Time Series HW # 1

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1) Let $Z_t = 2 + 0.5t + a_t$ where $\{a_t\}$ is a white noise sequence with mean zero and variance σ_a^2. Determine the mean and variance functions for the process $\{Z_t\}$. Is the process stationary?

We can compute the expectation as:

$$\begin{aligned} \mathbb{E}(Z_t) &= \mathbb{E}(2 + 0.5t + a_t) \\ &= \mathbb{E}(2) + \mathbb{E}(0.5t) + \mathbb{E}(a_t) && \text{Expectations are linear} \\ &= 2 + 0.5t && \text{Only non-constant is } a_t\text{, and } \mathbb{E}(a_t) = 0 \end{aligned}$$

We can compute the variance as:

$$\begin{aligned} \mathbb{V}(Z_t) &= \mathbb{V}(2 + 0.5t + a_t) \\ &= V(a_t) && \text{Since } \mathbb{V}(c + Y) = \mathbb{V}(Y) \text{ for constants } c \\ &= \sigma_a^2 \end{aligned}$$

The mean of Z_t changes over time, so by definition, the process can not be stationary.

2) Henceforth let $\tilde{Z}_t = Z_t - \mu$ be a mean zero process with $\{a_t\}$ a white noise sequence with variance σ_a^2. For each of the following, determine if the process is stationary and if it is invertible. If it is stationary, compute the ACF and PACF for lags of $k = 0, 1, 2, 3, 4$.

An AR process $a_t = \pi(B)\widetilde{Z}_t = \sum_{j=0}^{\infty} \pi_j \tilde{Z}_{t-j}$ is invertible if it is absolutely summable, i.e. $\sum_{j=1}^{\infty} |\pi_j| < \infty$. Since all the below processes have finitely many non-zero coefficients π_i, all the processes are automatically invertible (a finite sum of finite values is finite).

The Yule-Walker equations give us an easy way to compute the autocovariances (and thus autocorrelations) of an AR(p) process. We have $\gamma_k = \pi_1\gamma_{k-1} + \cdots + \pi_p\gamma_{k-p}$ for all lags $k \in \mathbb{N}$.

In general, we can compute the partial autocorrelations as $\phi_{k,k} = \dfrac{\left| \begin{bmatrix} 1 & \rho_1 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & 1 & \cdots & \rho_{k-3} & \rho_2 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & \rho_1 & \rho_k \end{bmatrix} \right|}{\left| \begin{bmatrix} 1 & \rho_1 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & \rho_1 & 1 \end{bmatrix} \right|}$.

The partial autocorrelation at lag 1 is the same as the autocorrelation. For an AR(p) process, the partial autocorrelation of a lag $k > P$ is zero. Of course, if we had a way to write our model as $\tilde{Z}_{t+k} = \phi_{k,1}\tilde{Z}_{t+k-1} + \phi_{k,2}\tilde{Z}_{t+k-2} + \cdots + \phi_{k,k}\tilde{Z}_t + a_{t+k}$, then we could simply pick off the autocorrelation as the coefficient to the last term.

a. $\tilde{Z}_t - 0.5\tilde{Z}_{t-1} = a_t$

We can write the process as $(1 - 0.5B)\tilde{Z}_t = a_t$. Since the root of $(1 - 0.5B)$ is 2, which lies outside the complex unit circle, the process is stationary.

The process is an AR(1) and so the autocorrelation function from the Yule-Walker equation has just one term– $\rho_k = \pi_1\rho_{k-1} = 0.5\rho_{k-1}$. So $\rho_1 = 0.5, \rho_2 = 0.25, \rho_3 = 0.125,$ and $\rho_4 = 0.06125$.

The partial autocorrelation is 0.5 at a lag of 1 (it is an AR(1) process, so is the same as the autocorrelation), and 0 at every other non-zero k.

b. $\tilde{Z}_t - 0.9\tilde{Z}_{t-1} = a_t$

We can write the process as $(1 - 0.9B)\tilde{Z}_t = a_t$. Since the root of $(1 - 0.9B)$ is $\frac{10}{9}$, which lies outside the complex unit circle, the process is stationary.

For the same reasoning as part a, the autocorrelations are $\rho_1 = 0.9$, $\rho_2 = 0.81$, $\rho_3 = 0.729$, and $\rho_4 = 0.6561$. The partial autocorrelation is 0.9 at a lag of 1, and 0 at other non-zero k.

c. $\tilde{Z}_t - 1.3\tilde{Z}_{t-1} + 0.4\tilde{Z}_{t-2} = a_t$

We can write the process as $(1 - 1.3B + 0.4B^2)\tilde{Z}_t = a_t$. The roots of $(0.4B^2 - 1.3B + 1)$ are: $\frac{-(-1.3)\pm\sqrt{(-1.3)^2-4(0.4)(1)}}{2(0.4)} = \frac{1.3\pm\sqrt{1.69-1.6}}{0.8} = \frac{1.3\pm\sqrt{.09}}{0.8} = \frac{1.3\pm 0.3}{0.8} = \frac{5}{4}, 2$. These are real roots which both lie outside the complex unit circle, so the process is stationary.

