OpenPOTD

Contents

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Github/sj-bs/OpenPOTD_Public

This is a temporary document to let people who have alternative solutions contribute to the end of season write up. Solutions have already been written up for the entire season, and where possible I have tried to include the officially provided solution, adapted them if the problem statement is changed, and filled in the gaps as best I could. In many circumstances problems for this season have not come with official write-ups and thus have required me to provide my own - in which case I apologise for any mistakes (or fakesolves!) in advance. If any mistakes are found, feel free to DM me .19#9839 or submit a push. Similarly, if you would like to contribute an alternate solution to whatever is in this document, again, feel free to submit a push or just DM me a solution and I can type it up.

At the end of each day I'll add the question and solution to the document - the end of each day naturally being such that the question and solution being added is no longer an active one.

Thank you to:

AiYa#2278 (675537018868072458)

For contributing to solutions!

§1 A Sequence of 5's

Source: United Kingdom - Maclaurin, 2015 M1
Proposer: .19#9839 (434767660182405131)
Problem ID: 47
Date: 2020-11-16

Consider the sequence $5,\ 55,\ 555,\ 5555,\ldots$

How many digits does the smallest number in the sequence have which is divisible by 495?

Solution.

We require the term to be divisible by $5 \cdot 9 \cdot 11$. Hence we need only consider the sequence $1,\ 11,\ 111\ \ldots$ with respect to $9 \cdot 11$. Clearly for odd numbered terms in the sequence, 11 does not divide into it, by the well-known divisibility rule for 11. Therefore, we require an even numbered term in the sequence, which is divisible by 9. We know 9 divides a number iff its digital sum is also divisible by 9. Hence, the smallest such will be the 18th term in the sequence, which will naturally have $\boxed{18}$ digits. □

Solution. [Write up by AiYa#2278 (675537018868072458)]

Each of these numbers can be written as $5 \cdot 1\ldots1$,whee there are n total ones. This can be rewritten as $5 \cdot (10^{n-1} + 10^{n-2} + \cdots + 10^0) = \frac{5}{9}(10^n - 1)$. Note that $495 = 9 \cdot 11 \cdot 5$ so we want $9 | \frac{10^n-1}{9}$ and $10^n \equiv 1 \mod 11$. From the congruence $\mod 11$ we see that n must be even. Note that $10^n - 1 = 9\ldots9$, where there are n total nines; if n is a multiple of 9 then $\frac{10^n-1}{9} = 1\ldots1$ where there are n total ones; this is a multiple of 9. Since n must be even, the smallest such n is $\boxed{18}$. □

§2 Brainy's Happy Set

Source: British Mathematical Olympiad - Round 1, 2010/2011 P1
Proposer: .19#9839 (434767660182405131)
Problem ID: 40
Date: 2020-11-17

Brainy has a set of integers, from 1 to n, which he likes to play with. Tony Wang, upon seeing the happiness that this set of integers brings Brainy, decides to steal one of the numbers in it. Suppose the average number of the remaining elements in the set is $\frac{163}{4}$. What is the sum of the elements in Brainy's set multiplied by the element that Tony stole?

(A four-function calculator may be used)

Solution.

We can set up the problem statement as

$$\frac{\frac{n}{2}(n+1)-x}{n-1} = \frac{163}{4}$$

Where x is the number Tony has stolen. This simplifies to $4x = 2n^2 - 161n + 163$. Since x must be a number within the set $\{1, 2, \ldots, n-1, n\}$, we have that $1 \leq x \leq n \Rightarrow 4 \leq 2n^2 - 161n + 163 \leq 4n$. By considering the lower bound, we get $(2n-159)(n-1) \geq 0$. This means that $n \leq 1 \Rightarrow n = 1$, or $n \geq \frac{159}{2} \Rightarrow n \geq 80$. By similar methodology when considering the upper bound, we get $1 \leq n \leq 81$. Thus $n \in \{1, 80, 81\}$. Clearly, $n \neq 1$, so either $n = 80$ or $n = 81$. Notice that if n is even, then for $4x = 2n^2 - 161n + 163$ the parity of he RHS is Odd, while the LHS is even, thus a contradiction occurs. This means that $n = 81$ and so $x = 61$. Thus the answer is $\frac{81(82)}{2} \cdot 61 = \boxed{202581}$. □

§3 MODSbot's Escape!

Source: Mathematics Admissions Test, 2012 Q5
Proposer: .19#9839 (434767660182405131)
Problem ID: 48
Date: 2020-11-18

In his evil mechatronics laboratory, Brainy has built a physical manifestation of MODSbot. MODSbot's movement is defined by three inputs: **F** to move forward a unit distance, **L** to turn left 90°, and **R** to turn right 90°.

