矩阵分解习题

T1

$A=(x_1,x_2,x_3)$

$u_1=x_1=(0,1,1)^T, y_1=\frac{u_1}{\|u_1\|_2}=\frac{1}{\sqrt{2}}(0,1,1)^T$

$u_2=x_2-(x_2,y_1)y_1=(1,1,0)^T-\frac{1}{2}(0,1,1)^T=\frac{1}{2}(2,1,-1)^T, y_2=\frac{u_2}{\|u_2\|_2}=\frac{1}{\sqrt{6}}(2,1,-1)^T$

$u_3=x_3-(x_3,y_1)y_1-(x_3,y_2)y_2=(1,0,1)^T-\frac{1}{2}(0,1,1)^T-\frac{1}{6}(2,1,-1)^T=\frac{2}{3}(1,-1,1)^T$

$y_3=\frac{u_3}{\|u_3\|_2}=\frac{1}{\sqrt{3}}(1,-1,1)^T$

$x_1=\sqrt{2}y_1$

$x_2=\frac{1}{\sqrt{2}}y_1+\frac{\sqrt{6}}{2}y_2$

$x_3=\frac{1}{\sqrt{2}}y_1+\frac{1}{\sqrt{6}}y_2+\frac{2}{\sqrt{3}}y_3$

$$A=\begin{pmatrix}0&\frac{2}{\sqrt{6}}&\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{6}}&-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{3}}\end{pmatrix}\begin{pmatrix}\sqrt{2}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\0&\frac{\sqrt{6}}{2}&\frac{1}{\sqrt{6}}\\0&0&\frac{2}{\sqrt{3}}\end{pmatrix}=QR$$

T2

取 A 第一列 $x_1=(2,0,2)^T$, 则 $c=\frac{\sqrt{2}}{2}, s=\frac{\sqrt{2}}{2}$

$$T_{13}=\begin{pmatrix}\frac{\sqrt{2}}{2}&0&\frac{\sqrt{2}}{2}\\0&1&0\\-\frac{\sqrt{2}}{2}&0&\frac{\sqrt{2}}{2}\end{pmatrix}, T_{13}A=\begin{pmatrix}2\sqrt{2}&\frac{3\sqrt{2}}{2}&\frac{3\sqrt{2}}{2}\\0&2&2\\0&-\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\end{pmatrix}$$

取 $T_{13}A$ 第二列后两行 $x_2=\left(2,-\frac{\sqrt{2}}{2}\right)^T$, 则 $c=\frac{2\sqrt{2}}{3}, s=-\frac{1}{3}$

$$T_{23}=\begin{pmatrix}1&0&0\\0&\frac{2\sqrt{2}}{3}&-\frac{1}{3}\\0&\frac{1}{3}&\frac{2\sqrt{2}}{3}\end{pmatrix}, T_{23}T_{13}A=\begin{pmatrix}2\sqrt{2}&\frac{3\sqrt{2}}{2}&\frac{3\sqrt{2}}{2}\\0&\frac{3\sqrt{2}}{2}&\frac{7\sqrt{2}}{6}\\0&0&\frac{4}{3}\end{pmatrix}=R$$

$$Q=T_{13}^{-1}T_{23}^{-1}=T_{13}^TT_{23}^T=\begin{pmatrix}\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{6}&-\frac{2}{3}\\0&\frac{2\sqrt{2}}{3}&\frac{1}{3}\\\frac{\sqrt{2}}{2}&-\frac{\sqrt{2}}{6}&\frac{2}{3}\end{pmatrix}$$

$A=QR$

T3

取 A 第一列 $x_1=(0,1,0)^T$, 则 $u_1=\frac{x_1-\|x_1\|_2e_1}{\left\|x_1-\|x_1\|_2e_1\right\|_2}=\frac{1}{\sqrt{2}}(-1,1,0)^T$

$$H_1=I-2u_1u_1^T=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$$

$$H_1A=\begin{pmatrix}1&1&1\\0&4&1\\0&3&2\end{pmatrix}$$

取 H_1A 第二列后两行 $x_2=(4,3)^T$, 则 $u_2=\frac{x_2-\|x_2\|_2e_1}{\left\|x_2-\|x_2\|_2e_1\right\|_2}=\frac{1}{\sqrt{10}}(-1,3)^T$

$$\widetilde{H_2}=I-2u_2u_2^T=\begin{pmatrix}\frac{4}{5}&\frac{3}{5}\\\frac{3}{5}&-\frac{4}{5}\end{pmatrix}, H_2=\begin{pmatrix}1&0&0\\0&\frac{4}{5}&\frac{3}{5}\\0&\frac{3}{5}&-\frac{4}{5}\end{pmatrix}$$

$$H_2H_1A=\begin{pmatrix}1&1&1\\0&5&2\\0&0&-1\end{pmatrix}=R$$

$Q = H_1^{-1}H_2^{-1} = H_1H_2 = \begin{pmatrix} 0 & \frac{4}{5} & \frac{3}{5} \\ 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \end{pmatrix}$

$A = QR$

T4

(1)

由定义可知

$\|A\|_2 = \sqrt{\lambda_{\max}(A^TA)} = \max(\sigma) = \sigma_1$

(2)

