OpenPOTD

Contents

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Last updated: Tuesday 1st December, 2020
Github/sj-bs/OpenPOTD_Public

Introduction

Welcome to the OpenPOTD solutions booklet! Here you'll find answers & solutions to all past seasons..

Solutions for season one were entirely written up by Brainysmurfs#2860, while I (.19#9839) have overseen most of season two. Where possible, from season two onwards, I have tried to include the officially provided solutions to problems, or adapted them in line with any changes to the problem statement, and in most cases also and filled in the gaps as best I could, to make solutions more approachable to beginners. In many circumstances problems have not come with official write-ups and thus have required me to provide my own - in which case I apologise for any mistakes (or fakesolves!) in advance.

If any mistakes are found, feel free to DM me .19#9839 or submit a push to the Github. Similarly, if you would like to contribute an alternate solution, or improve an existing one in this document, again, feel free to submit a push or just DM me a solution and I can type it up. This applies to both recent problems and seasons, as well as older ones (no matter how old!).

Thank you (in no particular order) to the following contributers:[1]

AiYa#2278 (675537018868072458): Solution write-ups (2.1.1, 2.1.3, 2.1.4)
Tony Wang#1729 (541318134699786272): Original problem proposal (1.1.2)
Charge#3766(481250375786037258): Original problem proposal (1.1.6, 1.2.5)
bfan05#5219 (692851547062665317): Original problem proposal (1.1.7)
Kiesh#0917 (544960202101751838): Original problem proposal (1.2.3)
ChristopherPi#8528 (696497464621924394): Problem proposal (2.1.6)
Keegan#9109 (116217065978724357): Original problem proposal (2.2.3)
Slaschu#5267 (296304659059179520): Solution write-up (2.2.5)
Arjun#8974 (723413754800373780): Typo fixes

[1] The numbers are in the format: Season.Week.Problem

§1 Season 1

§1.1 Week 1

§1.1.1 Intersecting Circles

Source: United Kingdom Senior Mathematical Challenge, 2015 Q4
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 21
Date: 2020-10-27

Consider the positive integer N, and Two internally tangent circles Γ_1 and Γ_2 are given such that Γ_1 passes through the center of Γ_2. Find the fraction of the area of Γ_1 lying outside Γ_2. If this fraction is $\frac{a}{b}$ where $\gcd(a, b) = 1$, then find $a + b$.

Solution.

Suppose Γ_2 has radius $2r$. Since Γ_1 is internally tangent to Γ_2 and passes through its centre, the radius of Γ_1 is half the radius of Γ_2, i.e. just r. So fraction of the area of Γ_1 lying outside Γ_2 is $\dfrac{4\pi r^2 - \pi r^2}{4\pi r^2} = \dfrac{3}{4}$. Since $3 + 4 = 7$, the answer is $\boxed{7}$ □

§1.1.2 Guava Juice

Source: Original Problem
Proposer: Tony Wang#1729 (541318134699786272)
Problem ID: 22
Date: 2020-10-28

The ingredients list of a Guava Juice Drink is as follows:

> Water (80%), Guava Juice, Sugar, Fructose (3%), Sodium Carboxymethyl Cellulose, Citric Acid, Flavour, Vitamin C (0.04%)

Assuming only that the ingredients list is ordered by their constituent percentage in the drink (which are not necessarily distinct), find the maximum and minimum possible percentage of Guava Juice in the drink. If their difference is n, submit $100n$.

Solution.

Note there are 2 ingredients between water and fructose, and 3 between fructose and Vitamin C.

For the amount of guava juice to be maximised, everything should be minimised. In particular, there should be 0.04% of Sodium Carboxymethyl Cellulose, Citric Acid, and Flavour; and 3% of Sugar and Fructose. This gives us a Guava Juice percentage of 13.84%.

For the amount of guava juice to be minimised, everything else should be maximised. In particular, there should be 3% of Sodium Carboxymethyl Cellulose, Citric Acid, and Flavour; and equal amounts of sugar and guava juice. This gives us a guava juice percentage of 5.48%.

This gives us $13.84 - 3.98 = 9.86\%$. Multiplying this by 100 gives us $\boxed{986}$. □

§1.1.3 Complex Roots

Source: New South Wales Higher School Certificate '4U', 2020 Q2
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 23
Date: 2020-10-29

Given that $z = 3 + i$ is a root of $z^2 + pz + q = 0$, where p and q are real, find the values of p and q, and submit $p + q$.

Solution.

Applying the Complex Conjugate Theorem, $z = 3i$ is also a root of the quadratic. Expanding $(z-3-i)(z-3+i)$ gives us $z^2 - 6z + 10$. Thus $p = -6$ and $q = 10$, meaning $p + q = \boxed{4}$. $\square$

§1.1.4 Exponents

Source: Singapore Mathematical Olympiad Junior, Round 1 2001 Q4
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 25
Date: 2020-10-30

If a and b are positive reals such that $a^b = b^a$ and $b = 2a$, then the value of b^2 is?

Solution.

A direct search yields that $2^4 = 4^2$ and $4 = 2 \times 2$. The problem implies that such a unique value for b exists, hence $b^2 = \boxed{16}$. $\square$

§1.1.5 Volumes of Cubes

Source: New Zealand Senior Mathematics Competition Round 2 2019 Q7
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 26
Date: 2020-10-31

Two cubes with positive integer side lengths are such that the sum of their volumes is numerically equal to the difference of their surface areas. Find the sum of their volumes.

Solution.

A direct search yields that cubes of side length 4 and 2 satisfy the condition. The answer is thus $4^3 + 2^3 = \boxed{72}$. $\square$

§1.1.6 Complex Mess

Source: Original Problem
Proposer: Charge#3766(481250375786037258)
Problem ID: 24
Date: 2020-11-01

Suppose

$$\left(\sqrt{5}+i\sqrt{10-2\sqrt{5}}+1\right)^{2020}=a^b$$

for some $a, b \in \mathbb{Z}$ where b is maximised. Compute $a+b$.

