Nisiyama_Suzune's blog

[Tutorial] Math note — Möbius inversion

By Nisiyama_Suzune, 2 years ago,

If you've ever taken some lessons on competitive programming, chances are that you have already heard about one of the most famous formula: the Möbius inversion. This article is aimed to provide some basic insight on what is the Möbius inversion, as well as how to apply it in various programming tasks.

Prequisite

If you are not familiar with the linear sieve and multiplicative functions, it is recommended that you read about them first here.

I will introduce some frequently used notations and lemmas first.

Notation

1. $[P]$ refers to the boolean expression, i.e. $[P] = 1$ when P is true, and 0 otherwise.
2. $\rfloor x \lfloor$ refers to rounding x down to the nearest integer. Thus $\lfloor \frac{q}{p} \rfloor$ refers to the integer division.
3. $d|n$ means that d can divide n (without a remainder).

The following functions are all multiplicative functions, where p is a prime number and k is a positive integer.

1. The constant function $I(p^k) = 1$.
2. The identity function $Id(p^k) = p^k$.
3. The power function $Id_a(p^k) = p^{ak}$, where a is constant.
4. The unit function $\epsilon(p^k) = [p^k = 1]$.
5. The divisor function $\sigma_a(p^k) = \sum_{i=0}^{k} p^{ai}$, denoting the sum of the a-th powers of all the positive divisors of the number.
6. The Möbius function $\mu(p^k) = [k = 0] - [k = 1]$.
7. The Euler's totient function $\varphi(p^k) = p^k - p^{k-1}$.

Lemma

I have some my unofficial names for these frequently used conclusions. If you happen to know any more commonly used name for them, you are more than welcome to tell me.

- (**The integer division lemma**) For every positive integer p, q and r, $\lfloor \frac{\lfloor \frac{p}{q} \rfloor}{r} \rfloor = \lfloor \frac{p}{qr} \rfloor$.

Proof: We know that $\frac{p}{q} - \lfloor \frac{p}{q} \rfloor < 1$. Hence $\frac{p}{qr} - \frac{\lfloor \frac{p}{q} \rfloor}{r} < \frac{1}{r}$. Since the fraction part of $\frac{\lfloor \frac{p}{q} \rfloor}{r}$ cannot exceed $\frac{r-1}{r}$, we achieve the conclusion.

- (**The harmonic lemma**) Consider the harmonic sequence on integer division $\left(\lfloor \frac{n}{1} \rfloor, \lfloor \frac{n}{2} \rfloor, \ldots, \lfloor \frac{n}{n} \rfloor\right)$.

1. The sequence is non-increasing, and there are at most $2\sqrt{n}$ different elements.
2. The sum of the sequence is approximate to $O(n \log n)$.

Proof: Denote $d(i) = \lfloor \frac{n}{i} \rfloor$. For every i not exceeding $\sqrt{n}$, there will be no more than $\sqrt{n}$ values in the domain of $d(i)$, so there can be at most $\sqrt{n}$ different values for $d(i)$. For the rest part of i greater than $\sqrt{n}$, we can infer that $d(i) < \sqrt{n}$ and is a positive integer, so there can be at most $\sqrt{n}$ different values for $d(i)$. Therefore we know that there are at most $2\sqrt{n}$ different elements in the sequence.

Since the Euler–Mascheroni constant states that $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i} = \ln n + \gamma$, we know that $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{i} = n \ln n + n\gamma$, so the sum of the original sequence is approximate to $O(n \log n)$.

