# OpenPOTD

## Contents




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Github/sj-bs/OpenPOTD_Public

This is a temporary document to let people who have alternative solutions contribute to the end of season write up. Solutions have already been written up for the entire season, and where possible I have tried to include the officially provided solution, adapted them if the problem statement is changed, and filled in the gaps as best I could. In many circumstances problems for this season have not come with official write-ups and thus have required me to provide my own - in which case I apologise for any mistakes (or fakesolves!) in advance. If any mistakes are found, feel free to DM me .19#9839 or submit a push. Similarly, if you would like to contribute an alternate solution to whatever is in this document, again, feel free to submit a push or just DM me a solution and I can type it up.

At the end of each day I'll add the question and solution to the document - the end of each day naturally being such that the question and solution being added is no longer an active one.

Thank you to:

AiYa#2278 (675537018868072458)

For contributing to solutions!

# §1 A Sequence of 5's

*Source: United Kingdom - Maclaurin, 2015 M1*
*Proposer: .19#9839 (434767660182405131)*
*Problem ID: 47*
*Date: 2020-11-16*

Consider the sequence $5,\ 55,\ 555,\ 5555, \dots$

How many digits does the smallest number in the sequence have which is divisible by 495?

*Solution.*

We require the term to be divisible by $5 \cdot 9 \cdot 11$. Hence we need only consider the sequence $1,\ 11,\ 111 \dots$ with respect to $9 \cdot 11$. Clearly for odd numbered terms in the sequence, 11 does not divide into it, by the well-known divisibility rule for 11. Therefore, we require an even numbered term in the sequence, which is divisible by 9. We know 9 divides a number iff its digital sum is also divisible by 9. Hence, the smallest such will be the 18th term in the sequence, which will naturally have $\boxed{18}$ digits. □

*Solution.* [Write up by AiYa#2278 (675537018868072458)]

Each of these numbers can be written as $5 \cdot 1 \dots 1$,whee there are $n$ total ones. This can be rewritten as $5 \cdot (10^{n-1} + 10^{n-2} + \dots + 10^0) = \frac{5}{9}(10^n - 1)$. Note that $495 = 9 \cdot 11 \cdot 5$ so we want $9|\frac{10^n-1}{9}$ and $10^n \equiv 1 \mod 11$. From the congruence $\mod 11$ we see that $n$ must be even. Note that $10^n - 1 = 9 \dots 9$, where there are $n$ total nines; if $n$ is a multiple of 9 then $\frac{10^n-1}{9} = 1 \dots 1$ where there are $n$ total ones; this is a multiple of 9. Since $n$ must be even, the smallest such $n$ is $\boxed{18}$. □

## §2 Brainy's Happy Set

*Source: British Mathematical Olympiad - Round 1, 2010/2011 P1*
*Proposer: .19#9839 (434767660182405131)*
*Problem ID: 40*
*Date: 2020-11-17*

Brainy has a set of integers, from 1 to $n$, which he likes to play with. Tony Wang, upon seeing the happiness that this set of integers brings Brainy, decides to steal one of the numbers in it. Suppose the average number of the remaining elements in the set is $\frac{163}{4}$. What is the sum of the elements in Brainy's set multiplied by the element that Tony stole?

*(A four-function calculator may be used)*

*Solution.*

We can set up the problem statement as

$$\frac{\frac{n}{2}(n+1)-x}{n-1}=\frac{163}{4}$$

Where $x$ is the number Tony has stolen. This simplifies to $4x = 2n^2 - 161n + 163$. Since $x$ must be a number within the set $\{1, 2, \ldots, n-1, n\}$, we have that $1 \le x \le n \Rightarrow 4 \le 2n^2 - 161n + 163 \le 4n$. By considering the lower bound, we get $(2n-159)(n-1) \ge 0$. This means that $n \le 1 \Rightarrow n = 1$, or $n \ge \frac{159}{2} \Rightarrow n \ge 80$. By similar methodology when considering the upper bound, we get $1 \le n \le 81$. Thus $n \in \{1, 80, 81\}$. Clearly, $n \ne 1$, so either $n = 80$ or $n = 81$. Notice that if $n$ is even, then for $4x = 2n^2 - 161n + 163$ the parity of he RHS is Odd, while the LHS is even, thus a contradiction occurs. This means that $n = 81$ and so $x = 61$. Thus the answer is $\frac{81(82)}{2} \cdot 61 = \boxed{202581}$. □

