Exam IFM Formula Sheet

Forwards and Futures

Pricing forwards:

$$F_{0,T} = S_0 e^{rT} - \text{CumVal(Divs)}$$
$$F_{0,T} = S_0 e^{(r-\delta)T}$$

For currency options: Domestic risk-free rate serves as r; foreign risk-free rate serves as δ.

Prepaid forwards: $F^P_{0,T} = e^{-rT} F_{0,T}$

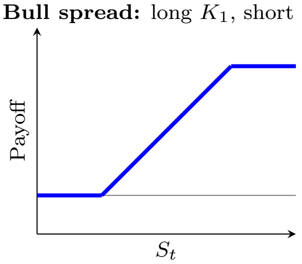
Futures are marked-to-market. If the margin account falls below the maintenance margin, the broker makes a margin call to the investor to provides funds to increase the margin account level to the initial margin level.

Option Strategies

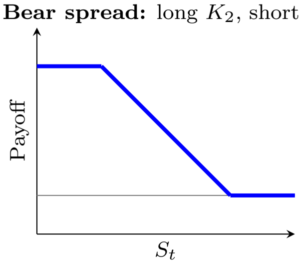
With underlying assets:

- Long put, long stock (**floor**)
- Long call, short stock (**cap**)
- Short call, long stock (**covered written call**)
- Short put, short stock (**covered written put**)

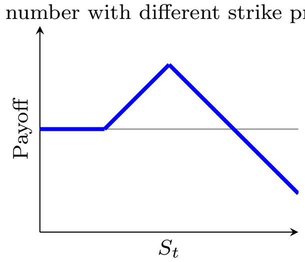
Bull spread: long K_1, short K_2 ($K_2 > K_1$)



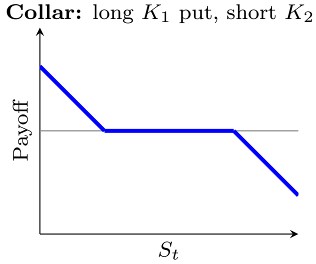
Bear spread: long K_2, short K_1 ($K_2 > K_1$)



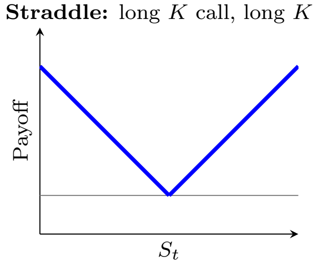
Ratio spread: long and short an unequal number with different strike prices



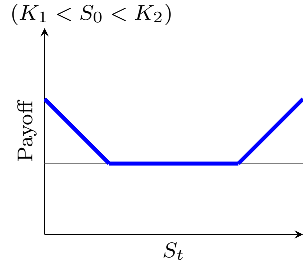
Collar: long K_1 put, short K_2 call ($K_2 > K_1$)



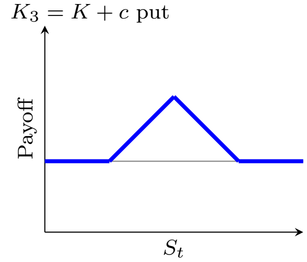
Straddle: long K call, long K put ($K = S_0$)



Strangle: long K_2 call, long K_1 put ($K_1 < S_0 < K_2$)



Symmetric butterfly spread: long 1 $K_1 = K - c$ put, short 2 $K_2 = K$ puts, long 1 $K_3 = K + c$ put



Asymmetric butterfly spread: Long n bull spreads (strikes K_1, K_2) and m bear spreads (strikes K_2, K_3), with

$$\frac{n}{m} = \frac{K_3 - K_2}{K_2 - K_1}$$

Put-Call Parity

Put-Call Parity equation:

$$\begin{aligned} C(S,K,T) - P(S,K,T) &= e^{-rT}(F_T(S) - K) \\ &= F^P_T(S) - Ke^{-rT} \end{aligned}$$

For stock,

$$C - P = S_0 - \text{PV}_{0,T}(\text{Divs}) - Ke^{-rT}$$
$$C - P = S_0 e^{-\delta T} - Ke^{-rT}$$

For exchange options,

$$C(A,B) - P(A,B) = F^P_T(A) - F^P_T(B)$$
$$C(A,B) = P(B,A) = \text{receive } A, \text{give up } B$$

Comparing Options

Bounds on option prices:

1. American options are worth at least as much as European ones.
2. A call option cannot be worth more than the underlying stock. A put option cannot be worth more than the strike price. European options cannot be worth more than the present values of these.
3. An option must be worth at least 0.
4. European options are worth at least as much as implied by put-call parity assuming the other option is worth 0.
5. American options are worth at least their exercise value.

