Derivation of Conservative MHD Equations

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April 10, 2006

1 Boltzman Equation

This derivation uses parts of [1] with input from [2, 3, 4, 5, 6]. The Boltzmann equation for the probability distribution function $f_a(\mathbf{x}, \mathbf{v}, t)$ of species a is:

$$\frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \nabla f_a + \mathbf{a} \cdot \nabla_v f_a = \left.\frac{\partial f_a}{\partial t}\right|_{col} \tag{1}$$

where ∇_v is a gradient in $\mathbf{v}$-space, and the right hand side is the collision term.

Multiplying this by some function $\chi(\mathbf{v})$ and integrating over $\mathbf{v}$-space, gives:

$$\int \chi \frac{\partial f_a}{\partial t} d^3\mathbf{v} + \int \chi \mathbf{v} \cdot \nabla f_a d^3\mathbf{v} + \int \chi \mathbf{a} \cdot \nabla_v f_a d^3\mathbf{v} = \int \chi \left.\frac{\partial f_a}{\partial t}\right|_{col} d^3\mathbf{v} \tag{2}$$

However,

$$\frac{\partial}{\partial t} \int \chi f_a d^3\mathbf{v} = \int \chi \frac{\partial f_a}{\partial t} d^3\mathbf{v} + \int f_a \frac{\partial \chi}{\partial t} d^3\mathbf{v}$$

But because $\chi(\mathbf{v})$ is a function of $\mathbf{v}$ only, $\frac{\partial \chi}{\partial t} = 0$, so the first term of Equation (2) can be written as:

$$\int \chi \frac{\partial f_a}{\partial t} d^3\mathbf{v} = \frac{\partial}{\partial t} \int \chi f_a d^3\mathbf{v}$$

Likewise with the second term of Equation (2):

$$\int \chi \mathbf{v} \cdot \nabla f_a d^3\mathbf{v} = \nabla \cdot \int \mathbf{v} \chi f_a d^3\mathbf{v} - \int f_a \mathbf{v} \cdot \nabla \chi d^3\mathbf{v} - \int f_a \chi \nabla \cdot \mathbf{v} d^3\mathbf{v}$$

Since $\chi(\mathbf{v})$ is a function of $\mathbf{v}$ only, $\nabla \chi = 0$. Similarly $\nabla \cdot \mathbf{v} = 0$, leaving:

$$\int \chi \mathbf{v} \cdot \nabla f_a d^3\mathbf{v} = \nabla \cdot \int \mathbf{v} \chi f_a d^3\mathbf{v}$$

The third term of Equation (2) can also be rewritten:

$$\int \chi \mathbf{a} \cdot \nabla_v f_a d^3\mathbf{v} = \int \nabla_v \cdot (\mathbf{a} \chi f_a) d^3\mathbf{v} - \int f_a (\mathbf{a} \cdot \nabla_v) \chi d^3\mathbf{v} - \int f_a \chi \nabla_v \cdot \mathbf{a} d^3\mathbf{v} \tag{3}$$

The first term on the right hand side of Equation (3) is an exact differential, so

$$\int \nabla_v \cdot (\mathbf{a}\chi f_a) d^3\mathbf{v} = \mathbf{a}\chi f_a$$

This vanishes if $f_a \to 0$ at $v \to \infty$.

The third term on the right hand side of Equation (3) contains $\nabla_v \cdot \mathbf{a}$, where

$$\mathbf{a} = \frac{\mathbf{F}}{m_a}$$

If the k-component of $\mathbf{F}$ is independent of v_k, then this term would be 0. The Lorentz force including gravity is

$$\mathbf{F} = q_a(\mathbf{E} + \mathbf{u} \times \mathbf{B}) + m\mathbf{g} \tag{4}$$

For forces that are constant, $\nabla_v \cdot \mathbf{a} = 0$. The $\mathbf{u} \times \mathbf{B}$ results in a force perpendicular to the motion, and so for that term also $\nabla_v \cdot \mathbf{a} = 0$. As a consequence the force in the k-direction does not depend on v_k, and the third term on the right hand side of Equation (3) can be ignored. This leaves:

$$\int \chi \mathbf{a} \cdot \nabla_v f_a d^3\mathbf{v} = -\int f_a(\mathbf{a} \cdot \nabla_v)\chi d^3\mathbf{v}$$

Finally, the collision term of Equation (2) can be written as

$$\int \chi \left.\frac{\partial f_a}{\partial t}\right|_{col} d^3\mathbf{v} = \frac{\partial}{\partial t}\int \chi \left. f_a\right|_{col} d^3\mathbf{v} - \int \left. f_a\right|_{col} \frac{\partial \chi}{\partial t} d^3\mathbf{v}$$

Because $\chi(v)$ is not a function of t, $\frac{\partial \chi}{\partial t} = 0$, leaving:

$$\int \chi \left.\frac{\partial f_a}{\partial t}\right|_{col} d^3\mathbf{v} = \frac{\partial}{\partial t}\int \chi \left. f_a\right|_{col} d^3\mathbf{v}$$

Putting all of this together, Equation (2) becomes:

$$\frac{\partial}{\partial t}\int \chi f_a d^3\mathbf{v} + \nabla \cdot \int \mathbf{v}\chi f_a d^3\mathbf{v} - \int f_a(\mathbf{a} \cdot \nabla_v)\chi d^3\mathbf{v} = \frac{\partial}{\partial t}\int \chi \left. f_a\right|_{col} d^3\mathbf{v} \tag{5}$$

With the following definition of the average value $< \chi >$ of a property χ:

$$n_a < \chi >_a = \int \chi f_a d^3\mathbf{v} \tag{6}$$

Equation (5) becomes the generalized transport equation:

$$\frac{\partial}{\partial t}(n_a < \chi >_a) + \nabla \cdot (n_a < \chi \mathbf{v} >_a) - n_a < (\mathbf{a} \cdot \nabla_v)\chi >_a = \frac{\partial}{\partial t} \left.(n_a < \chi >_a)\right|_{col} \tag{7}$$

