A Blood Cell Population Model

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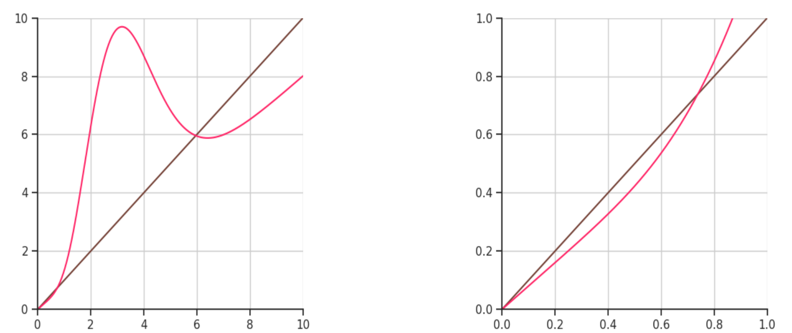
Model Analysis

An analysis of the function governing the dynamical process of the model developed in section 3 within the final project instructions takes place in section 4. Figure 1 shows the plotting we were asked to do in this section. The plots show on the same set of axes both the function $F(x) = (1-a)x + bx^r e^{-sx}$ and $y = x$ using the following values of the control parameter: $a = 0.2$, $b = 4$, $r = 6$, and $s = 2$.



(a) On the region $[0, 1] \times [0, 1]$.

(b) On the region $[0, 1] \times [0, 1]$.

Figure 1: $F(x)$ in pink; $y = x$ in purple

Observe the 3 fixed points for the function F on the region $[0, 10] \times [0, 10]$.

Topics For Investigation

This section goes over my investigation into each of the questions posed in instructions paper.

i. Is it true that for all $a \in (0, 1)$ the dynamical system has three fixed points?

   It is true that the dynamical system has three points for those conditions. Observe that $x = 0$ is always a fixed point for the function F for every value of a because $\forall a \in \mathbb{R}, F(0) = 0$. Besides $x = 0$, we see that there are always two other fixed points. When $x \neq 0$, $\frac{F(x)}{x} = 1$ for each fixed point. The behavior of the function F, for all values of a on the region $[0, 10] \times [0, 10]$, $F(x)$ starts out with a fixed point at $x = 0$. The function then briefly dips below the line $y = x$ before curving up to cross it again creating another fixed point. The function reaches a maximum to then dip below the $y = x$ line once more to create a third fixed point. The behavior of this function was observed experimentally using matplotlib for many $a \in (0, 1)$ including values of a really close to, but slightly larger than and smaller than both 0 and 1, respectively.

ii. Using the convention $p_0 = 0, p_0 < p_1 < p_2$, explain why we know that p_0 is attracting and p_1 is repelling for all $a \in (0, 1]$.

   For $p_0 = 0$, the stability is clearly attracting. The model analysis section of the instructions paper puts it nicely saying, "We expect p_0 to be attracting since, from a clinical point of view, a person cannot recover when the population of blood cells falls below a critical threshold." We are given the derivative of $F(x)$ in terms of the bifurcation parameter a, $F'(x) = 1+a(r-sx-1)$ in the paper. From this equation we can infer that p_1 is a repelling by the definition repelling fixed points. $\mid F'(x = p_1) \mid$ must be greater than 1 since $r > sp_1 + 1$ making p_1 a repelling fixed point.

iii. For which values of $a \in (0, 1)$ does F have a maximum point x_m such that $F_a(x_m) > x_m$? Note x_m depends on a. For these values of a, find an interval that is mapped onto itself by F_a.

   I did not investigate this question very thoroughly since it was said to be not important in the hints and tips video on BlackBoard.

iv. Find the value a_0 for which the derivative $F_{a_0}(p_2) = -1$

   To find this value, I used the root solving algorithm provided by Mathematica. But first, I used equation (9) to find p_2.

$$F'_{a_0}(p_2) = 1 + a_0(6 - 4p_2 - 1) = -1 \tag{1}$$

$$a_0(5 - 4p_2) = -2 \tag{2}$$

$$5a_0 - 4a_0p_2 = -2 \tag{3}$$

$$p_2 = \frac{-2 - 5a_0}{4a_0} \tag{4}$$

Then using calling $FindRoot\{p_2[a], \{a, 0.5\}$ returns 40000.6 which seems like a wrong answer. I have spent way too much time on this problem and will return to it when I'm done looking into the other problems if I have time before the deadline.