We define a program to be a sequence of commands. The program P_{n+1} (for $n \geq 0$) involves performing P_n, turning left, performing P_n again, then turning right:

$$P_{n+1} = P_n\mathbf{L}P_n\mathbf{R},\ P_0 = \mathbf{F}$$

Unbeknownst to Brainy, MODSbot, though limited in movement, is sentient and realises Brainy is just a small asian Frankenstein, whose intentions for them were nefarious and non-consensual. As a result, after Brainy goes home for the day, MODSbot makes its escape from Brainy's laboratory.

Let (x_n, y_n) be the position of the robot after performing the program P_n, so $(x_0, y_0) = (1, 0)$ and $(x_1, y_1) = (1, 1)$, etc.

How far away from the place Brainy left it does MODSbot make it after performingP_{24}?

Solution.

Note first that after each iteration of P_n MODSbot faces in the positive x direction, as each P_n contains as many **L**s as it does **R**s. Now, assuming MODSbot is at (x_n, y_n) after having performed P_n, we see the next iteration of P puts MODSbot at $(x_n - y_n, x_n + y_n)$. Note then that:

$$\begin{aligned}
(x_{n+2}, y_{n+2}) &= (x_{n+1} - y_{n+1}, x_{n+1} + y_{n+1}) = (-2y_n, 2x_n) \\
(x_{n+4}, y_{n+4}) &= (-2y_{n+2}, 2x_{n+2}) = (-4x_n, -4y_n) \\
(x_{n+8}, y_{n+8}) &= (-4x_{n+4}, 4y_{n+4}) = (16x_n, 16y_n)
\end{aligned}$$

Thus, we see that $(x_{8k}, y_{8k}) = (16^k, 0)$, and therefore that $|P_{24}| = \boxed{4096}$ □

Solution. [Write up by AiYa#2278 (675537018868072458)]

Observe that each program has the same amount of left and right turns, so MODSbot will always be facing the positive x-direction after each program. This means that $\mathbf{L}P_n$ is just the program P_n performed at a 90-degree counterclock-wise rotation. For instance P_1 moves MODSbot right 1 and up 1, so $\mathbf{L}P_1$ moves MODSbot up 1 and left 1 (right gets rotated 90 counterclockwise to up and up to left). This motivates us to work in the complex plane; let P_n be the complex-number representing MODSBOT's displacement after following P_n. Then $\mathbf{L}P_n = iP_n$, so $P_{n+1} = P_n + \mathbf{L}P_n = (1+i)P_n = \sqrt{2}e^{\frac{\pi i}{4}}P_n$. WIth $P_0 = 1$ we get $P_n = 2^{\frac{n}{2}}e^{\frac{\pi i n}{4}}$. So $|P_{24} = \boxed{4096}$ □

§4 Sides of a Polygon

Source: Folklaw[1]
Proposer: .19#9839 (434767660182405131)
Problem ID: 53
Date: 2020-11-19

Points $A,\ B,\ C,\ D$ are the consecutive vertices of a regular polygon, and the following relation holds:

$$\frac{1}{AB} = \frac{1}{AC} + \frac{1}{AD}$$

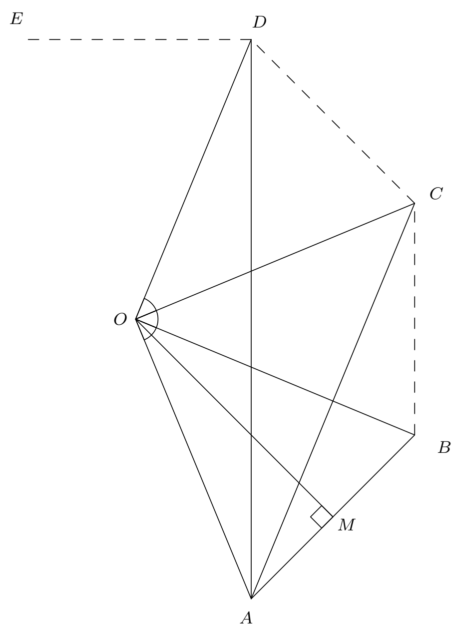
How many sides does this polygon have?

Solution.