1.1 Conservation of Mass

Using $\chi = m_a$, then $<\chi>= m_a$. Define the bulk velocity $\mathbf{u_a} =< \mathbf{v_a} >$, then $\mathbf{v} = \mathbf{u_a} + \mathbf{c_a}$, $< \mathbf{v_a} >=< \mathbf{u_a} + \mathbf{c_a} >$, and the fluctuation $< \mathbf{c_a} >= 0$. Then $< \chi\mathbf{v} >_a= m_a < \mathbf{v_a} >= m_a\mathbf{u_a}$. With this and the fact that $\nabla_v\chi = 0$, Equation (7) becomes:

$$\frac{\partial}{\partial t} n_a m_a + \nabla \cdot (n_a m_a \mathbf{u_a}) = m_a \int \left.\frac{\partial f_a}{\partial t}\right|_{col}$$

Define the collision term

$$S_a = \left(\frac{\partial \rho_a}{\partial t}\right)_{col}$$

and with $\rho_a = n_a m_a$, this becomes

$$\frac{\partial \rho_a}{\partial t} + \nabla \cdot (\rho_a \mathbf{u_a}) = \left(\frac{\partial \rho_a}{\partial t}\right)_{col} = S_a \tag{8}$$

1.2 Conservation of Momentum

With $\chi = m_a\mathbf{v}$, Equation (7) becomes:

$$\frac{\partial}{\partial t}(\rho_a < \mathbf{v} >_a) + \nabla \cdot (\rho_a < \mathbf{vv} >_a) - n_a < (\mathbf{F_a} \cdot \nabla_v)\chi >_a= m_a \int \mathbf{v} \left(\frac{\partial f_a}{\partial t}\right)_{col}$$

But if $\mathbf{v} = \mathbf{u_a} + \mathbf{c_a}$, with $< \mathbf{c_a} >= 0$, then

$$\frac{\partial}{\partial t}(\rho_a < \mathbf{v} >_a) = \frac{\partial}{\partial t}(\rho_a \mathbf{u_a})$$

and

$$\begin{aligned}\nabla \cdot (\rho_a < \mathbf{vv} >_a) &= \nabla \cdot [\rho_a(\mathbf{u_a u_a} + \mathbf{u_a} < \mathbf{c_a} > + < \mathbf{c_a} > \mathbf{u_a} + < \mathbf{c_a c_a} >)] \\ &= \nabla \cdot (\rho_a \mathbf{u_a u_a} + \rho_a < \mathbf{c_a c_a} >)\end{aligned}$$

Next

$$\begin{aligned}-n_a < (\mathbf{F_a} \cdot \nabla_v)\chi >_a &= -n_a < (\mathbf{F_x}\frac{\partial}{\partial v_x} + \mathbf{F_y}\frac{\partial}{\partial v_y} + \mathbf{F_z}\frac{\partial}{\partial v_z})\mathbf{v} >_a \\ &= -n_a < \mathbf{F_x i} + \mathbf{F_y j} + \mathbf{F_z k} >= -n_a < \mathbf{F} >\end{aligned}$$

Also define the collision term

$$\mathbf{A_a} = m_a \int \mathbf{v} \left(\frac{\partial f_a}{\partial t}\right)_{col} = \left(\frac{\partial < \mathbf{v_a} >}{\partial t}\right)_{col} = \left(\frac{\partial \mathbf{u_a}}{\partial t}\right)_{col}$$

Define the pressure tensor

$$P_a = \rho_a < \mathbf{c_a c_a} > \tag{9}$$

giving

$$\frac{\partial \rho_a \mathbf{u_a}}{\partial t} + \nabla \cdot (\rho_a \mathbf{u_a} \mathbf{u_a}) + \nabla \cdot (P_a) - n_a < \mathbf{F} >= \mathbf{A_a} \tag{10}$$

As an aside, this equation is sometimes written differently. The second term can be written out as

$$\begin{aligned}
\nabla \cdot (\rho_a \mathbf{u_a} \mathbf{u_a}) &= \frac{\partial}{\partial x}(\rho_a u_x \mathbf{u_a}) + \frac{\partial}{\partial y}(\rho_a u_y \mathbf{u_a}) + \frac{\partial}{\partial z}(\rho_a u_z \mathbf{u_a}) \\
&= \rho_a (u_x \frac{\partial \mathbf{u_a}}{\partial x} + u_y \frac{\partial \mathbf{u_a}}{\partial y} + u_z \frac{\partial \mathbf{u_a}}{\partial z}) + \mathbf{u}(\frac{\partial \rho \mathbf{u_a}}{\partial x} + \frac{\partial \rho \mathbf{u_a}}{\partial y} + \frac{\partial \rho \mathbf{u_a}}{\partial z}) \\
&= \rho_a (\mathbf{u_a} \cdot \nabla) \mathbf{u_a} + \mathbf{u_a} (\nabla \cdot \rho \mathbf{u_a})
\end{aligned}$$

then using Equation (8) gives

$$\rho_a (\frac{\partial \mathbf{u_a}}{\partial t} + \mathbf{u_a} \cdot \nabla \mathbf{u_a}) + \nabla \cdot P_a - n_a < \mathbf{F} >= \mathbf{A_a} - \mathbf{u_a} \mathbf{S_a} \tag{11}$$

or, with the Lorentz force of Equation (4) and total differential $\frac{D}{Dt} = (\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla)$ this equation can be written as

$$\rho_a \frac{D \mathbf{u_a}}{Dt} + \nabla \cdot P_a - n_a q_a (\mathbf{E} + \mathbf{u_a} \times \mathbf{B}) = \mathbf{A_a} - \mathbf{u_a} S_a + \rho_a \mathbf{g} \tag{12}$$

1.3 Conservation of Energy

With $\chi = \frac{1}{2} m_a v^2 = \frac{1}{2} m_a (\mathbf{v} \cdot \mathbf{v})$, then

$$\nabla_v \chi = \frac{1}{2} m_a \nabla_v (\mathbf{v} \cdot \mathbf{v}) = m_a (\mathbf{v} \cdot \nabla_v) \mathbf{v} = m_a \mathbf{v}$$

and Equation (7) becomes

$$\sum_a \frac{\partial}{\partial t} (\frac{1}{2} \rho_a < v^2 >_a) + \sum_a \nabla \cdot (\frac{1}{2} \rho_a < v^2 \mathbf{v} >_a) - \sum_a n_a < \mathbf{F} \cdot \mathbf{v} >_a = \frac{\partial}{\partial t} (\frac{1}{2} \rho_a < v^2 >_a) \Big|_{col}$$