Early exercise:

1. Early exercise is never optimal for call options on nondividend paying stocks.
2. For dividend paying stocks, the PV of the dividends must exceed the lost interest: $\text{PV}_{t,T}(\text{Divs}) \geq K\left(1 - e^{-r(T-t)}\right)$.
3. Early exercise for put options may be rational if $C_{\text{Eur}}(S_t, K, T-t) + \text{PV(Divs)} - K\left(1 - e^{-r(T-t)}\right) < 0$.

Time to expiry:

1. An American option (or European call option on nondividend paying stock) with expiry T and strike price K is worth at least as much as one with expiry $t < T$ and strike price K.
2. A European option on nondividend paying stock with expiry T and strike price $Ke^{r(T-t)}$ must cost at least as much with one with expiry t and strike price K.

Different strike prices:

$K_3 > K_2 > K_1$. All options are for the same underlying stock and have the same time to expiry.

Direction:

$$C(K_2) \leq C(K_1)$$
$$P(K_2) \geq P(K_1)$$

Slope:

$$C(K_1) - C(K_2) \leq K_2 - K_1$$
$$P(K_2) - P(K_1) \leq K_2 - K_1$$

Convexity:

$$\frac{C(K_3) - C(K_2)}{K_3 - K_2} \geq \frac{C(K_2) - C(K_1)}{K_2 - K_1}$$
$$\frac{P(K_3) - P(K_2)}{K_3 - K_2} \geq \frac{P(K_2) - P(K_1)}{K_2 - K_1}$$

Binomial Trees

Replicating portfolio: Buy Δ shares, lend B

$$\Delta = \left(\frac{C_u - C_d}{S(u-d)}\right) e^{-\delta h}$$
$$B = e^{-rh}\left(\frac{uC_d - dC_u}{u-d}\right)$$

Risk-neutral probability:

For a forward tree,

$$u = e^{(r-\delta)h+\sigma\sqrt{h}}, d = e^{(r-\delta)h-\sigma\sqrt{h}}$$

In general,

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d}$$
$$= \frac{1}{1+e^{\sigma\sqrt{h}}} \text{ for forward tree}$$

Premium:

$$C = \Delta S + B$$
$$= e^{-rh}\left(p^* C_u + (1-p^*)C_d\right)$$

Futures: Use $\delta = r$ in all formulas except for replicating portfolio formulas:

$$\Delta = \frac{C_u - C_d}{F(u-d)}$$
$$B = C$$

For an infinitely lived call option on a stock with $\sigma = 0$, exercise is optimal if $S\delta > Kr$.

Lognormal Distribution

The ratio S_t/S_0 is a lognormal random variable with parameters $m = (\alpha - \delta - 0.5\sigma^2)t$ and $v = \sigma\sqrt{t}$.

$$\hat{d}_1 = \frac{\ln(S_0/K) + (\alpha - \delta + 0.5\sigma^2)t}{\sigma\sqrt{t}}$$
$$\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{t}$$

$$\Pr(S_t > K) = N(\hat{d}_2)$$
$$\Pr(S_t < K) = N(-\hat{d}_2)$$

$$\mathbf{PE}[S_t \mid S_t > K] = S_0 e^{(\alpha-\delta)t} N(\hat{d}_1)$$
$$\mathbf{PE}[S_t \mid S_t < K] = S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)$$

$$E[X \mid Y] = \frac{\mathbf{PE}[X \mid Y]}{\Pr(Y)}$$

Expected option payoffs: Accumulate Black-Scholes premium, replace r with α:

Call: $S_0 e^{(\alpha-\delta)t} N(\hat{d}_1) - KN(\hat{d}_2)$

Put: $KN(-\hat{d}_2) - S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)$.

