

UNIVERSITY OF WEST BOHEMIA IN PILSEN
FACULTY OF ELECTRICAL ENGINEERING

Doctoral thesis

LARGE-SCALE NUMERICAL SIMULATIONS
OF MAGNETO-HYDRODYNAMICS
PHENOMENA IN ASTROPHYSICS

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TODO

do uvodu napsat vice o state-of-the-art, odkaz na Mirovo clanek, ukazat tam ty FD vysledky, ze je chceme nahradit a ze je tam hodne dulezita adaptivity
pak teda popsat az tam z toho budu davat vysledky, tak vykopirovat nejaky rovnice
IC / BC z clanku, jak se to implementuje, atd.
srovnani

adaptivita - ukazat na 2d prikladu z Hermesu co to je referencni reseni, co to je ||zprojektovane - presne|| (v dealu?), co to je tohle bez normy, zminka o distribuovanosti, ze musim napocitat nejaky thresholdy mapReducem, atd.
napsat pak algoritmus tedy cely, kde je casovy krok, adaptivita (ze se vraci casovy krok po zjemneni / zhrubeni, aby se neztracela informace), atd.

popis a vysledky Blastu, O-T
- bez adaptivity jen "pocita to"

- s adaptivitou "a navic tak presne to spoctu a usetrim 80% dofu"
- tohle ukazovat na rezech hustotou - bylo by fajn nejaký najít

2016

Abstract

The objective of this Doctoral Thesis was to develop, implement and test new algorithms for the large-scale solution of nonstationary compressible MHD equations based on higher-order discontinuous Galerkin (DG) methods. The basis for the new methods will be the discontinuous Galerkin methods and adaptive mesh refinement (AMR) algorithms. The new algorithms will be implemented and tested in the framework of the open source library deal.II, and they will be applied to selected problems of MHD in astrophysics, namely the magnetic reconnection phenomena.

Keywords

numerical simulation, finite element method, MHD equations, adaptivity, discontinuous Galerkin method, astrophysics, solar flares, CME, magnetic reconnection

Abstract [CZ]

Záměrem této práce je navrhnut, implementovat a otestovat nové algoritmy pro rozsáhlé simulace nestacionárních jevů spadajících do oblasti stlačitelné magnetohydrodynamiky. Vytvořený software bude založen na použití nespojité Galerkinovy metody (discontinuous Galerkin, DG) s vyššími řády přesnosti. Zároveň bude použita metoda automatického zjemňování výpočetní triangulace (automatic mesh refinement, AMR). Vytvořené algoritmy budou testovány ve frameworku deal.II a budou aplikovány na skutečné problémy v astrofyzice, jmenovitě na simulaci jevu magnetické rekonexe.

Keywords [CZ]

numerická simulace, metoda konečných prvků, MHD rovnice, adaptivní algoritmy, nespojitá Galerkinova metoda, astrofyzika, sluneční erupce, CME, magnetická rekonexe

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Statement

I am presenting this doctoral thesis, created during my doctoral studies at the Faculty of Electrical Engineering of the University of West Bohemia.

I confirm having prepared this work by my own, and having listed all used sources of information in the bibliography. All license conditions of all works of software and other nature were respected.

In Pilsen, June 18, 2018, Lukáš Korous

Contents

1	Introduction	1
1.1	Magnetohydrodynamics in Astrophysics	1
1.1.1	Magnetic flux tube model of plasma	1
1.1.2	Magnetic reconnection	4
1.2	State of the art	5
1.3	Notation	6
2	Mathematical model	9
2.1	Derivation of the mathematical model	9
2.1.1	Initial setup	9
2.1.2	Euler's equations of compressible flow	10
2.1.3	Maxwell's equations of electromagnetism	11
2.1.4	Derived relations between electromagnetic quantities	11
2.1.5	Simplifying assumptions	12
2.1.6	Adding the induction equation	13
2.1.7	Conservative form of the MHD equations	13
2.1.8	Solution considerations	14
2.2	Weak formulation of the problem	15
2.3	Boundary conditions	16
2.3.1	Essential (inflow) boundary conditions	16
2.3.2	Outflow (do-nothing) boundary conditions	16
2.3.3	Periodic boundary conditions	16
3	Numerical approach	17
3.1	Triangulation	17
3.1.1	Distributed triangulation	19
3.2	Discontinuous Galerkin method	19
3.2.1	Overview of the DG method	19
3.2.2	DG formulation of MHD equations	20
3.2.3	Numerical flux	22
3.2.4	Numerical handling of boundary conditions	30
3.3	Slope limiting	31
3.3.1	Vertex-based limiter	33
3.4	Divergence-free FE space	38
3.5	Discretization in time	40
3.5.1	Discrete problem	40
3.5.2	Time step length	40

Contents

3.6	Algebraic formulation	41
3.7	Numerical integration	42
3.8	Assembling the algebraic problem	45
3.9	Time-stepping and linearization	47
3.9.1	Time-stepping	47
3.10	Performance considerations	48
3.10.1	Parallelization	48
3.10.2	Vectorization	48
3.10.3	Distribution	49
4	Adaptive Mesh Refinement	50
4.1	Overview of the AMR	50
4.2	Adaptive-mesh refinement and DG	52
4.2.1	Relationship with slope limiters	53
4.3	Reference solution approach	53
5	Results	54
5.1	Benchmarks	54
5.1.1	MHD Blast	54
5.1.2	Orszag-Tang vortex	59
5.2	Flux tube eruption model	61
6	Conclusion, outlook	63
	Bibliography	63

1 Introduction

The term magnetohydrodynamics (MHD) covers all physical phenomena that involve both electromagnetic (EM) field and a fluid that carries the EM field. Such phenomena are very interesting, yet very complex to study. The behavior of such a fluid is utilized in some industrial applications - liquid-metal cooling of nuclear reactors, magnetic fluid in dampers, sensors for precise measuring of angular velocities, etc. Such phenomena occur in nature as well - the most significant of which are definitely the processes that take place inside and on the surface of stars - which is the topic of the next section.

1.1 Magnetohydrodynamics in Astrophysics

There are several phenomena in the universe that we can look at as magnetohydrodynamic in nature - planets consisting of metals, interplanetary space, but mainly - stars. If we talk about the nearest star - and the only one we are able to study well enough - the Sun - these phenomena include those that occur in the Sun's photosphere (the layer of Sun that is visible): Sun spots, but also phenomena that occur above the Sun (further from the center of the Sun): in Sun's chromosphere, or even corona (solar flares) - even phenomena that originate from the Sun, but then spread through our solar system - solar winds, space weather. All these phenomena have a large impact on the lives of all of us. For example solar flares (that are often followed by ejection of mass out of the Sun - the so called coronal mass ejections - CMEs) have impact on the Earth's magnetic field which in turn has impact on the electronic communication down on Earth (because the communication satellites used for transmissions may be damaged by the disturbances in the magnetic field). Also people operating at high altitudes, both in airplanes and manned space missions are exposed to the energetic particles coming from the Sun (this term is sometimes called *cosmic rays*). For all the above reasons, it is of great importance to understand the phenomena of space weather, and other MHD phenomena that occur in space.

1.1.1 Magnetic flux tube model of plasma

We are interested in the process of evolution of the flux tube eruption. Such an eruption was captured during observations by [13], and the captured image is shown in Figure 1.1 taken from [13] (Figure 2 in the article).

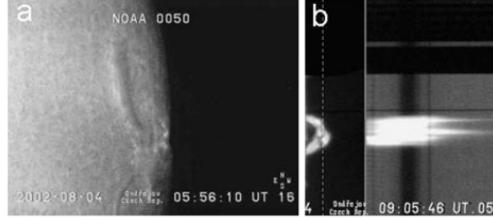


Figure 1.1: Observation of a limb event. $H\alpha$ slit-jaw is the middle (b) part, taken from [13].

For the modeling purposes, we are interested in the so-called $H\alpha$ slit-jaw which is the side-view of the magnetic flux tube during its evolution.