Drawing out the general shape of an n-gon - as seen in the figure - and letting $OA = OB = OC = OD \cdots = 1$, and $\angle MOA = x$. By the sine rule on $\triangle AMO$, and noting $AM = \frac{1}{2}AB$, we get $\frac{1}{AB} = \frac{2}{\sin x}$. By a similar procedure, this time on $\triangle ACO$, we see $\frac{1}{AB} = \frac{2}{\sin 2x}$, and again on $\triangle ADO$, we have $\frac{1}{AD} = \frac{2}{\sin 3x}$. Therefore we have the equality:

$$\frac{1}{\sin x} = \frac{1}{\sin 2x} + \frac{1}{\sin 3x}$$

Simplifying this yields $\sin x \sin \frac{x}{2} \sin \frac{7x}{2} = 0$. However, note that $x \neq \frac{k\pi}{2},\ \frac{k\pi}{3}$ for $k \in \mathbb{Z}$, otherwise, we have the issue of dividing by 0. Hence it must be the case that $\sin \frac{7x}{2} = 0 \Rightarrow x = \frac{\pi}{7} + \frac{2k\pi}{7}$. Clearly the n-gon is not a square, so trivially it must be the case that $x = \frac{\pi}{7}$. Therefore, the polygon must have $\boxed{7}$ sides. □



A regular n-gon

[1] This has appeared on a Polish MO, British MO 1966 P4, an 2018 NZ IMO handout, a WOOT handout, to name a few...

Solution. [Write up by AiYa#2278 (675537018868072458)]

Let d_k represent the diagonal from a point to the kth vertex adjacent to it. For example, d_1 is a side of the polygon, d_2 is $\overline{AC}, d_3$ is $\overline{AD}$ and note that $d_k = d_{n-k}$ where n is the number of sides of the polygon. Reassign D to be the vertex three vertices away from A but on the opposite side of B and C; in other words, reflect D over $\overline{OA}$. Then, $AB = BC = d_1, AC = d_2, AD = d_3, BD = d_4$, and $CD = d_5$. By Ptolemy's Theorem, we get

$$AB \cdot CD + BC \cdot AD = AC \cdot BD \iff d_1 (d_5 + d_3) = d_2 d_4.$$

Rearrange our given equation to get

$$\frac{1}{d_1} = \frac{1}{d_2} + \frac{1}{d_3} \iff d_1 (d_2 + d_3) = d_2 d_3.$$

For both of these equations to be true, we can have $d_5 + d_3 = d_2 + d_3 \iff d_5 = d_2$ and $d_3 = d_4$; this is true if $n = \boxed{7}$. □

§5 2p

Source: China Western MO 2003, Day 1 P2
Proposer: .19#9839 (434767660182405131)
Problem ID: 51
Date: 2020-11-20

Let $a_1, a_2, \ldots, a_{24n}$ be real numbers with $\sum_{i=1}^{24n-1} (a_{i+1} - a_i)^2 = 1$.

For some $n > 0$ the maximum value of $(a_{12n+1} + a_{12n+2} + \cdots + a_{24n}) - (a_1 + a_2 + \cdots + a_{12n})$ is twice that of a prime.

What is the sum of the value of that prime and the corresponding value of n?

Solution.

If we substitute $x_{i+1} = a_{i+1} - a_i$, we get $a_i = \sum_1^i x_r$, and thus our constraint becomes

$$\sum_{i=1}^{24n-1} (a_{i+1} - a_i)^2 = \sum_{i=1}^{24n} x_i^2 = 1$$

Putting the bit we wish to be maximising in terms of the substitution gives:

$$\sum_{i=1}^{12n} a_i = 12nx_1 + (12n-1)x_2 + \cdots + 2x_{12n-1} + x_{12n}$$
$$\sum_{i=12n+1}^{24n} a_i = 12n(x_1 + \cdots + x_{12n+1}) + (12n-1)x_{12n+2} + \cdots 2x_{24n-1} + x_{24n}$$

Hence,

$$\sum_{i=12n+1}^{24n} a_i - \sum_{i=1}^{12n} a_i = x_2 + \cdots + (12n-1)x_{12n} + 12nx_{12n+1} + (12n-1)x_{12n+2} + \cdots + x_{24n}$$

Then by Cauchy-Schwarz:

$$\sum_{i=12n+1}^{24n} a_i - \sum_{i=1}^{12n} a_i \leq \sqrt{\left((12n)^2 + 2\sum_{i=1}^{12n-1} i^2\right)\left(\sum_{i=1}^{24n} x_i^2\right)} = \sqrt{(12n)^2 + \frac{12n(12n-1)(24n-1)}{3}}$$
$$\leq \sqrt{4n(2(12n)^2+1)} = 2p$$

Now we want values of n such that $n(288n^2+1) = p^2$ for a prime p. Since trivially for all n, $n < 288n^2+1$, we have $n = 1$ and $288n^2 + 1 = p^2$, hence $p = 17$, so the answer is $\boxed{18}$. $\square$

§6 Slippery Rooks

Source: AMOC 2019 December School Prep Problems C5
Proposer: ChristopherPi#8528 (696497464621924394)
Problem ID: 50
Date: 2020-11-22

MODSbot is trying to get rich by scamming MODS members out of their money, so it's devised a chess game on a 2020×2020 chessboard for unsuspecting people to attempt before they can enter **.19's EPIC QoTD Party**. Suppose Brainy, Ishan, Nyxto, Adam, Bubble, Sharky and Christopher all get scammed by MODSbot, that is, MODSbot plays the chess game against all 7 at the same time on different boards.

