Question 8

Exercise 11.12. Prove that a binary tree with k leaves has height at least log k.

We can first prove the number of leaves in a tree of height h is no more than 2^h

Proof: I will prove it by induction.

Base Case:

For h = 0, the tree has only one node (the only one leave).

L(h) = 2^h = 1, so the claim is true in the base case. (L(h) is number of leaves in a tree with height h).

Induction assumption: the tree with height n has 2^n leaves.

Induction:

For a tree with height of (n+1), the height of right and left subtree is no more than n, so the number of leaves in each of them is no more than 2^n. Moreover, the total number of leaves in the tree is equal to the sum of leaves in right and left subtree.

Therefore, we have the following equation:

L(n+1) <= 2^n + 2^n = 2 ^ (n+1)

By Induction, we prove that the number of leaves in a tree of height h is no more than 2^h.

Assume the number of leaves is k, and the height is h. According to above lemma, we get the following equation:

k <= 2^h

Do a log operation in both side of the equation:

h >= log k

Therefore, we proved that a binary tree with k leaves has height at least log k.