Solution.
Notice that $\left(\sqrt{5}+i\sqrt{10-2\sqrt{5}}+1\right) = 4\text{cis}\frac{\pi}{5}$. In particular, by De Moirve's Theorem,

$$\left(\sqrt{5}+i\sqrt{10-2\sqrt{5}}+1\right)^{2020} = 4^{2020}\text{cis}404\pi = 2^{4040}$$

.

So the answer is $2+4040=\boxed{4042}$. □

§1.1.7 Paper Eating

Source: Original Problem
Proposer: bfan05#5219 (692851547062665317)
Problem ID: 31
Date: 2020-11-02

Tan and Wen are writing questions for CCCC at a rate of 1 per minute. Immediately after Tan writes a question, Wen eats his paper with probability $\frac{1}{7}$, so that Tan must restart.

After an extremely long time (assume infinite), Tony Wang walks in. What is the expected number of questions Tony Wang sees written on Tan's paper?

Solution.

Let E_n be the expected number of questions Tony Wang sees written on Tan's paper after n minutes.

Then the following recurrence holds:

$$E_{n+1} = \frac{6}{7}(E_n + 1) + \frac{1}{7} \cdot 0$$

because with $\frac{6}{7}$ probability, Tan writes another question without Wen eating it, and with $\frac{1}{7}$ probability, Wen eats all of Tan's questions.

Claim 1: $E_n < 6\ \forall n$.

Proof. The proof is by induction. Note $E_0 = 0$. If $E_k < 6$, then $E_{k+1} = \frac{6}{7}(E_k + 1) < \frac{6}{7}(6 + 1) = 6$. This completes the proof. ■

Claim 2: $E_{n+1} > E_n\ \forall n$.

Proof. Note that since $E_k < 6\ \forall k$, we have $E_{k+1} = \frac{6}{7}(E_k + 1) > \frac{6}{7}(E_k + \frac{1}{6}E_k) = E_k$. ■

Now by the Monotone Bounded Convergence Theorem, the sequence (E_n) converges to a limit. Suppose $\lim E_n = L$. Then note $\lim E_{n+1} = L$ since changing the first terms of a sequence does not change the overall convergence. So since $E_{n+1} = \frac{6}{7}(E_n + 1)$, $\lim E_{n+1} = \lim \frac{6}{7}(E_n + 1)$. So $L = \frac{6}{7}(L + 1)$.

Solving this, we obtain $L = \boxed{6}$. □

§1.2 Week 2

§1.2.1 Regenerative Watermelons

Source: British Mathematical Olympiad Round 1, 2018 Q2
Proposer: .19#9839 (434767660182405131)
Problem ID: 22
Date: 2020-11-03

Out of 100 regenerative watermelons, each of six friends eats exactly 75 watermelons. There are n watermelons eaten by at least five of the friends. What is the sum of the largest and smallest possible values of n?

Note: The watermelons are regenerative to allow multiple people to eat the same watermelon. (But the same person cannot eat the same watermelon more than once)

Solution.

Maximum: The maximum is 90. Take the sum over all watermelons of how many times they were eaten. This is 450, since all $6 \cdot 75 = 450$. Then since $5n \leq 450$, $n \leq 90$. The construction for the maximum is: staff member k ($k = 1, 2, \cdots, 6$) eats all watermelons from 1 to 90 except those $k \pmod 6$.

Minimum: The minimum is 25. We intuit that in the minimum case all the watermelons were either eaten 6 times or 4. Then $6a + 4b = 450$, and $a + b = 75$. Solving yields $a = 25$ and $b = 75$.

The answer is $25 + 90 = \boxed{115}$. $\square$

§1.2.2 Largest Prime Factor

Source: United Kingdom Senior Mathematical Challenge, 2015 Q23
Proposer: Unknown
Problem ID: 27
Date: 2020-11-04

Given four different non-zero digits, it is possible to form 24 different numbers containing each of these four digits. What is the largest prime factor of the sum of the 24 numbers?

Solution.

Say the digits are a, b, c, and d. For each digit, it appears in the units place 6 times, the tens place 6 times, the hundreds place 6 times, and the thousands place 6 times.

Hence the sum of the 24 numbers is $6666(a+b+c+d) = 2 \cdot 3 \cdot 11 \cdot 101(a+b+c+d)$. Since $a+b+c+d < 40$, it cannot have any prime factors larger than 101.

The largest prime factor of the sum of the 24 numbers is thus $\boxed{101}$. $\square$

§1.2.3 Real Roots

Source: Original Problem
Proposer: Kiesh#0917 (544960202101751838)
Problem ID: 28
Date: 2020-11-05

If the polynomial $x^2 + bx + 101 = 0$ has integer roots m and n, where $|m| > |n|$, then what is the sum of the positive integer divisors of $\frac{m}{n}$?

Solution.

By Vieta's Formula $101 = mn$. Since 101 is prime, $m, n = \pm 101, \pm 1$. So $\frac{m}{n} = \pm 101$. The positive divisors of ± 101 are 1 and 101, so their sum is $1 + 101 = \boxed{102}$. □

§1.2.4 The Meme factor

Source:

Original Problem
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 32
Date: 2020-11-06

What is the sum of all integers x such that $\frac{6969}{x}$ is an integer?

Solution.

Note that if $\frac{6969}{x}$ is an integer, so is $\frac{6969}{-x}$. So the sum is $\boxed{0}$. $\square$

§1.2.5 Human Wolfram

Source: Original Problem
Proposer: Charge#3766(481250375786037258)
Problem ID: 33
Date: 2020-11-07

Compute the number between 1000 and 2000 that divides

$$69^{69} - 5^{69} + 6^{69}.$$

Solution.

Let $N = 69^{69} - 5^{69} + 6^{69}$.

Claim 1: $64 \mid N$.

Proof. Note that $69 \equiv 5 \pmod{64} \Rightarrow 69^{69} \equiv 5^{69} \pmod{64}$. Further, $6^{69} = 64 \cdot 3^6 \cdot 6^{63}$. So $64 \mid N$. ■

Claim 2: $25 \mid N$.