In order to model this phenomena physically, and geometrically, we utilize the magnetic field model by Titov and Demoulin ([27]) which describes a twisted flux tube as part of a torus with minor radius a (being the radius of the tube, not shown in Figure 1.2) and major radius R submerged below the photosphere of the Sun by $d < R$, oriented as in Figure 1.2 (taken from [27]) with total current I .

The magnetic configuration is kept in a global equilibrium by the action of the Lorentz force due to the overlying magnetic field. The sources of this ambient field are modeled by a sub-photospheric line current I_0 and a pair of magnetic charges $+q, -q$ located at distance L from the center of the torus, all located at the major axis of the torus.

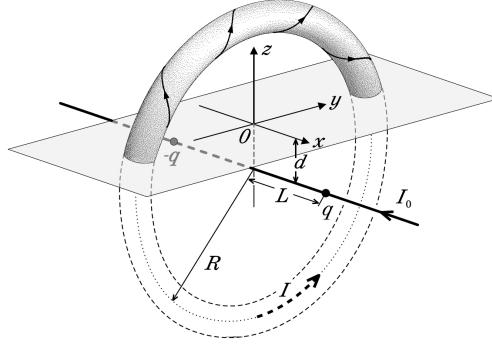


Figure 1.2: The magnetic field under study, taken from [27].

This geometrical/physical model needs to be properly modeled mathematically (including proper boundary conditions), then approached numerically, and finally

computed using software that - in order to achieve reasonable accuracy of the solution - needs to be precisely implemented, must utilize approaches common in the high performance computing (HPC) field, and must be heavily optimized.

In the article [13] which is a reference paper for this work, the model was set up according to observations, and numerical approach of Finite Difference Method (FDM) was used. The results obtained there are presented in Figure 1.3, Figure 1.4, and in order to compare them with Figure 1.1, also in the form in Figure 1.5.

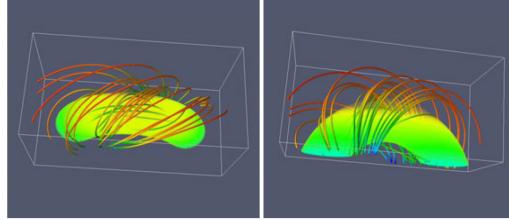


Figure 1.3: Results from [13], initial state, density volume and magnetic field isolines - top & side view.

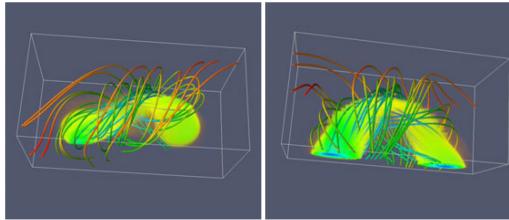


Figure 1.4: Results from [13], state at $t = 14$, density volume and magnetic field isolines - top & side view.

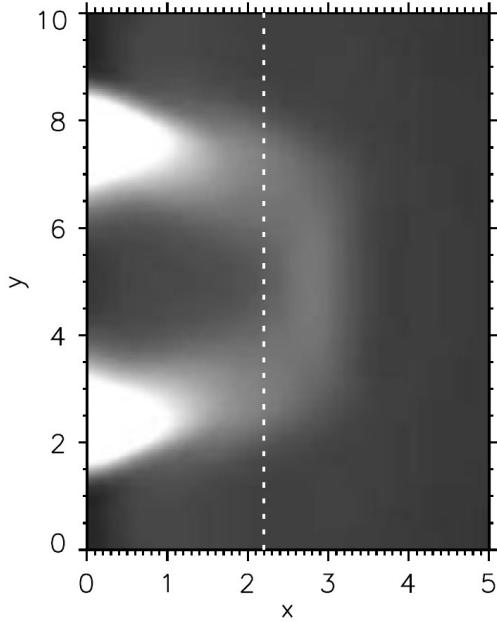


Figure 1.5: Results from [13], time step chosen for best fit with Figure 1.1.

The aim of this work is to be able to compute results of such problems with much higher resolution, and more importantly to implement a generic solver which can handle any similar problem with arbitrary geometric and physical parameters.

1.1.2 Magnetic reconnection

An additional topic, interesting from the astrophysical point of view and related to magnetohydrodynamics, is the magnetic reconnection.

Magnetic reconnection occurs within electrically charged gases called plasmas. These charged particles interact strongly with the magnetic field, but at the same time their motions modify the magnetic field. Under normal conditions, the magnetic field lines inside plasmas don't break or merge with other field lines. But sometimes, as field lines get close to each other, the entire pattern changes and everything realign into a new configuration. The amount of energy released can be formidable. Magnetic reconnection taps into the stored energy of the magnetic field, converting it into heat and kinetic energy that sends particles streaming out along the field lines. Solar flares, which are among the phenomena which are the most important to study, are driven by magnetic reconnection - and thus studying magnetic reconnection is of great importance.

From the numerical perspective, to investigate such phenomena is to bring the complexity of multiple scales present in the physical world - magnetic reconnection occurs at substantially different scales than e.g. the solar flares. To handle this numerically with adequate resolution, and reasonable computational costs, the implementation must be able to limit the number of degrees of freedom used for the discretization to such that yield the largest accuracy increase - namely, in this work, this is achieved via a state-of-the-art Adaptive Mesh Refinement approach.

1.2 State of the art

Only recently, the scientific computation community, due to the advances in computer and supercomputer capabilities, has started with non-trivial numerical simulations of such complex physical phenomena that the MHD model describes. Since both for industrial applications, and obviously for astrophysical application of the MHD model, it is quite expensive (or downright impossible) to perform any experiments, the benefit of being able to simulate the phenomena on a computer is very large.

There exist several available numerical simulation codes, such as [25], [19], [11], [23]. These codes have been successfully applied to a range of problems in astrophysics. There are many numerical methods implemented in these codes, such as the finite difference method ([23]), finite volume method ([11]), and the (continuous) finite element method ([22]).

There have been some attempts to employ also the discontinuous Galerkin method ([21], [18]), but so far no open-source generic software employing this method is available. Moreover, as the astrophysical interest lies in multi-scale problem, a software that can handle such problems would be much more beneficial. An approach that can achieve this capability of solving multi-scale problems is the Adaptive Mesh Refinement technique (AMR). What we understand under this term is not only a mesh refinement that is local (i.e. non-uniform), but a mesh refinement that does not originate in the problem description, and neither is invoked programmatically with user input. The term 'adaptivity' means that through a predefined *refinement indicator*, which is a function operating on the set of elements of the triangulation, elements to be refined are chosen automatically. This *refinement indicator* is calculated from the solution, and in effect makes the triangulation 'adapt' to the solution - hence, AMR.

The reason for the development of a new code is two-fold. First, there is a unique collaboration between the Astronomical Institute of the Czech Academy of Sciences and the University of West Bohemia, where astrophysicists work together with electrical engineers (from theoretical and numerical modeling backgrounds), and the developed code will be usable for both simulating of astrophysical MHD phenomena, and industrial MHD applications.

Notation

Second, the newly developed code is based on locally-adaptive Discontinuous Galerkin method, which yields several advantages over the existing codes (which use e.g. finite difference, or finite volumes methods) developed at institutions of such high quality as *Princeton* - [25], [19]. The advantages are especially of performance, and automation nature - method of higher order together with AMR yields results qualitatively and quantitatively comparable to low order uniform mesh methods, but with computational cost that can easily be an order of magnitude smaller. Automation is mentioned here related to the AMR, which, without user interaction, can optimize the computational triangulation for a particular time instance in the evolution of the modeled phenomena.

Another benefit (namely over [19]) of the newly created software are the use of modern object-oriented programming techniques and experience gained on creating finite element software ([24], [16], [12]). The implementation related to this work is written in the C++ language, with the use of existing software packages that are proven, and used by a wide community of researchers all over the world - deal.II ([2]), Trilinos ([?]), P4EST ([?]), Intel Parallel Studio ([?]), UMFPACK ([5]), Paraview([?]), and others.

