Frieze Patterns and Cluster Algebras

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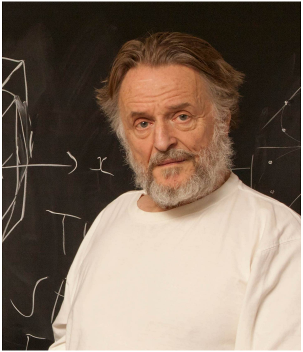
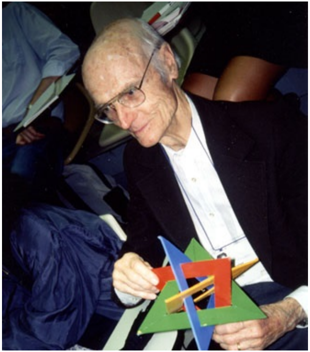
Conway and Coxeter



J.H. Conway
Photo: Denise Applewhite



H.S.M. Coxeter
Photo: Marion Walter

Frieze Patterns

A frieze pattern of height n consists of $n+2$ rows of positive integers

$$
\begin{array}{ccccccccccccccccccc}
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \cdots \\
 & 3 & & 1 & & 2 & & 3 & & 2 & & 2 & & 2 & & 1 & & 5 & \\
\cdots & & 2 & & 1 & & 5 & & 5 & & 3 & & 3 & & 1 & & 4 & & \cdots \\
 & 9 & & 1 & & 2 & & 8 & & 7 & & 4 & & 1 & & 3 & & 11 & \\
\cdots & & 4 & & 1 & & 3 & & 11 & & 9 & & 1 & & 2 & & 8 & & \cdots \\
 & 3 & & 3 & & 1 & & 4 & & 14 & & 2 & & 1 & & 5 & & 5 & \\
\cdots & & 2 & & 2 & & 1 & & 5 & & 3 & & 1 & & 2 & & 3 & & \cdots \\
 & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} &
\end{array}
$$

such that

(1) every entry in the first and last row is 1, and

(2) the entries satisfy the SL_2 diamond rule, meaning that every local configuration $\begin{smallmatrix} & b & \\ a & & d \\ & c & \end{smallmatrix}$ satisfies $ad - bc = 1$.

Lightning Bolts

$$\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix} \quad \implies \quad ad - bc = 1 \quad \implies \quad d = \frac{1 + bc}{a}$$

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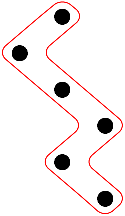
$$\mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \cdots$$



$$\cdots \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1}$$

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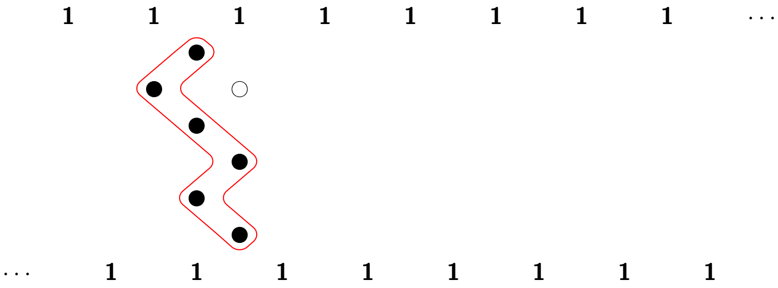

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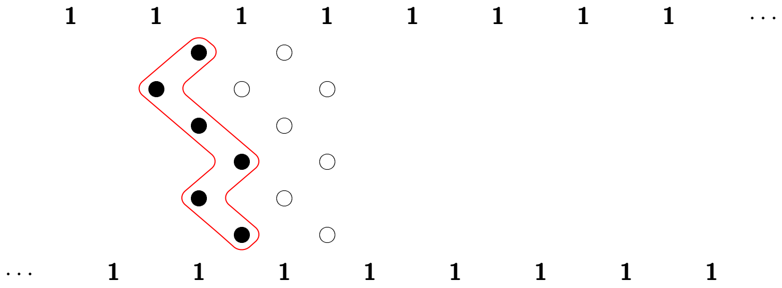