Proof. Note that $69 \equiv (-6) \pmod{25} \Rightarrow 69^{69} \equiv (-6)^{69} \pmod{25}$, since 69 is odd. Thus $25 \mid N$ as required. ■

Since $64 \cdot 25 = 1600$ and the question implies there is only one answer, we obtain that the required number is $\boxed{1600}$. □

§1.2.6 Guess the Config

Source: British Mathematical Olympiad Round 2, 2008 Q2
Proposer: .19#9839 (434767660182405131)
Problem ID: 34
Date: 2020-11-08

Let triangle ABC have incentre I and circumcenter O. Suppose that $\angle AIO = 90°$ and $\angle CIO = 45°$. Suppose the ratio $AB : BC : CA$ can be expressed as $a : b : c$ where $\gcd(a, b, c) = 1$. Find $a + b + c$.

Solution.

Place a $3 - 4 - 5$ triangle on the coordinate plane with $A = (0, 0)$, $B = (3, 0)$ and $C = (0, 4)$. We can compute that $I = (1, 1)$ and $O = (1.5, 2)$. This arrangement of points satisfies the question's constraints, and so the answer is $3 + 4 + 5 = \boxed{12}$. □

§1.2.7 A Quadratic Mess

Source: Singapore Mathematical Olympiad, Open Section Round 2, 2004 Q2
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 35
Date: 2020-11-09

Find the number of ordered pairs (a, b) of integers, where $1 \le a, b \le 2004$, such that

$$x^2 + ax + b = 167y$$

has integer solutions in x and y.

Note: You are allowed a four-function calculator.

Solution.

Note that 167 is prime. So

$$\begin{aligned} x^2 + ax + b &\equiv 0 \pmod{167} \\ \iff (x + \frac{a}{2})^2 - \frac{a^2}{4} + b &\equiv 0 \pmod{167} \\ \iff (2x + a)^2 - a^2 + 4b &\equiv 0 \pmod{167} \end{aligned}$$

So $a^2 - 4b$ is a quadratic residue mod 167. Because 167 is an odd prime, there are exactly $\frac{167+1}{2} = 84$ such quadratic residues. This means that for each choice of a, for which there are 2004, there are 84 choices of b between 1 and 167. Since $2004 = 12 \cdot 167$, there are $12 \cdot 84$ choices of b in total.
So the answer is $2004 \cdot 12 \cdot 84 = \boxed{2020032}$. □

§2 Season 2 (.19's Season)

§2.1 Week 1

§2.1.1 A Sequence of 5's

Source: United Kingdom - Maclaurin, 2015 M1
Proposer: .19#9839 (434767660182405131)
Problem ID: 47
Date: 2020-11-16

Consider the sequence $5,\ 55,\ 555,\ 5555, \ldots$

How many digits does the smallest number in the sequence have which is divisible by 495?

Solution.

We require the term to be divisible by $5 \cdot 9 \cdot 11$. Hence we need only consider the sequence $1,\ 11,\ 111 \ldots$ with respect to $9 \cdot 11$. Clearly for odd numbered terms in the sequence, 11 does not divide into it, by the well-known divisibility rule for 11. Therefore, we require an even numbered term in the sequence, which is divisible by 9. We know 9 divides a number iff its digital sum is also divisible by 9. Hence, the smallest such will be the 18th term in the sequence, which will naturally have $\boxed{18}$ digits. □

Solution. [Write up by AiYa#2278 (675537018868072458)]

Each of these numbers can be written as $5 \cdot 1 \ldots 1$,whee there are n total ones. This can be rewritten as $5 \cdot (10^{n-1} + 10^{n-2} + \cdots + 10^0) = \frac{5}{9}(10^n - 1)$. Note that $495 = 9 \cdot 11 \cdot 5$ so we want $9 | \frac{10^n-1}{9}$ and $10^n \equiv 1 \mod 11$. From the congruence $\mod 11$ we see that n must be even. Note that $10^n - 1 = 9 \ldots 9$, where there are n total nines; if n is a multiple of 9 then $\frac{10^n-1}{9} = 1 \ldots 1$ where there are n total ones; this is a multiple of 9. Since n must be even, the smallest such n is $\boxed{18}$. □

§2.1.2 Brainy's Happy Set

Source: British Mathematical Olympiad - Round 1, 2010/2011 P1
Proposer: .19#9839 (434767660182405131)
Problem ID: 40
Date: 2020-11-17

Brainy has a set of integers, from 1 to n, which he likes to play with. Tony Wang, upon seeing the happiness that this set of integers brings Brainy, decides to steal one of the numbers in it. Suppose the average number of the remaining elements in the set is $\frac{163}{4}$. What is the sum of the elements in Brainy's set multiplied by the element that Tony stole?

(A four-function calculator may be used)

Solution.

We can set up the problem statement as

$$\frac{\frac{n}{2}(n+1)-x}{n-1}=\frac{163}{4}$$

Where x is the number Tony has stolen. This simplifies to $4x = 2n^2 - 161n + 163$. Since x must be a number within the set $\{1, 2, \ldots, n-1, n\}$, we have that $1 \le x \le n \Rightarrow 4 \le 2n^2 - 161n + 163 \le 4n$. By considering the lower bound, we get $(2n-159)(n-1) \ge 0$. This means that $n \le 1 \Rightarrow n = 1$, or $n \ge \frac{159}{2} \Rightarrow n \ge 80$. By similar methodology when considering the upper bound, we get $1 \le n \le 81$. Thus $n \in \{1, 80, 81\}$. Clearly, $n \ne 1$, so either $n = 80$ or $n = 81$. Notice that if n is even, then for $4x = 2n^2 - 161n + 163$ the parity of he RHS is Odd, while the LHS is even, thus a contradiction occurs. This means that $n = 81$ and so $x = 61$. Thus the answer is $\frac{81(82)}{2} \cdot 61 = \boxed{202581}$. □

§2.1.3 MODSbot's Escape!