The state of the art of numerical simulation of magnetohydrodynamics can be summarized as a state when the mathematical theory of the equations is quite solid, but the methods to solve the equations numerically in the most optimal and fast way are still being improved. The numerical solution is not merely about theoretical convergence rates and attributes of the particular method, but also the actual implementation plays an important role - i.e. programming, hardware, and software, and execution, both before, during, and very importantly after the actual method invocation (of the so-called *postprocessing* of results). In all aspects of implementation, there is space for new approaches, new ideas, new milestones, that can expand the capabilities of today's numerical solution of magnetohydrodynamics phenomena.

1.3 Notation

Overview of notation used in the text is given in Table 1.1.

Notation

Symbol	Meaning
a	a scalar quantity "a"
\mathbf{a}	a 3-element vector "a"
a_i	the i-th component of a vector \mathbf{a}
\mathbf{A}	an 8-element vector "A"
ρ	Density
$\boldsymbol{\pi}$	Momentum
μ	Permeability
μ_0	Permeability of vacuum
t	Time variable
$\mathbf{x} = (x, y, z)$	Space variable
p	Pressure
\mathbf{u}	Velocity
u	Magnitude of velocity, i.e. $ \mathbf{u} $
\mathbf{B}	Magnetic flux density
\mathbf{J}	Current density
B	Magnitude of magnetic flux density, i.e. $ \mathbf{B} $
\mathbf{E}	Electric field
e	Internal energy density
c	Speed of light
\mathbf{g}	Gravitational acceleration
σ	Conductivity, $\sigma = \frac{1}{\eta}$
η	Electrical resistivity, $\eta = \frac{1}{\sigma}$
c_v	Specific heat at constant volume
c_p	Specific heat at constant pressure
γ	Poisson adiabatic constant
θ	Absolute temperature
\mathbf{q}	Heat flux
q	Electric charge
\mathbf{f}	Density of force acting on fluid
\mathbf{F}_L	Lorentz force, $\mathbf{F}_L = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$
\mathbf{f}_L	Lorentz force density, $\mathbf{f}_L = \rho_q \mathbf{E} + \mathbf{J} \times \mathbf{B}$
ε	Electrical permittivity
ε_0	Electrical permittivity of vacuum
ρ_q	Charge density
U_k	Kinetic energy
U_m	Magnetic energy
U	Total energy, $U = U_k + U_m + \rho e$
p_T	Total pressure
\tilde{U}	Hydrodynamic energy, $\tilde{U} = U - U_m$
\mathbf{n}	Unit outer normal

Table 1.1: Notation

Notation

Here

$$U_m(\mathbf{x}, t) = \frac{1}{2} \sum_{i=1}^3 \mathbf{B}_i(\mathbf{x}, t) \quad (1.1)$$

is the magnetic energy,

$$U_k(\mathbf{x}, t) = \frac{1}{2} \sum_{i=1}^3 \mathbf{u}_i(\mathbf{x}, t) \quad (1.2)$$

is the kinetic energy, and the relationship between the Total energy U and pressure p reads

$$U(\mathbf{x}, t) = \frac{p(\mathbf{x}, t)}{\gamma - 1} + U_m(\mathbf{x}, t) + U_k(\mathbf{x}, t). \quad (1.3)$$

Moreover, the total pressure p_T is defined as

$$p_T(\mathbf{x}, t) = p(\mathbf{x}, t) + U_m(\mathbf{x}, t). \quad (1.4)$$

Note that in the text, subscripts denoting spatial dimension 1, 2, 3, and x, y, z will be used interchangeably.

2 Mathematical model

In this chapter, the mathematical model will be derived from basic equations governing the studied physical phenomena, its weak formulation will be presented, and the complete mathematical problem which is solved will be described.

2.1 Derivation of the mathematical model

In order to derive the mathematical model, we first need to establish the basics - time-space frame (Section 2.1.1), assumptions (Section 2.1.1), then we will start with the known Euler equations of compressible flow (Section 2.1.2, derivation of which is given e.g. in [?]), and by adding the Maxwell's equations (Section 2.1.3), we will derive the mathematical model for Magnetyhodrodynamics in the complexity needed for handling problems presented in the introduction.

2.1.1 Initial setup

We consider a time interval $(0, T)$ and space domain $\Omega_t \subset \mathbb{R}^3$ occupied by a fluid at time t . By \mathcal{M} we denote the space-time domain in consideration:

$$\mathcal{M} = \{(\mathbf{x}, t) ; \mathbf{x} \in \Omega_t, t \in (0, T)\}. \quad (2.1)$$

Moreover we assume that \mathcal{M} is an open set.

Assumptions

When dealing with MHD phenomena in plasma, the following rules apply

- We assume that the fluid is inviscid.
- We assume that the fluid is compressible.
- We consider only the so-called *perfect gas* or *ideal gas* whose state variables satisfy the following *equation of state*

$$p = R\theta\rho, \quad (2.2)$$

where ρ denotes density, θ denotes the absolute temperature, and R is the *gas constant*, which is defined as

$$R = c_p - c_v. \quad (2.3)$$

In the above, c_p, c_v are specific heats at constant pressure, and at constant volume.

2.1.2 Euler's equations of compressible flow

This system of equation reads

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\boldsymbol{\pi}) = 0 \quad (2.4)$$

$$\frac{\partial \boldsymbol{\pi}}{\partial t} + \nabla \cdot (\boldsymbol{\pi} \otimes \mathbf{u}) = \rho \mathbf{f} - \nabla p, \quad (2.5)$$

$$\frac{\partial \tilde{U}}{\partial t} + \nabla \cdot (\tilde{U} \mathbf{u}) = \rho \mathbf{f} \cdot \mathbf{u} - \nabla \cdot (p \mathbf{u}) + \nabla \cdot \mathbf{q}, \quad (2.6)$$

where $\boldsymbol{\pi}$ is momentum, p pressure, U total energy. Moreover, \mathbf{u} denotes velocity, \mathbf{f} density of the force acting on the fluid, and \mathbf{q} is the heat flux. By \otimes , we denote the *tensor product*:

$$\mathbf{a} \otimes \mathbf{b} = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{pmatrix}.$$

Moreover, the following relations hold:

$$\tilde{U} = \rho e + U_k, \quad (2.7)$$

$$p = (\gamma - 1) (\tilde{U} - U_k), \quad (2.8)$$

$$\theta = (\tilde{U}/\rho - |\mathbf{u}|^2/2)/c_v. \quad (2.9)$$

This system is simply called the *compressible Euler equations* for a heat-conductive perfect gas. The individual equations are called the *continuity equation* (Equation (2.4)), the *Navier-Stokes equations* (Equation (2.5)), and the *energy equation* (Equation (2.6)).

For the force density \mathbf{f} we assume that only the Lorentz force and gravity act upon the fluid:

$$\mathbf{f} = \mathbf{f}_L + \mathbf{g}.$$

2.1.3 Maxwell's equations of electromagnetism

In this work, we use Maxwell's equations with the assumption of constant electrical permittivity, and constant permeability. This system of equation reads

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \quad (2.10)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2.11)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\varepsilon_0} \quad (2.12)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2.13)$$

where \mathbf{B} denotes magnetic flux density, \mathbf{E} denotes electric field, \mathbf{J} denotes current density, ε_0 is permittivity of vacuum, and ρ_q is electric charge density. The individual equations are known as Faraday's law(Equation (2.11)), Ampere's law(Equation (2.10)), and Gauss's laws(Equation (2.12), Equation (2.13)).