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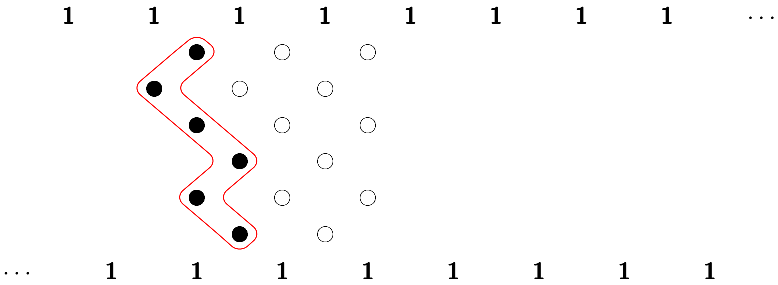

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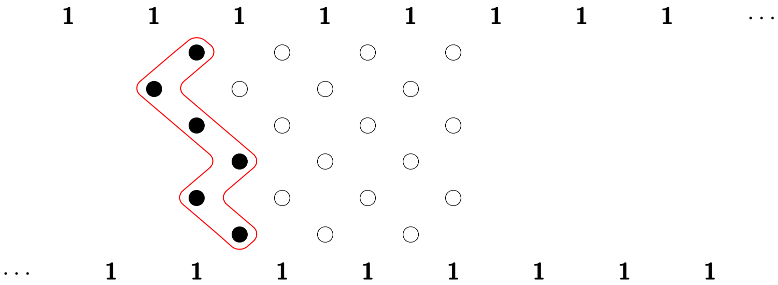

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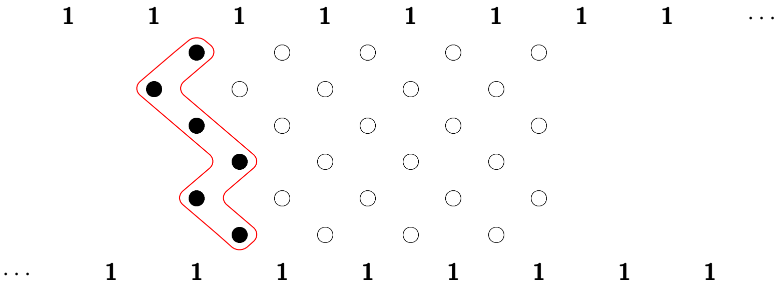

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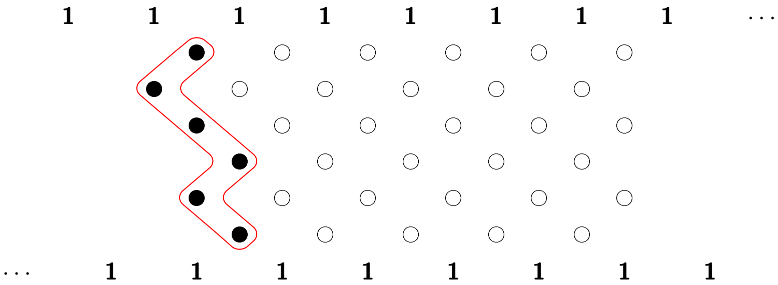

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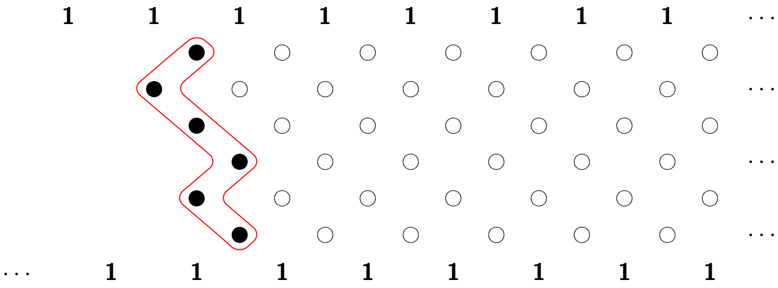

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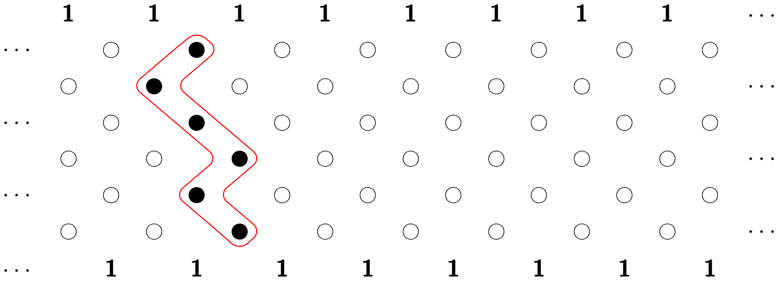