Moreover, it is actually possible to exploit the property that the sequence has at most $2\sqrt{n}$ different elements, and write a loop that runs in $O(\sqrt{n})$ complexity to iterate through every possible value of $d(i)$, using the fact that the greatest integer x satisfying $d(x) = d(i)$ is $\lfloor \frac{n}{\lfloor \frac{n}{i} \rfloor} \rfloor$. The piece of code below demonstrates one way to program it.

```
for (int i = 1, la; i <= n; i = la + 1) {
	la = n / (n / i);
	//n / x yields the same value for i <= x <= la.
}
```

- (**The transitive property of multiplicative functions**)

1. If both $f(x)$ and $g(x)$ are multiplicative, then $h(x) = f(x)g(x)$ is also multiplicative.
2. If both $f(x)$ and $g(x)$ are multiplicative, then their Dirichlet convolution $f * g(x) = \sum_{d|x} f(d)g(\frac{x}{d})$ is also multiplicative. Specifically, if $f(x)$ is multiplicative, $h(x) = \sum_{d|x} f(d)$ is also multiplicative.

The proof is more mathematical, which I will skip here.

What is the Möbius inversion?

According to Wikipedia, the Möbius inversion states that:

If $g(n) = \sum_{d|n} f(d)$ for every positive integer n, then $f(n) = \sum_{d|n} g(d)\mu(\frac{n}{d})$, where $\mu(x)$ is the Möbius function, which is multiplicative and satisfies $f(1) = 1, f(p) = -1, f(p^k) = 0$ for any prime number p and any integer $k \geq 2$. (*It is worth noting that you can pre-compute all values of the Möbius function from 1 to n using the linear sieve.*)

However, the definition alone does not mean much (unless the problem statement explicitly gives you something like $g(n) = \sum_{d|n} f(d)$. In that case, well...). There is one important property that is probably more useful than the definition:

$$\sum_{d|n} \mu(d) = \epsilon(n) = [n = 1]$$

In order to prove this property, we have to use the transitive property of multiplicative functions to show that $f(n) = \sum_{d|n} \mu(d)$ is multiplicative. After that, we can see $f(1) = 1$ and $f(p^k) = 0$, which means $f(x) = \varepsilon(x)$. Now you may ask: why is this property important? I think it is best to show the reasons in some basic examples below.

Example 1. Find out the number of co-prime pairs of integer (x, y) in range $[1, n]$.

Solution 1. Notice that the question is the same as asking you the value of

$$f(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} [gcd(i, j) = 1]$$

Now apply the Möbius inversion on $[gcd(i, j)] = 1$, we have

$$f(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{d|gcd(i,j)} \mu(d)$$

which is the same as

$$f(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{d=1}^{n} [d|gcd(i, j)]\mu(d)$$

Notice that $[d|gcd(i, j)] = [d|i][d|j]$. Therefore

$$f(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{d=1}^{n} [d|i][d|j]\mu(d)$$

We can change the order of summing things up, so

$$f(n) = \sum_{d=1}^{n} \mu(d) \left(\sum_{i=1}^{n} [d|i]\right) \left(\sum_{j=1}^{n} [d|j]\right)$$

We know that $\sum_{i=1}^{n} [d|i] = \sum_{j=1}^{n} [d|j] = \lfloor \frac{n}{d} \rfloor$. Thus

$$f(n) = \sum_{d=1}^{n} \mu(d) \lfloor \frac{n}{d} \rfloor^2$$

Then we can simply loop through 1 to n to compute the formula. While we can optimize this loop to $O(\sqrt{n})$ using the harmonic lemma, we still have to use the linear sieve to pre-compute the values of $\mu(d)$. Therefore, the overall complexity of the algorithm is $O(n)$.

Example 2. Find out the sum of $gcd(x, y)$ for every pair of integer (x, y) in range $[1, n]$ ($gcd(x, y)$ means the greatest common divisor of x and y).