Black-Scholes Formula

General Black-Scholes Formula:

$$C(S, K, \sigma, r, T, \delta) = F^P(S)N(d_1) - F^P(K)N(d_2)$$

where

$$d_1 = \frac{\ln\left(F^P(S)/F^P(K)\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

For stock,

$$C = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$
$$P = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$

where

$$d_1 = \frac{\ln(S/K) + \left(r - \delta + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

For futures, use $\delta = r$.

Greeks

The only useful formula not on the formula sheet is

$$\Delta_{\text{put}} = \Delta_{\text{call}} - e^{-\delta(T-t)}$$

Option elasticity is given by

$$\Omega = \frac{S\Delta}{C}$$

The **volatility** of an option is

$$\sigma_{\text{option}} = \sigma_{\text{stock}}\,|\Omega|$$

The **risk premium** of an option (letting γ be the rate of return on the option) is

$$\gamma - r = \Omega(\alpha - r)$$

For both the stock and the option, the **Sharpe ratio** is

$$\phi = \frac{\alpha - r}{\sigma_{\text{stock}}}$$

The Greek for a portfolio is the sum of the Greeks.

The elasticity for a portfolio is the weighted average of the elasticities.

Delta Hedging

Overnight profit on a delta-hedged portfolio consists of (1) the change in the value of the option, (2) Δ times the change in the stock price, and (3) interest. Overnight profit is 0 at

$$S \pm S\sigma\sqrt{h}$$

To hedge Greek(s), buy assets such that the Greek(s) for the entire portfolio are 0.

The **delta-gamma-theta approximation** for a change in stock price ϵ is

$$C(S_{t+h}) = C(S_t) + \Delta\epsilon + \tfrac{1}{2}\Gamma\epsilon^2 + h\theta$$

We can use this to approximate market-maker profit as

$$-\left(\tfrac{1}{2}\Gamma\epsilon^2 + \theta h + rh(\Delta S - C(S)) - \delta h \Delta S\right)$$

Exotic Options

Asian options:

Payoff	Average Price	Average Strike
Call	$\max(0, \overline{S} - K)$	$\max(0, S_T - \overline{S})$
Put	$\max(0, K - \overline{S})$	$\max(0, \overline{S} - S_T)$

Barrier options:

- *Knock-in:* goes into existence if barrier is hit
- *Knock-out:* goes out of existence if barrier is hit

Parity: Knock-in option + Knock-out option = Ordinary option

Compound options with strike x and expiry t_1 are options on options:

$$\text{CallOnCall} - \text{PutOnCall} = C - xe^{-rt_1}$$
$$\text{CallOnPut} - \text{PutOnPut} = P - xe^{-rt_1},$$

where C and P are the prices of the strike assets (standard call and puts).

Gap options pay according to the strike price K_1 if S_T is above (call) or below (put) the trigger K_2

- Use K_1 in formula for C and P
- Use K_2 in formula for d_1

Exchange option to receive S in return for Q:

- $\sigma^2 = \sigma_S^2 + \sigma_Q^2 - 2\rho\sigma_S\sigma_Q$
- Use Q and δ_Q for K and r, respectively

Chooser options:

$$V = C(K, T) + e^{-\delta(T-t)}P\left(Ke^{-(r-\delta)(T-t)}, t\right)$$

Forward start options offering a call option with strike price cS_t at time t expiring at time T:

$$V = Se^{-\delta T}N(d_1) - cSe^{-r(T-t)-\delta T}N(d_2)$$
$$d_1 = \frac{-\ln c + (r - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 = \sigma\sqrt{T-t}$$

Actuarial Applications of Options

Variable Annuities:

- *Guaranteed minimum death benefit (GMDB):* guarantees a minimum amount (usually the premium), regardless of the account value, upon death
  - *Earnings-enhanced death benefit:* pays a percentage of the excess of the account value over the premium
- *Guaranteed minimum accumulation benefit (GMAB):* guarantees a minimum value for the account at a specific time
- *Guaranteed minimum withdrawal benefit (GMWB):* guarantees that after a policyholder reaches a specified age, he may withdraw a certain amount every year for life
- *Guaranteed minimum income benefit (GMIB):* guarantees a whole life annuity purchase rate at specified ages

Mortgage guaranty insurance is purchased by the lender and pays the lender the outstanding loan balance plus settlement costs. The insurance company receives the proceeds of the foreclosure sale.

