

第四章习题

T1

证:

$$|\alpha + t\beta|^2 = \langle \alpha + t\beta, \alpha + t\beta \rangle = |\alpha|^2 + 2t\langle \alpha, \beta \rangle + t^2 |\beta|^2$$

则

$$|\alpha + t\beta|^2 \geq |\alpha|^2 \Leftrightarrow 2t\langle \alpha, \beta \rangle + t^2 |\beta|^2 \geq 0$$

不等式若要对任意 t 成立, 则 $\langle \alpha, \beta \rangle = 0$, 即 α, β 正交。

T2

(1)

设 $x_1, x_2 \in W$, 则 $\langle x_1 + x_2, \alpha \rangle = \langle x_1, \alpha \rangle + \langle x_2, \alpha \rangle = 0$

同理可证 $\langle x_1 + x_2, \beta \rangle = \langle x_1 + x_2, \gamma \rangle = 0$

于是 $x_1 + x_2 \in W$

$$\forall k \in R, \langle kx_1, \alpha \rangle = k\langle x_1, \alpha \rangle = 0$$

同理可证, $\langle kx_1, \beta \rangle = \langle kx_1, \gamma \rangle = 0$

于是 $kx_1 \in W$

综上, W 是 V 的子空间

(2)

设 α, β, γ 所构成的空间为 V_1 , 由 W 定义可知, W 是 V_1 的正交补

则有 $V = W \oplus V_1$, 于是有 $\dim V = \dim W + \dim V_1 \Rightarrow \dim W = n - 3$

T3

$$|A + B| = |A| |B^T + A^T| |B| = |A| |B| |A + B| = -|A|^2 |A + B|$$

$$\Rightarrow (1 + |A|^2) |A + B| = 0 \Rightarrow |A + B| = 0$$

所以 $A + B$ 不可逆

T4

设 $\alpha_1, \alpha_2, \dots, \alpha_m$ 生成的子空间为 V_1 , $\beta_1, \beta_2, \dots, \beta_m$ 生成的子空间为 V_2

设线性映射 T 使得 $T(\alpha_i) = \beta_i$

对于 $\forall x \in V_1, x = k_1\alpha_1 + k_2\alpha_2 + \dots + k_m\alpha_m$

$$\text{则 } T(x) = k_1T(\alpha_1) + k_2T(\alpha_2) + \dots + k_mT(\alpha_m) = k_1\beta_1 + k_2\beta_2 + \dots + k_m\beta_m$$

所以 T 保持线性关系

下证内积不变

$$\forall x, y \in V_1, x = k_1\alpha_1 + k_2\alpha_2 + \dots + k_m\alpha_m, y = c_1\alpha_1 + c_2\alpha_2 + \dots + c_m\alpha_m$$

$$\begin{aligned}
\langle x, y \rangle &= \langle k_1\alpha_1 + k_2\alpha_2 + \dots + k_m\alpha_m, c_1\alpha_1 + c_2\alpha_2 + \dots + c_m\alpha_m \rangle \\
&= k_1c_1\langle \alpha_1, \alpha_1 \rangle + k_2c_2\langle \alpha_2, \alpha_2 \rangle + \dots + k_mc_m\langle \alpha_m, \alpha_m \rangle \\
\langle T(x), T(y) \rangle &= \langle k_1\beta_1 + k_2\beta_2 + \dots + k_m\beta_m, c_1\beta_1 + c_2\beta_2 + \dots + c_m\beta_m \rangle \\
&= k_1c_1\langle \beta_1, \beta_1 \rangle + k_2c_2\langle \beta_2, \beta_2 \rangle + \dots + k_mc_m\langle \beta_m, \beta_m \rangle
\end{aligned}$$

由 $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle$, 可得 $\langle x, y \rangle = \langle T(x), T(y) \rangle$

于是 T 是从 V_1 到 V_2 的同构映射, 即 V_1 和 V_2 同构

T5

由 T4 证明过程可知, T 是从 V_1 到 V_2 的正交变换

T6

由题意有

$$(\sigma(\varepsilon_1), \sigma(\varepsilon_2), \dots, \sigma(\varepsilon_n)) = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)A, (\tau(\varepsilon_1), \tau(\varepsilon_2), \dots, \tau(\varepsilon_n)) = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)B$$

$$\forall \alpha \in V$$

$$\alpha = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)X$$

$$\text{则 } \sigma(\alpha) = (\sigma(\varepsilon_1), \sigma(\varepsilon_2), \dots, \sigma(\varepsilon_n)) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)AX$$

$$\text{同理, } \tau(\alpha) = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)BX$$

$$\text{由 } |\sigma(\alpha)| = |\tau(\alpha)| \Leftrightarrow \langle \sigma(\alpha), \sigma(\alpha) \rangle = \langle \tau(\alpha), \tau(\alpha) \rangle \text{ 有}$$

$$X^T A^T \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix} (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) AX = X^T A^T AX = X^T B^T BX \Rightarrow A^T A = B^T B$$

TODO:

T7

(1)

$$\forall x_1, x_2 \in R^n, k_1, k_2 \in R$$

$$\begin{aligned}
\sigma(k_1x_1 + k_2x_2) &= k_1x_1 + k_2x_2 - k\langle k_1x_1 + k_2x_2, \varepsilon \rangle \varepsilon \\
&= k_1x_1 + k_2x_2 - k\langle k_1x_1, \varepsilon \rangle \varepsilon - k\langle k_2x_2, \varepsilon \rangle \varepsilon \\
&= k_1x_1 - k\langle k_1x_1, \varepsilon \rangle \varepsilon + k_2x_2 - k\langle k_2x_2, \varepsilon \rangle \varepsilon \\
&= k_1\sigma(x_1) + k_2\sigma(x_2)
\end{aligned}$$

故 σ 是线性变换

(2)

$$\forall x, y \in R^n$$

$$\begin{aligned}
\langle \sigma(x), \sigma(y) \rangle &= \langle x - k\langle x, \varepsilon \rangle \varepsilon, y - k\langle y, \varepsilon \rangle \varepsilon \rangle \\
&= \langle x, y \rangle - k\langle x, \varepsilon \rangle \langle \varepsilon, y \rangle - k\langle y, \varepsilon \rangle \langle \varepsilon, x \rangle + k^2 \langle x, \varepsilon \rangle \langle y, \varepsilon \rangle \\
&= \langle x, y \rangle - 2k\langle x, \varepsilon \rangle \langle y, \varepsilon \rangle + k^2 \langle x, \varepsilon \rangle \langle y, \varepsilon \rangle
\end{aligned}$$

要证 $\langle \sigma(x), \sigma(y) \rangle = \langle x, y \rangle$

只需证 $2k\langle x, \varepsilon \rangle \langle y, \varepsilon \rangle = k^2 \langle x, \varepsilon \rangle \langle y, \varepsilon \rangle$, 即 $2k = k^2$

解得 $k = 2$ 或 $k = 0$, 此时 σ 是正交变换

T8

要证 σ 为恒等变换, 只需证 $A = I$

由 σ 为恒等变换, A 为对称矩阵

由 A 为正定矩阵, 可得 A 的特征值均为正实数

由 A 为正交矩阵, 可得 $AA^T = A^2 = I$, A 的特征值 λ 满足 $\lambda^2 = 1$

所以, A 的特征值均为 1, 即 $A = I$