The group decide to pool together their money which comes to a total of 4.20 BTC, and to play, they'll need to buy n batches of slippery rooks from MODSbot. A batch of slippery rooks contains one white and four black rooks, and each batch is sold at a price equivalent to 0.069 BTC per rook. Once the batches of rooks have been bought, the group may choose to distribute them in a way which allows all members to beat the game.

In the game, only one white rook may be placed on the board, and we define how slippery rooks move as follows: it slips along the row or column it's moved along and comes to rest on an empty square because it is obstructed by either the edge of the board or another rook. Initially, MODSbot places the rooks on the board randomly, and marks a square red. Then the person being scammed can choose any rook on each turn and move as allowed, and attempt to place the white rook on the red square in a finite number of moves.

The amount of money they have left over after buying the smallest n batches rooks to guarantee that they all succeed in beating MODSbots game is k BTC. What is the value of $1000k$?

Solution. [Write up by ChristopherPi]

Consider simply the case of one person. We prove that three rooks are required, one white and two black.

First we show that two are not enough: simply place the two rooks at corners of the board and mark any square not on the side of the board. It's clear that neither rook can ever move to a square not on the side of the board. Now we show three are enough.

Suppose square (a, b) is marked, where $(1, 1)$ is the bottom left corner and $(2020, 2020)$ is the top right corner. Trivially one can move the black rooks to $(1, 1)$ and $(2, 1)$ and the white rook to $(2020, 2020)$. Next, simply "loop" the black rooks as follows: take the rook further left, and move it up, right, down and left such that it moves to the right of the rook originally on its right, and repeat until you place a black rook at $(a - 1, 1)$. Now if b is odd, move the white rook to $(a, 1)$ and the black rook at $(a - 2, 1)$ to $(2020, 2020)$; if it's even, loop the leftmost black rook one more time to place it at $(a, 1)$.

Now move the rook at $(a - 1, 1)$ to $(a - 1, 2020)$, and move the rook at $(2020, 2020)$ left then down to $(a, 2)$. Next we describe another "looping" procedure: take the rook with first coordinate a and smaller second coordinate, and move it right, up, left and down, so it now has first coordinate a and second coordinate larger than the other rook with first coordinate a. Repeat this until you place a rook at (a, b) - since the colour of the rook placed at $(a, 1)$ is dependent on the parity of b, this ensures that the rook placed at (a, b) must be a white rook.

This procedure won't work if either of (a, b) is 1 or 2020, or both of a and b are either 2 or 2019. In the first case, rotate the board such that $a = 1$. Now place a black rook at $(2020, 2020)$. If b is odd, place the white rook at $(1, 1)$ and the other black rook at $(2, 1)$; else place the other black rook at $(1, 1)$ and the white rook at $(2, 1)$. Now use the first looping procedure until the white rook is placed at $(1, b)$ as required - since the position of the white rook depends on the parity of b this is certain to work. In the second case, rotate the board such that $(a, b) = (2, 2)$. Now you can trivially move the white rook to $(1, 1)$ and the black rooks to $(2, 1)$ and $(1, 2020)$. Now move the white rook up, right, up, left and down to place it at $(2, 2)$ as required.

This shows that three is sufficient for one person. Hence the group must buy 7 batches because each of them needs a white rook, and one batch contains one white rook. Therefore, the answer is $1000(4.2 - 7 \cdot 5 \cdot 0.069) = \boxed{1785}$. $\square$

§7 Sets of Integer Solutions

China Mathematical Olympiad, 2005 Day 2 P6
Proposer: .19#9839 (434767660182405131)
Problem ID: 46
Date: 2020-11-19

Define functions f and g such that $f(a,b) = 2^a 3^b$, and $g(c,d) = 5^c 7^d$, for $a,b,c,d \in \mathbb{Z}_{\geq 0}$.

Given $f(a,b) = 1 + g(c,d)$, what is the sum of all valid b's, c's and d's, multiplied by the sum of all valid a's?