Define the scalar pressure p_a as

$$p_a = \frac{1}{d} \sum_{ij} P_{aij} \delta_{ij} = \frac{1}{d} \sum_i P_{aii} \tag{13}$$

Where d is the dimensionality of the space, usually $d = 3$. Since P has been defined above in Equation (9), the scalar pressure can also be written as

$$p_a = \frac{1}{d} \rho_a \sum_i < c_{ai}^2 >= \frac{1}{d} \rho_a < \sum_i c_{ai}^2 >= \frac{1}{d} \rho_a < c_a^2 >$$

or, with $\gamma = \frac{d+2}{d}$

$$p_a = \frac{1}{d} \rho_a < c_a^2 >= \frac{\gamma - 1}{2} \rho_a < c_a^2 > \tag{14}$$

Therefore

$$n_a < \chi >_a = \frac{1}{2}\rho_a < c_a^2 > + \frac{1}{2}\rho_a u_a^2 = \frac{1}{\gamma - 1} p_a + \frac{1}{2}\rho_a u_a^2$$

It is convenient to define this latter quantity as the energy density

$$\epsilon_a = \frac{p_a}{\gamma - 1} + \frac{1}{2}\rho_a u_a^2 \tag{15}$$

Next

$$\begin{aligned}
\nabla \cdot (n_a < \chi \mathbf{v} >_a) &= \nabla \cdot (\frac{1}{2}\rho_a < (\mathbf{v} \cdot \mathbf{v})\mathbf{v} >_a) \\
&= \nabla \cdot (\frac{1}{2}\rho_a < ((\mathbf{u_a} + \mathbf{c_a}) \cdot (\mathbf{u_a} + \mathbf{c_a}))(\mathbf{u_a} + \mathbf{c_a}) >) \\
&= \nabla \cdot (\frac{1}{2}\rho_a < (u_a^2 + 2\mathbf{u_a} \cdot \mathbf{c_a} + c_a^2)(\mathbf{u_a} + \mathbf{c_a}) >) \\
&= \nabla \cdot (\frac{\rho_a}{2} u_a^2 \mathbf{u_a} + \frac{\rho_a}{2} < c_a^2 > \mathbf{u_a} + \rho_a < \mathbf{c_a}\mathbf{c_a} > \cdot \mathbf{u_a} + \frac{\rho_a}{2} < c_a^2 \mathbf{c_a} >) \\
&= \nabla \cdot (\epsilon_a \mathbf{u_a} + P_a \cdot \mathbf{u_a} + \frac{1}{2}\rho_a < c_a^2 \mathbf{c_a} >)
\end{aligned}$$

With the last term defined as the heat flux

$$\mathbf{q_a} = \frac{1}{2}\rho_a < c_a^2 \mathbf{c_a} > \tag{16}$$

then

$$\nabla \cdot (n_a < \chi \mathbf{v} >_a) = \nabla \cdot (\epsilon_a \mathbf{u_a} + P_a \cdot \mathbf{u_a} + \mathbf{q_a})$$

The third term of Equation (7) becomes:

$$-n_a < \mathbf{a} \cdot \nabla_v \chi >_a = -n_a < \frac{\mathbf{F}}{m_a} \cdot (m_a \mathbf{v}) >_a = -n_a < \mathbf{F} \cdot \mathbf{v} >_a$$

Define the collision term

$$M_a = \frac{1}{2} m_a \int v^2 \left. \frac{\partial f_a}{\partial t} \right|_{col} d^3 v = \frac{\partial}{\partial t}(\frac{1}{2}\rho_a < v^2 >_a)$$

and Equation (7) becomes:

$$\frac{\partial \epsilon_a}{\partial t} + \nabla \cdot (\epsilon_a \mathbf{u_a}) + \nabla \cdot (P_a \cdot \mathbf{u_a}) + \nabla \cdot \mathbf{q_a} - n_a < \mathbf{F} \cdot \mathbf{v} >_a = M_a \tag{17}$$

Now,

$$< \mathbf{F} \cdot \mathbf{v} > = < \mathbf{F} \cdot (\mathbf{u_a} + \mathbf{c_a}) > = < \mathbf{F} > \cdot \mathbf{u_a} + < \mathbf{F} \cdot \mathbf{c_a} >$$

For all velocity-independent forces $\mathbf{F}$,

$$< \mathbf{F} \cdot \mathbf{c_a} > = \mathbf{F} \cdot < \mathbf{c_a} > = 0$$

For the one velocity-dependent force of interest, the Lorentz force, and assuming that $\mathbf{E}$ has no dependence on velocity

$$
\begin{aligned}
< \mathbf{F} \cdot \mathbf{c_a} > &=< (q_a(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + m\mathbf{g}) \cdot \mathbf{c_a} > \\
&= q_a < (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{c_a} > + q_a < \mathbf{E} \cdot \mathbf{c_a} > + m < \mathbf{g} \cdot \mathbf{c_a} > \\
&= q_a < ((\mathbf{u_a} + \mathbf{c_a}) \times \mathbf{B}) \cdot \mathbf{c_a} > \\
&= q_a \mathbf{u_a} \times \mathbf{B} \cdot < \mathbf{c_a} > + q_a < (\mathbf{c_a} \times \mathbf{B}) \cdot \mathbf{c_a} >= 0
\end{aligned}
$$

Of course $\mathbf{u_a} \cdot (\mathbf{u_a} \times \mathbf{B}) = 0$, so if there are no other velocity-dependent forces to consider, the energy equation reduces to

$$
\frac{\partial \epsilon_a}{\partial t} + \nabla \cdot (\epsilon_a \mathbf{u_a}) + \nabla \cdot (P_a \cdot \mathbf{u_a}) + \nabla \cdot \mathbf{q_a} - n_a q_a \mathbf{u_a} \cdot \mathbf{E} - \rho_a \mathbf{u_a} \cdot \mathbf{g} = M_a \tag{18}
$$

