(6,3) Code Example

For the (6,3) code:

$n = 6$
$k = 3$
$m = 3$

So, $r = k/n = 0.5$
$BW\_exp = 1/r = 2.0$

The parity symbols are defined by:

$p_1 = m_1 + m_3$
$p_2 = m_1 + m_2$
$p_3 = m_2 + m_3$

The generator matrix is $G = [I \;\; P]$:

```
G =

     1     0     0     1     1     0
     0     1     0     0     1     1
     0     0     1     1     0     1
```

The parity-check matrix is $H = [P^T \;\; I]$:

```
H =

     1     0     1     1     0     0
     1     1     0     0     1     0
     0     1     1     0     0     1
```

Check orthogonality:

```
>> mod(G * H', 2)

ans =

     0     0     0
     0     0     0
     0     0     0
```

Since k = 3 there are $2^k = 2^3 = 8$ 3-tuple source messages:

```
M =

     0     0     0
     0     0     1
     0     1     0
     0     1     1
     1     0     0
     1     0     1
     1     1     0
     1     1     1
```

The eight valid 6-tuple code words are:

```
>> C = mod(M * G, 2)

C =

     0     0     0     0     0     0
     0     0     1     1     0     1
     0     1     0     0     1     1
     0     1     1     1     1     0
     1     0     0     1     1     0
     1     0     1     0     1     1
     1     1     0     1     0     1
     1     1     1     0     0     0
```

The remaining $2^n - 2^k = 2^6 - 2^3 = 64 - 8 = 56$ 6-tuples are invalid code words.

A 64-entry lookup table could be constructed that maps each of the possible received 6-tuples to the nearest valid codeword and corresponding 3-tuple source message.

Instead construct a $2^m = 2^3 = 8$ row standard array. The first (header) row is the all-zero syndrome followed by all the valid codewords:

000   000000 001101 010011 011110 100110 101011 110101 111000

For maximum likelihood decoding the entries in the second row are determined by any remaining invalid codeword of minimum weight. In this example there are six of them, all with only a single 1. Choose 000001 and call it e_1. The corresponding syndrome is $S_1 = e_1H^T = 001$. Construct the second row with S_1 followed by column entries that are the error pattern e_1 (coset leader) added to each of the valid codewords (coset):

000 000000 001101 010011 011110 100110 101011 110101
111000 001 000001 001100 010010 011111 100111 101010
110100 111001

The process is repeated. There are still five invalid codewords remaining that have weight 1. Choose $e_2 = 000010$ with syndrome $S_2 = e_2H^T = 010$:

000 000000 001101 010011 011110 100110 101011 110101
111000 001 000001 001100 010010 011111 100111 101010
110100 111001
010 000010 001111 010001 011100 100100 101001 110111
111010

Now there are four invalid codewords with weight 1. For the fourth row choose $e_3 = 000100$ with syndrome $S_3 = e_3H^T = 100$:

000 000000 001101 010011 011110 100110 101011 110101
111000
001 000001 001100 010010 011111 100111 101010 110100
111001
010 000010 001111 010001 011100 100100 101001 110111
111010
100 000100 001001 010111 011010 100010 101111 110001
111100

Construct the fifth, sixth and seventh rows with the remaining three invalid codewords of weight 1:

$e_4 = 001000$ $S_4 = e_4H^T = 101$
$e_5 = 010000$ $S_5 = e_5H^T = 011$
$e_6 = 100000$ $S_6 = e_6H^T = 110$

000 000000 001101 010011 011110 100110 101011 110101
111000
001 000001 001100 010010 011111 100111 101010 110100
111001
010 000010 001111 010001 011100 100100 101001 110111
111010

100 000100 001001 010111 011010 100010 101111 110001 111100
101 001000 000101 011011 010110 101110 100011 111101 110000
011 010000 011101 000011 001110 110110 111011 100101 101000
110 100000 101101 110011 111110 000110 001011 010101 011000

At this point there are eight remaining invalid codewords:

000111 001010 010100 011001 100001 101100 110010 111111

This means in addition to correcting all single symbol error patterns this (6,3) code can correct some multiple symbol error patterns as well. Since double symbol error patterns are more likely than triple or higher symbol error patterns for maximum likelihood decoding we should choose one of the remaining invalid codewords with weight 2. Arbitrarily select e_7 = 100001 with syndrome $S_7 = e_7H^T$ = 111.

The complete standard array is then:

000	000000	001101	010011	011110	100110	101011	110101	111000
001	000001	001100	010010	011111	100111	101010	110100	111001
010	000010	001111	010001	011100	100100	101001	110111	111010
100	000100	001001	010111	011010	100010	101111	110001	111100
101	001000	000101	011011	010110	101110	100011	111101	110000
011	010000	011101	000011	001110	110110	111011	100101	101000
110	100000	101101	110011	111110	000110	001011	010101	011000
111	100001	101100	110010	111111	000111	001010	010100	011001

As an example assume we want to send the source message m = [1 0 1]. The corresponding codeword from above is c = [1 0 1 0 1 1]. Suppose after this codeword is sent through a noisy channel we receive r = [1 1 1 0 1 1]. Can we recover the original source message?

Using MATLAB:

```
>> m = [1 0 1]

m =

     1     0     1


>> c = mod(m * G, 2)

c =

     1     0     1     0     1     1


>> r = [1 1 1 0 1 1]
```

```
r =

     1     1     1     0     1     1
```

Calculate the syndrome $S = rH^T$:

```
>> S = mod(r * H', 2)

S =

     0     1     1
```

Use the syndrome to locate the corresponding error pattern e_5 in the standard array:

```
>> e5 = [0 1 0 0 0 0]

e5 =

     0     1     0     0     0     0
```

Decode the received message as $\hat{c} = r + e_5$:

```
>> c_hat = mod(r + e5, 2)

c_hat =

     1     0     1     0     1     1
```

The estimate of the original source message $\hat{m}$ is the first three symbols of $\hat{c}$:

```
>> m_hat = c_hat(1:3)

m_hat =

     1     0     1
```

This is in fact the original source message.