Golay Codes

The generator matrix for a systematic extended Golay(24,12) code is:

$$G=[I \quad A]$$

where I is the 12×12 identity matrix and A is the 12×12 parity array matrix:

$$A=\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

This parity array matrix has the following two useful properties:

$$A=A^T$$

$$A\,A^T=I$$

The Golay(24,12) code is self-dual meaning that the generator matrix and the parity-check matrix are the same:

$$G=H$$

The decoding algorithm determines the error pattern e such that:

$$r=c+e$$

The syndrome is calculated as usual:

$$s=r\,H^T=(c+e)\,H^T=e\,H^T$$

We split the error pattern into two equal length 12-tuples $e=[x \quad y]$. Then:

$$s=e\,H^T=[x \quad y]\begin{bmatrix} I \\ \text{¿}A \end{bmatrix}=x+yA$$

Post-multiply both sides of the above equation by A^T and rearrange to find:

$y = (s + x)A$
The Golay(24,12) code corrects up to three errors. For correctable error patterns:

$w(e) = w(x) + w(y) \leq 3$

There are four possibilities:

$w(x) \leq 3$ and $w(y) = 0$

$w(x) \leq 2$ and $w(y) = 1$

$w(x) \leq 1$ and $w(y) = 2$

$w(x) = 0$ and $w(y) = 3$

These four possibilities define four different types of correctable error patterns:

- If $w(y) = 0$ then $y = 0$ where 0 is the all-zero 12-tuple. From this:

$x = s.$
For this case $w(s) = w(x) \leq 3$ and the correctable error pattern is:

$e = [s \quad 0]$

- If $w(y) = 1$ then $y = u_j$ where u_j is the unit 12-tuple with a one in the jth position.

$s = x + u_j A = x + A_j$
where A_j denotes the jth row of the matrix A. Rearranging,

$x = s + A_j$

Now $w(s + A_j) = w(x) \leq 2$. The correctable error pattern is:

$e = [s + A_j \quad u_j]$

- If $w(y) = 2$ or $w(y) = 3$ and $w(x) = 0$ then $x = 0$. It follows that:

$y = sA$

Here $w(sA) = w(y) \leq 3$ and the correctable error pattern becomes:

$e = [0 \quad sA]$

- Finally if $w(y)=2$ and $w(x)=1$ then $x=u_j$. This gives:

$$y=(s+u_j)A=sA+A_j$$

For this possibility $w(sA+A_j)=w(y)=2$. The correctable error pattern is:

$$e=[u_j \quad sA+A_j]$$

A decoding algorithm can be devised for the Golay(24,12) code based on this analysis:

Step 1: Compute the syndrome $s=rH^T$.

Step 2: If $w(s)\le 3$ then set $e=[s \quad 0]$.
Go to Step 8.

Step 3: If $w(s+A_j)\le 2$ for some row A_j of A then set $e=[s+A_j \quad u_j]$.
Go to Step 8.

Step 4: Compute the second syndrome $s_2=sA$.

