# Introduction

The goal of this document is to provide a mathematical model for the game CrossFade. The game is played on a square grid of tiles, which we will call the board. In principle, the board can be any size. In examples, we will use small integers, but when doing work in the abstract, we will discuss an arbitrary board of size $n > 1$, which holds $n^2$ tiles. Each tile has a light which can be on or off. A configuration of the board is a collection of on/off values corresponding to each tile.

The player moves by selecting tiles (also called "moving on" a tile). Selecting a tile flips the on/off value of every tile in the selected tile's row and column (including the selected tile). The player's goal is to turn off all the lights. Specifically, a game of CrossFade consists of a starting board configuration and a sequence of moves that results in the all-off board configuration, which we will call the blank board. We refer to starting configurations where a win is possible as winnable configurations, or solvable configurations.

Ideally, our model will provide insights as to how to design effective CrossFade levels. With this in mind, we state the following questions in advance.

- Are all initial configurations solvable? If not, what characterizes solvable configurations?
- Does order matter when selecting tiles?
- Does the size of the board change the character of the set of solvable states?

# 1 The matrix model

To attain a more concrete mathematical model for CrossFade, we will represent board configurations with matrices. Specifically, we will use matrices whose entries are 1's and 0's: 1 for lights that are on, 2 for lights that are off.

Matrices of this type have a natural entry-by-entry addition, but with a twist: if two 1ś are added, the resulting entry is a 0, not a 1. In truth, for a given board side-length $n$, configurations of the board are elements of the additive matrix group $M_{n\times n}(\mathbb{Z}/2\mathbb{Z})$; that is, $n$ by $n$ matrices over the field $\mathbb{Z}/2\mathbb{Z}$. This is fancy algebra-speak for "$n$ by $n$ integer matrices mod 2." The significance of being able to add two configurations will be explored later. For now, we will use matrix addition to model making moves.

One thing to note about matrices in $M_{n\times n}(\mathbb{Z}/2\mathbb{Z})$ before we begin modelling moves: they are each their own additive inverse. That is, add any matrix to itself, and you will get a matrix that has 0's wherever the original had 0's and 2's wherever the original had 1's. Since we our integer arithmetic is mod 2, all the 2's are actually 0's. In other words, the result of adding a matrix to itself is the identity matrix, $I$, which corresponds to the blank board state. This allows us to simplify sums of matrices a fair bit. Any term that appears twice in a sum cancels itself out, so any matrix sum can be reduced to a sum of unique terms. This will be useful later!

## 2 Modelling moves and games

Let's talk about modelling moves. Moving on the tile in row $i$, column $j$ flips the value of all the tiles in the $i^{\text{th}}$ row and $j^{\text{th}}$ column. In terms of matrix addition, this means adding a 1 to the entries in that row and column. Moving on the $(i, j)^{\text{th}}$ tile, then, is equivalent to adding a matrix that is 0's everywhere except the $i^{\text{th}}$ row and $j^{\text{th}}$ column, which are all 1's. We will refer to such "move matricies" as $\Gamma_{(i,j)}$, or simply $\Gamma$ when speaking generally. The set of all move matrices for an $n$ by $n$ given board will be referred to as $\{\Gamma\}_n$.

A note on notation: for most of this document, we are, tacitly or explicitly, discussing an $n$ by $n$ board. When we use notation such as $\Gamma_{(i,j)}$, $\Gamma$, we are talking about $n$ by $n$ matrices for an arbitrary but fixed $n$. We never need to compare move or configuration matrices that are of different sizes.

Now that we have a model for moves, we can apply it to entire CrossFade games. Given an initial board configuration represented by some matrix $M_0$, a winning game of CrossFade would consist of a series of moves, each represented by $\Gamma_{(i,j)}$ for some $i, j < n$. We can represent this series of moves by a sum that yields the identiy matrix:

$$M_0 + \Gamma_1 + \Gamma_2 + \cdots + \Gamma_m = I \tag{1}$$

Immediately, we can draw several conclusions from this model. First, note that matrix addition is *commutative*, meaning we can freely rearrange the terms, just as we can with a finite sum of integers. We have an answer to one of our questions: it does *not* matter the order in which moves are made. A collection of moves may be made in any order and it will produce the same result. Also recall the special property of matrix addition in $M_{n\times n}(\mathbb{Z}/2\mathbb{Z})$: any sum may be reduced to a sum of unique terms. This gives us another insight about CrossFade games: in the shortest potential winning game, any given tile is moved on at most once.