Source: Mathematics Admissions Test, 2012 Q5
Proposer: .19#9839 (434767660182405131)
Problem ID: 48
Date: 2020-11-18

In his evil mechatronics laboratory, Brainy has built a physical manifestation of MODSbot. MODSbot's movement is defined by three inputs: **F** to move forward a unit distance, **L** to turn left 90°, and **R** to turn right 90°.

We define a program to be a sequence of commands. The program P_{n+1} (for $n \geq 0$) involves performing P_n, turning left, performing P_n again, then turning right:

$$P_{n+1} = P_n\mathbf{L}P_n\mathbf{R},\ \ P_0 = \mathbf{F}$$

Unbeknownst to Brainy, MODSbot, though limited in movement, is sentient and realises Brainy is just a small asian Frankenstein, whose intentions for them were nefarious and non-consensual. As a result, after Brainy goes home for the day, MODSbot makes its escape from Brainy's laboratory.

Let (x_n, y_n) be the position of the robot after performing the program P_n, so $(x_0, y_0) = (1, 0)$ and $(x_1, y_1) = (1, 1)$, etc.

How far away from the place Brainy left it does MODSbot make it after performingP_{24}?

Solution.

Note first that after each iteration of P_n MODSbot faces in the positive x direction, as each P_n contains as many **L**s as it does **R**s. Now, assuming MODSbot is at (x_n, y_n) after having performed P_n, we see the next iteration of P puts MODSbot at $(x_n - y_n, x_n + y_n)$. Note then that:

$$\begin{aligned}
(x_{n+2}, y_{n+2}) &= (x_{n+1} - y_{n+1}, x_{n+1} + y_{n+1}) = (-2y_n, 2x_n)\\
(x_{n+4}, y_{n+4}) &= (-2y_{n+2}, 2x_{n+2}) = (-4x_n, -4y_n)\\
(x_{n+8}, y_{n+8}) &= (-4x_{n+4}, 4y_{n+4}) = (16x_n, 16y_n)
\end{aligned}$$

Thus, we see that $(x_{8k}, y_{8k}) = (16^k, 0)$, and therefore that $|P_{24}| = \boxed{4096}$ □

Solution. [Write up by AiYa#2278 (675537018868072458)]

Observe that each program has the same amount of left and right turns, so MODSbot will always be facing the positive x-direction after each program. This means that $\mathbf{L}P_n$ is just the program P_n performed at a 90-degree counterclock-wise rotation. For instance P_1 moves MODSbot right 1 and up 1, so $\mathbf{L}P_1$ moves MODSbot up 1 and left 1 (right gets rotated 90 counterclockwise to up and up to left). This motivates us to work in the complex plane; let P_n be the complex-number representing MODSBOT's displacement after following P_n. Then $\mathbf{L}P_n = iP_n$, so $P_{n+1} = P_n + \mathbf{L}P_n = (1+i)P_n = \sqrt{2}e^{\frac{\pi i}{4}}P_n$. WIth $P_0 = 1$ we get $P_n = 2^{\frac{n}{2}}e^{\frac{\pi i n}{4}}$. So $|P_{24} = \boxed{4096}$ □

§2.1.4 Sides of a Polygon

Source: China Western MO 2003, Day 1 P2
Proposer: .19#9839 (434767660182405131)
Problem ID: 51
Date: 2020-11-20

Let $a_1, a_2, \ldots, a_{24n}$ be real numbers with $\sum_{i=1}^{24n-1}(a_{i+1}-a_i)^2 = 1$.

For some $n > 0$ the maximum value of $(a_{12n+1} + a_{12n+2} + \cdots + a_{24n}) - (a_1 + a_2 + \cdots + a_{12n})$ is twice that of a prime.

What is the sum of the value of that prime and the corresponding value of n?

Solution.

If we substitute $x_{i+1} = a_{i+1} - a_i$, we get $a_i = \sum_1^i x_r$, and thus our constraint becomes

$$\sum_{i=1}^{24n-1}(a_{i+1}-a_i)^2 = \sum_{i=1}^{24n} x_i^2 = 1$$

Putting the bit we wish to be maximising in terms of the substitution gives:

$$\sum_{i=1}^{12n} a_i = 12nx_1 + (12n-1)x_2 + \cdots + 2x_{12n-1} + x_{12n}$$
$$\sum_{i=12n+1}^{24n} a_i = 12n(x_1 + \cdots + x_{12n+1}) + (12n-1)x_{12n+2} + \cdots 2x_{24n-1} + x_{24n}$$

Hence,

$$\sum_{i=12n+1}^{24n} a_i - \sum_{i=1}^{12n} a_i = x_2 + \cdots + (12n-1)x_{12n} + 12nx_{12n+1} + (12n-1)x_{12n+2} + \cdots + x_{24n}$$

Then by Cauchy-Schwarz:

$$\sum_{i=12n+1}^{24n} a_i - \sum_{i=1}^{12n} a_i \leq \sqrt{\left((12n)^2 + 2\sum_{i=1}^{12n-1} i^2\right)\left(\sum_{i=1}^{24n} x_i^2\right)} = \sqrt{(12n)^2 + \frac{12n(12n-1)(24n-1)}{3}}$$
$$\leq \sqrt{4n(2(12n)^2+1)} = 2p$$

Now we want values of n such that $n(288n^2+1) = p^2$ for a prime p. Since trivially for all n, $n < 288n^2+1$, we have $n = 1$ and $288n^2 + 1 = p^2$, hence $p = 17$, so the answer is $\boxed{18}$. □

§2.1.5 2p

Source: China Western MO 2003, Day 1 P2
Proposer: .19#9839 (434767660182405131)
Problem ID: 51
Date: 2020-11-20

Let $a_1, a_2, \ldots, a_{24n}$ be real numbers with $\displaystyle\sum_{i=1}^{24n-1}(a_{i+1}-a_i)^2 = 1$.

For some $n > 0$ the maximum value of $(a_{12n+1} + a_{12n+2} + \cdots + a_{24n}) - (a_1 + a_2 + \cdots + a_{12n})$ is twice that of a prime.

