

第三章习题

T1

由题意有

$$T(e_1, e_2, e_3) = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = (e_1, e_2, e_3)A \Rightarrow A = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

T2

$$D(x_1) = ae^{at} \cos bt - be^{at} \sin bt = ax_1 - bx_2$$

$$D(x_2) = ae^{at} \sin bt + be^{at} \cos bt = ax_2 + bx_1$$

$$D(x_3) = e^{at} \cos bt + tD(x_1) = x_1 + ax_3 - bx_4$$

$$D(x_4) = e^{at} \sin bt + tD(x_2) = x_2 + ax_4 + bx_3$$

$$D(x_5) = te^{at} \cos bt + \frac{1}{2}t^2 D(x_1) = x_3 + ax_5 - bx_6$$

$$D(x_6) = te^{at} \sin bt + \frac{1}{2}t^2 D(x_2) = x_4 + ax_6 + bx_5$$

则有

$$A = \begin{pmatrix} a & b & 1 & 0 & 0 & 0 \\ -b & a & 0 & 1 & 0 & 0 \\ 0 & 0 & a & b & 1 & 0 \\ 0 & 0 & -b & a & 0 & 1 \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & -b & a \end{pmatrix}$$

T3

(1)

$$T_1 E_{11} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} E_{11} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = aE_{11} + cE_{21}$$

$$T_1 E_{12} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} E_{12} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} = aE_{12} + cE_{22}$$

$$T_1 E_{13} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} E_{13} = \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} = bE_{11} + dE_{21}$$

$$T_1 E_{14} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} E_{14} = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} = bE_{12} + dE_{22}$$

则有

$$A_1 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

(2)

$$T_2 E_{11} = E_{11} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = aE_{11} + bE_{12}$$

$$T_2 E_{12} = E_{12} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = cE_{11} + dE_{12}$$

$$T_2 E_{21} = E_{21} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} = aE_{21} + bE_{22}$$

$$T_2 E_{22} = E_{22} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} = cE_{21} + dE_{22}$$

则有

$$A_2 = \begin{pmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{pmatrix}$$

(3)

由 $T_1X = XA_1, T_2X = XA_2$, 有

$$T_3X = (T_1 \circ T_2)X = XA_3 \Rightarrow T_3X = T_2(T_1X) = T_2(XA_1) = XA_1A_2 \Rightarrow A_3 = A_1A_2$$

则

$$A_3 = \begin{pmatrix} a^2 & ac & ab & bc \\ ab & ad & b^2 & bd \\ ac & c^2 & ad & cd \\ bc & cd & bd & d^2 \end{pmatrix}$$

T4

设有一组数 $a_i, i = 1, \dots, k$, 使得 $a_1x + a_2Tx + \dots + a_kT^{k-1}x = 0$

上式作用一次 T , 有 $a_1Tx + a_2T^2x + \dots + a_{k-1}T^{k-1}x = 0$

总共作用 $k-1$ 次 T 后, 有 $a_1T^{k-1}x = 0$

又因为 $T^{k-1}x \neq 0$, 所以 $a_1 = 0$

同理可证 $a_2 = a_3 = \dots = a_k = 0$

所以 $x, Tx, \dots, T^{k-1}x$ 线性无关

T5

(1)

记过渡矩阵为 C

则有 $(y_1, y_2, y_3) = (x_1, x_2, x_3)C$

$$\Rightarrow C = (x_1, x_2, x_3)^{-1}(y_1, y_2, y_3) = \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & -1 \\ -1 & -1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -4 & -3 & 3 \\ 2 & 3 & 3 \\ 2 & -1 & -5 \end{pmatrix}$$

(2)

$$T(x_1, x_2, x_3) = (y_1, y_2, y_3) = (x_1, x_2, x_3)C$$

T 在基 x_1, x_2, x_3 下的过渡矩阵为 C

(3)

T 在基 y_1, y_2, y_3 下的过渡矩阵为 $C^{-1}CC = C$

T6

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & -1 & 0 \\ 4 & \lambda+1 & 0 \\ -4 & 8 & \lambda+2 \end{vmatrix} = (\lambda+2)(\lambda-1)^2$$