The autocorrelations can be computed from the Yule-Walker equations with $p = 2$. Recalling $\rho_0 = 1$ and the symmetry of the correlations, we compute:

$$\begin{aligned}
\rho_1 &= \pi_1\rho_{1-1} + \pi_2\rho_{1-2} = (1.3)(1) + (-0.4)(\rho_1) \implies \rho_1 = \frac{1.3}{1+0.4} \approx 0.93 \\
\rho_2 &= \pi_1\rho_{2-1} + \pi_2\rho_{2-2} = (1.3)(\rho_1) + (-0.4)(1) \implies \rho_2 = \frac{1.3\cdot 1.3}{1.4} - 0.4 \approx 0.81 \\
\rho_3 &= \pi_1\rho_{3-1} + \pi_2\rho_{3-2} = (1.3)(\rho_2) + (-0.4)(\rho_1) \implies \rho_3 \approx 0.68 \\
\rho_4 &= \pi_1\rho_{4-1} + \pi_2\rho_{4-2} = (1.3)(\rho_3) + (-0.4)(\rho_2) \implies \rho_4 \approx 0.56
\end{aligned}$$

For the partial autocorrelations, we see:

$$\phi_{1,1} = \rho_1 \approx 0.93$$

$$\phi_{2,2} = \frac{\left|\begin{bmatrix}1 & \rho_1\\ \rho_1 & \rho_2\end{bmatrix}\right|}{\left|\begin{bmatrix}1 & \rho_1\\ \rho_1 & 1\end{bmatrix}\right|} = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2} \approx \frac{0.81 - 0.93^2}{1 - 0.93} \approx -0.4$$

Double-checking our work:

```
> ar_coeff=c(1.3, -.4)
> round(ARMAacf(ar=ar_coeff, lag.max=5),3)
    0     1     2     3     4     5
1.000 0.929 0.807 0.678 0.558 0.455
> round(ARMAacf(ar=ar_coeff, lag.max=5, pacf=TRUE),3)
[1]  0.929 -0.400  0.000  0.000  0.000
```

d. $\tilde{Z}_t - 1.2\tilde{Z}_{t-1} + 0.8\tilde{Z}_{t-2} = a_t$

We can write the process as $(1 - 1.2B + 0.8B^2)\tilde{Z}_t = a_t$. The roots of $(0.8B^2 - 1.2B + 1)$ are: $\frac{-(1.2)\pm\sqrt{(-1.2)^2-4(0.8)(1)}}{2(0.8)} = \frac{-1.2\pm\sqrt{1.44-3.2}}{1.6} = \frac{-1.2\pm\sqrt{-1.76}}{1.6}$. We see there are complex roots, and with the help of software see that the modulus of these roots lie outside the complex unit circle, so the process is stationary.

```
> coef=c(1,-1.2,0.8)
> Mod(polyroot(coef))
[1] 1.118034 1.118034
```

The autocorrelations can be computed from the Yule-Walker equations with $p = 2$. Recalling $\rho_0 = 1$ and the symmetry of the correlations, we compute:

$$\rho_1 = \pi_1\rho_{1-1} + \pi_2\rho_{1-2} = (1.2)(1) + (-0.8)(\rho_1) \implies \rho_1 = \frac{1.2}{1+0.8} = \frac{2}{3}$$
$$\rho_2 = \pi_1\rho_{2-1} + \pi_2\rho_{2-2} = (1.2)(\rho_1) + (-0.8)(1) \implies \rho_2 = \frac{1.2\cdot 2}{3} - 0.8 = 0$$
$$\rho_3 = \pi_1\rho_{3-1} + \pi_2\rho_{3-2} = (1.2)(\rho_2) + (-0.8)(\rho_1) \implies \rho_3 = -0.8\frac{2}{3} \approx -0.53$$
$$\rho_4 = \pi_1\rho_{4-1} + \pi_2\rho_{4-2} = (1.2)(\rho_3) + (-0.8)(\rho_2) \implies \rho_4 \approx -0.64$$