What is the sum of the value of that prime and the corresponding value of n?

Solution.

If we substitute $x_{i+1} = a_{i+1} - a_i$, we get $a_i = \sum_1^i x_r$, and thus our constraint becomes

$$\sum_{i=1}^{24n-1}(a_{i+1}-a_i)^2 = \sum_{i=1}^{24n} x_i^2 = 1$$

Putting the bit we wish to be maximising in terms of the substitution gives:

$$\sum_{i=1}^{12n} a_i = 12nx_1 + (12n-1)x_2 + \cdots + 2x_{12n-1} + x_{12n}$$
$$\sum_{i=12n+1}^{24n} a_i = 12n(x_1 + \cdots + x_{12n+1}) + (12n-1)x_{12n+2} + \cdots 2x_{24n-1} + x_{24n}$$

Hence,

$$\sum_{i=12n+1}^{24n} a_i - \sum_{i=1}^{12n} a_i = x_2 + \cdots + (12n-1)x_{12n} + 12nx_{12n+1} + (12n-1)x_{12n+2} + \cdots + x_{24n}$$

Then by Cauchy-Schwarz:

$$\sum_{i=12n+1}^{24n} a_i - \sum_{i=1}^{12n} a_i \leq \sqrt{\left((12n)^2 + 2\sum_{i=1}^{12n-1} i^2\right)\left(\sum_{i=1}^{24n} x_i^2\right)} = \sqrt{(12n)^2 + \frac{12n(12n-1)(24n-1)}{3}}$$
$$\leq \sqrt{4n(2(12n)^2+1)} = 2p$$

Now we want values of n such that $n(288n^2+1) = p^2$ for a prime p. Since trivially for all n, $n < 288n^2+1$, we have $n = 1$ and $288n^2 + 1 = p^2$, hence $p = 17$, so the answer is $\boxed{18}$. □

§2.1.6 Slippery Rooks

Source: AMOC 2019 December School Prep Problems C5
Proposer: ChristopherPi#8528 (696497464621924394)
Problem ID: 54
Date: 2020-11-22

MODSbot is trying to get rich by scamming MODS members out of their money, so it's devised a chess game on a 2020×2020 chessboard for unsuspecting people to attempt before they can enter **.19's EPIC QoTD Party**. Suppose Brainy, Ishan, Nyxto, Adam, Bubble, Sharky and Christopher all get scammed by MODSbot, that is, MODSbot plays the chess game against all 7 at the same time on different boards.

The group decide to pool together their money which comes to a total of 4.20 BTC, and to play, they'll need to buy n batches of slippery rooks from MODSbot. A batch of slippery rooks contains one white and four black rooks, and each batch is sold at a price equivalent to 0.069 BTC per rook. Once the batches of rooks have been bought, the group may choose to distribute them in a way which allows all members to beat the game.

In the game, only one white rook may be placed on the board, and we define how slippery rooks move as follows: it slips along the row or column it's moved along and comes to rest on an empty square because it is obstructed by either the edge of the board or another rook. Initially, MODSbot places the rooks on the board randomly, and marks a square red. Then the person being scammed can choose any rook on each turn and move as allowed, and attempt to place the white rook on the red square in a finite number of moves.

The amount of money they have left over after buying the smallest n batches rooks to guarantee that they all succeed in beating MODSbots game is k BTC. What is the value of $1000k$?

Solution. [Write up by ChristopherPi]

Consider simply the case of one person. We prove that three rooks are required, one white and two black.

First we show that two are not enough: simply place the two rooks at corners of the board and mark any square not on the side of the board. It's clear that neither rook can ever move to a square not on the side of the board. Now we show three are enough.

Suppose square (a, b) is marked, where $(1, 1)$ is the bottom left corner and $(2020, 2020)$ is the top right corner. Trivially one can move the black rooks to $(1, 1)$ and $(2, 1)$ and the white rook to $(2020, 2020)$. Next, simply "loop" the black rooks as follows: take the rook further left, and move it up, right, down and left such that it moves to the right of the rook originally on its right, and repeat until you place a black rook at $(a - 1, 1)$. Now if b is odd, move the white rook to $(a, 1)$ and the black rook at $(a - 2, 1)$ to $(2020, 2020)$; if it's even, loop the leftmost black rook one more time to place it at $(a, 1)$.

Now move the rook at $(a - 1, 1)$ to $(a - 1, 2020)$, and move the rook at $(2020, 2020)$ left then down to $(a, 2)$. Next we describe another "looping" procedure: take the rook with first coordinate a and smaller second coordinate, and move it right, up, left and down, so it now has first coordinate a and second coordinate larger than the other rook with first coordinate a. Repeat this until you place a rook at (a, b) - since the colour of the rook placed at $(a, 1)$ is dependent on the parity of b, this ensures that the rook placed at (a, b) must be a white rook.

This procedure won't work if either of (a, b) is 1 or 2020, or both of a and b are either 2 or 2019. In the first case, rotate the board such that $a = 1$. Now place a black rook at $(2020, 2020)$. If b is odd, place the white rook at $(1, 1)$ and the other black rook at $(2, 1)$; else place the other black rook at $(1, 1)$ and the white rook at $(2, 1)$. Now use the first looping procedure until the white rook is placed at $(1, b)$ as required - since the position of the white rook depends on the parity of b this is certain to work. In the second case, rotate the board such that $(a, b) = (2, 2)$. Now you can trivially move the white rook to $(1, 1)$ and the black rooks to $(2, 1)$ and $(1, 2020)$. Now move the white rook up, right, up, left and down to place it at $(2, 2)$ as required.

This shows that three is sufficient for one person. Hence the group must buy 7 batches because each of them needs a white rook, and one batch contains one white rook. Therefore, the answer is $1000(4.2 - 7 \cdot 5 \cdot 0.069) = \boxed{1785}$. □

§2.1.7 Sets of Integer Solutions

China Mathematical Olympiad, 2005 Day 2 P6
Proposer: .19#9839 (434767660182405131)
Problem ID: 55
Date: 2020-11-23

Define functions f and g such that $f(a,b) = 2^a 3^b$, and $g(c,d) = 5^c 7^d$, for $a,b,c,d \in \mathbb{Z}_{\geq 0}$.