Okay, so after playing with cobweb diagrams in problem 5, I'm convinced that the correct a_0 is somewhere in the range of $(0.314, 0.346)$ since it is in that range where 2-cycles start to occur.

v. Establish that for $a > a_0$ the system has a periodic orbit of period 2. Find the orbit numerically for values of a close to a_0 and analyze the stability numerically. Note that this level represents the onset of a disease state in an organism.

Because I failed to find a value for a_0 that makes sense in the previous question, I am going to choose a value for a that's somewhat close to 1 that hopefully is greater than a_0. I'm going to assume $a = 0.9 > a_0$. Well it seems this value is too high. I'm going to iterate over a range of possible a values to search for some cycles!

Without knowing the answer to the previous question it's hard not to get stuck in the basin of attraction of the fixed point $p_0 = 0$. Using $x = 5$ as an initial seed value, I iterated through 15 values of $a \in (0, 1)$ to find different two cycles using cobweb diagrams. When $a > 0.85$, the orbit gets stuck in the basin of attraction at p_0. But there are two-cycles to be found for a values in the range $(0.37, 0.8)$. I've included a picture of my cobweb plot below for $a = 0.4357$ as an example of a two-cycle I found.

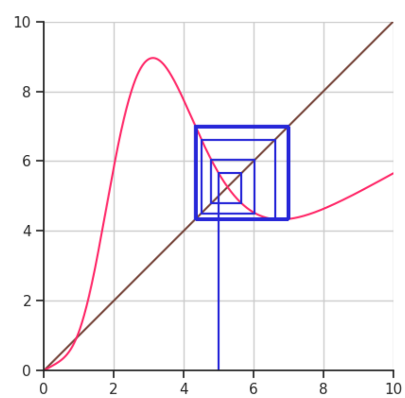


Figure 2: Cobweb plot two-cycle; $seed = 5, a = 0.4357$

vi. Are there values of a for which the system has a periodic orbit of period 3?

Yes, for example, when $a = 0.805$ there is the 3-cycle shown below. A three-cycle also exists for $a = 0.821$. There probably are three-cycles for a values between and likely surrounding these two values.

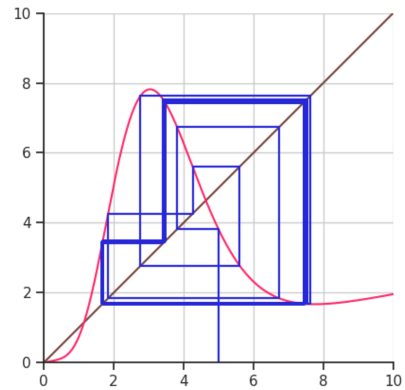


Figure 3: Cobweb plot three-cycle, $seed = 5, a = 0.805$

vii. Find the value of a for which the system appears to be chaotic. Show time series plots to support your claim.

I found a values for which the system appears to be chaotic for a-values in the range $(0.836, 0.868)$. For example, the graph below has a seed of $x = 5$ and $a = 0.853$.

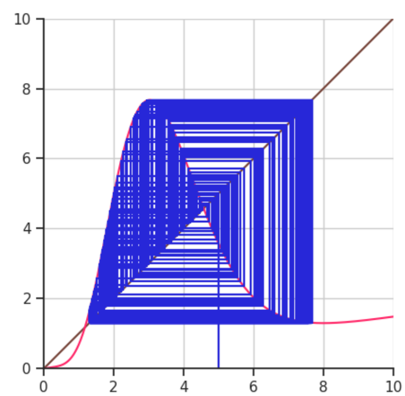
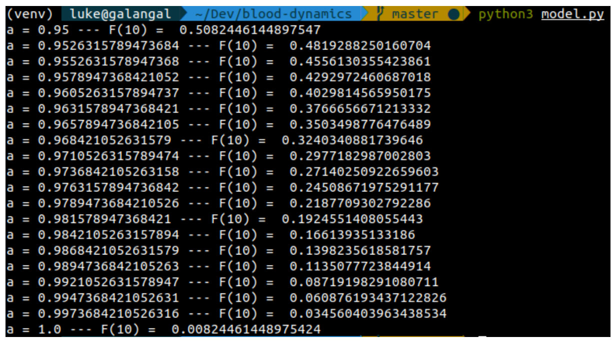


