Charles University in Prague
Faculty of Mathematics and Physics

MASTER THESIS

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Adaptive hp Discontinuous Galerkin Method for Nonstationary Compressible Euler Equations

Department of Numerical Mathematics

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Study branch: Numerical and Computational Mathematics

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I confirm having prepared the master thesis by my own, and having listed all used sources of information in the bibliography. I agree with lending the master thesis.

In Prague Lukáš Korous

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Název práce: Adaptivní hp nespojitá Galerkinova metoda pro nestacionární stlacitelné Eulerovy rovnice
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Abstrakt: Stlačitelné Eulerovy rovnice popisují pohyb stlačitelných nevazkých tekutin jako jsou plyny nebo vzduch. Používají se v mnoha oblastech leteckého, automobilového a jaderného inženýrství, chemie, ekologie, klimatologie, a v dalších. Matematicky, stlačitelné Eulerovy rovnice jsou hyperbolický systém skládající se z několika nelineárních parciálních diferenciálních rovnic (zákony zachování). Tyto rovnice jsou řešeny nejčasteji pomocí metody konečných objemů (MKO), a metody konečných prvků (FEM) nízkého řádu. Nicméně, oba tyto přístupy nedosahují vyššího řádu přesnosti, a navíc je dobře známo, že spojitá metoda konečných prvku není optimální nástroj pro diskretizaci rovnic prvního řádu. Nejnadějnější přístup k přibližnému řešení stlačitelných Eulerových rovnic je Nespojitá Galerkinova metoda, která kombinuje stabilitu MKO s vynikajícími aproximačními vlastnostmi FEM vyššího řádu. Cílem této diplomové práce byl vývoj, implementace a testování nových algoritmů pro adaptivní řešení nestacionárních stlačitelných Eulerovových rovnic na základě vyššího řádu nespojité Galerkinovy metody (hp-DG). Základem pro nové metody byli Nespojitá Galerkinova metoda a časoprostorové hp-FEM algoritmy na dynamických sítích pro nestacionární problémy druhého řádu. Nové algoritmy byly implementovány a testovány v rámci open source knihovny Hermes, a vztahují se na vybrané problémy transonického proudění, a toky s nízkým Machovým číslem v klimatologii.

Klíčová slova: numerické simulace, metoda konečných prvku, Eulerovy rovnice, hp-adaptivita, Nespojitá Galerkinova metoda

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Abstract: The compressible Euler equations describe the motion of compressible inviscid fluids such as gases or the air. They are used in many areas ranging from aerospace, automotive, and nuclear engineering to chemistry, ecology, climatology, and others. Mathematically, the compressible Euler equations are a hyperbolic

system consisting of several nonlinear partial differential equations (conservation laws). These equations are solved most frequently by means of Finite Volume Methods (FVM) and low-order Finite Element Methods (FEM). However, both these approaches are lacking higher order accuracy and moreover, it is well known that continuous FEM is not the optimal tool for the discretization of first-order equations. The most promissing approach to the approximate solution of compressible Euler equations is the discontinuous Galerkin method that combines the stability of FVM, with excellent approximation properties of higher-order FEM. The objective of this Master's Thesis was to develop, implement and test new adaptive algorithms for nonstationary compressible Euler equations based on higher-order discontinuous Galerkin (hp-DG) methods. The basis for the new methods will be the discontinuous Galerkin methods and space-time adaptive hp-FEM algorithms on dynamical meshes for nonstationary second-order problems. The new algorithms will be implemented and tested in the framework of the open source library Hermes, and they will be applied to selected problems of transonic flow and low Mach number flows in climatology.

Keywords: numerical simulation, finite element method, Euler equations, hp-adaptivity, Discontinuous Galerkin method

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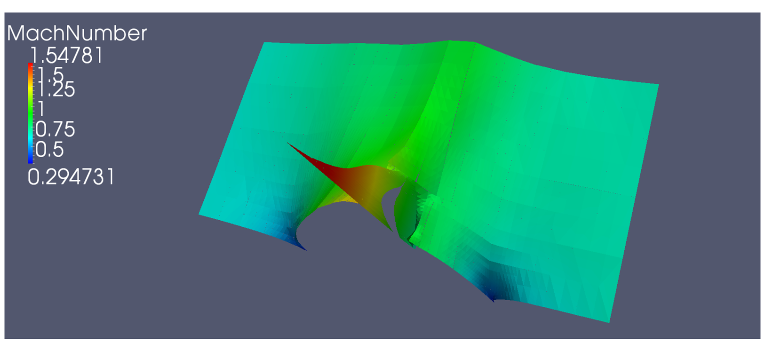

