 1 \cdot 0.01 \cdot 0.05 \cdot (\vec{n} \cdot \vec{\omega})$ photons. Note that are two possible normal vectors to the line (one in each direction), and the choice of which affects the sign of the result.

4. Conservation of Photons

A motivating question: what is $\frac{\mathrm{d}}{\mathrm{d}t} N(V, \Omega, t)$? That is, how quickly does the number of photons in a given region and direction change? This question can be answered by calculating the rate photons enter $V \times \Omega$, and subtracting the rate they leave $V \times \Omega$. What are some reasons for each of these changes? We might have:

- Photons are created, e.g. by being emitted from a light source.
- Photons are destroyed, e.g. by being absorbed by smoke or water.
- Photons cross ∂V with direction in Ω (photon streaming).
- Photons scatter across $\partial\Omega$ inside V, e.g. due to smoke.

Note that photon-photon interactions are ignored, so we assume photons can't hit each other. Under these assumptions, $\frac{\mathrm{d}}{\mathrm{d}t} N(V, \Omega, t)$ can be calculated by adding up these rates.

Let's start with emission. For rendering, photon emission only happens because of light sources in the scene, which is fully specified. Suppose the light emission in the scene is defined by the function $q(\vec{x}, \vec{\omega}, t)$. In $V \times \Omega$, the rate photons are produced is then $\displaystyle\int_V \int_\Omega q(\vec{x}, \vec{\omega}, t)\ \mathrm{d}\vec{\omega}\ \mathrm{d}\vec{x}$.

Next, absorption. Like with emission, absorption is due to elements of the scene, which is fully specified. Let $\sigma_a(\vec{x}, \vec{\omega})$ be the probability that a photon at point $\vec{x}$, travelling in direction $\vec{\omega}$, is absorbed per length travelled. To get the absorption rate across $V \times \Omega$, we'll integrate the probability of each photon being absorbed times the flux of photons: $\displaystyle\int_V \int_\Omega \sigma_a(\vec{x}, \vec{\omega}) u(\vec{x}, \vec{\omega}, t)\ \mathrm{d}\vec{\omega}\ \mathrm{d}\vec{x}$.

Next, streaming. Here, we don't care about what happens on the interior of V; we only care about the flow through the boundary. That flow is measured by u, so the rate across the whole boundary is $\displaystyle\int_\Omega \int_{\partial V} u(\vec{x}, \vec{\omega}, t) \vec{n}(\vec{x}) \cdot \vec{\omega}\ \mathrm{d}\vec{x}\ \mathrm{d}\vec{\omega}$, where $\vec{n}(\vec{x})$ is the outward-pointing normal vector of ∂V at point $\vec{x}$.

Finally, scattering. Depending on the reason for the scattering, photons may be more likely to turn only a little bit or turn a lot. Let $k(\vec{x}, \vec{\omega}, \vec{\omega}')$ be the probability of scattering from direction $\vec{\omega}$ to $\vec{\omega}'$ at point $\vec{x}$ per length travelled. For inscattering, we want to count scattering from outside Ω into Ω, and for outscattering we want the opposite. Inscattering can then be counted by the integral:

$$\int_V \int_{S^2 \setminus \Omega} \int_\Omega k(\vec{x}, \vec{\omega}', \vec{\omega}) u(\vec{x}, \vec{\omega}', t) \; \mathrm{d}\vec{\omega} \; \mathrm{d}\vec{\omega}' \; \mathrm{d}\vec{x}$$

Similarly, outscattering is counted by the integral:

$$\int_V \int_{S^2 \setminus \Omega} \int_\Omega k(\vec{x}, \vec{\omega}, \vec{\omega}') u(\vec{x}, \vec{\omega}, t) \; \mathrm{d}\vec{\omega} \; \mathrm{d}\vec{\omega}' \; \mathrm{d}\vec{x}$$