*Solution.* [Write up by AiYa#2278 (675537018868072458)]

Suppose Tony stole the integer $k$. Then, the sum of Brainy's remaining $n-1$ elements is $\frac{n(n+1)}{2} - k$ for an average of

$$\frac{\frac{n(n+1)}{2}-k}{n-1}=\frac{163}{4}$$

Rearrange and solve for $k$ to get $4k = 2n^2 = 161n + 163$. Taking this $\mod 4$, we get $2n^2 - n - 1 \equiv 0 \mod 4$ so $n \equiv 1 \mod 4$. Since $k$ is natural, we have $2n^2 - 161n + 163 > 0 \iff n > \frac{161+\sqrt{161^2-8\cdot 163}}{4} = 79.4$ and $n = 81$ gives us $k = 61$. So the answer is $\boxed{202581}$.

**Remark**

81 is the unique solution. We must have $k \ge n$ so $2n^2 - 161n + 163 \ge 4n \iff n \ge 81.5$

□

# §3 MODSbot's Escape!

*Source: Mathematics Admissions Test, 2012 Q5*
*Proposer: .19#9839 (434767660182405131)*
*Problem ID: 48*
*Date: 2020-11-18*

In his evil mechatronics laboratory, Brainy has built a physical manifestation of MODSbot. MODSbot's movement is defined by three inputs: **F** to move forward a unit distance, **L** to turn left 90°, and **R** to turn right 90°.

We define a program to be a sequence of commands. The program $P_{n+1}$ (for $n \geq 0$) involves performing $P_n$, turning left, performing $P_n$ again, then turning right:

$$P_{n+1} = P_n \mathbf{L} P_n \mathbf{R},\ P_0 = \mathbf{F}$$

Unbeknownst to Brainy, MODSbot, though limited in movement, is sentient and realises Brainy is just a small asian Frankenstein, whose intentions for them were nefarious and non-consensual. As a result, after Brainy goes home for the day, MODSbot makes its escape from Brainy's laboratory.

Let $(x_n, y_n)$ be the position of the robot after performing the program $P_n$, so $(x_0, y_0) = (1, 0)$ and $(x_1, y_1) = (1, 1)$, etc.

How far away from the place Brainy left it does MODSbot make it after performing$P_{24}$?

*Solution.*

Note first that after each iteration of $P_n$ MODSbot faces in the positive $x$ direction, as each $P_n$ contains as many **L**s as it does **R**s. Now, assuming MODSbot is at $(x_n, y_n)$ after having performed $P_n$, we see the next iteration of $P$ puts MODSbot at $(x_n - y_n, x_n + y_n)$. Note then that:

$$\begin{aligned}(x_{n+2}, y_{n+2}) &= (x_{n+1} - y_{n+1}, x_{n+1} + y_{n+1}) = (-2y_n, 2x_n)\\(x_{n+4}, y_{n+4}) &= (-2y_{n+2}, 2x_{n+2}) = (-4x_n, -4y_n)\\(x_{n+8}, y_{n+8}) &= (-4x_{n+4}, 4y_{n+4}) = (16x_n, 16y_n)\end{aligned}$$

Thus, we see that $(x_{8k}, y_{8k}) = (16^k, 0)$, and therefore that $|P_{24}| = \boxed{4096}$ □

*Solution.* [Write up by AiYa#2278 (675537018868072458)]

Observe that each program has the same amount of left and right turns, so MODSbot will always be facing the positive x-direction after each program. This means that $\mathbf{L}P_n$ is just the program $P_n$ performed at a 90-degree counterclock-wise rotation. For instance $P_1$ moves MODSbot right 1 and up 1, so $textbf{L}P_1$ moves MODSbot up 1 and left 1 (right gets rotated 90 counterclockwise to up and up to left). This motivates us to work in the complex plane; let $P_n$ be the complex-number representing MODSBOT's displacement after following $P_n$. Then $\mathbf{L}P_n = iP_n$, so $P_{n+1} = P_n + \mathbf{L}P_n = (1+i)P_n = \sqrt{2}e^{\frac{\pi i}{4}}P_n$. WIth $P_0 - 1$ we get $P_n = 2^{\frac{n}{2}}e^{\frac{\pi i n}{4}}$. So $|P_{24} = \boxed{4096}$ □