Solution 2. Similar as above, notice that the question is the same as asking you the value of

$$f(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} gcd(i, j)$$

Let $k = gcd(i, j)$. We can then loop for k first.

$$f(n) = \sum_{k=1}^{n} k \sum_{i=1}^{n} \sum_{j=1}^{n} [gcd(i, j) = k]$$

Now assume $i = ak, j = bk$, and then

$$f(n) = \sum_{k=1}^{n} k \sum_{a=1}^{\lfloor \frac{n}{k} \rfloor} \sum_{b=1}^{\lfloor \frac{n}{k} \rfloor} [gcd(a, b) = 1]$$

It can be observed that the last part of the formula is the same as the one in example 1. Applying the same procedure, we have

$$f(n) = \sum_{k=1}^{n} k \sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} \mu(d) \lfloor \frac{n}{kd} \rfloor^2$$

According to the harmonic lemma, $O(\sum_{k=1}^{n} \lfloor \frac{n}{k} \rfloor) = O(n \log n)$. Therefore, if we compute the sum above with brute force, the overall complexity will be $O(n \log n)$.

This is, however, not the best complexity we can achieve with the Möbius inversion. Now let $l = kd$, and we can loop for l first.

$$f(n) = \sum_{l=1}^{n} \lfloor \frac{n}{l} \rfloor^2 \sum_{k|l} k\mu(\frac{l}{k})$$

Applying the transitive property of multiplicative functions, we can see that $g(l) = \sum_{k|l} k\mu(\frac{l}{k})$ is multiplicative. Moreover, we have $g(p^k) = p^k - p^{k-1}$. If you have somewhat studied about the number theory, you may figure out that $g(l)$ is actually the Euler's totient function. Regardless of whether you know about the coincidence or not, $g(l)$ can be pre-computed with the linear sieve, and $f(n) = \sum_{l=1}^{n} \lfloor \frac{n}{l} \rfloor^2 g(l)$, which can be simply computed in $O(n)$ complexity.

Example 3. Find out the sum of $lcm(x, y)$ for every pair of integer (x, y) in range $[1, n]$ ($lcm(x, y)$ means the least common multiple of x and y).

Solution 3. Well, we should be pretty familiar with the techniques by now. First of all, the answer should be

$$f(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} lcm(i, j)$$

Since $lcm(i, j) = \frac{ij}{gcd(i,j)}$, let $k = gcd(i, j)$. We can then loop for k first.

$$f(n) = \sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} [gcd(i, j) = k] \frac{ij}{k}$$

Now assume $i = ak, j = bk$, and then

$$\begin{aligned} f(n) &= \sum_{k=1}^{n} \sum_{a=1}^{\lfloor \frac{n}{k} \rfloor} \sum_{b=1}^{\lfloor \frac{n}{k} \rfloor} [gcd(a, b) = 1]abk \\ &= \sum_{k=1}^{n} k \sum_{a=1}^{\lfloor \frac{n}{k} \rfloor} a \sum_{b=1}^{\lfloor \frac{n}{k} \rfloor} b[gcd(a, b) = 1] \\ &= \sum_{k=1}^{n} k \sum_{a=1}^{\lfloor \frac{n}{k} \rfloor} a \sum_{b=1}^{\lfloor \frac{n}{k} \rfloor} b \sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} [d|a][d|b]\mu(d) \\ &= \sum_{k=1}^{n} k \sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} \mu(d) \left(\sum_{a=1}^{\lfloor \frac{n}{k} \rfloor} [d|a]a\right) \left(\sum_{b=1}^{\lfloor \frac{n}{k} \rfloor} [d|b]b\right) \end{aligned}$$

Now let's proceed to assume $a = dp, b = dq$, then

$$\begin{aligned} f(n) &= \sum_{k=1}^{n} k \sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} \mu(d) \left(\sum_{p=1}^{\lfloor \frac{n}{kd} \rfloor} dp\right) \left(\sum_{q=1}^{\lfloor \frac{n}{kd} \rfloor} dq\right) \\ &= \sum_{k=1}^{n} k \sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} \mu(d)d^2 \left(\sum_{p=1}^{\lfloor \frac{n}{kd} \rfloor} p\right) \left(\sum_{q=1}^{\lfloor \frac{n}{kd} \rfloor} q\right) \end{aligned}$$