2.1.4 Derived relations between electromagnetic quantities

Further relations that are useful when deriving the MHD equations are:

$$\frac{d}{dt} U_m = \frac{1}{\mu_0} \nabla \cdot (\mathbf{B} \times \mathbf{E}) - \mathbf{E} \cdot \mathbf{J}, \quad (2.14)$$

$$\mathbf{E} = -u \times \mathbf{B} + \frac{\eta}{\mu_0} \mathbf{J}, \quad (2.15)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{u}) = -\frac{1}{\mu_0} \nabla \times (\eta \nabla \times \mathbf{B}), \quad (2.16)$$

where μ_0 denotes permeability of vacuum, η is resistivity, and U_m is magnetic energy. The equation Equation (2.15) is the differential form of the *Ohm's law*, the equation Equation (2.16) is the *induction equation*.

Applying now Section 2.1.2 to Equation (2.5), Equation (2.6), we obtain:

$$\frac{\partial \boldsymbol{\pi}}{\partial t} + \nabla \cdot (\boldsymbol{\pi} \otimes \mathbf{u}) = \rho_q \mathbf{E} + \mathbf{J} \times \mathbf{B} + \rho \mathbf{g} - \nabla p, \quad (2.17)$$

$$\frac{\partial \tilde{U}}{\partial t} + \nabla \cdot (\tilde{U} \mathbf{u}) = \rho_q \mathbf{E} \cdot \mathbf{u} + \mathbf{J} \times \mathbf{B} \cdot \mathbf{u} + \rho \mathbf{g} \cdot \mathbf{u} - \nabla \cdot (p \mathbf{u}) + \nabla \cdot \mathbf{q}. \quad (2.18)$$

The adjusted energy equation Equation (2.18) does not include the magnetic energy U_m , which we do want to include in the MHD equations. To achieve this, we employ Equation (2.14) - Equation (2.16) and rearrange. After rearranging we obtain

$$\frac{\partial U}{\partial t} = -\nabla \cdot \left[\boldsymbol{\pi} \left(\frac{u^2}{2} + e \right) + p \mathbf{u} - \frac{1}{\mu_0} \mathbf{B} \times \mathbf{E} \right] + \rho \mathbf{g} \cdot \mathbf{u} + \nabla \cdot \mathbf{q}. \quad (2.19)$$

2.1.5 Simplifying assumptions

In what follows, we will make several simplifying assumptions, according to which we will get a system of equations that will adequately respect the physical model, yet will be easier to be solved.

Negligible time derivative of electric field

For the time increments that we are concerned with, the time derivative in Equation (2.10) (the so-called *Maxwell's displacement current*) is very small. To estimate the minimum time increment value τ which would allow us to neglect the derivative, take the ratio of the two terms on the right hand side of Equation (2.10):

$$\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \approx \frac{\varepsilon_0 \mathbf{E}}{\sigma \tau} \approx \frac{\varepsilon_0}{\sigma \tau} \approx \frac{10^{-11}}{\tau}. \quad (2.20)$$

This means for time scales much greater than 10^{-11} seconds, the time derivative of \mathbf{E} can be neglected. As a consequence Equation Equation (2.10) can be written as:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \quad (2.21)$$

Using Equation (2.21), we can write

$$\mu_0 \mathbf{J} \times \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla \frac{B^2}{2} = \nabla \cdot (\mathbf{B} \mathbf{B}) - \nabla \frac{B^2}{2}, \quad (2.22)$$

where the last equality comes from Equation (2.13). And using Equation (2.22) we can rewrite Equation (2.17) as

$$\frac{\partial \boldsymbol{\pi}}{\partial t} + \nabla \cdot (\boldsymbol{\pi} \otimes \mathbf{u}) = q \mathbf{E} + \nabla \cdot \left(\frac{1}{\mu_0} \mathbf{B} \mathbf{B} - \frac{B^2}{2} \mathbf{I} \right) + \rho \mathbf{g} - \nabla p \quad (2.23)$$

Negligible electric field in the Navier-Stokes equations

The magnitude of electric field is smaller than the magnetic field by the factor $\frac{u^2}{c^2}$, so we can neglect the term $\rho_q \mathbf{E}$ on the right-hand-side of Equation (2.23).

Negligible heat fluxes

Since the heat transfer accounts for a negligible contribution to the overall energy transfer, we neglect the heat flux terms, i.e. we set $\mathbf{q} = \mathbf{0}$ in the energy equation Equation (2.19).

2.1.6 Adding the induction equation

The induction equation Equation (2.16) can be rearranged in the following way:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{u}) = -\frac{1}{\mu_0} \nabla \times (\eta \nabla \times \mathbf{B}) \quad (2.24)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}) + \frac{1}{\mu_0 \sigma} (\nabla^2 \mathbf{B}). \quad (2.25)$$

Now we can form the system of MHD equations:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\boldsymbol{\pi}), \quad (2.26)$$

$$\frac{\partial \boldsymbol{\pi}}{\partial t} = -\nabla \cdot (\boldsymbol{\pi} \otimes \mathbf{u}) + \nabla \cdot \left(\frac{1}{\mu_0} \mathbf{B} \mathbf{B} - \frac{B^2}{2} \mathbf{I} \right) + \rho \mathbf{g} - \nabla p, \quad (2.27)$$

$$\frac{\partial U}{\partial t} = -\nabla \cdot \left[\boldsymbol{\pi} \left(\frac{u^2}{2} + e \right) + p \mathbf{u} - \frac{1}{\mu_0} \mathbf{B} \times \mathbf{E} \right] + \rho \mathbf{g} \cdot \mathbf{u} \quad (2.28)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}) + \frac{1}{\mu_0 \sigma} (\nabla^2 \mathbf{B}). \quad (2.29)$$

This form suggests that rewriting these equations into a more suitable (for numerical calculations) conservative form shall be possible.

2.1.7 Conservative form of the MHD equations

A conservative form of a system of equations takes the form of

$$\frac{\partial \boldsymbol{\Psi}}{\partial t} + \nabla \cdot \mathbf{F}(\boldsymbol{\Psi}) = \mathbf{S}, \quad (2.30)$$

where $\boldsymbol{\Psi}$ is the so-called *state vector*, \mathbf{F}_i , $i = 1, 2, 3$ are the so-called *fluxes*, and \mathbf{S} is the so-called *source term*.

Rewriting the system of equations Equation (2.26) - Equation (2.29) to the form Equation (2.30) is fairly straightforward. We obtain the following:

$$\Psi = \begin{pmatrix} \rho \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ U \\ B_1 \\ B_2 \\ B_3 \end{pmatrix}, \quad (2.31)$$

$$\mathbf{F}_i = \begin{pmatrix} \pi_i \\ \frac{\pi_1\pi_i}{\rho} - B_1B_i + \frac{1}{2}\delta_{1i}(p + U_m) \\ \frac{\pi_2\pi_i}{\rho} - B_1B_i + \frac{1}{2}\delta_{2i}(p + U_m) \\ \frac{\pi_3\pi_i}{\rho} - B_1B_i + \frac{1}{2}\delta_{3i}(p + U_m) \\ \frac{\pi_i}{\rho} \left(\frac{\gamma}{\gamma-1}p + U_k \right) + 2\eta\varepsilon_{ijk}J_jB_k + \frac{2}{\rho}\varepsilon_{ijk}(\pi_kB_i - \pi_iB_k)B_j \\ \frac{\pi_iB_1 - \pi_1B_i}{\rho} + \eta\varepsilon_{1ij}J_j \\ \frac{\pi_iB_2 - \pi_2B_i}{\rho} + \eta\varepsilon_{2ij}J_j \\ \frac{\pi_iB_3 - \pi_3B_i}{\rho} + \eta\varepsilon_{3ij}J_j \end{pmatrix}, \quad (2.32)$$

$$\mathbf{S} = \begin{pmatrix} 0 \\ \rho g_1 \\ \rho g_2 \\ \rho g_3 \\ \boldsymbol{\pi} \cdot \mathbf{g} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (2.33)$$

where $J_j = (\nabla \times \mathbf{B})_j$, ε_{ijk} is the Levi-Civita symbol, and δ_{ij} the Kronecker delta.

