Time Series HW # 3

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1) Consider the following models:

$$\begin{aligned}
&\boldsymbol{A: (1-B)\tilde{Z}_t = (1-1.5B)a_t}\\
&\boldsymbol{B: (1-0.8B)\tilde{Z}_t = (1-0.5B)a_t}\\
&\boldsymbol{C: (1-1.1B+0.8B^2)\tilde{Z}_t = (1-1.7B+0.72B^2)a_t}\\
&\boldsymbol{D: (1-0.6B)\tilde{Z}_t = (1-1.2B+0.2B^2)a_t}
\end{aligned}$$

a. Verify whether or not the model for $\boldsymbol{Z_t}$ is stationary and/or invertible.

An ARMA model is stationary when all the roots of it's AR polynomial lie outside the complex unit circle. From inspection, that means models B and D are stationary. We can find the roots of the AR polynomial for model C as $\frac{1.1\pm\sqrt{1.1^2-4(0.8)(1)}}{2(0.8)} = \frac{1.1\pm\sqrt{-1.99}}{1.6} = \frac{11}{16}\pm\frac{i\sqrt{1.99}}{1.6}$ and so the complex modulus is $\sqrt{\frac{121}{256}+\frac{199}{256}} > \sqrt{1} = 1$, which means model C is also stationary. All told, models B, C, and D are stationary.

An ARMA model is invertible if all the roots of it's MA polynomial lie outside the complex unit circle. From inspection, that means model B is invertible. We can find the roots of the MA polynomial for model C as $\frac{1.7\pm\sqrt{1.7^2-4(0.72)(1)}}{2(0.72)} = \frac{1.7\pm\sqrt{2.89-2.88}}{1.44} = \frac{1.7\pm0.1}{1.44} > 1$ and so model C is also invertible. We can find the roots of the MA polynomial for model D as $\frac{1.2\pm\sqrt{1.2^2-4(0.2)(1)}}{2(0.2)} = \frac{1.2\pm0.8}{0.4}$; one of the roots lies on the unit circle and so model D is not invertible. All told, models B and C are invertible.

b. Express the model as an infinite MA if the process is stationary.

We equate the coefficients of the backshift operator to get the expression. In the case of Model B, we have:

$$(1-0.8B)\tilde{Z}_t = (1-0.5B)a_t$$
$$(1-0.8B)\left(1+\psi_1 B+\psi_2 B^2+\cdots\right)a_t = (1-0.5B)a_t$$

And so:

$$\psi_1 B - 0.8B = -0.5B \implies \psi_1 = 0.3$$
$$\psi_2 B^2 - 0.8\psi_1 B^2 = 0B^2 \implies \psi_2 = 0.24$$
$$\psi_3 B^3 - 0.8\psi_2 B^3 = 0B^3 \implies \psi_3 = 0.8(0.24)$$

Continuing in this fashion, we have: $\psi_{B_k} = \begin{cases} 0.3, & k=1 \\ 0.8(0.3)^{k-1}, & k>1 \end{cases}$

In the case of model C, we have:

$$(1-1.1B+0.8B^2)\tilde{Z}_t = (1-1.7B+0.72B^2)a_t$$
$$(1-1.1B+0.8B^2)\left(1+\psi_1 B+\psi_2 B^2+\cdots\right) = (1-1.7B+0.72B^2)a_t$$

And so:

$$\psi_1 B - 1.1B = -1.7B \implies \psi_1 = -0.6$$
$$\psi_2 B^2 - 1.1\psi_1 B^2 + 0.8B^2 = 0.72B^2 \implies \psi_2 = 0.72 + (1.1\cdot -0.6) - 0.8 = -0.74$$
$$\psi_3 B^3 - 1.1\psi_2 B^3 + 0.8\psi_1 B^3 = 0B^3 \implies \psi_3 = 1.1(-0.74) - 0.8(-0.6)$$

Continuing in this way, we achieve the recursion $\psi_{C_k} = 1.1(\psi_{C_{k-1}}) - 0.8(\psi_{C_{k-2}})$ for $k \geq 3$ where $\psi_{C_{k_1}} = -0.6$ and $\psi_{C_{k_2}} = -0.74$.

In the case of model D, we have:

$$(1-0.6B)\tilde{Z}_t = (1-1.2B+0.2B^2)a_t$$
$$(1-0.6B)\left(1+\psi_1 B+\psi_2 B^2+\cdots\right) = (1-1.2B+0.2B^2)a_t$$

And so:

$$\psi_1 B - 0.6B = -1.2B \implies \psi_1 = -0.6$$
$$\psi_2 B^2 - 0.6\psi_1 B^2 = 0.2B^2 \implies \psi_2 = 0.2 + 0.6(-0.6) = -0.16$$
$$\psi_3 B^3 - 0.6\psi_2 B^3 = 0 \implies \psi_3 = 0.6(-0.16)$$

Continuing in this way, we see $\psi_{D_k} = 0.6^{k-2}(-0.16)$ for $k \geq 3$ while $\psi_{D_1} = -0.6$ and $\psi_{D_2} = -0.16$.

c. Express the model as an infinite AR representation if the process is invertible.