Figure 4: Cobweb plot chaos, $seed = 5, a = 0.853$

viii. Investigate numerically what is happening when a is close to 1. Provide graphs and a paragraph or two of explanations.

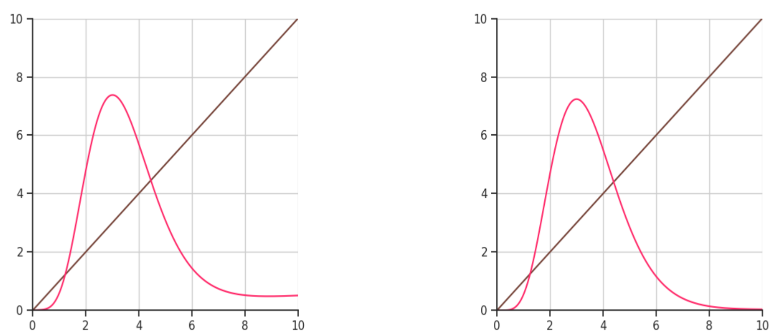
As a approaches 1, $F(x)$ becomes asymptotically close to 0 when x is a bit higher than 8. To figure out this behavior numerically, I plotted $F(x)$ with 20 values of a in the range $(0.95, 1.0)$ and printed $F(10)$ for each of these a values (shown below).



```
(venv) luke@galangal ~/Dev/blood-dynamics master python3 model.py
a = 0.95 --- F(10) =  0.5082446144897547
a = 0.9526315789473684 --- F(10) =  0.4819288250160704
a = 0.9552631578947368 --- F(10) =  0.4556130355423861
a = 0.9578947368421052 --- F(10) =  0.4292972460687018
a = 0.9605263157894737 --- F(10) =  0.4029814565950175
a = 0.9631578947368421 --- F(10) =  0.3766656671213332
a = 0.9657894736842105 --- F(10) =  0.3503498776476489
a = 0.968421052631579 --- F(10) =  0.3240340881739646
a = 0.9710526315789474 --- F(10) =  0.2977182987002803
a = 0.9736842105263158 --- F(10) =  0.27140250922659603
a = 0.9763157894736842 --- F(10) =  0.24508671975291177
a = 0.9789473684210526 --- F(10) =  0.2187709302792286
a = 0.981578947368421 --- F(10) =  0.1924551408055443
a = 0.9842105263157894 --- F(10) =  0.16613935133186
a = 0.9868421052631579 --- F(10) =  0.1398235618581757
a = 0.9894736842105263 --- F(10) =  0.1135077723844914
a = 0.9921052631578947 --- F(10) =  0.08719198291080711
a = 0.9947368421052631 --- F(10) =  0.060876193437122826
a = 0.9973684210526316 --- F(10) =  0.034560403963438534
a = 1.0 --- F(10) =  0.00824461448975424
```

Figure 5: Terminal output showing $F(10) \to 0$, as $a \to 1$

We also see that there is a lag in the graph where $F(x)$ is almost stuck at 0 until $x > 0.35$ with a values close to 1. This is apparent in the graphs below.



(a) Plot of $F(x)$ when $a = 0.95$

(b) Plot of $F(x)$ when $a = 0.997$

Figure 6: cool plots!

This is the end of my journey learning LaTeX this semester! I'm not a math major, so I struggled with some of the non-graphical aspects of this assignment, notably question iv. I've enjoyed learning about dynamical systems! Thanks for the great course.