2.1.8 Solution considerations

For mathematical clarity, we should state, that the solution to the equations Equation (2.30) is such a function

$$\Psi \in C^1((0, T), [C^1(\Omega_t)]^8); \quad \Psi_i \in C^1((0, T), C^2(\Omega_t)), \quad i = 6, 7, 8, \quad (2.34)$$

for which Equation (2.30) holds for all $\mathbf{x} \in \Omega_t$, $t \in (0, T)$. The kind of spaces we used in the definition is called the Bochner spaces.

Because such a requirement on the solution Ψ is rather strong, we shall instead look for a so-called *weak solution*, which is specified in the coming sections.

2.2 Weak formulation of the problem

The DG method is defined by first considering the weak formulation of the equations Equation (2.30) obtained by multiplying the equations at every time instance t by a (vector-valued) test function $\mathbf{v} \in W_t$, where W_t is a suitable space of (vector-valued) functions $W_t = (W_1, \dots, W_8)$, integrating over the space domain Ω_t , and performing integration by parts. We start with multiplying Equation (2.30) by a test function:

$$\frac{\partial \Psi}{\partial t} \mathbf{v} + (\nabla \cdot \mathbf{F}(\Psi)) \mathbf{v} = \mathbf{S}\mathbf{v}, \quad (2.35)$$

then we integrate over Ω_t :

$$\int_{\Omega_t} \frac{\partial \Psi}{\partial t} \mathbf{v} + \int_{\Omega_t} (\nabla \cdot \mathbf{F}(\Psi)) \mathbf{v} = \int_{\Omega_t} \mathbf{S}\mathbf{v}, \quad (2.36)$$

and finally we integrate by parts:

$$\int_{\Omega_t} \frac{\partial \Psi}{\partial t} \mathbf{v} - \int_{\Omega_t} \mathbf{F}(\Psi) (\nabla \cdot \mathbf{v}) + \int_{\partial \Omega_t} (\mathbf{F}(\Psi) \cdot \mathbf{n}) \mathbf{v} = \int_{\Omega_t} \mathbf{S}\mathbf{v}, \quad (2.37)$$

where the terms $\mathbf{F}(\Psi) (\nabla \cdot \mathbf{v}) ; (\mathbf{F}(\Psi) \cdot \mathbf{n}) \mathbf{v}$ are meant as a component-wise multiplication.

Now without going into detail, we can conclude, that a suitable space W would be

$$W_t = [H^1(\Omega_t)]^8, \quad (2.38)$$

and we shall look for a solution to the Equation (2.37) in the same space W_t at every time instance t . We shall not relax the requirement for continuity of time-derivatives we imposed for the *hard solution* Equation (2.34) for reasons discussed later, and we arrive at the following Bochner space in which we are looking for a weak solution of Equation (2.30):

$$W = C^1((0, T), W_t). \quad (2.39)$$

To sum up we can define the *weak solution* $\Psi = \Psi((t, \mathbf{x}))$ of MHD equations Equation (2.30) as

$\Psi((t, \mathbf{x})) \in W$ defined in Equation (2.39).
 Equation (2.37) holds for all $t \in (0, T)$, and all $\mathbf{v} \in W_t$.
 $\Psi(0, \mathbf{x}) = \Pi\Psi^0(\mathbf{x})$.

Table 2.1: Notation

In Table 2.1, Π is a projection of the initial condition Ψ^0 onto W_0 .

2.3 Boundary conditions

For the problem of finding the solution Table 2.1 to be complete, we need to specify the proper boundary conditions.

2.3.1 Essential (inflow) boundary conditions

Since the solution Table 2.1 of the problem specified in Equation (2.37) is not influenced by its values on the boundary Ω_t , we are not able to employ standard essential boundary conditions of the form $\mathbf{u}(x, y, z) = \mathbf{u}_D(x, y, z)$ with a known \mathbf{u}_D . If such a condition is required from the physical nature of the described phenomenon (as often is the case), it is only implied by a correct definition of fluxes at the boundary - as one can see in the last integrand $(\mathbf{F}(\Psi) \cdot \mathbf{n}) \mathbf{v}$ in Equation (2.37). We shall see how that is numerically handled in later chapters.

2.3.2 Outflow (do-nothing) boundary conditions

If a particular boundary is only present in the numerical model (representing e.g. an "outer" boundary that is there to limit the size of the computation to the area of interest), that is, a free boundary, that is not in any way present as an actual physical boundary or interface, this is achieved by specifying that values of Ψ do not change through the boundary in Equation (2.37):

$$\frac{\partial \Psi}{\partial \mathbf{n}} = 0. \quad (2.40)$$

2.3.3 Periodic boundary conditions

Periodic boundary condition is always specified on two parts Γ_1, Γ_2 of the domain boundary that share the bijection mapping between points:

$$[x_1, y_1, z_1] \leftrightarrow [x_2, y_2, z_2] \forall [x_1, y_1, z_1] \in \Gamma_1, \forall [x_2, y_2, z_2] \in \Gamma_2, \quad (2.41)$$

so that for each pair of related points $[x_1, y_1, z_1] \leftrightarrow [x_2, y_2, z_2]$, the values must be the same:

$$u([x_1, y_1, z_1]) = u([x_2, y_2, z_2]) \forall [x_1, y_1, z_1] \in \Gamma_1, \forall [x_2, y_2, z_2] \in \Gamma_2 : [x_1, y_1, z_1] \leftrightarrow [x_2, y_2, z_2]. \quad (2.42)$$

3 Numerical approach

The weak formulation of the problem we obtained in Table 2.1 still posses a problematic attribute - the space defined in Equation (2.39) is of infinite dimension, and therefore we would need to employ analytical methods to find the solution Table 2.1 in such a space. The equation Equation (2.37) is however rather impossible to be solved analytically, and we have to utilize some sort of numerical simulation - which in turn needs to operate on finite-dimensional spaces. But we need to make sure that the simplifying (reducing) assumptions we make on the way to the numerical model are acceptable so that the numerical solution we obtain converges (as we reduce the discretization size) to the solution defined in Table 2.1.

In this chapter we shall consider that $\Omega_t = \Omega \forall t \in (0, T)$, i.e. the computational domain does not change with respect to time. There are approaches to numerical simulation of MHD phenomena without this condition in place, which utilize the exact same general approach described in this work plus they add additional steps in the algorithm. These are outside of the scope of this work. Also, we always take $\Omega \subset \mathbb{R}^3$.

3.1 Triangulation

We start with leaving the time-derivative untouched, and focus on the discretization in space for now - we are performing a *space semidiscretization*.

First step in the process of the discretization is to divide the computational domain $\overline{\Omega}$ into a finite number of subsets with properties described below. These subsets form the set, further denoted by T_h , called the *triangulation or mesh of the domain* Ω . Please note that the terms *triangulation* and *mesh* shall be used in the text interchangeably. The parameter $h > 0$ of the triangulation usually represents maximum of diameters of all elements $K \in T_h$. The elements $K \in T_h$ are in the context of the finite volume method called *finite volumes*.

Properties of T_h :

1. Each $K \in T_h$ is closed and connected with its interior $K^\circ \neq \emptyset$.
2. Each $K \in T_h$ has a Lipschitz boundary.
3. $\cup_{K \in T_h} K = \overline{\Omega}$

Triangulation

4. If $K_1, K_2 \in T_h$, $K_1 \neq K_2$, then $K_1^\circ \cap K_2^\circ = \emptyset$.

In our case of the three-dimensional problem, we assume that the domain Ω is obtained as an approximation of the original computational domain (also denoted by Ω), and the triangulation is chosen accordingly to the following attributes:

- A) Each $K \in T_h$ is a closed rectangular hexahedron, possibly with curved edges.
- B) For $K_1, K_2 \in T_h$, $K_1 \neq K_2$ we have either $K_1 \cap K_2 = \emptyset$ or K_1, K_2 share one edge (if the shared edge is a whole common edge, we call the triangulation *regular*), or K_1, K_2 share one vertex, or K_1, K_2 share one face.
- C) $\cup_{K \in T_h} K = \overline{\Omega}$.