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 & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \cdots \\
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\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} &
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 & & & & 1 & & & & & & & & & & & & & \\
 & & & & & 1 & & & & & & & & & & & & \\
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} &
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 & & & & 1 & & 3 & & & & & & & & & & & \\
 & & & & & 1 & & & & & & & & & & & & \\
 & & & & 1 & & 2 & & & & & & & & & & & \\
 & & & & & 1 & & & & & & & & & & & & \\
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} &
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 & & & & 1 & & 3 & & & & & & & & & & & \\
 & & & & & 1 & & 7 & & & & & & & & & & \\
 & & & & 1 & & 2 & & & & & & & & & & & \\
 & & & & & 1 & & 3 & & & & & & & & & & \\
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} &
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 & & & & 1 & & 3 & & 2 & & & & & & & & & \\
 & & & 1 & & 2 & & 5 & & & & & & & & & & \\
 & & & & 1 & & 3 & & 12 & & & & & & & & & \\
 & & & & & 1 & & 7 & & & & & & & & & & \\
 & & & & 1 & & 2 & & 11 & & & & & & & & & \\
 & & & & & 1 & & 3 & & & & & & & & & & \\
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} &
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 & & & & 1 & & 3 & & 2 & & 3 & & & & & & & \\
 & & & 1 & & 2 & & 5 & & 5 & & & & & & & & \\
 & & & & 1 & & 3 & & 12 & & 8 & & & & & & & \\
 & & & & & 1 & & 7 & & 19 & & & & & & & & \\
 & & & & 1 & & 2 & & 11 & & 7 & & & & & & & \\
 & & & & & 1 & & 3 & & 4 & & & & & & & & \\
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} &
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 & & & & 1 & & 3 & & 2 & & 3 & & 2 & & & & & \\
 & & & 1 & & 2 & & 5 & & 5 & & 5 & & & & & & \\
 & & & & 1 & & 3 & & 12 & & 8 & & 2 & & & & & \\
 & & & & & 1 & & 7 & & 19 & & 3 & & & & & & \\
 & & & & 1 & & 2 & & 11 & & 7 & & 1 & & & & & \\
 & & & & & 1 & & 3 & & 4 & & 2 & & & & & & \\
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} &
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 & & & & 1 & & 3 & & 2 & & 3 & & 2 & & 1 & & & \\
 & & & 1 & & 2 & & 5 & & 5 & & 5 & & 1 & & & & \\
 & & & & 1 & & 3 & & 12 & & 8 & & 2 & & 1 & & & \\
 & & & & & 1 & & 7 & & 19 & & 3 & & 1 & & & & \\
 & & & & 1 & & 2 & & 11 & & 7 & & 1 & & 2 & & & \\
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 & & & & 1 & & 3 & & 2 & & 3 & & 2 & & 1 & & 3 & \\
 & & & 1 & & 2 & & 5 & & 5 & & 5 & & 1 & & 2 & & \cdots \\
 & & & & 1 & & 3 & & 12 & & 8 & & 2 & & 1 & & 7 & \\
 & & & & & 1 & & 7 & & 19 & & 3 & & 1 & & 3 & & \cdots \\
 & & & & 1 & & 2 & & 11 & & 7 & & 1 & & 2 & & 5 & \\
 & & & & & 1 & & 3 & & 4 & & 2 & & 1 & & 3 & & \cdots \\
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} &
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 & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \cdots \\
\cdots & & 2 & & 1 & & 3 & & 2 & & 3 & & 2 & & 1 & & 3 & \\
 & 7 & & 1 & & 2 & & 5 & & 5 & & 5 & & 1 & & 2 & & \cdots \\
\cdots & & 3 & & 1 & & 3 & & 12 & & 8 & & 2 & & 1 & & 7 & \\
 & 8 & & 2 & & 1 & & 7 & & 19 & & 3 & & 1 & & 3 & & \cdots \\
\cdots & & 5 & & 1 & & 2 & & 11 & & 7 & & 1 & & 2 & & 5 & \\
 & 3 & & 2 & & 1 & & 3 & & 4 & & 2 & & 1 & & 3 & & \cdots \\
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} &
\end{array}$$