计算可得 A 的特征值及对应特征向量为:

$$\lambda_1 = -2, \xi_1 = k_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, k_1 \neq 0, k_1 \in F$$

$$\lambda_2 = \lambda_3 = 1, \xi_2 = k_2 \begin{pmatrix} 3 \\ -6 \\ 20 \end{pmatrix}, k_2 \neq 0, k_2 \in F$$

于是 T 的特征值及对应特征向量为:

$$\lambda_1 = -2, \eta_1 = X\xi_1 = k_1x_3, k_1 \neq 0, k_1 \in F$$

$$\lambda_2 = \lambda_3 = 1, \eta_2 = X\xi_2 = k_2(3x_1 - 6x_2 + 20x_3), k_2 \neq 0, k_2 \in F$$

T7

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & -1 & 0 \\ 4 & \lambda-3 & 0 \\ -1 & 0 & \lambda-2 \end{vmatrix} = (\lambda-2)(\lambda-1)^2$$

取 $\lambda = 2$, 对应特征向量 $p_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\text{令 } p_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, p_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{可得 } P_3 = (p_1, p_2, p_3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$AP_3 = P_3 \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -4 & 3 \end{pmatrix}$$

$$\text{记 } A_2 = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}, \det(\lambda I - A_2) = (\lambda-1)^2$$

其特征值为 $\lambda = 1$, 对应特征向量为 $q_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\text{令 } q_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{可得 } P_2 = (q_1, q_2) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$A_2 P_2 = P_2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

则

$$A = P_3 \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -4 & 3 \end{pmatrix} P_3^{-1} = \tilde{P}_3 \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tilde{P}_3^{-1}$$

$$\text{其中 } \tilde{P}_3 = P_3 \begin{pmatrix} 1 & & \\ & P_2 & \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

T8

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 0 & -2 \\ 0 & \lambda+1 & -1 \\ 0 & -1 & \lambda \end{vmatrix} = \lambda^3 - 2\lambda + 1$$

则

$$2A^8 - 3A^5 + A^4 + A^2 - 4E_3 = (A^3 - 2A + 1)(2A^5 + 4A^3 - 5A^2 + 9A - 14E) + 24A^2 - 37A + 10E$$

于是

$$2A^8 - 3A^5 + A^4 + A^2 - 4E_3 = 24A^2 - 37A + 10E = \begin{pmatrix} -3 & 48 & -26 \\ 0 & 95 & -61 \\ 0 & -61 & 34 \end{pmatrix}$$

T9

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-7 & -4 & 4 \\ -4 & \lambda+8 & 1 \\ 4 & 1 & \lambda+8 \end{vmatrix} = (\lambda-9)(\lambda+9)^2$$

又 $(A-9I)(A+9I) = O$, 则最小多项式为 $(\lambda-9)(\lambda+9)$

T10

$$\text{因为 } |\lambda I - A| = |(\lambda I - A)^T| = |\lambda I - A^T|$$

所以 A 和 A^T 有相同的特征多项式和最小多项式

T11

(1)

$$\det(\lambda I - A) = \lambda^3 - \lambda^2 + \lambda - 1 = (\lambda - 1)(\lambda^2 + 1)$$

所以 A 的特征值为 $1, i, -i$

则 Jordan 标准型为 $\begin{pmatrix} 1 & & \\ & i & \\ & & -i \end{pmatrix}$

(2)

$$\det(\lambda I - A) = (\lambda - 1)^4$$

所以 A 的特征值为 1

由 $(A - I)X = 0$ 可得对应特征向量 $\xi = (0, 0, -1, 1)^T$

则 $\lambda = 1$ 的几何重数为 1, $J(1)$ 只包含一个 Jordan 块

故 A 的 Jordan 标准型为 $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

T12

设 A 的 Jordan 标准型为 $J = \begin{pmatrix} J_1(\lambda_1) & & \\ & \ddots & \\ & & J_s(\lambda_s) \end{pmatrix}$, $J_i(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & \dots & 0 \\ 0 & \lambda_i & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \dots & \lambda_i \end{pmatrix}$

于是存在可逆矩阵 P 使得 $A = PJP^{-1}$

则 $J^m = P^{-1}A^mP = I \Rightarrow [J_i(\lambda_i)]^m = I_p \Rightarrow p = 1$

则 $J_i(\lambda_i)$ 中的 Jordan 块为 1 阶, 即 $J_i(\lambda_i) = \lambda_i$

故 A 相似于对角矩阵