Orthogonality

The concept of orthogonality relates to perpendicularity.

- For Euclidean vectors the inner or dot product is zero if the two vectors are orthogonal:

  $f \cdot g = 0$

  Example:

  $(1,3,2) \cdot (3,-1,0) = 3 - 3 + 0 = 0$

- For continuous functions that are orthogonal:

  $$\int_a^b f(x) g(x) w(x)\, dx = 0$$

  where $w(x)$ is a weighting function. Often $w(x) = 1$.

  Example:

  $$\int_0^{2\pi} \sin x \cos x \, dx = 0$$

- For two equal length orthogonal binary sequences a_i and a_j the cross-correlation is zero:

  $$z_{ij} = \frac{\text{number of symbols the same} - \text{number of symbols different}}{\text{total number of symbols}} = 0$$

  If the zeros and ones in the binary sequence are represented by -1 and +1:

  $$z_{ij} = \frac{1}{n} \sum_{k=1}^{n} a_{ik} a_{jk} = 0$$

  where n is the number of symbols in a sequence.

  If the sequences are anti-correlated then $z_{ij} = -1$ and the sequences are said to be *antipodal*.
  Example:

  $a_i = [0\,0\,0\,0\,1\,1\,1\,1]$
  $a_j = [0\,1\,0\,1\,1\,0\,1\,0]$

$$z_{ij} = \frac{4-4}{8} = 0$$

Waveform Coding

Waveform coding transforms a message waveform set into an improved coded waveform set that provides improved received message symbol error rate. The coding procedure endeavors to make each of the coded waveforms as unalike as possible rendering the cross-correlation z_{ij} among all pairs of coded waveforms as small as possible.

The cross-correlation between two coded waveforms is a measure of the *distance* between them.

The most common waveform codes are *orthogonal*, *bi-orthogonal* and *trans-orthogonal* or *simplex* codes.

<u>Orthogonal</u>

For orthogonal waveform coding the coded waveforms are orthogonal and:

$$z_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The coded waveforms are generated from the *Hadamard* matrix:

$$H_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

This matrix is used directly to create the codewords for a one-bit symbol sequence. For a k-bit symbol sequence the Hadamard matrix is extended as follows:

$$H_k = \begin{bmatrix} H_{k-1} & H_{k-1} \\ H_{k-1} & \acute{H}_{k-1} \end{bmatrix}$$

Where $\acute{H}_{k-1}$ denotes the bit complement of H_{k-1}.

The number of orthogonal codewords needed to encode a k-bit sequence is $M = 2^k$. The number of symbols in each orthogonal codeword is $n = M$.

To encode a sequence a table look-up is employed. To decode a received codeword cross-correlation coefficients with each valid codeword are calculated and the valid codeword with the highest cross-coefficient is chosen as the most likely.

Example:

Suppose we want to send the 3-bit sequence [1 0 0] using an orthogonal waveform code. The correct codeword from the H_3 matrix is [0 0 0 0 1 1 1 1]. Suppose further this codeword is sent through a noisy channel and the sequence [0 0 1 0 1 1 1 1] is received.

To determine the most likely original message sequence we calculate the eight cross-correlation coefficients:

$$z_0 = \frac{3-5}{8} = \frac{-1}{4}$$

$$z_1 = \frac{3-5}{8} = \frac{-1}{4}$$

$$z_2 = \frac{5-3}{8} = \frac{1}{4}$$

$$z_3 = \frac{4-4}{8} = 0$$

$$z_4 = \frac{7-1}{8} = \frac{3}{4}$$

$$z_5 = \frac{3-5}{8} = \frac{-1}{4}$$

$$z_6 = \frac{5-3}{8} = \frac{1}{4}$$

$$z_7 = \frac{5-3}{8} = \frac{1}{4}$$

The cross-correlation coefficient z_4 is the largest indicating that the sequence most likely sent is [1 0 0] which is in fact the case.

Bi-Orthogonal

For *bi-orthogonal* waveform coding the coded waveforms are a mixture of orthogonal and antipodal codewords. The matrix that generates the bi-orthogonal codewords for a k-bit sequence is:

$$B_k = \begin{bmatrix} H_{k-1} \\ \acute{H}_{k-1} \end{bmatrix}$$

The number of bi-orthogonal codewords needed to encode a k-bit sequence is again $M = 2^k$. The number of symbols in each bi-orthogonal codeword is $n = M/2$, half that of the corresponding orthogonal code.

The combination of orthogonal and antipodal codewords results in a code set that has a greater average distance relative to orthogonal coding. Bi-orthogonal coding therefore has better error performance. Since each bi-orthogonal codeword is half the length of the corresponding orthogonal codeword, bi-orthogonal coding requires half the bandwidth.

Trans-Orthogonal

The first symbol of an orthogonal code is always 0. It can be deleted resulting in a slight bandwidth efficiency gain. The number of trans-orthogonal codewords needed to encode a k-bit sequence is once again $M = 2^k$. The number of symbols in each trans-orthogonal codeword is $n = M - 1$.

Conclusions

Error performance for all three waveform coding schemes improves as k and $M = 2^k$ are increased. That said, the level of error performance is essentially the same for all three for large values of M. Bi-orthogonal coding requires half the bandwidth compared to the other two. All three suffer from an exponential growth in bandwidth requirements and system complexity (number of parallel correlators) as M is increased.

For example suppose we choose $k = 5$ for these three waveform codes. In all cases $M = 32$ and the probability of a bit error is about 10^{-10} for $\frac{E_b}{N_0} = 9.6$ dB compared to 10^{-5} for uncoded.

	n	r	BW Expansion
Orthogonal	32	0.1563	6.4
Bi-Orthogonal	16	0.3125	3.2
Trans-Orthogonal	31	0.1613	6.2