Formal Variables in Frieze Patterns

$$\begin{array}{cccccccc}
\cdots & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & \mathbf{1} & \cdots \\
& x_1 & & & & & & \\
\cdots & & x_2 & & & & & \\
& & & x_3 & & & & \\
\cdots & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & \mathbf{1} & \cdots
\end{array}$$

Formal Variables in Frieze Patterns

$$
\begin{array}{cccccccc}
\cdots & & \mathbf{1} & & \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots \\
 & x_1 & & \frac{1+x_2}{x_1} & & & & \\
\cdots & & x_2 & & & & & \\
 & \frac{1+x_2}{x_3} & & x_3 & & & & \\
\cdots & & \mathbf{1} & & \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots
\end{array}
$$

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$$
\begin{array}{cccccccc}
\cdots & & \mathbf{1} & & \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots \\
 & x_1 & & \frac{1+x_2}{x_1} & & & & \\
\cdots & & x_2 & & \frac{x_1+x_3+x_2x_3}{x_1x_2} & & & \\
 & \frac{1+x_2}{x_3} & & x_3 & & & & \\
\cdots & & \mathbf{1} & & \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots
\end{array}
$$

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$$
\begin{array}{ccccccccc}
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & \mathbf{1} & \cdots \\
& x_1 & & \frac{1+x_2}{x_1} & & \frac{x_1+x_3}{x_2} & & & \\
\cdots & & x_2 & & \frac{x_1+x_3+x_2x_3}{x_1x_2} & & & & \\
& \frac{1+x_2}{x_3} & & x_3 & & \frac{x_1+x_3+x_1x_2+x_2x_3}{x_1x_2x_3} & & & \\
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & \mathbf{1} & \cdots
\end{array}
$$

$$
\frac{1+\frac{x_1+x_3+x_2x_3}{x_1x_2}}{\frac{1+x_2}{x_1}} = \frac{x_1+x_3+x_1x_2+x_2x_3}{x_2(1+x_2)} = \frac{(x_1+x_3)(1+x_2)}{x_2(1+x_2)} = \frac{x_1+x_3}{x_2}
$$

Formal Variables in Frieze Patterns

$$
\begin{array}{ccccccccc}
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & \mathbf{1} & \cdots \\
 & x_1 & & \frac{1+x_2}{x_1} & & \frac{x_1+x_3}{x_2} & & & \\
\cdots & & x_2 & & \frac{x_1+x_3+x_2x_3}{x_1x_2} & & \frac{x_1x_2+x_1+x_3}{x_2x_3} & & \\
 & \frac{1+x_2}{x_3} & & x_3 & & \frac{x_1+x_3+x_1x_2+x_2x_3}{x_1x_2x_3} & & & \\
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & \mathbf{1} & \cdots
\end{array}
$$

$$
\begin{aligned}
\frac{1+\frac{x_1+x_3}{x_2}\cdot\frac{x_1+x_3+x_1x_2+x_2x_3}{x_1x_2x_3}}{\frac{x_1+x_3+x_2x_3}{x_1x_2}} &= \frac{x_1x_2^2x_3+(x_1+x_3)(x_1+x_3+x_1x_2+x_2x_3)}{x_2x_3(x_1+x_3+x_2x_3)} \\
&= \frac{x_1x_2^2x_3+(x_1^2x_2+x_1x_2x_3)+(x_1+x_3)(x_1+x_3+x_2x_3)}{x_2x_3(x_1+x_3+x_2x_3)} \\
&= \frac{(x_1x_2+x_1+x_3)(x_1+x_3+x_2x_3)}{x_2x_3(x_1+x_3+x_2x_3)} = \frac{x_1+x_3+x_1x_2}{x_2x_3}
\end{aligned}
$$

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$$
\begin{array}{cccccccccc}
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & \cdots \\
 & x_1 & & \frac{1+x_2}{x_1} & & \frac{x_1+x_3}{x_2} & & \frac{1+x_2}{x_3} & & \\
\cdots & & x_2 & & \frac{x_1+x_3+x_2x_3}{x_1x_2} & & \frac{x_1x_2+x_1+x_3}{x_2x_3} & & & \\
 & \frac{1+x_2}{x_3} & & x_3 & & \frac{x_1+x_3+x_1x_2+x_2x_3}{x_1x_2x_3} & & x_1 & & \\
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & \cdots
\end{array}
$$