Furthermore

$$T_h = \{K_i, i \in I\}, \quad (3.1)$$

where $I \subset Z^+ = \{0, 1, 2, \dots\}$ is a suitable index set.

By Γ_{ij} we denote a common face between two neighboring elements K_i and K_j . We set

$$s(i) = \{j \in I; K_j \text{ is adjacent to } K_i\}.$$

The boundary $\partial\Omega$ is formed by a finite number of faces of elements K_i adjacent to $\partial\Omega$. We denote all these boundary faces by S_j , where $j \in I_b \subset Z^- = \{-1, -2, \dots\}$. Now we set

$$\gamma(i) = \{j \in I_b; S_j \text{ is a face of } K_i \in T_h\}$$

and

$$\Gamma_{ij} = S_j \text{ for } K_i \in T_h \text{ such that } S_j \subset \partial K_i, j \in I_b.$$

For K_i not containing any boundary face S_j we set $\gamma(i) = \emptyset$.

Obviously, $s(i) \cap \gamma(i) = \emptyset$ for all $i \in I$. If we write $S(i) = s(i) \cup \gamma(i)$, we have

$$\partial K_i = \cup_{j \in S(i)} \Gamma_{ij}, \quad \partial K_i \cap \partial\Omega = \cup_{j \in \gamma(i)} \Gamma_{ij}.$$

Furthermore we define the set of internal (i.e. not lying on the boundary $\partial\Omega$) edges as

$$\Gamma_I = \cup_{i \in I} \cup_{j \notin \gamma(i)} \Gamma_{ij}, \quad (3.2)$$

and the set of boundary (i.e. lying on the boundary $\partial\Omega$) edges as

$$\Gamma_B = \cup_{i \in I} \cup_{j \in \gamma(i)} \Gamma_{ij}. \quad (3.3)$$

Note If we were to use not $\Omega \subset \mathbb{R}^3$, but rather $\Omega \subset \mathbb{R}^4$, we may just employ the following machinery also to the time-derivative - this is not an uncommon approach. Why the approach described in this work is favored by the author is twofold:

- Data (in a general sense - e.g. algebraic systems, function bases, etc.) are smaller when using a separate handling for time-derivative

Discontinuous Galerkin method

- The dependency on time and space may (and usually does) vary a lot for physical phenomena - to have a separate approach is therefore beneficial

3.1.1 Distributed triangulation

The standard approach to handle large problems that are impossible to be calculated on a single processor in mesh-based numerical simulations (such as Discontinuous Galerkin method) is to employ a *domain decomposition* method, where each of the processors on which the simulation runs holds data about a subset of elements of the mesh T . Consequently, also the matrix and vector assembly (described in Algorithm 2), the linear problem solution, slope limiting, and AMR procedures are performed by all processors using data they have available. The aim here is not to go into deep technical details of distributing data, etc.

TODO Add pictures of domain decomposition

3.2 Discontinuous Galerkin method

For complex problems of compressible flow, and of course for even more complex problems of compressible MHD, there has been a number of attempts to use standard and well known Finite Element Methods that replace the spaces defined in Equation (2.39) with finite-dimension spaces with bases formed by continuous piecewise polynomial functions. These attempts struggled with a common problem of spurious oscillations appearing in the solution - the origin of which is the lack of "stabilization", provided by the second-order terms in elliptic equations. Solution to these problems is the application of stabilization techniques, that usually introduce some sort of artificial diffusion (the second-order term), all of which are non-physical, and generally involve "magical" numbers - constants that are of pure computational nature (not a part of the physical description) or even worse are problem-specific.

3.2.1 Overview of the DG method

Due to this reason, there was an effort to develop methods which would not need such stabilization techniques, and would still offer reasonable resolution of shock-waves, boundary and interior layers, and steep gradients without exhibiting spurious oscillations in the approximate solutions. The approach taken here is based on the idea to combine finite volume and Finite element methods leading to the so-called *discontinuous Galerkin finite element method (DGFEM, DG)*. Here we shall derive and analyze DG for our equations. Let T_h be a triangulation of Ω . For each $K \in T_h$ we introduce the notation

$$\partial K^- = \{x \in \partial K; \beta(x) \cdot \mathbf{n}(x) < 0\}, \quad (3.4)$$

$$\partial K^+ = \{x \in \partial K; \beta(x) \cdot \mathbf{n}(x) \geq 0\}. \quad (3.5)$$

Discontinuous Galerkin method

By $H^1(\Omega, T_h)$ we denote the so-called *broken Sobolev space*:

$$H^1(\Omega, T_h) = \{v \in L^2(\Omega); v|_K \in H^1(K) \forall K \in T_h\}. \quad (3.6)$$

This space is an approximation of the space defined in Equation (2.38), but it contains functions that are discontinuous on element interfaces Γ_{ij} .

For $u \in H^1(\Omega, T_h)$ we set

$$u_K^+ = \text{trace of } u|_K \text{ on } \partial K \quad (3.7)$$

(i.e. the interior trace of u on ∂K). For each face $E \subset \partial K \setminus \Gamma$ of K , there exists $K' \neq K$, $K' \in T_h$, adjacent to E from the opposite side than K . Then we put

$$u_K^- = \text{trace of } u|_{K'} \text{ on } E. \quad (3.8)$$

In this way we obtain the exterior trace u_K^- of u on $\partial K \setminus \Gamma$ and define the jump of u on $\partial K \setminus \Gamma$:

$$[u]_K = u_K^+ - u_K^-. \quad (3.9)$$

Approximation of the broken Sobolev space

Let the domain Ω be covered with a mesh $T_h = \{K_1, K_2, \dots, K_M\}$ where each element K_m carries an arbitrary polynomial degree $1 \leq p_m, \forall m = 1, 2, \dots, M$. The broken Sobolev space $H^1(\Omega, T_h)$ will be approximated by a finite-dimensional space of piecewise-polynomial functions

$$V_h = \{v \in L^2(\Omega); v|_{K_m} \in P^{p_m}(K_m) \text{ for all } 1 \leq m \leq M\} \quad (3.10)$$

where P^p is defined as

$$P^p = \text{span}\{\sum_{\substack{0 \leq i,j,k \leq p \\ i+j+k \leq p}} \alpha_i x_1^i x_2^j x_3^k, \alpha_i \in \mathbb{R}\}.$$

3.2.2 DG formulation of MHD equations

Although the resulting system will look very similar to the weak formulation Equation (2.37), the derivation makes more sense to be done starting with the Equation (2.30).

As stated in Section 3.1, at this point we will discretize the problem in space, and leave the time-derivative untouched. The approximate solution will be sought at each time instant t as an element of the finite-dimensional space

$$[V_h]^8, \quad (3.11)$$

where V_h is defined in Equation (3.10). Functions

$$\mathbf{v}_h \in [V_h]^8 \approx [H^1(\Omega, T_h)]^8, \quad (3.12)$$

where $H^1(\Omega, T_h)$ is defined in Equation (3.6), are in general discontinuous on interfaces Γ_{ij} . By $\mathbf{v}_h|_{ij}$ and $\mathbf{v}_h|_{ji}$ we denote the values of \mathbf{v}_h on Γ_{ij} considered from the interior and the exterior of K_i , respectively. The symbols

$$\langle \mathbf{v}_h \rangle_{ij} = \frac{1}{2} (\mathbf{v}_h|_{ij} + \mathbf{v}_h|_{ji}), \quad [\mathbf{v}_h]_{ij} = \mathbf{v}_h|_{ij} - \mathbf{v}_h|_{ji}$$

denote the average and jump of a function \mathbf{v}_h on Γ_{ij} . In order to derive the discrete problem, we multiply Equation (2.30) by a test function $\mathbf{v}_h \in [V_h]^8$ in a component-wise fashion, integrate over any element $K_i \in T_h$, apply Green's theorem and sum over all $i \in I$, where I is defined in Equation (3.1):

$$\int_{\Omega_t} \frac{\partial \Psi_h}{\partial t} \mathbf{v}_h - \sum_{K_i \in T_h} \int_{K_i} \mathbf{F}(\Psi_h) (\nabla \cdot \mathbf{v}_h) + \sum_{K_i \in T_h} \sum_{j \in s_i} \int_{\Gamma_{ij}} (\mathbf{F}(\Psi_h) \cdot \mathbf{n}_{ij}) \mathbf{v}_h = \int_{\Omega_t} \mathbf{S} \mathbf{v}_h, \quad (3.13)$$

where \mathbf{n}_{ij} is the unit outer normal to Γ_{ij} . Now, the term

$$\int_{\Gamma_{ij}} \mathbf{F}(\Psi_h) \cdot \mathbf{n}_{ij} \mathbf{v}_h \quad (3.14)$$

is problematic, because the value of Ψ_h on Γ_{ij} is not unique - we have two values:

- $\Psi_h|_{ij}$ - which is the value of Ψ_h on Γ_{ij} considered from the element K_i ,
- $\Psi_h|_{ji}$ - which is the value of Ψ_h on Γ_{ij} considered from the element K_j .

