矩阵分解习题

T1

$A = (x_1, x_2, x_3)$

$u_1 = x_1 = (0,1,1)^T, y_1 = \frac{u_1}{\|u_1\|_2} = \frac{1}{\sqrt{2}}(0,1,1)^T$

$u_2 = x_2 - (x_2, y_1)y_1 = (1,1,0)^T - \frac{1}{2}(0,1,1)^T = \frac{1}{2}(2,1,-1)^T, y_2 = \frac{u_2}{\|u_2\|_2} = \frac{1}{\sqrt{6}}(2,1,-1)^T$

$u_3 = x_3 - (x_3, y_1)y_1 - (x_3, y_2)y_2 = (1,0,1)^T - \frac{1}{2}(0,1,1)^T - \frac{1}{6}(2,1,-1)^T = \frac{2}{3}(1,-1,1)^T$

$y_3 = \frac{u_3}{\|u_3\|_2} = \frac{1}{\sqrt{3}}(1,-1,1)^T$

$x_1 = \sqrt{2}y_1$

$x_2 = \frac{1}{\sqrt{2}}y_1 + \frac{\sqrt{6}}{2}y_2$

$x_3 = \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{6}}y_2 + \frac{2}{\sqrt{3}}y_3$

$A = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{6}}{2} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix} = QR$

T2

取 A 第一列 $x_1 = (2,0,2)^T$, 则 $c = \frac{\sqrt{2}}{2}, s = \frac{\sqrt{2}}{2}$

$T_{13} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}, T_{13}A = \begin{pmatrix} 2\sqrt{2} & \frac{3\sqrt{2}}{2} & \frac{3\sqrt{2}}{2} \\ 0 & 2 & 2 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

取 $T_{13}A$ 第二列后两行 $x_2 = \left(2, -\frac{\sqrt{2}}{2}\right)^T$, 则 $c = \frac{2\sqrt{2}}{3}, s = -\frac{1}{3}$

$T_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2\sqrt{2}}{3} & -\frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2\sqrt{2}}{3} \end{pmatrix}, T_{23}T_{13}A = \begin{pmatrix} 2\sqrt{2} & \frac{3\sqrt{2}}{2} & \frac{3\sqrt{2}}{2} \\ 0 & \frac{3\sqrt{2}}{2} & \frac{7\sqrt{2}}{6} \\ 0 & 0 & \frac{4}{3} \end{pmatrix} = R$

$Q = T_{13}^{-1}T_{23}^{-1} = T_{13}^T T_{23}^T = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} & -\frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{6} & \frac{2}{3} \end{pmatrix}$

$A = QR$

T3

取 A 第一列 $x_1 = (0,1,0)^T$, 则 $u_1 = \frac{x_1 - \|x_1\|_2 e_1}{\|x_1 - \|x_1\|_2 e_1\|_2} = \frac{1}{\sqrt{2}}(-1,1,0)^T$

$H_1 = I - 2u_1 u_1^T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$H_1 A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 3 & 2 \end{pmatrix}$

取 $H_1 A$ 第二列后两行 $x_2 = (4,3)^T$, 则 $u_2 = \frac{x_2 - \|x_2\|_2 e_1}{\|x_2 - \|x_2\|_2 e_1\|_2} = \frac{1}{\sqrt{10}}(-1,3)^T$

$\widetilde{H_2} = I - 2u_2 u_2^T = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{pmatrix}, H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{3}{5} \\ 0 & \frac{3}{5} & -\frac{4}{5} \end{pmatrix}$

$H_2 H_1 A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 2 \\ 0 & 0 & -1 \end{pmatrix} = R$

$$Q = H_1^{-1} H_2^{-1} = H_1 H_2 = \begin{pmatrix} 0 & \frac{4}{5} & \frac{3}{5} \\ 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \end{pmatrix}$$

$$A = QR$$

T4

(1)
由定义可知

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)} = \max(\sigma) = \sigma_1$$

(2)
$$\|A^{-1}\|_2 = \sqrt{\lambda_{\max}\left[(A^{-1})^T A^{-1}\right]} = \sqrt{\lambda_{\max}\left[(AA^T)^{-1}\right]} = \sqrt{\frac{1}{\lambda_{\min}(A^T A)}} = \frac{1}{\min(\sigma)} = \sigma_2$$

