Rectangular Codes

Like Hamming codes rectangular codes have a minimum distance of 3 so they can correct a single error or detect two errors. The message bits are arranged in a rectangular array of M rows and N columns. A horizontal parity bit is appended to each row and a vertical parity bit is appended to each column. The code parameters are then:

$$k = MN$$

$$n = (M+1)(N+1)$$

$$r = \frac{k}{n} = \frac{MN}{(M+1)(N+1)}$$

For a discussion of the construction of the generator and parity-check matrices associated with rectangular codes consider as an example the 3×4 rectangular code:

m_1	m_2	m_3	m_4	p_1
m_5	m_6	m_7	m_8	p_2
m_9	m_{10}	m_{11}	m_{12}	p_3
p_4	p_5	p_6	p_7	

The parity bit equations are:

$$p_1 = m_1 + m_2 + m_3 + m_4$$

$$p_2 = m_5 + m_6 + m_7 + m_8$$

$$p_3 = m_9 + m_{10} + m_{11} + m_{12}$$

$$p_4 = m_1 + m_5 + m_9$$

$$p_5 = m_2 + m_6 + m_{10}$$

$$p_6 = m_3 + m_7 + m_{11}$$

$$p_7 = m_4 + m_8 + m_{12}$$

They result in the following parity array matrix:

$$P=\begin{bmatrix}1&0&0&1&0&0&0\\1&0&0&0&1&0&0\\1&0&0&0&0&1&0\\1&0&0&0&0&0&1\\0&1&0&1&0&0&0\\0&1&0&0&1&0&0\\0&1&0&0&0&1&0\\0&1&0&0&0&0&1\\0&0&1&1&0&0&0\\0&0&1&0&1&0&0\\0&0&1&0&0&1&0\\0&0&1&0&0&0&1\end{bmatrix}$$

The generator and parity-check matrices are then:

$$G=\begin{bmatrix}I & P\end{bmatrix}$$

$$H=\begin{bmatrix}P^T & I\end{bmatrix}$$

The syndrome is calculated in the usual manner:

$$s=r\,H^T$$

The syndrome is $M+N$ bits long. The first M bits are row parity-check bits and the remaining N bits are column parity-check bits. If a single row or column parity check fails then that row or column parity bit is in error and it is flipped to determine the received code word. If both a single row and a single column parity check fails then the message bit located by those row and column parity bits is in error. That message bit is flipped to determine the received code word. If more than one row or column parity check fails then the received code word cannot be reliably corrected.

Suppose [0 0 0 0 1 0 0 0 1 1 1 0 0 1 1 0 1 1 0] is received for the 3×4 rectangular code under discussion. The syndrome is [0 0 0 0 0 0 0] indicating no error and the received code word is therefore valid.

If [0 0 0 0 1 0 0 0 1 1 1 0 0 1 1 0 1 1 1] is received the syndrome is [0 0 0 0 0 0 1] indicating that the last column parity bit p_7 is in error. Flipping that bit results in the correct code word.

If instead [0 0 0 0 1 1 0 0 1 1 1 0 0 1 1 0 1 1 0] is received the syndrome becomes [0 1 0 0 1 0 0] indicating an error in the second row and second column of the message array which is m_6. That bit is flipped and the correct code word is [0 0 0 0 1 0 0 0 1 1 1 0 0 1 1 0 1 1 0] as before.

Finally if [1 0 0 0 0 0 0 0 1 1 1 0 0 1 1 0 1 1 0] is received the syndrome is [1 1 0 0 0 0 0]. Two row parity checks failed. This received code word cannot be corrected.