Guaranteed replacement cost coverage is a rider to a property insurance contract that provides for paying the replacement cost for the property rather than the actual cash value before the loss.

Inflation indexing: Let I be the CPI. For an indexed pension,

$$P_t = \max_{0 \le s \le t} (P_0 \times (I_s/I_0)).$$

Static hedging:

- *Lookback options:* Let M be the maximum price of the underlying over the option period and m the minimum price. The payoffs of lookback options are
  - Standard lookback call: $S_T - m$
  - Standard lookback put: $M - S_T$
  - Extrema lookback call: $\max(M - K, 0)$
  - Extrema lookback put: $\max(K - m, 0)$
- *Shout options:* The purchaser "shouts" on a date where the price is S^*. The payoffs of shout options are
  - Shout call option: $\max(0, S_T - K, S^* - K)$
  - Shout put option: $\max(0, K - S_T, K - S^*)$
- *Rainbow options:* For two assets S, T, the payoffs of rainbow options are
  - Rainbow call option: $\max(0, S_T - K, Q_T - K)$
  - Rainbow put option: $\max(0, K - S_T, K - Q_T)$

Hedging catastrophe risk

- *Reinsurance*
- *Weather derivatives* pay based on whether a defined weather event occurs
- *Catastrophe bonds* pay (higher) interest and principal, but the the company is allowed to default if a catastrophe occurs.

Project Analysis

Downside semi-variance:

$$\sigma^2_{SV} = \mathbf{E}[\min(0, (R - \mu))^2]$$
$$\hat{\sigma}^2_{SV} = \frac{1}{n}\sum_{i=1}^{n} \min(0, (R_i - \overline{R}))^2$$

Value-at-risk:

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha)$$

Tail value-at-risk:

$$\mathrm{TVaR}_\alpha(x) = \frac{\int_{-\infty}^{\mathrm{VaR}_\alpha(x)} x f(x)dx}{\alpha} \text{ (downside)}$$
$$\mathrm{TVaR}_\alpha(x) = \frac{\int_{\mathrm{VaR}_\alpha(x)}^{\infty} x f(x)dx}{1-\alpha} \text{ (upside)}$$

Monte Carlo Simulation

Inversion method: For a uniform number $u \in (0, 1)$ and a random variable X, we can generate a random sample with $x = F_X^{-1}(u)$.

Efficient Markets Hypothesis

Form	**Prices reflect**	**Evidence**
Weak	Past information	Stock prices follow random walk
Semi-strong	The above, and all publicly or easily available information	Takeover announcements reflected immediately in price
Strong	The above, and private, hard-to-get, or hard-to-interpret information	Most fund managers don't beat the market

Mean-Variance Portfolio Theory

Estimating volatility:

$$\sigma^2 = \frac{\sum (R_i - \overline{R})^2}{n-1} = \frac{n}{n-1}\left(\frac{\sum R_i^2}{n} - \overline{R}^2\right) = \frac{\sum R_i^2 - n\overline{R}^2}{n-1}$$

Standard error of sample mean: $\frac{\sigma}{\sqrt{n}}$

Estimating covariance (j is the year):

$$\mathrm{Cov}(R_1, R_2) = \frac{\sum (R_{1j} - \overline{R}_1)(R_{2j} - \overline{R}_2)}{n-1} = \frac{n}{n-1}\left(\frac{\sum R_{1j}R_{2j}}{n} - \overline{R}_1\overline{R}_2\right) = \frac{\sum R_{1j}R_{2j} - n\overline{R}_1\overline{R}_2}{n-1}$$

Variance of a portfolio (x_i are the weights):

$$\mathrm{Var}(R) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \,\mathrm{Cov}(R_i, R_j)$$

For an equally weighted portfolio,

$$\mathrm{Var}(R) = \frac{1}{n}\mathrm{AvgVar} + \frac{n-1}{n}\mathrm{AvgCov}$$

Volatility of a portfolio:

$$\sigma_P = \sum_{i=1}^{n} x_i \rho_{P,i}\sigma_i$$

Sharpe Ratio:

$$\phi = \frac{\alpha - r}{\sigma}$$

The most efficient portfolio is the one with the highest Sharpe ratio.