Given $f(a,b) = 1 + g(c,d)$, what is the sum of all valid b's, c's and d's, multiplied by the sum of all valid a's?

For example if we had valid solutions of $(a,b,c,d) = (1,1,2,4), (5,1,6,2), (0,0,0,0)$

Then the answer would be $(\underbrace{1+1+0}_{b's} + \underbrace{2+6+0}_{c's} + \underbrace{4+2+0}_{d's}) \times (\underbrace{1+5+0}_{a's}) = 96$

Solution.

We proceed by considering parity, for $2^a 3^b = 5^c 7^d + 1$, we have the RHS as even, thus we must have $a \geq 1$. If we let $b = 0$, then for $2^a - 5^c \cdot 7^d = 1$, we have $2^a \equiv 1 \mod 5$ for $c \neq 0$. This gives $a \equiv 0 \mod 4$, so $2^a - 1 \equiv 0 \mod 3$. But this clearly cannot be the case so we must have $c = 0$ when $b = 0$.
Hence, we consider $2^a - 7^d = 1$. Bashing gives $(a,d) = (1,0),\ (3,1)$. hese are the only such solutions as for $a > 4,\ 7^d \equiv -1 \mod 16$, but this is impossible. So for the case of $b = 0$ all possible non-negative integer solutions are $(1,0,0,0),\ (3,0,0,1)$.

Now let $b > 0$ and $a = 1$, so we now consider $2 \cdot 3^b - 5^c \cdot 7^d = 1$ under $\mod 3$, which gives $-5^c 7^d \mod 3$. Since $7^d \equiv 1 \mod 3$, for all $d \geq 0$, we are left with $(-1)^c 5^c \equiv 1 \mod 3$. Now $5^c = \{1,2\} \mod 3$, thus we see that we must have c being odd. Under $\mod 5$, we see that $2 \cdot 3^b \equiv 1 \mod 5$, $3^{b-1} \equiv 1 \mod 5$. As we observe that $3^b \equiv \{3,4,2,1\} \mod 5$, we must have $b \equiv 1 \mod 4$. If $d \neq 0$, then $2 \cdot 3^b \equiv 1 \mod 7$. Again observe that $3^b \equiv \{3,2,6,4,5,1\} \mod 7$, we see $b \equiv 4 \mod 6$. But $b \equiv 1 \mod 4$, so a contradiction arises, and thus $d = 0$ and hence $2 \cdot 3^b - 5^c = 1$. For $b = 1$, clearly $c = 1$. So if $b \geq 2$, then $5^c \equiv -1 \mod 9 \Rightarrow c \equiv 3 \mod 6$. Therefore $5^c + 1 \equiv 0 \mod (5^3 + 1) \Rightarrow 5^c + 1 \equiv 0 \mod 7$, but this contradicts the fact that $5^c + 1 = 2 \cdot 3^b$. Hence in this case we only have one solution $(a,b,c,d) = (1,1,1,0)$.

Finally, consider the case where $b > 0$, and $a \geq 0$. Then we have $5^c 7^d \equiv -1 \mod 4$, and $5^c 7^d \equiv -1 \mod 3$, i.e. $(-1)^d \equiv -1 \mod 4$ and $(-1)^c \equiv -1 \mod 3$. Therefore we have that both c and d being odd. Thus, $2^a 3^b = 5^c 7^d + 1 \equiv 4 \mod 8$. So $a = 2$ and thus $4 \cdot 3^b \equiv 1 \mod 5$ and $4 \cdot 3^b \equiv 1 \mod 7$. This gives $b \equiv 2 \mod 12$. Substituting $b = 12k + 2$ for $k \in \mathbb{Z}_{\geq 0}$, then $5^c 7^d = (2 \cdot 3^{6k+1} - 1)(2 \cdot 3^{6k+1} + 1)$.
Now as $\gcd(2 \cdot 3^{6k+1} + 1, 2 \cdot 3^{6k+1} - 1)$, $2 \cdot 3^{6k+1} - 1 \equiv 0 \mod 5$, therefore $2 \cdot 3^{6k+1} - 1 = 5^a$ and $2 \cdot 3^{6k+1} = 7^d$. If $k \geq 1$, then $5^c \equiv -1 \mod 9$. But this is impossible, so if $k = 0$, then $b = 2,\ c = 1$, and $d = 1$. Thus in this case, we have only one solution: $(a,b,c,d) = (2,2,1,1)$.
Hence we can conclude all non-negative integer solutions are

$$(a,b,c,d) = \begin{cases} (1,0,0,0) \\ (3,0,0,1) \\ (1,1,1,0) \\ (2,2,1,1) \end{cases}$$

This then gives us an answer of $\underbrace{(0+0+0+0+0+1+1+1+0+2+1+1)}_{7} \times \underbrace{(1+3+1+2)}_{7} = \boxed{49}$

$\square$

§2.2 Week 2

§2.2.1 A Game of Deductions

Source: Mathematics Admissions Test, 2014 Q6
Proposer: .19#9839 (434767660182405131)
Problem ID: 56
Date: 2020-11-24

CircleThm plays two rounds of a deduction game with Wen and Tan. In each round, CircleThm picks two integers x and y so that $1 \leq x \leq y$. He then whispers the sum of the two chosen integers to Wen, and the product of the two integers to Tan. Neither Wen nor Tan knows what CircleThm told the other. In the game, Tan and Wen must try to work out what the numbers x and y are using logical deductions.

In the first round, suppose the product of the two chosen numbers, x_1 and y_1 is 8.

Tan says *"I don't know what x_1 and y_1 are"*

Wen then says *"I already knew that"*

Tan then says *"I now know x_1 and y_1"*

In the second round, suppose the sum of the two chosen numbers x_2 and y_2 is 5.