Now notice $\sum_{p=1}^{\lfloor \frac{n}{kd} \rfloor} p = \frac{(1+\lfloor \frac{n}{kd} \rfloor)(\lfloor \frac{n}{kd} \rfloor)}{2}$, and we have

$$f(n) = \sum_{k=1}^{n} k \sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} \mu(d)d^2 \left(\frac{(1 + \lfloor \frac{n}{kd} \rfloor)(\lfloor \frac{n}{kd} \rfloor)}{2}\right)^2$$

Now following the example above, let $l = kd$, and then

$$f(n) = \sum_{l=1}^{n} \left(\frac{(1 + \lfloor \frac{n}{l} \rfloor)(\lfloor \frac{n}{l} \rfloor)}{2}\right)^2 \sum_{d|l} \mu(d)ld$$

Notice that $g(l) = \sum_{d|l} \mu(d)ld$ is also multiplicative, and $g(p^k) = p^k - p^{k+1}$. We can pre-compute the values of $g(l)$ via the linear sieve, and $f(n) = \sum_{l=1}^{n} \left(\frac{(1+\lfloor \frac{n}{l} \rfloor)(\lfloor \frac{n}{l} \rfloor)}{2}\right)^2 g(l)$. The overall complexity will be $O(n)$.

Practice Problems

Simple Sum

(Thanks _threat_ for showing me this one!)

Link

GCD

Link

Sky Code

Link

Extensions

It is known that all examples above can be further optimized to $O(n^{\frac{2}{3}})$. The exact technique will (probably) be introduced in the upcoming article.

Update: The article introducing the optimization is ready here.

Finally, thanks Tommyr7 for his effort in the article!

math, mobius inversion formula, mobius function, tutorial

+202 Nisiyama_Suzune 2 years ago 35

Comments (35)

Write comment?

ghoshsai5000 — 2 years ago, # | 0

Excellent tutorial. If you don't mind can you just write down in 1-2 sentences how to precompute the Mobius function using linear sieve ?

→ Reply

akali — 2 years ago, # ^ | 0

```
m[1] = 1;
for (int i = 2; i < MAXN; ++i) {
    if (!lp[i]) for (int j = i; j < MAXN; j += i)
        if (!lp[j]) lp[j] = i;
    m[i] = [](int x) {
        int cnt = 0;
        while (x > 1) {
            int k = 0, d = lp[x];
            while (x % d == 0) {
                x /= d;
                ++k;
                if (k > 1) return 0;
            }
            ++cnt;
        }
        if (cnt & 1) return -1;
        return 1;
    }(i);
}
```

→ Reply

Hansuzu — 14 months ago, # ^ | +6

See the example for the Euler's totient function in the linear sieve tutorial.

For the Möbius function the three cases are:
1) if x is prime: mobius[x]=-1 (obiously the number of prime factors of a prime number is odd)
2) x=ip and i%p!=0: mobius[x]=mobius[i]*mobius[j] (since the function is multiplicative)
3) x=ip and i%p==0: mobius[x]=0 (p^2 divides x)

→ Reply

r_pi — 12 months ago, # ^ | -13

→ Reply

The comment is hidden because of too negative feedback, click here to view it

noob__coder — 2 years ago, # | 0

Can you explain how to use Mobius inversion in The GCD problem link given here. I am having problem in using the theory given the that problem.

→ Reply

newCEA — 23 months ago, # ^ | 0

Step 1: $\gcd(x, y) = k$ if and only if $k \mid x, k \mid y$ and $\gcd(x/k, y/k) = 1$. Using this we can reduce the problem of finding $0 < x \le b, 0 < y \le d$ such that $\gcd(x, y) = k$ to the problem of example 1.

$$\sum_{x=1}^{b} \sum_{y=1}^{d} [\gcd(x, y) = k] = \sum_{x'=1}^{\lfloor \frac{b}{k} \rfloor} \sum_{y'=1}^{\lfloor \frac{d}{k} \rfloor} [\gcd(x, y) = 1] = \sum_{v=1}^{\max(\lfloor \frac{b}{k} \rfloor, \lfloor \frac{d}{k} \rfloor)} \left\lfloor \frac{b}{vk} \right\rfloor \cdot \left\lfloor \frac{d}{vk} \right\rfloor \mu(v)$$

Step 2: The sum over the interval $[a, b] \times [c, d]$ is the sum over $[1, b] \times [1, d]$ minus $[1, a-1] \times [1, d]$ minus $[1, b] \times [1, c-1]$ plus $[1, a-1] \times [1, c-1]$.