1.4 Summary

The conservation laws for specials a are summarized

$$
\frac{\partial \rho_a}{\partial t} + \nabla \cdot (\rho_a \mathbf{u_a}) = \left(\frac{\partial \rho_a}{\partial t} \right)_{col} = S_a \tag{19a}
$$

$$
\frac{\partial \rho_a \mathbf{u_a}}{\partial t} + \nabla \cdot (\rho_a \mathbf{u_a} \mathbf{u_a}) + \nabla \cdot (P_a) - n_a < \mathbf{F} >= \mathbf{A_a} \tag{19b}
$$

$$
\frac{\partial \epsilon_a}{\partial t} + \nabla \cdot (\epsilon_a \mathbf{u_a}) + \nabla \cdot (P_a \cdot \mathbf{u_a}) - n_a q_a \mathbf{u_a} \cdot \mathbf{E} - \rho_a \mathbf{u_a} \cdot \mathbf{g} = M_a \tag{19c}
$$

2 Equations of the Entire Fluid

The above equations apply to each individual species in the plasma. They may be summed over all species to provide a set of equations that describe the fluid as a whole. Define the following properties summed over all particle types

$$
\begin{aligned}
\rho &= \sum_a n_a m_a & \rho \mathbf{u} &= \sum_a n_a m_a \mathbf{u_a} \\
\rho_q &= \sum_a n_a q_a & \mathbf{J} &= \sum_a n_a q_a \mathbf{u_a}
\end{aligned} \tag{20}
$$

2.1 Conservation of Mass

Equation (8) becomes

$$
\sum_a \frac{\partial \rho_a}{\partial t} + \sum_a \nabla \cdot (\rho_a \mathbf{u_a}) = \sum_a S_a
$$

If total mass is conserved in the system, the sum of the collision terms equals 0, giving

$$
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{21}
$$

2.2 Conservation of Momentum

Next, summing Equation (10) with the Lorentz force substituted from Equation (4) gives

$$\sum_a \frac{\partial \rho_a \mathbf{u_a}}{\partial t} + \sum_a \nabla \cdot (\rho_a \mathbf{u_a} \mathbf{u_a}) + \sum_a \nabla \cdot (P_a) - \sum_a n_a (q_a (\mathbf{E} + \mathbf{u_a} \times \mathbf{B}) + m_a \mathbf{g}) = \sum_a \mathbf{A_a}$$

The sum of the collision terms is 0 if total momentum is conserved in the system, giving

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \sum_a \rho_a \mathbf{u_a} \mathbf{u_a} + \nabla \cdot \sum_a P_a - \rho_q \mathbf{E} - \mathbf{J} \times \mathbf{B} - \rho \mathbf{g} = 0 \tag{22}$$

The pressure tensor for each species was $P_a = \rho_a < \mathbf{c_a} \mathbf{c_a} >$, where $\mathbf{c_a} = \mathbf{v} - \mathbf{u_a}$. This is relative to the mean velocity of each species. It is useful to define a total pressure tensor relative to the global mean velocity

$$P = \sum_a \rho_a < (\mathbf{v} - \mathbf{u})(\mathbf{v} - \mathbf{u}) > \tag{23}$$

Define $\mathbf{w_a} = \mathbf{u_a} - \mathbf{u}$, then $\mathbf{v} - \mathbf{u} = \mathbf{v} - (\mathbf{u_a} - \mathbf{w_a}) = \mathbf{c_a} + \mathbf{w_a}$. Since $\mathbf{w_a}$ is single value for each species, $< \mathbf{c_a} \mathbf{w_a} > = < \mathbf{c_a} > \mathbf{w_a} = 0$ giving

$$P = \sum_a \rho_a < (\mathbf{c_a} + \mathbf{w_a})(\mathbf{c_a} + \mathbf{w_a}) > = \sum_a \rho_a < \mathbf{c_a} \mathbf{c_a} > + \sum_a \rho_a < \mathbf{w_a} \mathbf{w_a} >$$

or

$$\sum_a P_a = P - \sum_a \rho_a \mathbf{w_a} \mathbf{w_a} \tag{24}$$

Next, given that $\mathbf{u_a} = \mathbf{u} + \mathbf{w_a}$, and

$$\sum_a (\rho \mathbf{w_a} \mathbf{u}) = \mathbf{u} \sum_a \rho_a \mathbf{w_a} = \mathbf{u} \sum_a \rho_a (\mathbf{u_a} - \mathbf{u}) = \mathbf{u}(\rho \mathbf{u} - \rho \mathbf{u}) = 0$$

Then

$$\sum \rho_a \mathbf{u_a} \mathbf{u_a} = \sum \rho_a (\mathbf{u} + \mathbf{w_a})(\mathbf{u} + \mathbf{w_a}) = \rho \mathbf{u} \mathbf{u} + \sum_a \rho_a \mathbf{w_a} \mathbf{w_a}$$

Using this with Equation (22) gives

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) + \nabla \cdot P - \rho_q \mathbf{E} - \mathbf{J} \times \mathbf{B} - \rho \mathbf{g} = 0 \tag{25}$$

2.3 Conservation of Energy

Summing up the energy equation, with total energy conserved, gives

$$\sum_a \frac{\partial \epsilon_a}{\partial t} + \sum_a \nabla \cdot (\epsilon_a \mathbf{u_a} + P_a \cdot \mathbf{u_a} + \mathbf{q_a}) - \sum_a n_a q_a \mathbf{u_a} \cdot \mathbf{E} - \sum_a \rho_a \mathbf{u_a} \cdot \mathbf{g} = 0 \tag{26}$$

Like before,

$$\sum \rho_a u_a^2 = \sum \rho_a (u + w_a) \cdot (u + w_a) = \rho u^2 + \sum_a \rho_a w_a^2$$

Likewise define the total scalar pressure

$$p = \frac{1}{d} \sum_i P_{ii} = \frac{\gamma - 1}{2} \sum_a \rho_a < (c_a + w_a)^2 >= \sum_a p_a + \frac{\gamma - 1}{2} \sum_a \rho_a w_a^2 \tag{27}$$

Therefore

$$\begin{aligned}
\sum_a \epsilon_a &= \sum_a (\frac{1}{\gamma - 1} p_a + \frac{1}{2} \rho_a u_a^2) \\
&= \frac{p}{\gamma - 1} - \frac{1}{\gamma - 1} \frac{\gamma - 1}{2} \sum_a \rho_a w_a^2 + \frac{1}{2} \rho u^2 + \frac{1}{2} \sum_a \rho_a w_a^2 \\
&= \frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2
\end{aligned}$$