Tan says *"I don't know what x_2 and y_2 are"*

Wen then says *"I don't know what x_2 and y_2 are"*

Tan then says *"I don't know what x_2 and y_2 are"*

Wen then says *"I now know what x_2 and y_2 are"*

What is $(x_1x_2 + y_1y_2)^3$?

Solution.

The first thing to observe is that Tan can only immediately deduce the values of $\{x, y\}$ if, and only if, the prime factorisation of that number is unique - i.e. xy is prime.
If the product of $\{x_1, y_1\}$ is 8, then the decomposition can be either $\{1, 8\}$ or $\{2, 4\}$. However, if the decomposition was $\{2, 4\}$, then Wen would have a sum of 6, so from their point of view the decomposition could potentially have been $\{1, 5\}$, in which case Wen would have known that Tan would have known the decomposition as well - as the only way to achieve a product of 5 is from $\{1, 5\}$. Therefore the decomposition must have been $\{1, 8\}$.
For the second part, the decomposition's allowed are $\{1, 4\}$ and $\{2, 3\}$. Assume that it is $\{1, 4\}$. Then, Tan only knows the product is 4, which mean Tan believes the decomposition is either $\{1, 4\}$ or $\{2, 2\}$. If the decomposition was indeed $\{2, 2\}$, then Wen would know that the sum is also 4, and thus that Wen would think that Tan sees a composition of $\{1, 3\}$ or $\{2, 2\}$. Tan's first statement would show Wen that the decomposition was not $\{1, 3\}$ (as then Tan would instantly know the decomposition)- in which Wen should know that the decomposition is $\{2, 2\}$. By Wen's first statement Tan then should know by their second statement that the decomposition is $\{1, 4\}$; by Tan saying in their second statement that they don't know what the decomposition is, Wen then knows it must be $\{2, 3\}$. Thus the solution is $(1 \cdot 2 + 8 \cdot 3)^3 = \boxed{17576}$ □

§2.2.2 Maximising Exponents

Source: Sixth Term Examination Paper - III, 1996 Q4
Proposer: .19#9839 (434767660182405131)
Problem ID: 57
Date: 2020-11-25

Consider the positive integer N, and let $\mathcal{Q}(N)$ denote the maximised product of integers that sum to N.

What is the sum of the exponents of the prime factorisation of $\mathcal{Q}(1262)$?

For example: $\mathcal{Q}(6) = \cdot 3^2$, and $\mathcal{Q}(4) = 2^2$, in the respective cases the sum of the exponents is 2, so the answer you would submit is 2.

Solution.

Let us work in the general case by first constructing a methodology which maximises product while keeping the sum constant. Consider $N = n_1 + n_2 + \cdots + n_k$, and $P(N) = n_1 n_2 \cdots n_k$. For any $n_i \geq 4$, clearly we can replace it with $(n_i - 2) + 2$, which keeps the sum constant and increases the product (since $n_i \leq 2(n_i - 2)$). Hence W.O.L.G assume all $n_i < 4$. This means that we can maximise the product of integers that sum to N by arranging it into some combination of 2's and 3's. If $N \equiv 0 \mod 3$ trivially we set all n_i's equal 3. So $\mathcal{Q}(3k) = 3^{\frac{N}{3}}$ for an integer k. In the case of $N \equiv 1 \mod 3$, consider$\mathcal{Q}(3k+1)$. We have $\frac{N}{3}$ 3's in n_i, and then a 1, or $\frac{N}{3} - 1$ 3's, and then a 2^2. Clearly in the latter case, the product is maximised. Hence $\mathcal{Q}(3k+1) = 2^2 \cdot 3^{\frac{N-4}{3}}$. A similar train of thought yields $\mathcal{Q}(3k+2) = 2 \cdot 3^{\frac{N-2}{3}}$ for $N \equiv 2 \mod 3$

Therefore, we have the following result:

$$\mathcal{Q}(N) = \begin{cases} 3^{\frac{N}{3}} & \text{if } N \equiv 0 \mod 3 \\ 2^2 \cdot 3^{\frac{N-4}{3}} & \text{if } N \equiv 1 \mod 3 \\ 2 \cdot 3^{\frac{N-2}{3}} & \text{if } N \equiv 2 \mod 3 \end{cases}$$

Since $1262 \equiv 2 \mod 3$, we have $\mathcal{Q}(1262) = 2 \cdot 3^{\frac{1262-2}{3}}$, hence the sum of the exponents is$1 + 420$, so the answer is $\boxed{421}$. $\square$

§2.2.3 Colourful Problem

Source: Original
Proposer: Keegan#9109 (116217065978724357)
Problem ID: 59
Date: 2020-11-26

Let n be a positive integer.
The p-value of n, denoted $p(n)$:

The number of digit-sums needed to reduce n to a single digit.
Examples:

$69 \to 6+9 \to 15 \to 1+5 \to 6$ needs two digit-sums, so $p(69) = 2$.
$203 \to 2+0+3 \to 5$ needs only a single digit-sum, so $p(203) = 1$.
Clearly $p(5) = 0$.

Let P_k be the set of all n such that $p(n) = k$. Given that $a, b, c \in \mathbb{N}$, and

$$\min(P_5) = a \times 10^b - c$$

What is the value of $\min(a+b+c) \pmod{\min(P_3)}$?

Solution. □

§2.2.4 Combinatoral Addition

Source: Folklore / Classic Problem
Proposer: .19#9839 (434767660182405131)
Problem ID: 58
Date: 2020-11-27

Let x_1, x_2, x_3, x_4 be integers such that $1 \leq x_1, x_2, x_3, x_4 \leq 9$.

How many solutions are there to $x_1 + x_2 + x_3 + x_4 = 26$?

(A four-function calculator may be used)

Solution.