We use the same strategy of equating coefficients. In the case of model B, we have:

$$(1-0.8B)\tilde{Z}_t = (1-0.5B)a_t$$
$$(1-0.8B)\tilde{Z}_t = (1-0.5B)(1+\pi_1 B+\pi_2 B^2+\cdots)\tilde{Z}_t$$

And so:

$$-0.8B = \pi_1 B - 0.5B \implies \pi_1 = -0.3B$$
$$0 = \pi_2 B^2 - 0.5\pi_1 B^2 \implies \pi_2 = (0.5)(-0.3)$$

Continuing in this way, we have $\pi_{B_1} = -0.3$ and $\pi_{B_k} = 0.5\pi_{B_{k-1}}$ for $k \geq 2$.

In the case of model C, we have:

$$(1-1.1B+0.8B^2)\tilde{Z}_t = (1-1.7B+0.72B^2)a_t$$
$$(1-1.1B+0.8B^2)\tilde{Z}_t = (1-1.7B+0.72B^2)(1+\pi_1 B+\pi_2 B^2+\cdots)\tilde{Z}_t$$

And so:

$$-1.1B = \pi_1 B - 1.7B \implies \pi_1 = 0.6$$
$$0.8B^2 = \pi_2 B^2 - 1.7\pi_1 B^2 + 0.72B^2 \implies \pi_2 = 0.8 - 0.72 + 1.7(0.6) = 1.1$$
$$0 = \pi_3 B^3 - 1.7\pi_2 B^3 + +0.72\pi_1 B^3 \implies \pi_3 = 1.7(1.1) + 0.72(0.6) = 2.302$$

Continuing in this fashion, we see $\pi_{C_k} = \begin{cases} 0.6, & k=1 \\ 1.1, & k=2 \\ 1.7\pi_{C_{k-1}} - 0.72\pi_{C_{k-2}}, & k \geq 3 \end{cases}$

2) Consider the model $\tilde{Z}_t = 0.3\tilde{Z}_{t-1} + 0.34\tilde{Z}_{t-2} - 0.12\tilde{Z}_{t-3} + a_t - 0.7a_{t-1} + 0.12a_{t-2}$. Determine whether or not the model is in reduced form, and if it is not, find the reduced form.

The model is in reduced form if the AR and MA polynomials share no common factors. We go directly for the definition. The model is $(1 - 0.3B - 0.34B^2 + 0.12B^3)\tilde{Z}_t = (1 - 0.7B + 0.12B^2)a_t$. The quadratic factors as $(1 - 0.4B)(1 - 0.3B)$. Dividing the first factor into the cubic, we see $(0.12B^3 - 0.34B^2 - 0.3B + 1) = (1 - 0.4B)(-0.3B^2 + 0.1B + 1)$. Then we can write our model as $(-0.3B^2 + 0.1B + 1)\widetilde{Z}_t = (1 - 0.3B)a_t$.

3) Consider the model $(1-B)^2 Z_t = (1-0.3B-0.5B^2)a_t$.

a. Is the model for Z_t a stationary model? Why or why not?

Z_t is not stationary since it's AR model has two roots the lie *on* the complex unit circle.

b. Is the model for $W_t = (1-B)^2 Z_t$ a stationary model? Why or why not?

W_t is stationary since it is a finite MA model, and finite MA's are always stationary.

c. Determine the autocorrelation function for W_t.

The autocovariance function is $\mathbb{E}(W_t W_{t-k})$:

$$\begin{aligned}\mathbb{E}(W_t W_{t-k}) =& \mathbb{E}\left((a_t - 0.3a_{t-1} - 0.5a_{t-2})(a_{t-k} - 0.3a_{t-k-1} - 0.5a_{t-k-2})\right)\\ =& \mathbb{E}(a_t a_{t-k}) - 0.3\mathbb{E}(a_t a_{t-k-1}) - 0.5\mathbb{E}(a_t a_{t-k-2})\\ &- 0.3\mathbb{E}(a_{t-1}a_{t-k}) + 0.09\mathbb{E}(a_{t-1}a_{t-k-1}) + 0.15\mathbb{E}(a_{t-1}a_{t-k-2})\\ &- 0.5\mathbb{E}(a_{t-2}a_{t-k}) + 0.15\mathbb{E}(a_{t-2}a_{t-k-1}) + 0.25\mathbb{E}(a_{t-2}a_{t-k-2})\end{aligned}$$

By the properties of white noise, $\mathbb{E}(a_t a_{t-k}) = \begin{cases}\sigma_a^2, & k=0\\ 0, & k>0\end{cases}$. So:

$$\begin{aligned}\gamma_0 &= \sigma_a^2(1+0.09+0.25) = 1.34\sigma_a^2\\ \gamma_1 &= \sigma_a^2(-0.3+0.15) = -0.15\sigma_a^2\\ \gamma_2 &= \sigma_a^2(-0.5) = -0.5\sigma_a^2\\ \gamma_3 &= \sigma_a^2()\\ \gamma_4 &= 0\end{aligned}$$

. Dividing by γ_0 gives us the correlation function (here $s>2$):

$$\begin{aligned}\rho_0 &= 1\\ \rho_1 &= \frac{-0.15}{1.34} \approx -0.11\\ \rho_2 &= \frac{-0.5}{1.34} \approx -0.37\\ \rho_s &= 0\end{aligned}$$