Lesson 12 Appendix: Proofs and summary of important results.

Property 1. (Slide 15) Sum of all vertex degrees = 2m.

Proof. Every edge increases the degree of exactly two vertices by 1. Thus every edge increases the sum of all vertex degrees by 2. If there are m edges, sum of all vertex degrees = 2m.

Property 2. (Slide 15) $m \le n(n-1) / 2 = \binom{n}{2}$

Proof. Given n, there are n(n-1)/2 pairs possible. Each can be an edge. Hence the result.

Exercise on Slide 16. $m = n(n-1) / 2 = \binom{n}{2}$ for K_n, the complete graph on n vertices.

Proof. Given n, there are n(n-1)/2 edges possible. The complete graph has all possible edges. Thus m = n(n-1)/2.

Exercise on Slide 17. A graph is bipartite if and only if it has no odd (simple) cycle.

Proof. Case 1: There is no odd cycle. We can label adjacent vertices using labels 1 and 2 without any conflict. After labeling, let V_1 denote the set of all vertices labeled as 1 and let V_2 denote the set of all vertices labeled as 2. Clearly, V is partitioned into two subsets V_1 and V_2 such that all edges have one vertex in V_1 and the other vertex V_2. Therefore, the graph is bipartite.

Case 2: There is an odd cycle. Consider an odd cycle on 2n + 1 vertices. Start labeling all 1 and 2 starting from any vertex in the cycle. This will result in 1 and 2n + 1 having label as 1. Since this is a cycle, there is an edge between 1 and 2n + 1. Note that both 1 and 2n + 1 belongs to the same partition of V. From the definition of bipartite graph, we cannot have an edge between 1 and 2n + 1. Therefore, graph is not bipartite.

Exercise on Slide 19. If m > $\binom{n-1}{2}$, then G is connected.

Proof. If there are n vertices, keep one vertex apart from all other n -1 vertices and try placing edges connecting those n -1 vertices . The maximum number of edges possible is $\binom{n-1}{2}$. Therefore if there is one more edge, graph will become connected.

Exercise on Slide 20. If G is disconnected, then G^c is connected.

Proof. Assume G is disconnected. We want to show that G^c is connected. Suppose u and v are vertices. We need to show there is path from u to v in G^c.

Case 1: (u, v) is not an edge in G. Then it is an edge in G^c and so we have a path uv from u to v in G^c.

Case 2: (u, v) is an edge in G. This means u and v are in the same component of G. Since G is disconnected, we can find a vertex w in a different component. Note that (u, w) and (v, w) are not in G. Hence (u, w) and (v, w) are edges in G^c. Thus, uwv is a path from u to v in G^c.

Exercise on Slide 22. Show that every tree having 2 or more vertices has at least two vertices of degree one.

Proof. Consider a tree T having n > 1 vertices. Since it is a tree, m = n – 1. Now by property 1, sum of all vertex degrees is 2m = 2(n-1) = 2n – 2.

Since T is a tree, it is connected. Hence every vertex must have at least 1 vertex degree. Now you are left with only (2n – 2) – n = n-2 vertex degrees. You have n vertices and n – 2 leftover vertex degrees. Thus there will be at least two vertices with vertex degree 1.

Q1. What is the minimum number of edges required to guarantee the graph is connected?

$$\binom{n-1}{2} + 1$$

Q2. If the graph is connected what is the minimum number of edges it has?

$$n - 1$$

Q3. What is the minimum number of edges required to guarantee a graph has a cycle?

$$n$$

Let C be a connected component with n vertices and m edges.

(a) The minimum number of edges it has

$$n - 1$$

(a) If m = n - 1 then C is tree.
(b) If m >= n, then C has a cycle.
(c) If m = n and deg(v) = 2 for all vertices in v in C, then C is a cycle.