For the partial autocorrelations, we see:

$$\phi_{1,1} = \rho_1 = \frac{2}{3}$$
$$\phi_{2,2} = \frac{\left|\begin{bmatrix}1 & \rho_1\\ \rho_1 & \rho_2\end{bmatrix}\right|}{\left|\begin{bmatrix}1 & \rho_1\\ \rho_1 & 1\end{bmatrix}\right|} = \frac{0 - \frac{2}{3}^2}{1 - \frac{2}{3}^2} = \frac{-\frac{4}{9}}{\frac{5}{9}} = \frac{-4}{5}$$

Double-checking our work:

```
> ar_coeff=c(1.2, -.8)
> round(ARMAacf(ar=ar_coeff, lag.max=5),3)
     0      1      2      3      4      5
 1.000  0.667  0.000 -0.533 -0.640 -0.341
> round(ARMAacf(ar=ar_coeff, lag.max=5, pacf=TRUE),3)
[1]  0.667 -0.800  0.000  0.000  0.000
```

3) Compute the AR and MA processes for the below representations.

a. Find the AR representation of the MA(1) process $\tilde{\boldsymbol{Z}}_t = \boldsymbol{a}_t - 0.4\boldsymbol{a}_{t-1}$.

In general, the MA representation is $\tilde{Z}_t = \psi(B)a_t$ and AR representation is $\pi(B)\tilde{Z}_t = a_t$ for some functions ψ and π of B. Left multiplying the AR representation by $\psi(B)$, we see that $\psi(B)\pi(B)\tilde{Z}_t = \psi(B)a_t = \tilde{Z}_t$, or that $\psi(B) = \pi^{-1}(B)$.

Applying this to the problem at hand, $\psi(B) = (1 - 0.4B)$ and so, by the formula for a geometric series (since $0.4 < 1$), $\pi(B) = (1 - 0.4B)^{-1} = \sum_{n=0}^{\infty} 0.4B^n$. Our AR representation is then $\tilde{Z}_t + 0.4\tilde{Z}_{t-1} + 0.16\tilde{Z}_{t-2} + \cdots = a_t$, or more compactly $a_t = \sum_{n=0}^{\infty} 0.4^n \tilde{Z}_{t-n}$.

b. Find the MA representation of the AR(2) process $\tilde{\boldsymbol{Z}}_t = 0.2\tilde{\boldsymbol{Z}}_{t-1} + 0.4\tilde{\boldsymbol{Z}}_{t-2} + \boldsymbol{a}_t$.

For notational ease we can rewrite the above as $\tilde{Z}_t - 0.2\tilde{Z}_{t-1} - 0.4\tilde{Z}_{t-2} = a_t$ and then read off $\pi(B)$ as $(1 - 0.2B - 0.4B^2)$. Here we hit a dead-end since we can't factor the polynomial into the form $(1 - xB)(1 - yB)$ and use the same geometric series trick in part a. Instead, we try substitution.

First, we substitute for $\tilde{Z}_{t-1}$ to eliminate the $\tilde{Z}_{t-1}$ term and add a a_{t-1} term in our representation:

$$\begin{aligned}
\tilde{Z}_t &= 0.2[0.2\tilde{Z}_{t-2} + 0.4\tilde{Z}_{t-3} + a_{t-1}] + 0.4\tilde{Z}_{t-2} + a_t \\
&= 0.04\tilde{Z}_{t-2} + 0.08\tilde{Z}_{t-3} + 0.2a_{t-1} + 0.4\tilde{Z}_{t-2} + a_t \\
&= 0.44\tilde{Z}_{t-2} + 0.08\tilde{Z}_{t-3} + a_t + 0.2a_{t-1}
\end{aligned}$$

Next, we substitute for $\tilde{Z}_{t-2}$ for the same reason:

$$\begin{aligned}
\tilde{Z}_t &= 0.44[0.2\tilde{Z}_{t-3} + 0.4\tilde{Z}_{t-4} + a_{t-2}] + 0.08\tilde{Z}_{t-3} + a_t + 0.2a_{t-1} \\
&= 0.088\tilde{Z}_{t-3} + 0.176\tilde{Z}_{t-4} + 0.44a_{t-2} + 0.08\tilde{Z}_{t-3} + a_t + 0.2a_{t-1} \\
&= 0.168\tilde{Z}_{t-3} + 0.176\tilde{Z}_{t-4} + a_t + 0.2a_{t-1} + 0.44a_{t-2}
\end{aligned}$$