For example if we had valid solutions of $(a,b,c,d) = (1,1,2,4), (5,1,6,2), (0,0,0,0)$

Then the answer would be $(\underbrace{1+1+0}_{b's}+\underbrace{2+6+0}_{c's}+\underbrace{4+2+0}_{d's}) \times (\underbrace{1+5+0}_{a's}) = 96$

Solution.

We proceed by considering parity, for $2^a 3^b = 5^c 7^d + 1$, we have the RHS as even, thus we must have $a \geq 1$. If we let $b = 0$, then for $2^a - 5^c \cdot 7^d = 1$, we have $2^a \equiv 1 \mod 5$ for $c \neq 0$. This gives $a \equiv 0 \mod 4$, so $2^a - 1 \equiv 0 \mod 3$. But this clearly cannot be the case so we must have $c = 0$ when $b = 0$.
Hence, we consider $2^a - 7^d = 1$. Bashing gives $(a,d) = (1,0),\ (3,1)$. These are the only such solutions as for $a > 4,\ 7^d \equiv -1 \mod 16$, but this is impossible. So for the case of $b = 0$ all possible non-negative integer solutions are $(1,0,0,0),\ (3,0,0,1)$.

Now let $b > 0$ and $a = 1$, so we now consider $2 \cdot 3^b - 5^c \cdot 7^d = 1$ under $\mod 3$, which gives $-5^c 7^d \mod 3$. Since $7^d \equiv 1 \mod 3$, for all $d \geq 0$, we are left with $(-1)^c 5^c \equiv 1 \mod 3$. Now $5^c = \{1,2\} \mod 3$, thus we see that we must have c being odd. Under $\mod 5$, we see that $2 \cdot 3^b \equiv 1 \mod 5$, $3^{b-1} \equiv 1 \mod 5$. As we observe that $3^b \equiv \{3,4,2,1\} \mod 5$, we must have $b \equiv 1 \mod 4$. If $d \neq 0$, then $2 \cdot 3^b \equiv 1 \mod 7$. Again observe that $3^b \equiv \{3,2,6,4,5,1\} \mod 7$, we see $b \equiv 4 \mod 6$. But $b \equiv 1 \mod 4$, so a contradiction arises, and thus $d = 0$ and hence $2 \cdot 3^b - 5^c = 1$. For $b = 1$, clearly $c = 1$. So if $b \geq 2$, then $5^c \equiv -1 \mod 9 \Rightarrow c \equiv 3 \mod 6$. Therefore $5^c + 1 \equiv 0 \mod (5^3+1) \Rightarrow 5^c + 1 \equiv 0 \mod 7$, but this contradicts the fact that $5^c + 1 = 2 \cdot 3^b$. Hence in this case we only have one solution $(a,b,c,d) = (1,1,1,0)$.

Finally, consider the case where $b > 0$, and $a \geq 0$. Then we have $5^c 7^d \equiv -1 \mod 4$, and $5^c 7^d \equiv -1 \mod 3$, i.e. $(-1)^d \equiv -1 \mod 4$ and $(-1)^c \equiv -1 \mod 3$. Therefore we have that both c and d being odd. Thus, $2^a 3^b = 5^c 7^d + 1 \equiv 4 \mod 8$. So $a = 2$ and thus $4 \cdot 3^b \equiv 1 \mod 5$ and $4 \cdot 3^b \equiv 1 \mod 7$. This gives $b \equiv 2 \mod 12$. Substituting $b = 12k + 2$ for $k \in \mathbb{Z}_{\geq 0}$, then $5^c 7^d = (2 \cdot 3^{6k+1} - 1)(2 \cdot 3^{6k+1} + 1)$.
Now as $\gcd(2 \cdot 3^{6k+1} + 1, 2 \cdot 3^{6k+1} - 1)$, $2 \cdot 3^{6k+1} - 1 \equiv 0 \mod 5$, therefore $2 \cdot 3^{6k+1} - 1 = 5^a$ and $2 \cdot 3^{6k+1} = 7^d$. If $k \geq 1$, then $5^c \equiv -1 \mod 9$. But this is impossible, so if $k = 0$, then $b = 2,\ c = 1$, and $d = 1$. Thus in this case, we have only one solution: $(a,b,c,d) = (2,2,1,1)$.
Hence we can conclude all non-negative integer solutions are

$$(a,b,c,d) = \begin{cases} (1,0,0,0) \\ (3,0,0,1) \\ (1,1,1,0) \\ (2,2,1,1) \end{cases}$$

This then gives us an answer of $\underbrace{(0+0+0+0+0+1+1+1+0+2+1+1)}_{7} \times \underbrace{(1+3+1+2)}_{7} = \boxed{49}$ □