Next,

$$\sum_a p_a \mathbf{u_a} = \sum_a p_a (\mathbf{u} + \mathbf{w_a}) = \mathbf{u} \sum_a p_a + \sum_a p_a \mathbf{w_a} = p\mathbf{u} - \mathbf{u} \frac{\gamma - 1}{2} \sum_a \rho_a w_a^2 + \sum_a p_a \mathbf{w_a}$$

Next, since $\sum \rho_a \mathbf{w_a} = 0$

$$\begin{aligned}
&\sum_a \rho_a u_a^2 \mathbf{u_a} = \sum_a \rho_a (u + w_a)^2 (\mathbf{u} + \mathbf{w_a}) \\
&= \sum_a \rho_a (u^2 \mathbf{u} + 2(\mathbf{w_a} \cdot \mathbf{u})\mathbf{u} + w_a^2 \mathbf{u} + u^2 \mathbf{w_a} + 2(\mathbf{u} \cdot \mathbf{w_a})\mathbf{w_a} + w_a^2 \mathbf{w_a}) \\
&= \rho u^2 \mathbf{u} + 0 + \mathbf{u} \sum \rho_a w_a^2 + 0 + 2\mathbf{u} \cdot \sum \rho_a \mathbf{w_a} \mathbf{w_a} + \sum \rho_a w_a^2 \mathbf{w_a}
\end{aligned}$$

Next define the total heat flux

$$\mathbf{q} = \frac{1}{2} \sum_a \rho_a < (c_a + w_a)^2 (\mathbf{c_a} + \mathbf{w_a}) > \tag{28}$$

Because $< \mathbf{c_a} >= 0$, the latter can be written out as

$$\begin{aligned}
\mathbf{q} = \frac{1}{2} \sum_a \rho_a (&< c_a^2 \mathbf{c_a} > + 2 < (\mathbf{w_a} \cdot \mathbf{c_a}) \mathbf{c_a} > + w_a^2 < \mathbf{c_a} > \\
&+ < c_a^2 > \mathbf{w_a} + 2(< \mathbf{c_a} > \cdot \mathbf{w_a}) \mathbf{w_a} + w_a^2 \mathbf{w_a}) \\
&= \sum_a (\mathbf{q_a} + \mathbf{w_a} \cdot P_a + \frac{1}{\gamma - 1} p_a \mathbf{w_a} + \frac{1}{2} \rho_a w_a^2 \mathbf{w_a})
\end{aligned}$$

Therefore, recalling that $\sum_a P_a = P - \sum_a \rho_a \mathbf{w_a w_a}$, the gradient term in Equation (26) is

$$
\begin{aligned}
&\sum_a ((\frac{p_a}{\gamma - 1} + \frac{1}{2}\rho_a u_a^2)\mathbf{u_a} + P_a \cdot \mathbf{u_a} + \mathbf{q_a}) \\
&= \frac{1}{\gamma - 1}\sum_a p_a \mathbf{u_a} + \frac{1}{2}\sum_a \rho_a u_a^2 \mathbf{u_a} + \sum_a P_a \cdot \mathbf{u_a} + \sum_a \mathbf{q_a} \\
&= \frac{1}{\gamma - 1}(p\mathbf{u} - \mathbf{u}\frac{\gamma - 1}{2}\sum_a \rho_a w_a^2 + \sum_a p_a \mathbf{w_a}) \\
&+ \frac{1}{2}(\rho u^2 \mathbf{u} + \mathbf{u}\sum_a \rho_a w_a^2 + 2\mathbf{u} \cdot \sum_a \rho_a \mathbf{w_a w_a} + \sum_a \rho_a w_a^2 \mathbf{w_a}) \\
&+ \mathbf{u} \cdot (P - \sum_a \rho_a \mathbf{w_a w_a}) + \sum_a P_a \cdot \mathbf{w_a} \\
&+ \mathbf{q} - \sum_a \mathbf{w_a} \cdot P_a - \frac{1}{\gamma - 1}\sum_a p_a \mathbf{w_a} - \frac{1}{2}\sum_a \rho_a w_a^2 \mathbf{w_a} \\
&= \frac{p\mathbf{u}}{\gamma - 1} + \frac{1}{2}\rho u^2 \mathbf{u} + \mathbf{u} \cdot P + \mathbf{q} = \epsilon \mathbf{u} + P \cdot \mathbf{u} + \mathbf{q}
\end{aligned}
$$

Next,

$$\sum_a n_a q_a \mathbf{u_a} \cdot \mathbf{E} + \sum_a \rho_a \mathbf{u_a} \cdot \mathbf{g} = \mathbf{J} \cdot \mathbf{E} + \rho \mathbf{u} \cdot \mathbf{g}$$

Therefore the combined energy equation looks like

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot (\epsilon \mathbf{u} + P \cdot \mathbf{u} + \mathbf{q}) - \mathbf{J} \cdot \mathbf{E} - \rho \mathbf{u} \cdot \mathbf{g} = 0 \tag{29}$$

2.4 Maxwell's Equations

Maxwell's equations relate **E**, **J**, and **B** in the previous plasma equations

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0} \tag{30a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{30b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{30c}$$

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}) \tag{30d}$$

where μ_0 is permeability, ϵ_0 is the permittivity, **E** is the electric field, **B** is the magnetic field, **J** is the current, and ρ_q is the charge density.