→ Reply

JustAni — 21 month(s) ago, # ^ | 0

```
Please notice that, (x=5, y=7) and (x=7, y=5) are considered to be the same.
```

How you dealt with the condition that there should be unique pairs?

→ Reply

roll_no_1 — 11 months ago, # ^ | ← Rev. 3 0

Firstly, note that a and c are both equal to 1 in the problem statement. Now, let's first handle the corner cases:

Let $m = min(b, d)$. If $k = 0$ or $m < k$, the answer is clearly 0.

Now, firstly we obtain the answer without worrying about duplicate entries. It'll be computed by the formula given by newCEA.

After this, now let's eliminate all the duplicate values. Firstly, note that the duplicate pairs must have both the values lying within the range $[1, m]$ because the remaining nos. from the 2 given ranges that are not lying within this range must be present in exactly one of the ranges and therefore cannot contribute to duplicates. Let X be the answer to the original problem that includes the duplicates entries, i.e., the no. of co-prime pairs between the ranges $[1, b]$ and $[1, d]$. Now compute Y = no. of co-prime pairs where both numbers lie between the range $[1, m]$. Note that there must be exactly one pair (x, y) where $x = y$, and that is (k, k). For all other pairs $x \neq y$, so (x, y) and (y, x) must both appear in the answer as a co-prime pairs, so there are only $\frac{(Y-1)}{2}$ duplicate values and these are also the only duplicate values present within the ranges $[1, b]$ and $[1, d]$. So, the final answer is $X - \frac{(Y-1)}{2}$. Here is my solution for this problem as a reference.

→ Reply

roll_no_1 — 11 months ago, # ^ | 0

In your formula, the last must term must be

$$\sum_{v=1}^{min(\lfloor \frac{b}{k} \rfloor, \lfloor \frac{d}{k} \rfloor)} \left\lfloor \frac{b}{vk} \right\rfloor \left\lfloor \frac{d}{vk} \right\rfloor \mu(v)$$

The max must be replaced with min.

→ Reply

milanovic — 22 months ago, # | ← Rev. 2 0

this article is good but i have a big problem : in your example you have range from [1,n] but in many problem we have an array of element.if in example1 we have an array of element (1<=A(i)<=1000000) now how we can find number of Co-prime pair in array with mobius ? how use mobius in problem that given array of element? ***actually i want you prove this with array of element such your proving in three examples that you mentioned above?

→ Reply

kaldiuo — 22 months ago, # | +1

how is the greatest integer x satisfying d(x) = d(i) is n/(n/i)?

→ Reply

ffern — 21 month(s) ago, # ^ | +6

We know that for two positive integers a, n: $\lfloor \frac{n}{a} \rfloor <= \frac{n}{a} < \lfloor \frac{n}{a} \rfloor + 1$

$a * \lfloor \frac{n}{a} \rfloor <= n < a * \lfloor \frac{n}{a} \rfloor + a$

Let's define $\lfloor \frac{n}{a} \rfloor = k$

$a * k$ <= n < $a * k + a$

$a * k$ <= n and $a * (k + 1)$ > n

$a <= \lfloor \frac{n}{k} \rfloor$ and $a > \lfloor \frac{n}{k+1} \rfloor$

So, if for a positive integer x, $\lfloor \frac{n}{x} \rfloor = k \rightarrow \lfloor \frac{n}{k+1} \rfloor$ < x <= $\lfloor \frac{n}{k} \rfloor$