To see where we are going, we can substitute one more time:

$$\begin{aligned}
\tilde{Z}_t &= 0.168[0.2\tilde{Z}_{t-4} + 0.4\tilde{Z}_{t-5} + a_{t-3}] + 0.176\tilde{Z}_{t-4} + a_t + 0.2a_{t-1} + 0.44a_{t-2} \\
&= 0.0336\tilde{Z}_{t-4} + 0.0672\tilde{Z}_{t-5} + 0.168a_{t-3} + 0.176\tilde{Z}_{t-4} + a_t + 0.2a_{t-1} + 0.44a_{t-2} \\
&= 0.2096\tilde{Z}_{t-4} + 0.0672\tilde{Z}_{t-5} + a_t + 0.2a_{t-1} + 0.44a_{t-2} + 0.168a_{t-3}
\end{aligned}$$

Continuing in this fashion, we can continually add an a_{t-n} term. Carefully backtracking the calculation of the α_{t-3} term, we see it is the coefficient of the $\tilde{Z}_{t-3}$ term, which itself is the sum of the product of 0.44 (the a_{t-2} term) and 0.2 with 0.08 (the product of 0.2– the a_{t-1} term– and 0.4). So generically, for $n \geq 3$, we have $\psi_{t-n} = 0.2\psi_{t-n+1} + 0.4\psi_{t-n+2}$. Compactly, we have $\tilde{Z}_t = a_t + 0.2a_{t-1} + 0.44a_{t-2} + \sum_{j=3}^{\infty}(0.2\psi_{t-j+1} + 0.4\psi_{t-j+2})$.

4) Consider the AR(3) process $(1-0.4B)(1-0.2B+0.6B^2)\tilde{Z}_t = a_t$. Let $\sigma_a^2 = 1$. Determine the roots of $\phi(B) = 0$ and then answer the following: Is the process $\tilde{Z}_t$ stationary? Invertible? Why or why not? If the process is stationary, determine its autocorrelation function for integer values of k (you may give recursive equations for ρ_k for $k > 3$).

The roots of $\phi(B)$ are $\frac{5}{2}$ (read off from the first factor) and about $0.16 \pm 1.28i$ from the quadratic factor (as calculated by software, see below). Since all these roots have a modulus outside the complex unit circle, the process is stationary. Any AR(P) process with finite P is invertible as explained in question 2.

```
> coef=c(1,-.2,0.6)
> polyroot(coef)
[1] 0.1666667+1.280191i 0.1666667-1.280191i
> Mod(polyroot(coef))
[1] 1.290994 1.290994
```

To compute the autocorrelations, we can expand out the product, write our process in full, and then use the Yule-Walker equations to compute the correlations. We have $\pi(B) = 1-0.2B+0.6B^2-0.4B+0.08B^2-0.24B^3$ or $1-0.6B+0.68B^2-0.24B^3$. In full, our process is $\tilde{Z}_t = 0.6\tilde{Z}_{t-1} - 0.68\tilde{Z}_{t-2} + 0.24\tilde{Z}_{t-3} + a_t$. We calculate as follows:

$$\rho_1 = \pi_1\rho_{1-1} + \pi_2\rho_{1-2} + \pi_3\rho_{1-3} = (0.6)\rho_0 + (-0.68)\rho_1 + (0.24)\rho_2 = (0.6) + (-0.68)\rho_1 + (0.24)\rho_2$$
$$\rho_2 = \pi_1\rho_{2-1} + \pi_2\rho_{2-2} + \pi_3\rho_{2-3} = (0.6)\rho_1 + (-0.68)\rho_0 + (0.24)\rho_1 = (0.84)\rho_1 + (-0.68)$$
$$\rho_3 = \pi_1\rho_{3-1} + \pi_2\rho_{3-2} + \pi_3\rho_{3-3} = (0.6)\rho_2 + (-0.68)\rho_1 + (0.24)\rho_0 = (0.6)\rho_2 + (-0.68)\rho_1 + (0.24)$$

Substituting the second equation into the first, we see:

$$\rho_1 = (0.6) + (-0.68)\rho_1 + (0.24)\big((0.84)\rho_1 + (-0.68)\big) = 0.4368 - 0.4784\rho_1$$

So $\rho_1 = \frac{0.4368}{1+0.4784} \approx 0.295$, and then $\rho_2 = 0.84(\frac{0.4368}{1.4784}) - 0.68 \approx -0.432$, and finally $\rho_3 = 0.6(0.84(\frac{0.4368}{1.4784}) - 0.68) + (-0.68)(\frac{0.4368}{1.4784}) + 0.24 \approx -0.22$. For $k > 3$, we get the recursion $\rho_k = 0.6\rho_{k-1} - 0.68\rho_{k-2} + 0.24\rho_{k-3}$. As always, we get the negative "lags" by appealing to symmetry; $\rho_k = \rho_{-k}$.

Double checking our work:

```
> ar_coeffs=c(0.6,-.68, .24)
> round(ARMAacf(ar=ar_coeffs),3)
    0     1      2      3
1.000 0.295 -0.432 -0.220
```