Take the divergence of Equation (30d), giving

$$\nabla \cdot \nabla \times \mathbf{B} = \mu_0 \nabla \cdot \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \mathbf{E})$$

Using Equation (30a), and that the divergence of a curl is 0, results in the equation of current conservation

$$\frac{\partial \rho_q}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{31}$$

Take the momentum equation, multiply by $\frac{q_a}{m_a}$ and sum over species to get

$$\frac{\partial}{\partial t}\sum_a n_a q_a \mathbf{u_a} + \nabla \cdot (\sum_a n_a q_a \mathbf{u_a}\mathbf{u_a}) + \nabla \cdot (\sum_a \frac{q_a}{m_a} P_a) - \sum_a n_a \frac{q_a}{m_a} < \mathbf{F} >= \sum_a \frac{q_a}{m_a}\mathbf{A_a}$$

Note that

$$\mathbf{J} = \sum_a n_a q_a \mathbf{u_a} = \sum_a n_a q_a \mathbf{u} + \sum_a n_a q_a \mathbf{w_a} = \rho_q \mathbf{u} + \sum_a n_a q_a \mathbf{w_a}$$

This divides the current density $\mathbf{J}$ into a convection current density moving with $\mathbf{u}$, and a conduction current density in the frame moving with the plasma. Sometimes this latter quantity is defined as $\mathbf{J'} = \sum_a n_a q_a \mathbf{w_a}$

Then the sum in the second term can be written out as

$$\sum_a n_a q_a \mathbf{u_a}\mathbf{u_a} = \sum_a n_a q_a \mathbf{u_a}\mathbf{u} + \sum_a n_a q_a \mathbf{u}\mathbf{w_a} + \sum_a n_a q_a \mathbf{w_a}\mathbf{w_a}$$
$$= \mathbf{J}\mathbf{u} + \mathbf{u}(\mathbf{J} - \rho_q \mathbf{u}) + \sum_a n_a q_a \mathbf{w_a}\mathbf{w_a} = \mathbf{J}\mathbf{u} + \mathbf{u}\mathbf{J} - \rho_{\mathbf{q}}\mathbf{u}\mathbf{u} + \sum_a n_a q_a \mathbf{w_a}\mathbf{w_a}$$

Similarly define and electric pressure

$$P_{qa} = \frac{q_a}{m_a} P_a = n_a q_a < \mathbf{c_a}\mathbf{c_a} > \tag{32}$$

and like the total pressure

$$P_q = \sum_a P_{qa} + \sum_a n_a q_a \mathbf{w_a}\mathbf{w_a}$$

Putting this together gives

$$\frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot (\mathbf{J}\mathbf{u} + \mathbf{u}\mathbf{J} - \rho_{\mathbf{q}}\mathbf{u}\mathbf{u} + \mathbf{P_q}) - \sum_a n_a \frac{q_a}{m_a} < \mathbf{F} >= \sum_a \frac{q_a}{m_a}\mathbf{A_a} \tag{33}$$

2.5 Summary

The total plasma equations end up similar to the species equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{34a}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) + \nabla \cdot P - \rho_q \mathbf{E} - \mathbf{J} \times \mathbf{B} - \rho \mathbf{g} = 0 \tag{34b}$$

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot (\epsilon \mathbf{u} + P \cdot \mathbf{u} + \mathbf{q}) - \mathbf{J} \cdot \mathbf{E} - \rho \mathbf{u} \cdot \mathbf{g} = 0 \tag{34c}$$

$$\frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot (\mathbf{J}\mathbf{u} + \mathbf{u}\mathbf{J} - \rho_{\mathbf{q}} \mathbf{u}\mathbf{u} + \mathbf{P_q}) - \sum_a n_a \frac{q_a}{m_a} < \mathbf{F} >= \sum_a \frac{q_a}{m_a} \mathbf{A_a} \tag{34d}$$

$$\epsilon = \frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2 \tag{34e}$$

3 Simplifying Assumptions

3.1 Time derivative of E small

For sufficiently large time increments, the time derivative in Equation (30d) is small. To estimate a sufficiently large time τ, take the ratio of the two terms on the right hand side

$$\frac{\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}}{\mathbf{J}} \approx \frac{\frac{\epsilon_0 \mathbf{E}}{\tau}}{\sigma \mathbf{E}} \approx \frac{\epsilon_0}{\sigma \tau} \approx \frac{10^{-11}}{\tau}$$

This means for time scales much greater than 10^{-11} seconds, the time derivative of $\mathbf{E}$ can be neglected. As a consequence Equation (30d) becomes

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{35}$$

and $\mathbf{J}$ reduces to a simple function of $\mathbf{B}$. The term $\mathbf{J} \times \mathbf{B}$ in the Equation (34b) can be expanded in the following way (written out for clarity)

$$\begin{aligned}
&\mu_0 \mathbf{J} \times \mathbf{B} = (\nabla \times \mathbf{B}) \times \mathbf{B} \\
&= ((\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z})\mathbf{i} + (\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x})\mathbf{j} + (\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y})\mathbf{k}) \times (B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}) \\
&= (B_z \frac{\partial B_x}{\partial z} - B_z \frac{\partial B_z}{\partial x} - B_y \frac{\partial B_y}{\partial x} + B_y \frac{\partial B_x}{\partial y})\mathbf{i} + (B_x \frac{\partial B_x}{\partial x} - B_x \frac{\partial B_x}{\partial x})\mathbf{i} \\
&+ (B_x \frac{\partial B_y}{\partial x} - B_x \frac{\partial B_x}{\partial y} - B_z \frac{\partial B_z}{\partial y} + B_z \frac{\partial B_y}{\partial z})\mathbf{j} + (B_y \frac{\partial B_y}{\partial y} - B_y \frac{\partial B_y}{\partial y})\mathbf{j} \\
&+ (B_y \frac{\partial B_z}{\partial y} - B_y \frac{\partial B_y}{\partial z} - B_x \frac{\partial B_x}{\partial z} + B_x \frac{\partial B_z}{\partial x})\mathbf{k} + (B_z \frac{\partial B_z}{\partial z} - B_z \frac{\partial B_z}{\partial z})\mathbf{k}
\end{aligned}$$