Note: This corresponds to the notation set in Equation (3.7), Equation (3.8) - if we take K_i as the element at hand, we have

$$\Psi_h|_{ij} = \Psi_{hK_i}^+, \quad \Psi_h|_{ji} = \Psi_{hK_i}^-$$

Now, because of this non-uniqueness of the values, we replace the term Equation (3.14) with the so-called *numerical flux* $\mathbf{H} = \mathbf{H}(\mathbf{v}, \mathbf{w}, \mathbf{n})$ in the following fashion:

$$(\mathbf{F}(\Psi_h) \cdot \mathbf{n}_{ij}) \mathbf{v}_h \approx \mathbf{H}(\Psi_h|_{ij}, \Psi_h|_{ji}, \mathbf{n}_{ij}) \mathbf{v}_h. \quad (3.15)$$

We impose the following requirements on the numerical flux:

A) $\mathbf{H}(\mathbf{v}, \mathbf{w}, \mathbf{n})$ is defined and continuous on $\mathcal{D} \times \mathcal{D} \times \mathcal{S}_1$, where \mathcal{D} is the domain of definition of the flux \mathbf{F} and \mathcal{S}_1 is the unit sphere in \mathbb{R}^3 .

B) \mathbf{H} is *consistent*:

$$\mathbf{H}(\mathbf{v}, \mathbf{v}, \mathbf{n}) = \mathbf{F}(\mathbf{v}) \mathbf{n}, \quad \mathbf{v} \in \mathcal{D}, \quad \mathbf{n} \in \mathcal{S}_1. \quad (3.16)$$

C) \mathbf{H} is *conservative*:

$$\mathbf{H}(\mathbf{v}, \mathbf{w}, \mathbf{n}) = -\mathbf{H}(\mathbf{w}, \mathbf{v}, -\mathbf{n}), \quad \mathbf{v}, \mathbf{w} \in \mathcal{D}, \quad \mathbf{n} \in \mathcal{S}_1. \quad (3.17)$$

And using these properties of the numerical flux, we can rewrite Equation (3.13) as:

$$\begin{aligned} & \int_{\Omega_t} \frac{\partial \Psi_h}{\partial t} \mathbf{v}_h - \sum_{K_i \in T_h} \int_{K_i} \mathbf{F}(\Psi_h)(\nabla \cdot \mathbf{v}_h) \\ & + \sum_{\Gamma_{ij} \in \Gamma_I} \int_{\Gamma_{ij}} \mathbf{H}(\Psi_h|_{ij}, \Psi_h|_{ji}, \mathbf{n}_{ij}) \mathbf{v}_h = \int_{\Omega_t} \mathbf{S} \mathbf{v}_h, \end{aligned} \quad (3.18)$$

where we used the definition of Equation (3.2) on the page 18.

3.2.3 Numerical flux

Generally, the numerical flux function can be a non-differentiable (or even discontinuous) function. That renders it impossible to directly employ Newton's method to solve the resulting nonlinear problem, and either linearization (e.g. a semi-implicit scheme - [7], chapter 5.1) or explicit time discretization (used in this work) needs to be used.

Another complication arising from evaluation of numerical fluxes on element interfaces exists in distributed solver, where we need to make sure that all processors have relevant data (e.g. previous solution values) from all cells that neighbor any cells assembled on the processor at hand. This issue gets worse when local mesh refinement (there are more neighbor elements of the current cell across the interface at hand), as well as if periodic boundary conditions are used (the neighbor graph is more complex).

Lax-Friedrichs numerical flux

This is the most straightforward numerical flux satisfying Equation (3.16), and Equation (3.17) and is defined as follows:

$$\mathbf{H}_{LF}(\mathbf{v}, \mathbf{w}, \mathbf{n}) = \frac{1}{2} [\mathbf{F}(\mathbf{v}) + \mathbf{F}(\mathbf{w})] - \frac{\alpha}{2} (\mathbf{w} - \mathbf{v}), \quad (3.19)$$

where the parameter α is the so-called *stabilization parameter*, which in order for the solution to be stable needs to fulfill $\alpha < \frac{\Delta t}{\Delta x}$, where $\Delta t, \Delta x$ need to satisfy the CFL condition Equation (3.46). Now, this numerical flux is very sensitive to the choice of α - for larger values, it tends to be very diffusive, but for lower values, it is unstable and produces oscillations in the vicinity of shocks [10]. In this work it is primarily used for implementation verification purposes, as due to its simplicity, the risk of errors in its implementation is rather negligible.

Riemann problem for MHD

For the class of numerical fluxes, known as the *Godunov type fluxes*, the numerical flux across the element boundary is constructed as the flux \mathbf{F} of a solution of the following Riemann problem (Equation (3.20)) at $x = 0$: *Riemann problem*

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0, \quad (3.20)$$

where

$$\mathbf{U} = \begin{pmatrix} \rho \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ U \\ B_2 \\ B_3 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \pi_1 \\ \frac{\pi_1^2}{\rho} - B_1^2 + \frac{1}{2}(p + U_m) \\ \frac{\pi_2 \pi_1}{\rho} - B_1 B_2 \\ \frac{\pi_3 \pi_1}{\rho} - B_1 B_3 \\ \frac{\pi_1}{\rho} \left(\frac{\gamma}{\gamma-1} p + U_k \right) + \frac{2}{\rho} (\pi_k B_1 - \pi_1 B_k) B_1 \\ \frac{\pi_1 B_2 - \pi_2 B_1}{\rho} \\ \frac{\pi_1 B_3 - \pi_3 B_1}{\rho} \end{pmatrix}, \quad (3.21)$$

and where due to the divergence free condition $\nabla \cdot \mathbf{B} = 0$ of the magnetic field), B_1 is given as constant.

These equations (Equation (3.20)) have seven eigenvalues which correspond to two Alfvén waves ($\lambda_{2,6}$), two slow magneto-acoustic waves ($\lambda_{3,5}$), and two fast magneto-acoustic waves ($\lambda_{1,7}$), and one entropy wave (λ_4):

$$\lambda_{2,6} = x + c_a, \quad (3.22)$$

$$\lambda_{3,5} = t + c_s, \quad (3.23)$$

$$\lambda_{1,7} = \mathbf{u}_x + c_f, \quad (3.24)$$

$$\lambda_{2,6} = \mathbf{u}_x - c_a, \quad (3.25)$$

$$\lambda_{3,5} = \mathbf{u}_x - c_s, \quad (3.26)$$

$$\lambda_{1,7} = \mathbf{u}_x - c_f, \quad (3.27)$$

$$\lambda_4 = \mathbf{u}_x, \quad (3.28)$$

$$\text{where } c_a = \sqrt{\frac{B_1^2}{rho}}, \quad c_{s,f} = \left\{ \frac{\gamma p + |B|^2 \mp \sqrt{(\gamma p + |B|^2)^{\frac{1}{2}} - 4\gamma p B_1^2}}{2\rho} \right\}^{\frac{1}{2}}.$$

From this, it follows, that there are theoretically as many as 9 intermediate states (that is states between characteristics corresponding to the eigenvalues $\lambda_1, \dots, \lambda_7$) - see Figure 3.1. Unfortunately, for MHD equations, there is no exact solver of the Riemann problem across the element boundary, and therefore, approximate solvers are used. One instance of the derived numerical flux based on the solution at $x = 0$

of Equation (3.20) is described in the next section.