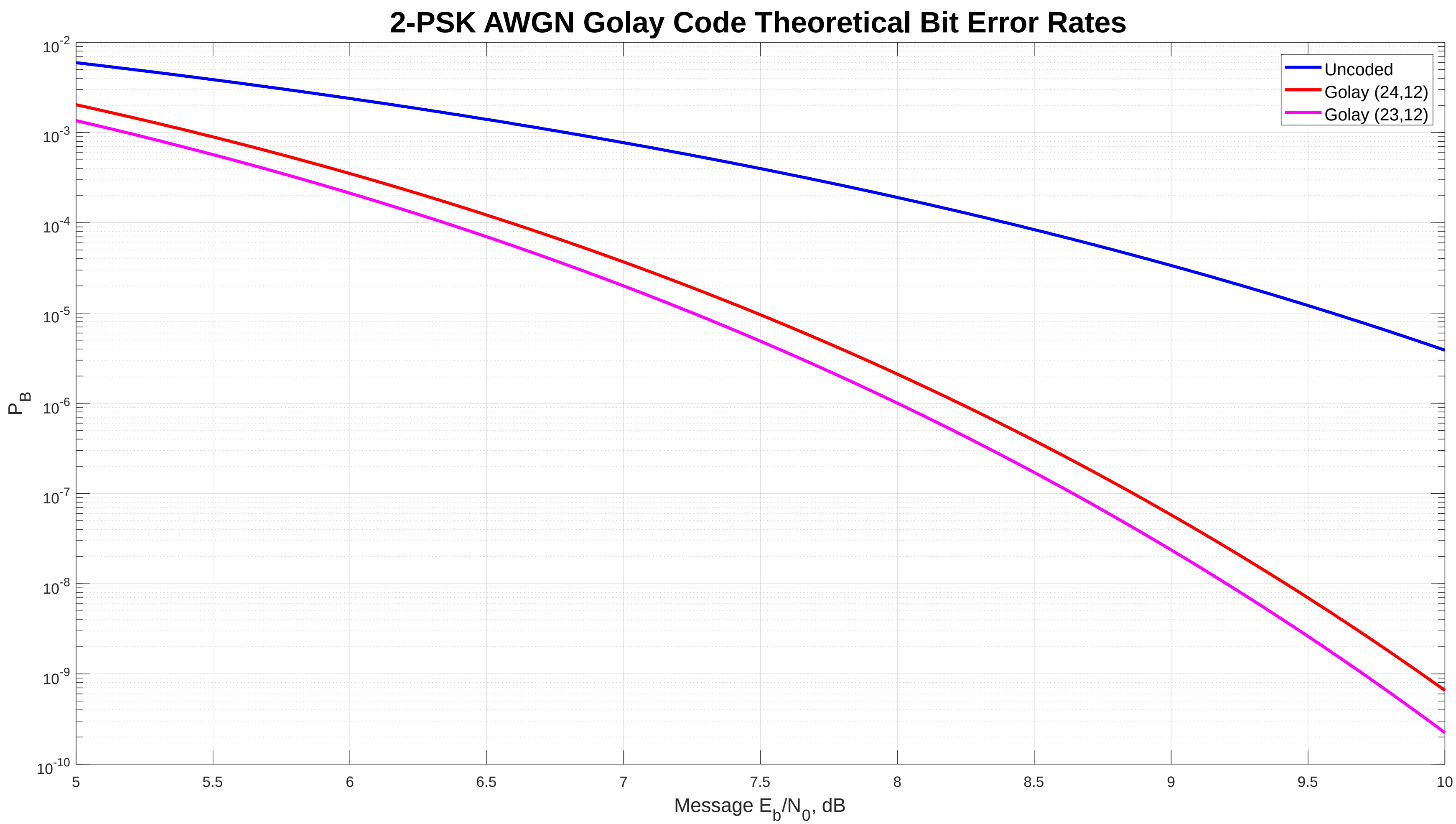
Step 5: If $w(s_2)\le 3$ then set $e=[0 \quad s_2]$.
Go to Step 8.

Step 6: If $w(s_2+A_j)=2$ for some row A_j of A then set $e=[u_j \quad s_2+A_j]$.
Go to Step 8.