This is another useful fact about matrix addition in $M_{n\times n}(\mathbb{Z}/2\mathbb{Z})$ that we noted earlier: every matrix is its own additive invers. This encapsulates the fact that any move in CrossFade may be undone by moving on the same tile again.Since any sequence of moves is reversible by repeating each move (it doesn't even matter in what order), we can equally easily think of winnable configurations as those from which we can reach the identiy matrix and those we can reach from the identity matrix by summing together some combination of $\Gamma$ matrices. This is often easier. Equation (1) can be restate as:

$$\Gamma_1 + \Gamma_2 + \cdots + \Gamma_m = M \tag{2}$$

That is, every winnable configuration matrix $M$ can be expressed as a sum of unique matrices $\Gamma$. This is a good start on characterizing winnable states. We may borrow more from algebra, and assert that the set of winnable states is the *subgroup generated by* $\{\Gamma\}_n$, written $\langle\Gamma\rangle_n$. This refers to the subset of $M_{n\times n}(\mathbb{Z}/2\mathbb{Z})$ consisting of matrices which can be expressed as a sum of $\Gamma$ matrices*. This set has group structure, meaning

*In fact, this is the motivation behind the name $\Gamma$. Group generators are traditionally represented by the lower-case gamma, it's closed under addition. In other words, *the sum of winnable states is winnable.*

One might reasonably ask what it even *means* to add two configurations. It makes sense as far as matrix arithmetic goes, and representing moves as matrix addition seems reasonable, but what is the game-relevance of summing two configurations? Since any winnable matrices can be expressed as a sum of $\Gamma$ matrices, we can restate any sum of winnable configurations as a longer sum of move matrices. This means that matrices in $\langle\Gamma\rangle_n$ represent not only winnable states, but also a handy shorthand for a sequence of moves. It will be very helpful to freely add together various winnable matrices, knowing that the sum itself is still a sum of $\Gamma$ matrices.

## 3 Studying the group $\langle\Gamma\rangle_n$

We now have a sound model for mathematically representing CrossFade board configurations, moves, and games. Already, this has provided some insights as to how the game functions. The biggest questions are yet to be answered: are all configurations winnable? If not, which ones are?

To answer these questions, we turn our attention to the subgroup $\langle\Gamma\rangle_n$. We know that inside the whole space of potential configurations of an $n$ by $n$ board, $M_{n\times n}(\mathbb{Z}/2\mathbb{Z})$, $\langle\Gamma\rangle_n$ is a subgroup that consists of all the winnable states. One question we might ask is how large is $\langle\Gamma\rangle_n$. This is difficult to determine, but we know that every element of $\langle\Gamma\rangle_n$ is, by definition, a sum of $\Gamma$ matrices. Moreover, we know that any such sum may be reduced to a sum where all the terms are unique. So how many unique combinations of $\Gamma$ matrices are there? We turn to the world of combinatorics.

To start, note that there are $n^2$ $\Gamma$ matrices: a $\Gamma_{(i,j)}$ for each tile in an $n$ by $n$ board. How many different ways can you combine $n^2$ items? The notation $\binom{n}{k}$ represents the number of $k$-sized collections of an $n$ element set. Applying this to our problem, $n^2 choose k$ would represent the number of $k$-term sums we could make with unique $\Gamma$ matrices. To count all the possible sums, we want to calculate $\binom{n^2}{k}$ for each $k$ from 0 to $n^2$.

$$\#\text{ of sums of unique }\Gamma = \sum_{k=0}^{n^2}\binom{n^2}{k} \tag{3}$$

To evaluate the sum on the right, we will use the *binomial theorem*, which states that for two real variables $x$ and $y$:

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} \tag{4}$$

The sum on the right in (4) looks suspiciously like the sum on the right in (3), but we cannot immediately use this to evaluate the latter. To do that, we first consider a special case of the binomial theorem where $x$