Capital Asset Pricing Model

Beta of investment i with respect to portfolio P (or the market, if P is not specified):

$$\beta_i^P = \frac{\text{Corr}(R_P, R_i)\,\text{SD}(R_i)}{\text{SD}(R_P)} = \frac{\text{Cov}(R_P, R_i)}{\text{Var}(R_P)}$$

Required return for investment i with respect to portfolio P:

$$\begin{aligned}\mathbf{E}[R_i] - r_f &> \phi_P\,\text{SD}(R_i)\,\text{Corr}(R_i, R_P)\\ &= \beta_i^P(\mathbf{E}[R_P] - r_f)\end{aligned}$$

Beta of portfolio in terms of its assets with weights x_i:

$$\beta_P = \sum x_i \beta_i$$

Capital Market Line: expected return of efficient portfolio vs. volatility

Security Market Line: Expected return of investment vs. beta

Fundamental approach to calculating market return:

$$R_{Mkt} = \frac{\text{Div}_1}{P_0} + g$$

Debt cost of capital:

$$r_d = (1-p)y + p(y-L) = y - pL$$

where p is the probability of default, the proportion of loss upon default is L, and the annual effective interest is y

Unlevered cost of capital:

$$r_U = \frac{E}{E+D}r_E + \frac{D}{E+D}r_D$$

Unlevered beta:

$$\beta_U = \frac{E}{E+D}\beta_E + \frac{D}{E+D}\beta_D$$

Arbitrage Pricing Theory

Arbitrage Pricing Theory Equation:

$$\mathbf{E}[R_s] - r_f = \sum_{n=1}^{N} \beta_s^{Fn}\left(\mathbf{E}[R_{Fn}] - r_f\right)$$

With self-financing portfolios:

$$\mathbf{E}[R_s] = \sum_{n=1}^{N} \beta_s^{Fn}\,\mathbf{E}[R_{Fn}]$$

Fama-French-Carhart factor specification:

$$\begin{aligned}\mathbf{E}[R_s] = r_f &+ \beta_s^{Mkt}\left(\mathbf{E}[R_{Mkt}] - r_f\right) + \beta_s^{SMB}\,\mathbf{E}[R_{SMB}]\\ &+ \beta_s^{HML}\,\mathbf{E}[R_{HML}] + \beta_s^{PR1YR}\,\mathbf{E}[R_{PR1YR}],\end{aligned}$$

where SMB is the small-minus-big portfolio, HML is the high-minus-low portfolio, and PR1YR is the prior year portfolio.

Capital Structure

Perfect Capital Market:

1. Competitive prices are available to all.
2. Transactions are efficient.
3. Capital structure provides no information.

Modigliani-Miller:

In a perfect market,

I. Capital structure does not affect firm value.

II. Cost of equity capital rises with leverage.

Cost of equity capital:

$$r_E = r_U + \frac{D}{E}(r_U - r_D)$$

Equity beta:

$$\beta_E = \beta_U + \frac{D}{E}(\beta_U - \beta_D)$$

Interest tax shield: $iD\tau_C$

Present value of interest tax shield:

$$V^L = V^U + \text{PV}(\text{Interest Tax Shield})$$

V^L is computed at the WACC and V^U is computed at the pretax WACC.

For permanent debt D, the present value of the interest tax shield is $D\tau_C$.

WACC:

$$\begin{aligned}r_{\text{WACC}} &= \frac{E}{D+E}r_E + \frac{D}{D+E}r_D(1-\tau_C)\\ &= r_{\text{pretax WACC}} - \left(\frac{D}{D+E}\right)r_D\tau_C\end{aligned}$$

Indirect costs of bankruptcy:

1. Loss of customers
2. Loss of suppliers
3. Loss of employees
4. Loss of receivables
5. Fire sale of assets
6. Inefficient liquidation
7. Costs to creditors

Trade-off theory:

$$\begin{aligned}V^L = V^U &+ \text{PV}(\text{tax shield})\\ &- \text{PV}(\text{financial distress costs})\\ &- \text{PV}(\text{agency costs})\\ &+ \text{PV}(\text{agency benefits})\end{aligned}$$

Asset substitution problem: Companies in distress substitute risky assets for non-risky ones.