With all the rates calculated, we can put all of them together to get the overall rate of photons into and out of $V \times \Omega$. If we're rendering a scene, that rate can be assumed to be zero, as the photons are in steady-state; objects do not get brighter or darker on human-viewable timescales. Then we can remove the t's, giving us the big equation:

$$\begin{aligned} 0 = &\int_V \int_\Omega q(\vec{x}, \vec{\omega}) \; \mathrm{d}\vec{\omega} \; \mathrm{d}\vec{x} - \int_V \int_\Omega \sigma_a(\vec{x}, \vec{\omega}) u(\vec{x}, \vec{\omega}) \; \mathrm{d}\vec{\omega} \; \mathrm{d}\vec{x} - \int_\Omega \int_{\partial V} u(\vec{x}, \vec{\omega}) \vec{n}(\vec{x}) \cdot \vec{\omega} \; \mathrm{d}\vec{x} \; \mathrm{d}\vec{\omega} \\ &+ \int_V \int_{S^2 \setminus \Omega} \int_\Omega k(\vec{x}, \vec{\omega}', \vec{\omega}) u(\vec{x}, \vec{\omega}') \; \mathrm{d}\vec{\omega} \; \mathrm{d}\vec{\omega}' \; \mathrm{d}\vec{x} - \int_V \int_{S^2 \setminus \Omega} \int_\Omega k(\vec{x}, \vec{\omega}, \vec{\omega}') u(\vec{x}, \vec{\omega}) \; \mathrm{d}\vec{\omega} \; \mathrm{d}\vec{\omega}' \; \mathrm{d}\vec{x} \end{aligned}$$

Just solve for u! To make any headway, though, we'll need to make some more simplifying assumptions. First, we'll assume scattering is symmetric, i.e. $k(\vec{x}, \vec{\omega}, \vec{\omega}') = k(\vec{x}, \vec{\omega}', \vec{\omega})$. This is reasonable for most physical reasons for scattering. Then the two scattering integrals can be combined into (after some more work):

$$\int_\Omega \int_V \int_{S^2} k(\vec{x}, \vec{\omega}', \vec{\omega}) u(\vec{x}, \vec{\omega}') \; \mathrm{d}\vec{\omega}' \; \mathrm{d}\vec{\omega} \; \mathrm{d}\vec{x}$$

Second, we can integrate by parts to simplify the streaming term:

$$\int_\Omega \int_{\partial V} u(\vec{x}, \vec{\omega}) \vec{n}(\vec{x}) \cdot \vec{\omega} \; \mathrm{d}\vec{x} \; \mathrm{d}\vec{\omega} = \int_\Omega \int_V \vec{\omega} \cdot \nabla u(\vec{x}, \vec{\omega}) \; \mathrm{d}\vec{x} \; \mathrm{d}\vec{\omega}$$

Applying these simplifications and rearranging gives:

$$\int_\Omega \int_V \vec{\omega} \cdot \nabla u(\vec{x}, \vec{\omega}) \; \mathrm{d}\vec{x} \; \mathrm{d}\vec{\omega} + \int_\Omega \int_V \sigma_a(\vec{x}, \vec{\omega}) u(\vec{x}, \vec{\omega}) \; \mathrm{d}\vec{x} \; \mathrm{d}\vec{\omega} = \int_\Omega \int_V \int_{S^2} k(\vec{x}, \vec{\omega}', \vec{\omega}) u(\vec{x}, \vec{\omega}') \; \mathrm{d}\vec{\omega}' \; \mathrm{d}\vec{\omega} \; \mathrm{d}\vec{x} + \int_V \int_\Omega q(\vec{x}, \vec{\omega}) \; \mathrm{d}\vec{\omega} \; \mathrm{d}\vec{x}$$

Throughout this derivation, we assumed nothing about V and Ω (except that V has a boundary). Since the integrals over V and Ω on both sides equal each other, and V and Ω were arbitrary, we can remove the integrals and still get an equality:

$$\vec{\omega} \cdot \nabla u(\vec{x}, \vec{\omega}) + \sigma_a(\vec{x}, \vec{\omega}) u(\vec{x}, \vec{\omega}) = \int_{S^2} k(\vec{x}, \vec{\omega}', \vec{\omega}) u(\vec{x}, \vec{\omega}') \, \mathrm{d}\vec{\omega}' + q(\vec{x}, \vec{\omega})$$

This is the radiative transport equation, which we will simplify further in future classes to get the rendering equation.