$$= (B_x \frac{\partial B_x}{\partial x} + B_y \frac{\partial B_x}{\partial y} + B_z \frac{\partial B_x}{\partial z} - B_x \frac{\partial B_x}{\partial x} - B_y \frac{\partial B_y}{\partial x} - B_z \frac{\partial B_z}{\partial x})\mathbf{i}$$
$$+ (B_x \frac{\partial B_y}{\partial x} + B_y \frac{\partial B_y}{\partial y} + B_z \frac{\partial B_y}{\partial z} - B_x \frac{\partial B_x}{\partial y} - B_y \frac{\partial B_y}{\partial y} - B_z \frac{\partial B_z}{\partial y})\mathbf{j}$$
$$+ (B_x \frac{\partial B_z}{\partial x} + B_y \frac{\partial B_z}{\partial y} + B_z \frac{\partial B_z}{\partial z} - B_x \frac{\partial B_x}{\partial z} - B_y \frac{\partial B_y}{\partial z} - B_z \frac{\partial B_z}{\partial z})\mathbf{k}$$
$$= ((\mathbf{B} \cdot \nabla)B_x - \frac{\partial}{\partial x}\frac{B^2}{2})\mathbf{i} + ((\mathbf{B} \cdot \nabla)B_y - \frac{\partial}{\partial y}\frac{B^2}{2})\mathbf{j} + ((\mathbf{B} \cdot \nabla)B_z - \frac{\partial}{\partial z}\frac{B^2}{2})\mathbf{k}$$
$$= (\mathbf{B} \cdot \nabla)(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) - (\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k})\frac{B^2}{2}$$
$$= (\mathbf{B} \cdot \nabla)\mathbf{B} - \nabla\frac{B^2}{2}$$

In order to write this as a divergence, note that because $\nabla \cdot \mathbf{B} = 0$,

$$\nabla \cdot (\mathbf{BB}) = (\mathbf{B} \cdot \nabla)\mathbf{B} + \mathbf{B}(\nabla \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{B}$$

Also

$$\nabla\frac{B^2}{2} = \nabla \cdot (\frac{B^2}{2}\mathbf{I})$$

Therefore Equation (25) can be written as

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{uu} + P + \frac{B^2}{2\mu_0}\mathbf{I} - \frac{1}{\mu_0}\mathbf{BB}) - \rho_q \mathbf{E} - \rho \mathbf{g} = 0 \tag{36}$$

3.2 Isotropic pressure

Replace the pressure tensor P with $p\mathbf{I}$, then $\nabla \cdot P = \nabla p$.

3.3 Charge neutrality

If the net charge everywhere balances, then $\rho_q = 0$.

3.4 Neglect small terms

Because $m_e << m_i$ for any ion, if the pressures of each species are about the same,

$$P_q = \sum_a \frac{q_a}{m_a} P_a = \sum_a \frac{q_a p_a}{m_a}\mathbf{I} = \sum_{a,ions} \frac{q_a p_a}{m_a} + \frac{e p_e}{m_e} \approx \frac{e p_e}{m_e}$$

If p_e itself is small, then terms involving P_q can be neglected.

3.5 Single ion flow with collision term approximation

In order to simplify the differential equation for the magnetic field down to something manageable, Equation (33) is applied to a plasma consisting of electrons and one ion type, allowing the following quantities to be written out

$$\mathbf{u} = \frac{\rho_e \mathbf{u_e} + \rho_i \mathbf{u_i}}{\rho_e + \rho_i}$$
$$\mathbf{J} = \sum_a n_a q_a \mathbf{u_a} = e(n_i \mathbf{u_i} - n_e \mathbf{u_e})$$
$$P_q = e(\frac{P_i}{m_i} - \frac{P_e}{m_e})$$

The force term can be written out as

$$\sum_a n_a \frac{q_a}{m_a} < \mathbf{F} >= e^2(\frac{n_i}{m_i} + \frac{n_e}{m_e})\mathbf{E} + e^2(\frac{n_i}{m_e} + \frac{n_e}{m_i})\mathbf{u} \times \mathbf{B} + e(\frac{1}{m_i} - \frac{1}{m_e})\mathbf{J} \times \mathbf{B}$$

Also the collision terms can be expressed as a linear approximation

$$\mathbf{A}_e = -\rho_e \nu_{ei}(\mathbf{u_e} - \mathbf{u_i})$$
$$\mathbf{A}_i = -\rho_i \nu_{ie}(\mathbf{u_i} - \mathbf{u_e})$$
$$\sum_a \mathbf{A}_i = (\rho_i \nu_{ie} - \rho_e \nu_{ei})(\mathbf{u_e} - \mathbf{u_i}) = 0$$
$$\rho_i \nu_{ie} = \rho_e \nu_{ei}$$

$\rho_q = 0$ implies that $n_e = n_i =$ n. Applying this and $m_e << m_i$,

$$\sum_a \frac{q_a}{m_a} \mathbf{A_a} = e\rho_e \nu_{ei}(\mathbf{u_e} - \mathbf{u_i})(\frac{1}{m_i} + \frac{1}{m_e}) \approx -\nu_{ei}\mathbf{J}$$

With these assumptions and approximations, and the definition

$$\sigma = \frac{ne^2}{m_e \nu_{ei}} \tag{37}$$

Equation (33) can be written as the generalized ohm's law

$$\frac{1}{\nu_{ei}}\frac{\partial \mathbf{J}}{\partial t} + \frac{1}{\nu_{ei}}\nabla \cdot (\mathbf{Ju} + \mathbf{uJ} - \rho_\mathbf{q}\mathbf{uu}) - \frac{\sigma}{ne}\nabla \cdot P_e = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \mathbf{J} - \frac{\sigma}{ne}\mathbf{J} \times \mathbf{B} \tag{38}$$

The left hand terms are typically neglected. If the last term, the Hall effect term, can also be neglected, this gives

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{39}$$

Using Equation (35) and taking the curl gives

$$\nabla \times \nabla \times \mathbf{B} = \mu_0 \sigma(\nabla \times \mathbf{E} + \nabla \times (\mathbf{u} \times \mathbf{B})) \tag{40}$$

Using Equation (30c) gives

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{\mu_0 \sigma} \nabla \times \nabla \times \mathbf{B} + \nabla \times (\mathbf{u} \times \mathbf{B})$$

Applying a vector identity and using $\nabla \cdot \mathbf{B} = 0$ gives

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu_0 \sigma} (\nabla^2 \mathbf{B}) + \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{41}$$

Note this is similar to the vorticity equation.

$$\frac{\partial \omega}{\partial t} = \nu (\nabla^2 \omega) + \nabla \times (\mathbf{u} \times \omega)$$

Another vector identity transforms Equation (41) into a replacement for Equation (33)