Debt overhang: Companies do not make positive-NPV investments because only creditors will benefit.

Approximate required NPV for equity holders to benefit for an investment of I:

$$\frac{\text{NPV}}{I} > \frac{\beta_D D}{\beta_E E}$$

Leverage ratchet effect: Presence of debt leads to issuing more debt.

Agency benefits:

1. Control of company in fewer hands.
2. Management has greater share of equity, discouraging waste.
3. No empire building.
4. Management more likely to be fired in financial distress.
5. Financial distress may lead to wage concessions.
6. More incentive to compete.

Credibility principle: Actions speak louder than words, when the words are in self-interest.

Adverse selection: Sellers with private information sell the least desirable items.

Lemons principle: Buyers discount price when seller has private information.

Pecking order hypothesis: Management prefers to finance first with retained earnings, then with debt, and only finally with equity.

Equity Financing

Pre-money valuation: value of the company before the funding round based on the price per share of the new series

Post-money valuation: value of the company after the funding round based on the price per share of the new series

Terms in financing agreements:

1. *Liquidation preference:* Minimum amount required if company is liquidated
2. *Seniority:* Get paid before earlier series, unless the series is *pari passu*
3. *Participation rights:* A clause may allow both participation in common stock dividends and liquidation preference.

4. *Anti-dilution protection:* If the funding round is a *down round* (price per share drops),
   - **Full ratchet protection:** Lowers conversion price to price of new series
   - **Broad-based weighted average protection:** The new conversion price is the old conversion price times

$$\frac{\text{Value of old shares at previous price} + \text{Cash received}}{\text{Value of old shares and new shares at previous price}}$$

5. *Board membership*

Debt Financing

Public debt:

1. *Notes:* unsecured, with terms less than 10 years
2. *Debentures:* unsecured, with terms of 10 years or more
3. *Mortgage bonds:* secured by real property
4. *Asset-based bonds:* secured by assets other than real property

International bonds:

1. *Domestic bonds:* issued domestically in local currency
2. *Foreign bonds:* issued locally by a foreign company in local currency (Known as *Yankee bonds* in the U.S.)
3. *Eurobonds:* not denominated in the currency of the country in which they are issued
4. *Global bonds:* sold in many countries simultaneously, each in its own currency

Private debt:

1. *Term loans:* loans by a bank or group of banks (usually investment grade)
2. *Private placements:* loans by a small group of investors

Sovereign debt: Debt issued by national governments. U.S. Treasuries are exempt from state and local taxes.

1. *Treasury bills:* No coupons, terms 1 year or less
2. *Treasury notes:* Semiannual coupons, terms over 1 year but no more than 10 years
3. *Treasury bonds:* Semiannual coupons, terms greater than 10 years
4. *TIPS:* Fixed coupon rates based on the inflation-adjusted principal, maturity value equal to the higher of the original face amount and the inflation-adjusted face amount

Municipal bonds: Debt issued by state and local governments. They are exempt from federal taxes (and the issuing state usually exempts them from its income tax)

1. *General obligation bonds:* payable from "full faith and credit" of the issuer
2. *Revenue bonds:* payable from specific revenue bonds
3. *Double-barreled bonds:* general obligation bond with a provision to set aside a particular revenue source

Asset-based securities:

- *Mortgage backed securities:*
  - *Ginnie Mae:* Government National Mortgage Association, explicitly guaranteed by the U.S. government
  - *Fannie Mae:* Federal National Mortgage Association
  - *Freddie Mac:* Federal Home Loan Mortgage Corporation
- *Sallie Mae:* Student Loan Marketing Association

Debts from asset-based loans can be packaged into **collateralized debt obligations (CDOs)**, with different **tranches** with different priorities.

Real Options

Option to wait is a call option.

Expenses during the year of waiting, or lost income, are treated as dividends. Free cash flows generated if the company does not wait is also a dividend, but discounted at the cost of capital rate.

Option to abandon is a put option.

Beta for an option with elasticity Ω is

$$\beta_{\text{option}} = \Omega\beta_{\text{stock}}$$