$\gamma$. Since matrices typically are given upper-case names, we a priori named our generators using an upper-case gamma, $\Gamma$

and $y$ are both 1. Substituting 1 in for both variables in (4) yields this equality:

$$2^n = \sum_{k=0}^{n} \binom{n}{k} \tag{5}$$

Now we are almost there. The only thing that remains is to make the somewhat confusing substitution of $n^2$ for $n$. (In the binomial theorem, the $n$ is just an arbitrary integer, whereas in our environment, $n$ represents board side-length.) Combining (3) and (5) with this substitution yields:

$$\begin{aligned} \#\text{ of sums of unique }\Gamma &= \sum_{k=0}^{n^2} \binom{n^2}{k} \\ &= 2^{n^2} \end{aligned} \tag{6}$$

There are $2^{n^2}$ possible sums that consist of elements of $\{\Gamma\}_n$ used at most once. This number by itself doesn't have much significance at a glance. Let's compare it to the size of the entire group $M_{n\times n}(\mathbb{Z}/2\mathbb{Z})$. What is the total number of configuration states of an $n$ by $n$ board? Well, there are $n^2$ tiles, and each can be on or off. The number of ways you can make $n$ choices with $k$ options each is $k^n$. So the number of ways you can make $n^2$ choices with 2 options each is... $2^{n^2}$. The same value as the number of sums of unique $\Gamma$ matrices. Coincidence? Not likely! Does this mean that the size of the subgroup $\langle\Gamma\rangle_n$ is the same as the size of the whole group, which would imply that it *is* the whole group, meaning all configurations are winnable?

Unfortunately, the answer is "not necessarily." We counted the number of sums you could make out of $\Gamma$ matrices where each term was unique, but that is merely an upper bound on the size of $\langle\Gamma\rangle_n$. The size of $\langle\Gamma\rangle_n$ is $2^{n^2}$ if and only if *each sum of unique* $\Gamma$ *produces a unique configuration.* In other words, if there are no sums of $\Gamma$ matrices that have different terms but add up to the same configuration.

So, is that true? Do distinct sums of unique $\Gamma$ matrices necessarily produce distinct results? The answer is, again, "not necessarily."

# 4 Uniqueness of representation in $\langle\Gamma\rangle_n$

Our goal now is to determine if and when it is true that each possible sum of unique $\Gamma$ matrices produces a distinct result. This question can be rephrased in the language of group theory: are representation of elements of $\langle\Gamma\rangle_n$ by it's generators unique? By definition, each matrix in $\langle\Gamma\rangle_n$ can be represented as a sum of $\Gamma$ matrices. Such a sum is a representation of that matrix by the generators of the group. If the representations by generators are unique, then every sum of generators is distinct and $\langle\Gamma\rangle_n$ contains every possible configuration. If those represenations are not unique, then $\langle\Gamma\rangle_n$ does not contain every possible configuration. Thus, the question of what configurations are solvable hinges on the question of unique representations through generators. It turns out that the answer to this question changes based on not the size of $n$, but whether $n$ is even or odd.

## 4.1 For odd $n$

We now consider only $n$ by $n$ boards where $n$ is odd. Consider the result of moving on every tile in a row or column. This is equivalent to the following sum (here of the $k^{\text{th}}$ row):

$$\Gamma_{(k,1)} + \Gamma_{(k,2)} + \cdots + \Gamma_{(k,n)} \tag{7}$$

What is the resulting configuration? First, let's consider a tile not on the $k^{\text{th}}$ row. For a tile at position $(i,j)$ where $i \neq k$ (or, equivalently, the entry at position $(i,j)$ in the resulting matrix of the sum in (7)). There is only one move in that set of moves that affects this tile: the move on tile $(k,j)$. All other moves are on other columns, and there are no moves on the $i^{\text{th}}$ row. Since only one move affects this tile, it will be set to on in the resulting configuration, assuming we began with a blank board. Now consider a tile on the $k^{\text{th}}$ row. How many moves affect this tile? Well, *every* move does, since every move was made in the $k^{\text{th}}$ row. Since $n$ is odd, an odd number of moves affect each tile in the $k^{\text{th}}$ row. This means that these tiles, too, will be on in the resulting configuration. In other words, the entire board is set to on by this sequence of moves.

Note that this is true for an arbitrary row $k$. It's also true for a column: if you just reverse all the coordinates in the above paragraph, the reasoning still holds. On an odd-sided board, *moving on each tile in a row or column will turn on every tile on the board.* Go ahead, try it! Open up CrossFade to an empty board and move on each tile in a row or column. The whole board lights up! Neat, huh?