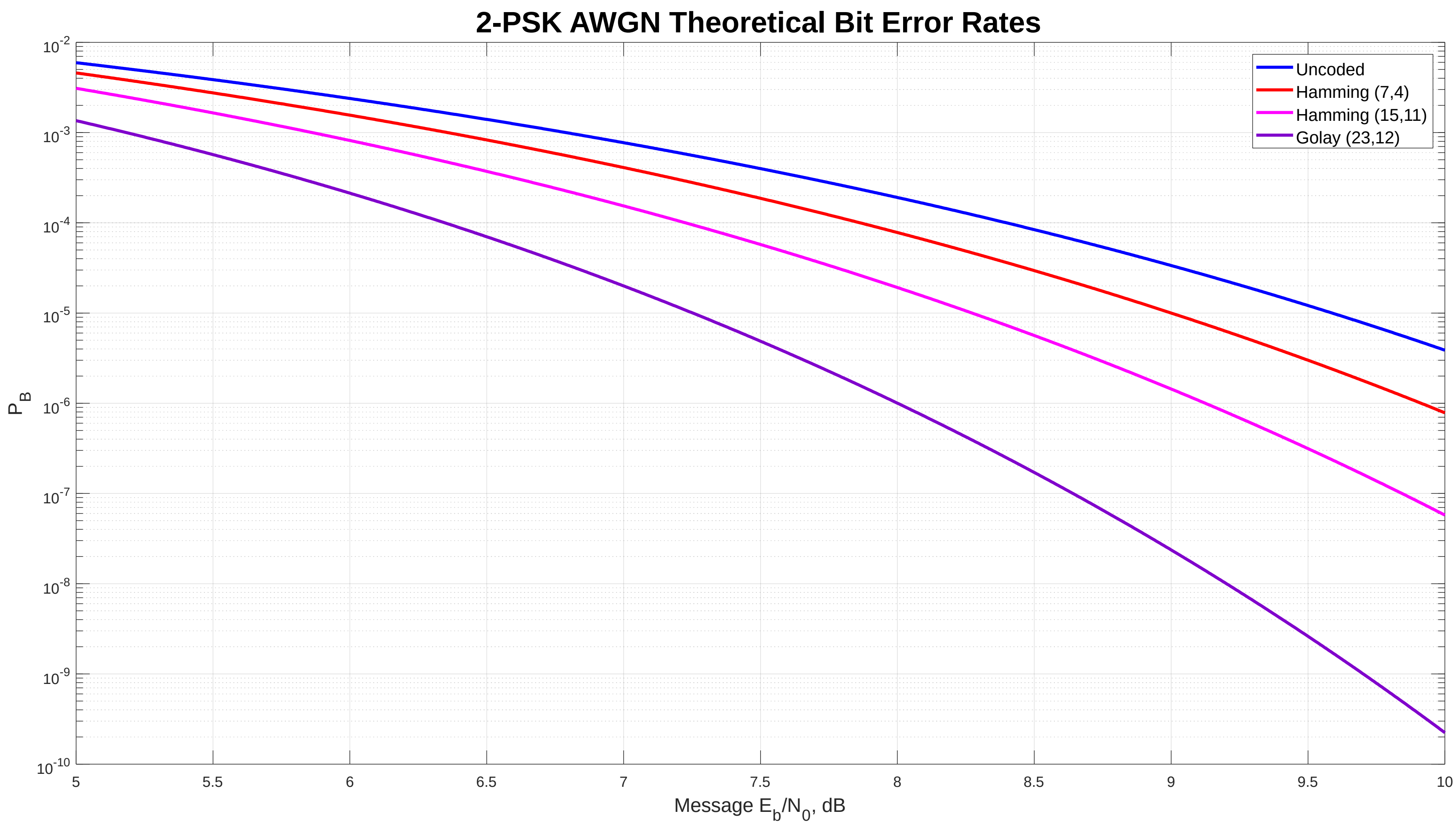
Step 7: The received sequence cannot be decoded with this algorithm.
Possible actions include:
- Declare a decoding failure
- Request retransmission
- Report the detection of a 4-bit error pattern
- Use the syndrome and a lookup table to decode a select subset of all the possible 4-bit error patterns

Step 8: The estimated code word is $\hat{c}=r+e$.

The Golay(23,12) code can be generated using the generator matrix for the extended Golay(24,12) code after deleting the last column. The decoding algorithm for the Golay(24,12) code can be used to decode the Golay(23,12) code if an overall parity bit is appended to the end of the 23-tuple using *odd* parity. This appended bit is discarded after decoding.



2-PSK AWGN Golay Code Theoretical Bit Error Rates
P_B
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
5
5.5
6
6.5
7
7.5
8
8.5
9
9.5
10
Message E_b/N_0, dB
Uncoded
Golay (24,12)
Golay (23,12)


2-PSK AWGN Theoretical Bit Error Rates
Uncoded
Hamming (7,4)
Hamming (15,11)
Golay (23,12)
P_B
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
5
5.5
6
6.5
7
7.5
8
8.5
9
9.5
10
Message E_b/N_0, dB