Solutions to Assignment 11

Exercise 1 [6 points]: Diffusion + chemistry: Turing instability

Consider the brusselator model

$$\begin{aligned}\partial_t u &= D_u \partial_x^2 u + a - (b+1)u + u^2 v,\\ \partial_t v &= D_v \partial_x^2 v + bu - u^2 v,\end{aligned}$$

where $u(x,t)$, $v(x,t)$ are concentration fields.

1. Determine the homogeneous stationary base state (with $u \neq 0, v \neq 0$).
2. Calculate the onset of the finite wavelength (Turing) instability, as discussed for the general case in the lecture.
   HINT: In the eigenvalue problem, it is again enough to study $\sigma = 0$. Solve for the control parameter $b(q)$, minimize with respect to q and show that you get $b_c^{\text{Turing}} = (1 + a\sqrt{D_u/D_v})^2$ and $q_c = \sqrt{\frac{a}{\sqrt{D_u D_v}}}$.
3. To observe the finite wavelength pattern, the control parameter b needs to be beyond b_c^{Turing} but below the threshold of the other instability occuring in the system, the oscillatory Hopf instability having $b_c^{\text{Hopf}} = 1 + a^2$ (see lecture). What restriction do b_c^{Turing} and b_c^{Hopf} hence imply for the diffusion coefficients of the two species?

SOLUTION:

1. For the stationary *homogeneous* state, we can put all derivatives to zero and have to solve the following two equations simultaneously:

$$\begin{aligned}0 &= f_u = a - (b+1)u + u^2 v \qquad (1)\\ 0 &= f_v = bu - u^2 v. \qquad (2)\end{aligned}$$

   The only solution we find is $\vec{w}_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} a \\ b/a \end{pmatrix}$.

2. The perturbation of the state is given by

$$\delta\dot{\vec{w}} = \begin{pmatrix} \delta\dot{u} \\ \delta\dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_u}{\partial u} - D_u q^2 & \frac{\partial f_u}{\partial v} \\ \frac{\partial f_v}{\partial u} & \frac{\partial f_v}{\partial v} - D_v q^2 \end{pmatrix} \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} = A \begin{pmatrix} \delta u \\ \delta v \end{pmatrix}, \qquad (3)$$

$$* \quad -(b-1)a^2 - D_v q^2(b-1) + D_u q^2 a^2 + D_u D_v q^4 + a^2 b = 0$$

$$= a^2 + q^2[D_u a^2 - D_v(b-1)] + D_u D_v q^4$$

$$\Rightarrow b(q) = \frac{a^2}{D_v q^2} + \frac{q^2[D_u a^2 + D_v]}{D_v q^2} + \frac{D_u D_v q^4}{D_v q^2} = \frac{a^2}{D_v q^2} + \frac{D_u}{D_v}a^2 + 1 + D_u q^2$$

with $\delta\vec{w} = \vec{w}_q e^{iqx+\sigma(q)t}$. Hence, a solution for σ can be found via the eigenvalue problem $\sigma\vec{w}_q = A\vec{w}_q$ and hence the equation for σ is:

$$\sigma_{1,2}(q) = \frac{1}{2}TrA \pm \frac{1}{2}\sqrt{(TrA)^2 - 4detA}. \tag{4}$$

Evaluating the homoegenous part of matrix A at $\vec{w}_0$ yields:

$$A|_{\vec{w}_0} = \begin{pmatrix} -(b+1) + 2u_0v_0 & u_0^2 \\ b - 2u_0v_0 & -u_0^2 \end{pmatrix}\Bigg|_{\vec{w}_0} = \begin{pmatrix} b-1 & a^2 \\ -b & -a^2 \end{pmatrix}, \tag{5}$$

At the onset of a possible finite q_c instability, $\sigma = 0$. Therefore, we just have to solve $detA = 0$ for the full system including the spacial derivatives. This leads to:

$$(b - 1 - D_uq^2)(-a^2 - D_vq^2) + a^2b = 0. \quad * \tag{6}$$

Solving this for $b(q)$ gives:

$$D_v + D_u a^2$$

$$b(q) = \frac{a^2}{D_vq^2} + \frac{\cancel{D_va^2D_u}}{D_v} + D_uq^2. \tag{7}$$

Minimizing the function gives:

$$0 = \partial_q b(q) = -2\frac{a^2}{D_vq^3} + 2D_uq \tag{8}$$

$$q_c = \sqrt{\frac{a}{\sqrt{D_uD_v}}} \tag{9}$$

and plugging this back into the equation for b gives:

$$\tag{10}$$

$$b_c = \left(1 + a\sqrt{\frac{D_u}{D_v}}\right)^2. \tag{11}$$

Note that for both instabilities $b > 1$ such that $A_{uu} = b - 1 > 0$ and $A_{vv} = -a^2 < 0$. This means that u is "activated" while v is inhibited around the stationary homogeneous solution. One can also say that due to the bu-term in the equation for v, u activates v if it is small (and hence $-u^2v$ is small).

3. b must be below the Hopf instability, i.e. one needs:

$$b < b_H = 1 + a^2. \tag{12}$$

For Turing one needs:

$$b > b_T = \left(1 + a\sqrt{\frac{D_u}{D_v}}\right)^2. \tag{13}$$

Hence we need $b_T < b < b_H$ which leads to: always < 1 !

$$\left(1 + a\sqrt{\frac{D_u}{D_v}}\right)^2 < 1 + a^2 \rightarrow \sqrt{\frac{D_u}{D_v}} < \frac{\sqrt{1+a^2} - 1}{a}. \tag{14}$$

The r.h.s. is of order a, hence finite and rather large, hence $D_u \ll D_v$, implying again short range activation and long range inhibition.

small a: $\sim a < 1$

large a: $\sim \frac{a-1}{a} \lesssim 1$

monotonous