But $\lfloor \frac{n}{x} \rfloor = \lfloor \frac{n}{i} \rfloor \rightarrow \lfloor \frac{n}{\lfloor \frac{n}{i} \rfloor + 1} \rfloor$ < x <= $\lfloor \frac{n}{\lfloor \frac{n}{i} \rfloor} \rfloor$

This tells us that $\lfloor \frac{n}{\lfloor \frac{n}{i} \rfloor} \rfloor$ is the maximum x for which $\lfloor \frac{n}{x} \rfloor = \lfloor \frac{n}{i} \rfloor$.

→ Reply

Valiors — 2 months ago, # ^ | ← Rev. 2 0

If for some x you have $\lfloor \frac{n}{x} \rfloor = k$ then the greatest x' such that $\lfloor \frac{n}{x'} \rfloor = k$ is $x' = \lfloor \frac{n}{k} \rfloor$ because $\frac{n}{\frac{n}{k}} = k$ and if you divide n by any bigger number than $\frac{n}{k}$ you get a smaller quotient than k.

→ Reply

mango_lassi — 22 months ago, # | 0

What is the intended complexity in Sky Code? My N d(N) q solution gets TLE immediately. (N = 10000)

→ Reply

roll_no_1 — 11 months ago, # ^ | +8

Precomputing the divisors of all numbers from 1 to 10000 gets AC on the problem. Using square root factorization gets TLE.

→ Reply

JustAni — 21 month(s) ago, # | ← Rev. 2 0

`Moreover, we have g(p^k) = p^k - p^(k - 1).` How did you conclude this?

Update: I solved it for prime **p** and then generalized it for some **p^k**. Is it the right way?

→ Reply

spellstaker — 18 months ago, # | +3

So in example 1, step 2, did he just say that f = f ∘ μ?

How does one conclude from

$$f(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} [gcd(i, j) = 1]$$

that

$$f(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{d|gcd(i,j)} \mu(d)$$

→ Reply

spellstaker — 18 months ago, # ^ | 0

Ah, I see, he just rewrote what was the dirichlet identity.

Duh doy, heh.

→ Reply

Dranoel321 — 15 months ago, # | 0

Very good tutorial. Thanks a lot!

→ Reply

roll_no_1 — 11 months ago, # | 0

Here is an easier version of the Sky Code problem. It asks to find co-prime triples, whereas in Sky Code, we are required to find co-prime quadruples (4-tuples).

→ Reply

yeeeet — 11 months ago, # | 0

now i know how bad my math is

→ Reply

lovppa — 10 months ago, # | 0

how to derive the formula for first question link that you provided..help plz

→ Reply

NothingReallyMattersToMe — 10 months ago, # | 0

Bob and Numbers How to solve this problem using the technique used in the above examples?

→ Reply

low_kii_savage — 9 months ago, # | 0

Can you explain the mobius inversion in the simple sum?

→ Reply

konstantinosk31 — 7 months ago, # ^ | +19

We have that:

$$\begin{aligned} f(n) &= \sum_{i=1}^{n} \frac{n}{gcd(i,n)} \\ &= \sum_{i=1}^{n} \sum_{k=1}^{n} [gcd(i, n) = k] \frac{n}{k} \end{aligned}$$

Rearranging the sums:

$$= \sum_{k=1}^{n} [k|n] \frac{n}{k} \sum_{i=1}^{n} [gcd(i, n) = k]$$

Let l = i/k and divide by k in the second sum:

$$= \sum_{k=1}^{n} [k|n] \frac{n}{k} \sum_{l=1}^{\frac{n}{k}} [gcd(l, \tfrac{n}{k}) = 1]$$