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\cdots & & x_2 & & \frac{x_1+x_3+x_2x_3}{x_1x_2} & & \frac{x_1x_2+x_1+x_3}{x_2x_3} & & x_2 & \\
& \frac{1+x_2}{x_3} & & x_3 & & \frac{x_1+x_3+x_1x_2+x_2x_3}{x_1x_2x_3} & & x_1 & & \\
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & \cdots
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Fomin and Zelevinsky

Sergey Fomin

Andrei Zelevinksy

Cluster Quivers

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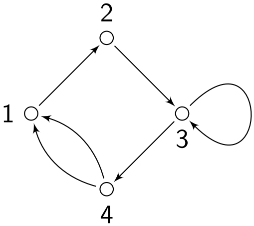

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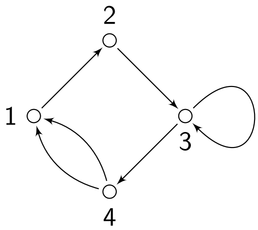

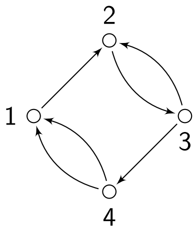

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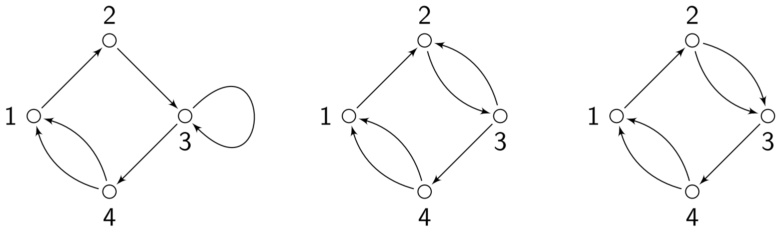

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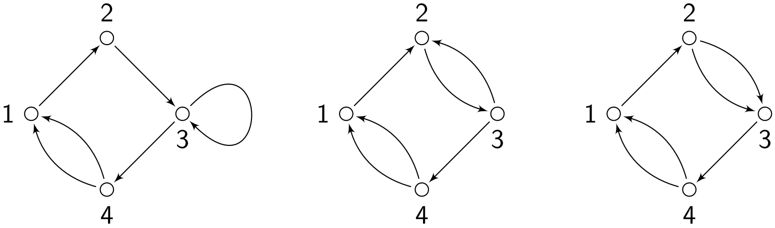

A cluster quiver is a quiver Q without oriented cycles of length 1 or 2.

Mutation of Cluster Quivers

Let Q be a cluster quiver and k be a vertex of Q. The mutation $\mu_k Q$ of Q at k is obtained by the following procedure:

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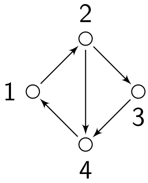

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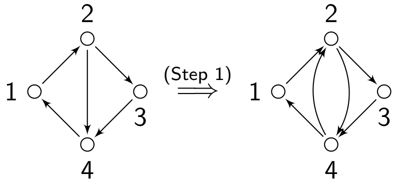

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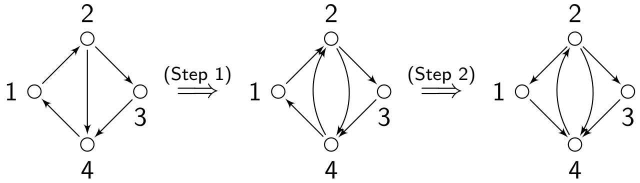

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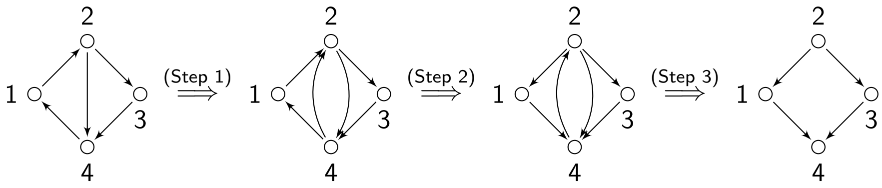

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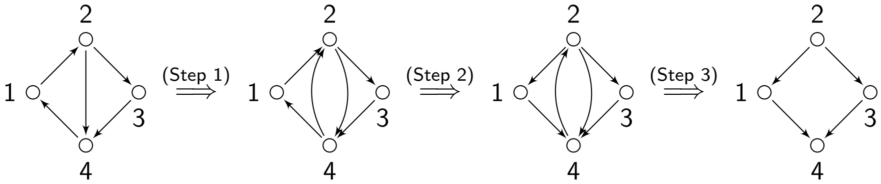

If we mutate at the same vertex twice, we get the original quiver back.