T5
$$A^H A V = V V^H A^H U U^H A V = V(U^H A V)^H U^H A V = V \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$$

故 V 的列向量是 $A^H A$ 的特征向量

$$A A^H U = U U^H A V V^H A^H U = U U^H A V (U^H A V)^H = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$$

故 U 的列向量是 $A A^H$ 的特征向量

T6
由定义可知

$$\|A\|_F^2 = \operatorname{tr}(A^T A) = \sum_{i=1}^{r} \sigma_i^2$$

T7
$$A^T A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \det(\lambda I - A^T A) = (\lambda - 1)(\lambda - 3)$$

$\lambda = 3$ 时，取单位化后的特征向量 $\xi_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T$

$\lambda = 1$ 时，取单位化后的特征向量 $\xi_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^T$

则 $V = (\xi_1, \xi_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \Sigma = \begin{pmatrix} \sqrt{3} & \\ & 1 \end{pmatrix}$

$$U_1 = A V \Sigma^{-1} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 \end{pmatrix}$$

取 $U_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)^T$，则 $U = (U_1, U_2)$

$$A = U \begin{pmatrix} \Sigma \\ O \end{pmatrix} V^T = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

T8
$B = \begin{pmatrix} A \\ A \end{pmatrix} \in C_r^{2m \times n}, B^H B = 2 A^H A$

又 $V^H A^H A V = \begin{pmatrix} \Sigma^2 & O \\ O & O \end{pmatrix} \Rightarrow 2 V^H A^H A V = V^H B^H B V = \begin{pmatrix} 2\Sigma^2 & O \\ O & O \end{pmatrix}$

记 $V = (V_1, V_2)$

则有

$BV_2 = O, V_2 \in C^{n \times (n-r)}$

$V_1^H B^H B V_1 = \left(\sqrt{2}\Sigma\right)^2$, 其中 $V_1 \in C^{n \times r}$

$\Rightarrow \frac{\sqrt{2}}{2}\Sigma^{-1} V_1^H B^H B V_1 \frac{\sqrt{2}}{2}\Sigma^{-1} = I_r$

记 $\widetilde{U}_1 = \frac{\sqrt{2}}{2} B V_1 \Sigma^{-1} \in C^{2m \times r}$

则有 $\widetilde{U}_1^H \widetilde{U}_1 = I_r, BV_1 = \sqrt{2}\widetilde{U}_1\Sigma$

取 $\widetilde{U}_2 \in C^{2m \times (2m-r)}, \tilde{U} = \left(\widetilde{U}_1, \widetilde{U}_2\right)$ 为 $2m$ 阶酉矩阵

$\widetilde{U}_2^H \widetilde{U}_1 = O, \widetilde{U}_2^H \widetilde{U}_2 = I_{2m-r}$

于是有

$\tilde{U}^H B V = \begin{pmatrix} \widetilde{U}_1^H \\ \widetilde{U}_2^H \end{pmatrix} B(V_1, V_2) = \begin{pmatrix} \widetilde{U}_1^H B V_1 & \widetilde{U}_1^H B V_2 \\ \widetilde{U}_2^H B V_1 & \widetilde{U}_2^H B V_2 \end{pmatrix} = \begin{pmatrix} \widetilde{U}_1^H \left(\sqrt{2}\widetilde{U}_1\Sigma\right) & O \\ \widetilde{U}_2^H \left(\sqrt{2}\widetilde{U}_1\Sigma\right) & O \end{pmatrix} = \begin{pmatrix} \sqrt{2}\Sigma & O \\ O & O \end{pmatrix}$

$B = \tilde{U} \begin{pmatrix} \sqrt{2}\Sigma & O \\ O & O \end{pmatrix} V^H$

T9

(1)

$\det(\lambda I - A) = (\lambda - 4)(\lambda - 2)$

$\lambda = 4$ 时，取单位化后的特征向量 $\xi_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)^T$

$\lambda = 2$ 时，取单位化后的特征向量 $\xi_2 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)^T, \xi_3 = (0, 1, 0)^T$

$E_1 = \xi_1 \xi_1^T = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$

$E_2 = \xi_2 \xi_2^T + \xi_3 \xi_3^T = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$

$A = 4E_1 + 2E_2$

(2)

$\det(\lambda I - A) = (\lambda - 4)(\lambda - 2) = \lambda^3 - 19\lambda^2 + 112\lambda - 200$