6.4.2 Approaches

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Numerical experiments

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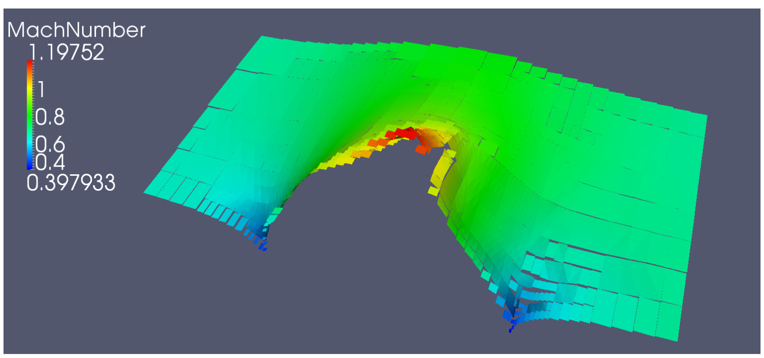
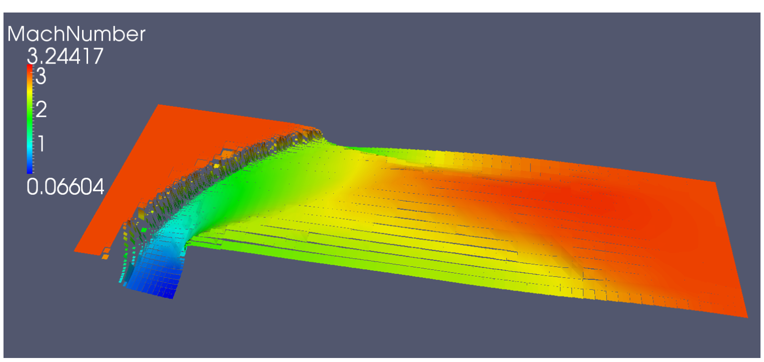


7.1.2 Forward step



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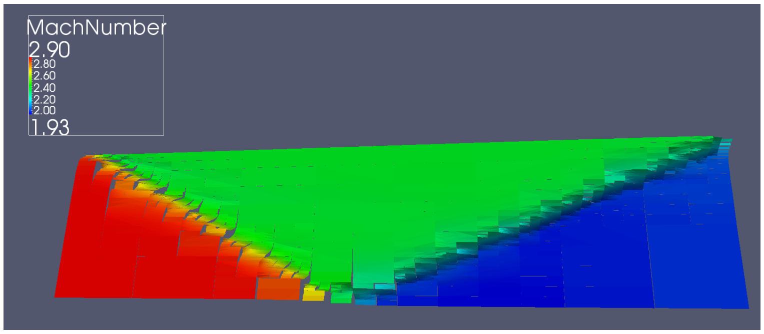
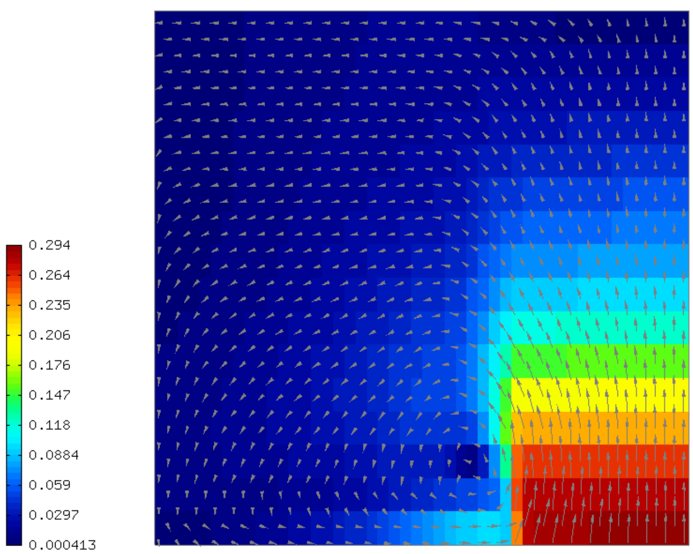


7.2 Low Mach number flows

7.2.1 Heating induced vortex



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