Mutation of Seeds

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A <u>seed</u> is a cluster quiver Q with vertex set $Q_0 = \{1, \ldots, n\}$, together with a free generating set $\{f_1, \ldots, f_n\} \subseteq \mathbb{Q}(x_1, \ldots, x_n)$ indexed by Q_0.

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Define mutation of a seed $(Q, \{f_i\})$ at vertex k to be $\mu_k(Q, \{f_i\}) = (\mu_k Q, \{f_i'\})$ where

$$f_i' = \begin{cases} f_i & i \neq k, \\ \dfrac{1}{f_k}\left(\displaystyle\prod_{k \to j} f_j + \prod_{\ell \to k} f_\ell\right) & i = k. \end{cases}$$

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If we mutate a seed twice at the same vertex, we recover the original seed.

Cluster Algebras

A cluster quiver Q with $Q_0 = \{1, \dots, n\}$ has initial seed $s_0 = (Q, \{x_i\})$.

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We call the rational functions appearing in seeds of $\mathcal{S}_Q$ cluster variables.

Laurent Phenomenon for Cluster Algebras

Theorem (Fomin-Zelevinksy 2002)

Let Q be a cluster quiver. Then every cluster variable in A_Q is a Laurent polynomial in the initial variables $\{x_1, \ldots, x_n\}$.

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Proof.

Fomin and Zelevinsky give a combinatorial proof in their original 2002 paper, using a more general definition of cluster algebras than we do. □

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Given a frieze of height n, the formulae expressing arbitrary entries in terms of those in a lightning bolt are given by cluster variables in A_Q for Q a quiver of type A_n.

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The rest of the talk focusses on explaining the proof of this theorem.

Choice of Quiver

Given a lightning bolt with n elements (so the frieze has n non-trivial rows), we define a quiver, Q, by having an arrow between points of the lightning bolt in adjacent rows and orienting the arrow to the right.

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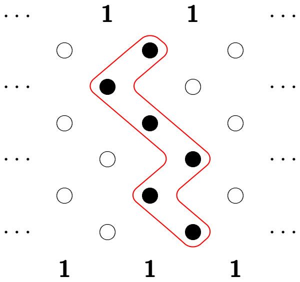

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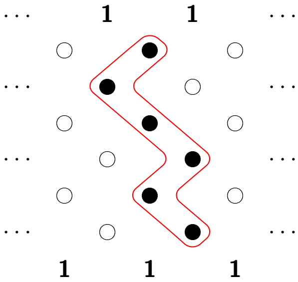

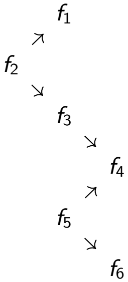

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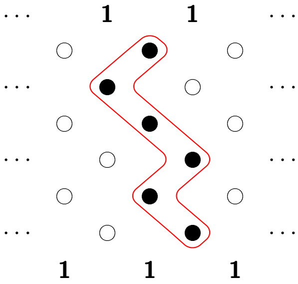

$$\begin{tikzcd}
& f_1 & \\
f_2 \arrow[ur] \arrow[dr] & & \\
& f_3 \arrow[dr] & \\
& & f_4 \\
& f_5 \arrow[ur] \arrow[dr] & \\
& & f_6
\end{tikzcd}$$

A source in Q corresponds to three entries on the lightning bolt (or in the trivial rows) forming the left-hand part of a diamond.

Mutation at Sources

If we mutate our quiver at a source i, this changes Q by reversing the arrows incident to i, which creates a sink. The new variable we obtain is:

$$f_i' = \frac{f_{i-1}f_{i+1}+1}{f_i}$$

where if $i=1$ (resp. $i=n$) we interpret f_{i-1} (resp. f_{i+1}) as 1.

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This then gives us a new lightning bolt of values, with the entry in the i-th row moved to the right by one step.

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Repeating this process as new sources are created, we calculate all of the entries to the right of the lightning bolt, and all of them are cluster variables in A_Q.

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