Using mobius inversion:

$$\begin{aligned} &= \sum_{k=1}^{n} [k|n] \frac{n}{k} \left(\sum_{l=1}^{\frac{n}{k}} \sum_{d|gcd(l,\frac{n}{k})} \mu(d) \right) \\ &= \sum_{k=1}^{n} [k|n] \frac{n}{k} \left(\sum_{l=1}^{\frac{n}{k}} \sum_{d=1}^{\frac{n}{k}} [d|l][d|\tfrac{n}{k}]\mu(d) \right) \end{aligned}$$

Rearranging the last 2 sums:

$$\begin{aligned} &= \sum_{k=1}^{n} [k|n] \frac{n}{k} \left(\sum_{d=1}^{\frac{n}{k}} [d|\tfrac{n}{k}]\mu(d) \left(\sum_{l=1}^{\frac{n}{k}} [d|l] \right) \right) \\ &= \sum_{k=1}^{n} [k|n] \frac{n}{k} \left(\sum_{d=1}^{\frac{n}{k}} [d|\tfrac{n}{k}]\mu(d) \frac{n}{kd} \right) \\ &= \sum_{k|n} \frac{n}{k} \left(\sum_{d|\frac{n}{k}} \mu(d) \frac{n}{kd} \right) \end{aligned}$$

Due to symmetry, we can replace k by n/k, getting:

$$= \sum_{k|n} k \left(\sum_{d|k} \mu(d) \frac{k}{d} \right)$$

Which as shown in the article is:

$$= \sum_{k|n} k\varphi(k)$$

Using properties mentioned in the article, we notice that k and φ(k) are multiplicative and therefore so is their product. Now since kφ(k) is multiplicative, so will its dirichlet convolution with the constant function be. In essence, f(n) is multiplicative and therefore we can precompute f(n) with a linear sieve.

Using the fact that φ(p^k) = p^k - p^(k-1), we can see that:

$$f(p^k) = \frac{p^{2k+1} + 1}{p + 1}$$

This editorial might also help, although it does not use the mobius function. That's it, hope I helped!

→ Reply

low_kii_savage — 7 months ago, # ^ | 0

Thanks

→ Reply

kazama460 — 6 months ago, # ^ | +1

can you explain LCM Sum using mobius function please? It would help a lot. Thank you.

→ Reply

konstantinosk31 — 6 months ago, # ^ | 0

Do you mean Example 3? If you do, since it's already explained, could you point out what you don't understand?

→ Reply

kazama460 — 6 months ago, # ^ | 0

Thank you for paying attention , i got the example 3 , the only problem i am getting is i am not able to find the formula(using mobius inversion) for problems in which

i is varying from 1 to n
j is varying from i+1 to n
like GCDEX2

if both are varying from 1 to n then there is no problem.

can you help me in this please?

thank you .

→ Reply

kazama460 — 6 months ago, # | 0

can anyone please explain how mobius inversion is used to convert [gcd(i, j)] = 1 to summation d|gcd(i , j) u(d) ? please help me , i am stuck on this from last 2-3 days

→ Reply

Nisiyama_Suzune — 6 months ago, # ^ | +11

$$\sum_{d|n} \mu(d) = \epsilon(n) = [n = 1]$$

Simply substitute $n = gcd(i, j)$ in this formula.

→ Reply

kazama460 — 6 months ago, # ^ | 0

thank you so much , it really helped

→ Reply

soul_voyage — 5 months ago, # | 0

How to interpret mobius inversion two forms equivalent to functions f(n) and g(n). For example in Vandermonde convolution we have a combinatorial interpretation which makes the resulting formula intuitive. Basically, I want to know why does the divisors comes into formulaes? Or does there exist a counting problem where counting them in two different ways results in the inversion by mobius? Nisiyama_Suzune

→ Reply

Wjex2 — 5 months ago, # ^ | 0

This may be helpful to get the intuition. (https://youtu.be/_noJI8UkTq8)

→ Reply

soul_voyage — 5 months ago, # ^ | 0

Oh, the famous necklace problem. No one ever mentions usage of mobius function while describing Burnside lemma.

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