This problem is a piece of PiE! The total number of possible solutions without restriction is going to be $\binom{26-1}{4-1} = \binom{25}{3}$, and we now must subtract all the solutions which do not fit the restriction on x_i. The number of solutions such that one of the x_i's is greater than 9 is going to be $\binom{26-1-9}{3}$. Similarly the number of solutions that two of the integers is going to be greater than 9 is going to be $\binom{26-9-9-1}{3} = \binom{7}{3}$. Note that there are no integers such that more than 3 of them are greater than 9 since $3 \cdot 9 > 26$. Now there are $\binom{4}{1}$ ways to select the x_i's such that one integer is greater than 9, and similarly there are $\binom{4}{2}$ ways to select two integers greater than 9 in the solution. Hence we have $\binom{25}{3} - \binom{4}{1}\binom{16}{4} + \binom{4}{2}\binom{7}{3} = \boxed{270}$ possible solutions. ☐

§2.2.5 Expected Values

Source: HMMT 2013 C6
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 60
Date: 2020-11-28

Values $a_1, a_2, \ldots, a_{2020}$ are chosen independently and at random from the set $1, 2, \ldots, 2020$. What is the floor of the expected number of distinct values in the set $a_1, a_2, \ldots, a_{2020}$?

(A scientific calculator may be used)

Solution. [Write up by Slaschu#5267 (296304659059179520)]

This problem may look daunting at first, 2020 numbers chosen out of a set of 2020 numbers is quite a handful. We can start the problem by considering a 2020-sided die instead, we are essentially rolling a die 2020 times then looking at the number we get. To simplify things a bit, and to better understand what is going on I tried the problem with a 6-sided die that is rolled 6 times instead.

Let us try finding the probability of getting a number apart from 1 after rolling 6 times:

$$\text{Firstroll}: \frac{5}{6}$$
$$\text{Secondroll}: \frac{5}{6} \cdot \frac{5}{6}$$
$$\ldots$$
$$\text{Sixthroll}: \left(\frac{5}{6}\right)^6$$

Therefore, there probability of getting the number 1 at least once is $1 - \left(\frac{5}{6}\right)^6$. Similarly, for the 2020-sided die we have a $1 - \left(\frac{2019}{2020}\right)^{2020}$ chance of getting 1 at least once. As this probability is the same for all the other numbers from 1 to 2020, we can say that the probability of getting a distinct value at least once is also $1 - \left(\frac{2019}{2020}\right)^{2020}$. Since we are trying to find the number of distinct values obtained from 2020 rolls, we compute the following: $2020\left(1 - \left(\frac{2019}{2020}\right)^{2020}\right)$. This results in our answer of $\boxed{1277}$.

□

§2.2.6 Trigonometric Product

Source: Unknown
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 61
Date: 2020-11-29

Evaluate the infinite product

$$690 \prod_{k=2}^{\infty} \left(1 - 4\sin^2 \frac{\pi}{3 \cdot 2^k}\right)$$

Solution.

By the double angle formula and difference of squares, we have

$$\begin{aligned} 1 - 4\sin^2(x) &= 2\cos(2x) - 1 \\ &= \frac{4\cos^2(2x) - 1}{2\cos(2x) + 1} \\ &= \frac{2\cos(4x) + 1}{2\cos(2x) + 1} \end{aligned}$$

Thus,

$$\begin{aligned} 690 \prod_{k=2}^{\infty} \left(1 - 4\sin^2 \left(\frac{\pi}{3 \cdot 2^k}\right)\right) &= 690 \prod_{k=0}^{\infty} \left(\frac{2\cos\left(\frac{\pi}{3 \cdot 2^k}\right) + 1}{2\cos\left(\frac{\pi}{6 \cdot 2^k}\right) + 1}\right) \\ &= 690 \frac{2\cos\left(\frac{\pi}{3}\right) + 1}{\lim\limits_{n \to 0} (2\cos(n) + 1)} \\ &= 690 \cdot \frac{2}{3} \end{aligned}$$

Hence, our answer is $\boxed{460}$ □

§2.2.7 Projective Geo

Source: OMO, Fall 2017 P28
Proposer: .19#9839 (434767660182405131)
Problem ID: 62
Date: 2020-11-30

Define a triangle ABC, with sides $AB : AC : BC = 7 : 9 : 10$. Further, for the circumcircle of ABC, ω, let the circumcenter be O, and the circumradius to be R. The tangets to ω at points B and C meet at X, and a variable line l passes through O. Define A_1 to be the projection of X onto l, and A_2 to be the reflection of A_1 over O. Suppose that there exists two points $Y,\ Z$ on l such that $\angle YAB + \angle YBC + \angle YCA = \angle ZAB + \angle ZCA = 90°$, where all angles are directed, furthermore that O lies inside segment YZ with $OYOZ = R^2$. Then there are several possible values for the sine of the angle at which the angle bisector of $\angle AA_2O$ meets BC. If the product of these values can be expressed in the form $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c, with b squarefree and a, c coprime, determine $a + b + c$.

Solution.

OMO Fall 2017 solutions (P28)

□

§3 Trigonometric Troubles (Season 3)

§3.1 Week 1

§3.1.1 Maximising Trig. Function

Source: Mathematics Admissions Test, 2020 Q1.D
Proposer: .19#9839 (434767660182405131)
Problem ID: 63
Date: 2020-11-30

The largest value achived by $3\cos^2 x + 2\cos x + 1$ can be represented as $\frac{m}{n}$ as a fraction in lowest terms. Find $m + n$.

Solution.

We proceed by using the identity $\cos^2 x = 1 - \sin^2 x$:

$$\begin{aligned} 3\cos^2 x + 2\sin x + 1 &= 3(1 - \sin^2 x) + 2\sin x + 1 \\ &= 4 + 2\sin x - 3\sin^2 x \end{aligned}$$

This is a quadratic in $\sin x$, specifically it is a convex parabola. Completing the square gives:

$$4 + 2\sin x - 3\sin^2 x = \frac{13}{3} - 3\left(\sin x - \frac{1}{3}\right)^2$$

Since for all values of $\sin x \neq \frac{1}{3}$, the function $f(x) = \frac{13}{3} - 3\left(\sin x - \frac{1}{3}\right)^2$ is clearly decresaing, we must have a maximum at $\sin x = \frac{1}{3}$, giving a value of $\frac{13}{3}$. So the answer is $\boxed{16}$.

$\square$