$\left\|A^{-1}\right\|_2 = \sqrt{\lambda_{\max}\left[\left(A^{-1}\right)^T A^{-1}\right]} = \sqrt{\lambda_{\max}\left[\left(AA^T\right)^{-1}\right]} = \sqrt{\frac{1}{\lambda_{\min}(A^TA)}} = \frac{1}{\min(\sigma)} = \sigma_2$

T5

$A^HAV = VV^HA^HUU^HAV = V\left(U^HAV\right)^H U^HAV = V\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$

故 V 的列向量是 A^HA 的特征向量

$AA^HU = UU^HAVV^HA^HU = UU^HAV\left(U^HAV\right)^H = U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$

故 U 的列向量是 AA^H 的特征向量

T6

由定义可知

$$\|A\|_F^2 = \operatorname{tr}(A^TA) = \sum_{i=1}^{r} \sigma_i^2$$

T7

$A^TA = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \det(\lambda I - A^TA) = (\lambda - 1)(\lambda - 3)$

$\lambda = 3$ 时，取单位化后的特征向量 $\xi_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T$

$\lambda = 1$ 时，取单位化后的特征向量 $\xi_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^T$

则 $V = (\xi_1, \xi_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \Sigma = \begin{pmatrix} \sqrt{3} & \\ & 1 \end{pmatrix}$

$U_1 = AV\Sigma^{-1} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 \end{pmatrix}$

取 $U_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)^T$，则 $U = (U_1, U_2)$

$A = U\begin{pmatrix} \Sigma \\ O \end{pmatrix}V^T = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

T8

$B = \begin{pmatrix} A \\ A \end{pmatrix} \in C_r^{2m\times n}, B^HB = 2A^HA$

又 $V^HA^HAV = \begin{pmatrix} \Sigma^2 & O \\ O & O \end{pmatrix} \Rightarrow 2V^HA^HAV = V^HB^HBV = \begin{pmatrix} 2\Sigma^2 & O \\ O & O \end{pmatrix}$

记 $V = (V_1, V_2)$

则有

$BV_2 = O, V_2 \in C^{n\times(n-r)}$

$V_1^H B^H BV_1 = \left(\sqrt{2}\Sigma\right)^2$, 其中 $V_1 \in C^{n\times r}$

$\Rightarrow \frac{\sqrt{2}}{2}\Sigma^{-1}V_1^H B^H BV_1 \frac{\sqrt{2}}{2}\Sigma^{-1} = I_r$

记 $\widetilde{U}_1 = \frac{\sqrt{2}}{2}BV_1\Sigma^{-1} \in C^{2m\times r}$

则有 $\widetilde{U_1}^H\widetilde{U_1} = I_r, BV_1 = \sqrt{2}\widetilde{U_1}\Sigma$

取 $\widetilde{U_2} \in C^{2m\times(2m-r)}, \tilde{U} = \left(\widetilde{U_1}, \widetilde{U_2}\right)$ 为 $2m$ 阶酉矩阵

$\widetilde{U_2}^H\widetilde{U_1} = O, \widetilde{U_2}^H\widetilde{U_2} = I_{2m-r}$

于是有

$\tilde{U}^H BV = \begin{pmatrix}\widetilde{U_1}^H\\ \widetilde{U_2}^H\end{pmatrix}B(V_1, V_2) = \begin{pmatrix}\widetilde{U_1}^H BV_1 & \widetilde{U_1}^H BV_2\\ \widetilde{U_2}^H BV_1 & \widetilde{U_2}^H BV_2\end{pmatrix} = \begin{pmatrix}\widetilde{U_1}^H\left(\sqrt{2}\widetilde{U_1}\Sigma\right) & O\\ \widetilde{U_2}^H\left(\sqrt{2}\widetilde{U_1}\Sigma\right) & O\end{pmatrix} = \begin{pmatrix}\sqrt{2}\Sigma & O\\ O & O\end{pmatrix}$

$B = \tilde{U}\begin{pmatrix}\sqrt{2}\Sigma & O\\ O & O\end{pmatrix}V^H$

T9

(1)

$\det(\lambda I - A) = (\lambda-4)(\lambda-2)$

$\lambda = 4$ 时，取单位化后的特征向量 $\xi_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)^T$

$\lambda = 2$ 时，取单位化后的特征向量 $\xi_2 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)^T, \xi_3 = (0,1,0)^T$

$E_1 = \xi_1\xi_1^T = \begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{2}\\ 0 & 0 & 0\\ \frac{1}{2} & 0 & \frac{1}{2}\end{pmatrix}$

$E_2 = \xi_2\xi_2^T + \xi_3\xi_3^T = \begin{pmatrix}\frac{1}{2} & 0 & -\frac{1}{2}\\ 0 & 0 & 0\\ -\frac{1}{2} & 0 & \frac{1}{2}\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & 0 & -\frac{1}{2}\\ 0 & 1 & 0\\ -\frac{1}{2} & 0 & \frac{1}{2}\end{pmatrix}$

$A = 4E_1 + 2E_2$

(2)

$\det(\lambda I - A) = \lambda^3 - 19\lambda^2 + 112\lambda - 200$