This means that the board state where all tiles are on can be represented as a sum of unique $\Gamma$ matrices in a number of ways. $2n$ ways, to be exact: one for each row and column. So for odd-sided boards, it is *not* true that representations by generators are unique. Thus, $\langle\Gamma\rangle_n$ is a proper subgroup of $M_{n\times n}(\mathbb{Z}/2\mathbb{Z})$, and not all configurations are solvable.

A final note on the process of moving on every tile in a row or column. We may think of this as a sequence of moves that turns on every light on a blank board, but we can also view it more generally as a sequence that will reverse the value at every tile on the board. In this sense, we can add the configuration that results from the sum in (7) (that is, a matrix of all 1's) to other configurations. Generally, if we know a representation of a configuration by the generators $\Gamma$, we can think of adding that configuration to the current configuration. This is part of the power of the matrix model. It allows us to think more than one move at a time; to view a board, and apply a more substantial change, such as reversing every value, knowing that we can deconstruct that change into a sequence of moves. More on this in section 5!

## 4.2 For even $n$

We have shown that representations by generators are not unique in $\langle\Gamma\rangle_n$ for odd $n$, but the oddness featured prominently in that demonstration. What about for even $n$? Again, we will consider the effect of a specific sequence of moves. What happens if we move on each tile in a certain row and a certain column (moving on the intersection once, not twice)? You can go ahead and test it on an even board in CrossFade, but we will consider it in the general case.

Fix a row $k$ and a column $l$, and consider the sum:

$$\Gamma_{(k,1)} + \Gamma_{(k,2)} + \cdots + \Gamma_{(k,n)} + \Gamma_{(1,l)} + \Gamma_{(2,l)} + \cdots \Gamma_{(k-1,l)} + \Gamma_{(k+1,l)} + \cdots + \Gamma_{(n,l)} \tag{8}$$

Note that the term $\Gamma_{(k,l)}$ is included in the first $n$ terms, but not in the second. We move on each tile in the $k^{\text{th}}$ row or $l^{\text{th}}$ column exactly once.

Again, we will consider the resulting value of various tiles assuming we began with a blank board. For some tile at position $(i,j)$ where $i \neq k$ and $j \neq l$, which moves affect that tile? Two: $\Gamma_{(k,j)}$ and $\Gamma_{(i,l)}$. Since it is affected twice, its resulting value will be off. Next, consider some tile on row $k$ but not on column $l$. How many moves affect this tile? Every move on row $k$ does, but none of the others. Since $n$ is even, such a tile is affected by an even number of moves, so its final value will also be off. This same logic can be applied to any tile on column $l$ but not row $k$: it is affected by the even-number of moves on column $l$ and no others.

So far, it seems like everything will remain off after this sequence. What about the tile at position $(k,l)$? How many moves affect this tile? There are $n$ moves on row $k$ and $n$ moves on column $l$, but counting that way double-counts the move on tile $(k,l)$ itself. The total number of moves that affect this tile, then, is $2n-1$. This number is guaranteed to be odd, thus the final value of the tile at position $(k,l)$ will be on.

This is very significant. Since we did this in generality, we can do this with any row/column pair to produce a sum of unique $\Gamma$ that results in a matrix that is all 0's except one position with a 1. Call such matrices $\beta_{(i,j)}{}^{\dagger}$. These matrices generate the entire group $M_{n\times n}(\mathbb{Z}/2\mathbb{Z})$. Every configuration matrix can be expressed as a unique sum of unique matrices $\beta_{(i,j)}$: for each 1 in a terget configuration's matrix, add the basis matrix with a 1 in the corresponding position to a sum. Every matrix is a sum of basis matrices, and basis matrices can be expressed as a sum of $\Gamma$ matrices. Therefore: every matrix can be expressed as a sum of $\Gamma$ matrices. For even $n$, all configurations are solvable, $\langle\Gamma\rangle_n$ does indeed have size $2^{n^2}$, and is equal to the entire group $M_{n\times n}(\mathbb{Z}/2\mathbb{Z})$.

In game-turns, this means total omnipotence. We may think of moving on each tile in a row and column (being sure not to double-move on the intersection) collectively as a transformation we can apply to the board that reverses the value of one tile (the intersection of the row and column), and leaves the rest of the board unchanged. Using this technique, we can freely move anywhere in the configuration space.

# 5 Applications in Design and Play

$^{\dagger}\beta$ for *basis*.