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu_0 \sigma} (\nabla^2 \mathbf{B}) + \nabla \cdot (\mathbf{B}\mathbf{u} - \mathbf{u}\mathbf{B})$$

or

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{B} - \mathbf{B}\mathbf{u}) = \frac{1}{\mu_0 \sigma} (\nabla^2 \mathbf{B}) \tag{42}$$

3.6 Perfect conductivity

If σ is extremely large, then the right hand side of Equation (42) can be neglected. Furthermore,

$$\begin{aligned} \mathbf{E} + \mathbf{u} \times \mathbf{B} &= \frac{\mathbf{J}}{\sigma} \approx 0 \\ \mathbf{E} &= -\mathbf{u} \times \mathbf{B} \end{aligned}$$

As a result, the $\mathbf{J} \cdot \mathbf{E}$ term in the energy equation can be written out using Equations (35) and (30c)

$$\begin{aligned} \mathbf{J} \cdot \mathbf{E} &= \frac{1}{\mu_0} \mathbf{E} \cdot (\nabla \times \mathbf{B}) \\ &= \frac{1}{\mu_0} (\mathbf{B} \cdot (\nabla \times \mathbf{E})) - \nabla \cdot (\mathbf{E} \times \mathbf{B})) \\ &= -\frac{1}{\mu_0} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} + \frac{1}{\mu_0} \nabla \cdot ((\mathbf{u} \times \mathbf{B}) \times \mathbf{B}) \\ &= -\frac{\partial}{\partial t} \frac{B^2}{2\mu_0} + \frac{1}{\mu_0} \nabla \cdot ((\mathbf{u} \cdot \mathbf{B})\mathbf{B} - B^2 \mathbf{u}) \end{aligned}$$

3.7 Summary of Ideal MHD Equations

To collect all the time derivatives in the energy equation, define

$$\epsilon = \frac{\rho u^2}{2} + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0} \tag{43}$$

Then the previous assumptions lead to the following ideal MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{44a}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + (p + \frac{B^2}{2\mu_0})\mathbf{I} - \frac{1}{\mu_0}\mathbf{B}\mathbf{B}) - \rho \mathbf{g} = 0 \tag{44b}$$

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot ((\epsilon + p + \frac{B^2}{2\mu_0})\mathbf{u} + \mathbf{q} - \frac{1}{\mu_0}(\mathbf{u} \cdot \mathbf{B})\mathbf{B}) - \rho \mathbf{u} \cdot \mathbf{g} = 0 \tag{44c}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{B} - \mathbf{B}\mathbf{u}) = 0 \tag{44d}$$

Often $\mathbf{q} = 0$ and $\mathbf{g} = 0$ resulting in a conservative form of the equations. If $\mathbf{g}$ is not zero, then it will generally be rewritten as a potential to maintain a conservative form.

4 Ideal MHD without $\nabla \cdot \mathbf{B} = 0$

The above equations can also be derived without assuming that $\nabla \cdot \mathbf{B} = 0$ everywhere [7]. The first consequence is that the $\mathbf{J} \times \mathbf{B}$ term in Equation (34b) becomes

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0}((\mathbf{B} \cdot \nabla)\mathbf{B} - \nabla \frac{B^2}{2}) = \frac{1}{\mu_0}(\nabla \cdot (\mathbf{B}\mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{B}))$$

The other consequence [8] is that Equation (30c) becomes

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{u}(\nabla \cdot \mathbf{B}) \tag{45}$$

The additional term expresses the change in $\mathbf{B}$ as a result of fluid flow. With this equation, then the $\mathbf{J} \cdot \mathbf{E}$ term of the energy equation becomes

$$\begin{aligned}
\mathbf{J} \cdot \mathbf{E} &= \frac{1}{\mu_0}\mathbf{E} \cdot (\nabla \times \mathbf{B}) \\
&= \frac{1}{\mu_0}(\mathbf{B} \cdot (\nabla \times \mathbf{E})) - \nabla \cdot (\mathbf{E} \times \mathbf{B})) \\
&= -\frac{1}{\mu_0}\mathbf{B} \cdot (\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}(\nabla \cdot \mathbf{B})) + \frac{1}{\mu_0}\nabla \cdot ((\mathbf{u} \times \mathbf{B}) \times \mathbf{B}) \\
&= -\frac{\partial}{\partial t}\frac{B^2}{2\mu_0} + \frac{1}{\mu_0}\nabla \cdot ((\mathbf{u} \cdot \mathbf{B})\mathbf{B} - B^2\mathbf{u}) - \frac{1}{\mu_0}\mathbf{B} \cdot \mathbf{u}(\nabla \cdot \mathbf{B})
\end{aligned}$$

Finally, Equation (40) with the modified Equation (45) looks like

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{\mu_0 \sigma}\nabla \times \nabla \times \mathbf{B} + \nabla \times (\mathbf{u} \times \mathbf{B}) - \mathbf{u}(\nabla \cdot \mathbf{B})$$

Note that the divergence of the above equation is

$$\frac{\partial(\nabla \cdot \mathbf{B})}{\partial t} + \nabla \cdot (\mathbf{u}(\nabla \cdot \mathbf{B})) = 0$$

This equation means that the quantity $(\nabla \cdot \mathbf{B})/\rho$ is carried by the flow along streamlines as a passive scalar. As long as $\nabla \cdot \mathbf{B} = 0$ as initial and boundary conditions, it will remain so on the differential equation level. The resulting equations result in the following weakly non-conservative differential equations for ideal MHD, the discrete solution of which is taken up in the following chapter.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{46a}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + (p + \frac{B^2}{2\mu_0})\mathbf{I} - \frac{1}{\mu_0}\mathbf{B}\mathbf{B}) = -\frac{1}{\mu_0}(\nabla \cdot \mathbf{B})\mathbf{B} \tag{46b}$$

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot ((\epsilon + p + \frac{B^2}{2\mu_0})\mathbf{u} - \frac{1}{\mu_0}(\mathbf{u} \cdot \mathbf{B})\mathbf{B}) = -\frac{1}{\mu_0}(\nabla \cdot \mathbf{B})(\mathbf{u} \cdot \mathbf{B}) \tag{46c}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{B} - \mathbf{B}\mathbf{u}) = -(\nabla \cdot \mathbf{B})\mathbf{u} \tag{46d}$$

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