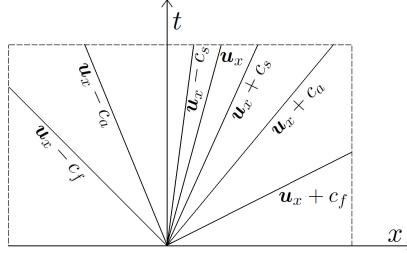


Figure 3.1: Characteristics corresponding to eigenvalues of Equation (3.20).

HLLD numerical flux

The abbreviation **HLLD** stands for Harten-Lax-van Leer (HLL) approximate Riemann solver, and **D** stands for Discontinuities. It is an extension of HLLC Riemann solver for the Euler equations ([?]). It divides the Riemann fan into 6 states as illustrated in ???. This particular numerical flux has been introduced in [17] and has been shown to be very suitable for the studied problems. HLLD is an approximate nonlinear solution of the MHD Riemann problem, and is algebraically derived in [17] under the assumptions that the normal velocity and the background potential magnetic field in the Riemann fan are constant. It follows from these assumptions, that 4 intermediate states are sufficient to resolve the Riemann problem in adequate manner ([17]).

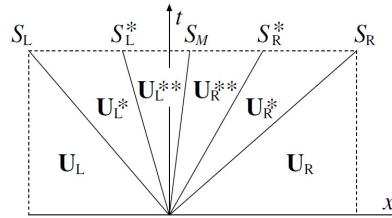


Figure 3.2: Six states considered in the definition of HLLD flux.

The HLLD flux according to its definition in [17] defines 4 intermediate (starred) states U_L^* , U_L^{**} , U_R^* , U_R^{**} and 2 outer states U_L , U_R corresponding to states of U defined in Equation (3.21) for the left ($x < 0$) and right ($x > 0$) parts of the time-space domain illustrated in Figure 3.2. Details of these states are given in [17]. The resulting fluxes are:

$$F_{HLLD} = \begin{cases} F_L & \text{if } S_L > 0, \\ F_L^* & \text{if } S_L \leq 0 \leq S_L^*, \\ F_L^{**} & \text{if } S_L^* \leq 0 \leq S_M, \\ F_R^{**} & \text{if } S_M \leq 0 \leq S_R^*, \\ F_R^* & \text{if } S_R^* \leq 0 \leq S_R, \\ F_R & \text{if } S_R < 0 \end{cases} . \quad (3.29)$$

These are derived using the assumptions mentioned earlier, that:

$$u_L^* = u_l^{**} = u_R^{**} = u_R^* = S_M \quad \text{normal velocity constant over the Riemann fan} \quad (3.30)$$

$$p_{T_L}^* = p_{T_L}^{**} = p_{T_R}^{**} = p_{T_R}^* = p_T^* \quad \text{total pressure constant over the Riemann fan,} \quad (3.31)$$

where p_T is the total pressure as defined in Equation (1.4), and

$$S_M \frac{(S_R - u_R) \rho_R u_R - (S_L - u_L) \rho_L u_L - p_{T_R} + p_{T_L}}{(S_R - u_R) \rho_R - (S_L - u_L) \rho_L}, \quad (3.32)$$

$$S_L = \min [\lambda_1(U_L), \lambda_1(U_R)], \quad (3.33)$$

$$S_R = \max [\lambda_7(U_L), \lambda_7(U_R)], \quad (3.34)$$

$$S_L^* = S_M - \frac{|B_x|}{\sqrt{\rho_L^*}} = S_M - \frac{|B_x|}{\sqrt{\rho_L \frac{S_L - u_L}{S_L - S_M}}}, \quad (3.35)$$

$$S_R^* = S_M + \frac{|B_x|}{\sqrt{\rho_R^*}} = S_M + \frac{|B_x|}{\sqrt{\rho_R \frac{S_R - u_R}{S_R - S_M}}}. \quad (3.36)$$

Numerical fluxes comparison

For the comparison, a simple version of the benchmark described in ?? was used. In the next figures, there are snapshots of density solution component at several time steps for the Lax-Friedrichs flux for two values of $\alpha = 0.5$, $\alpha = 1.5$, and for the HLLD flux. Please note that the results here are calculated with piecewise-constant functions.

Discontinuous Galerkin method

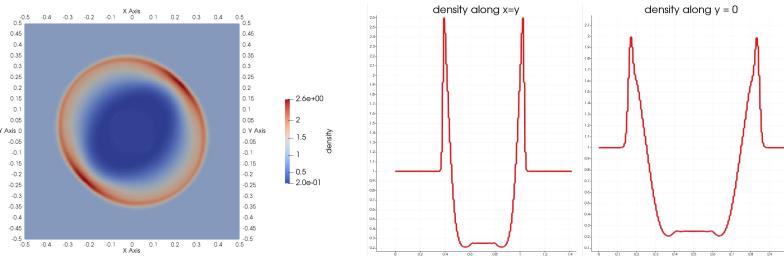


Figure 3.3: Density at time $t = 0.1$, Lax-Friedrichs flux with $\alpha = 0.5$

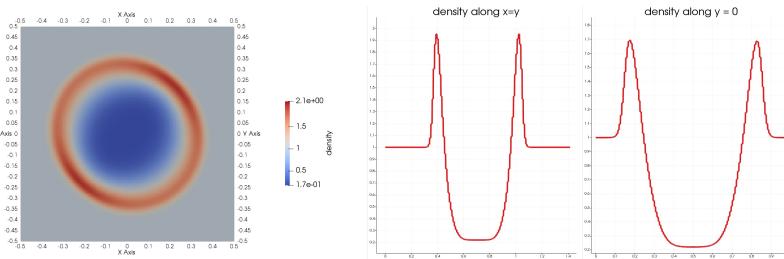


Figure 3.4: Density at time $t = 0.1$, Lax-Friedrichs flux with $\alpha = 1.5$

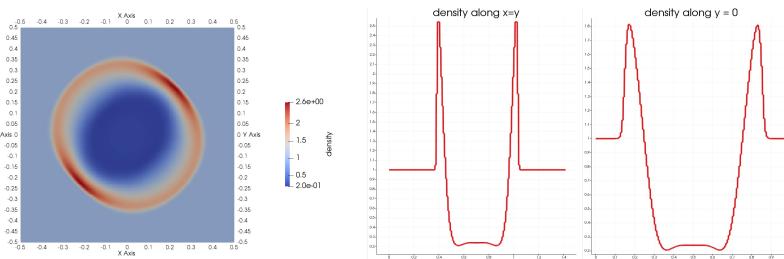


Figure 3.5: Density at time $t = 0.1$, HLLD flux

Discontinuous Galerkin method

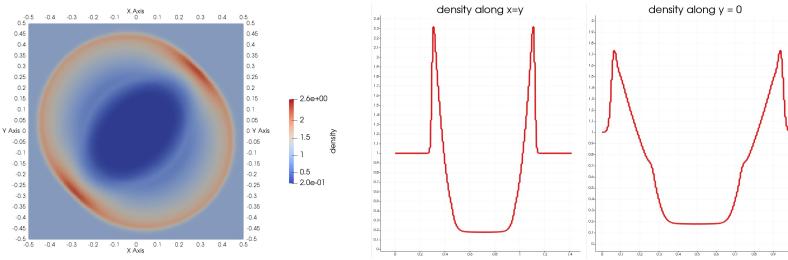


Figure 3.6: Density at time $t = 0.16$, Lax-Friedrichs flux with $\alpha = 0.5$

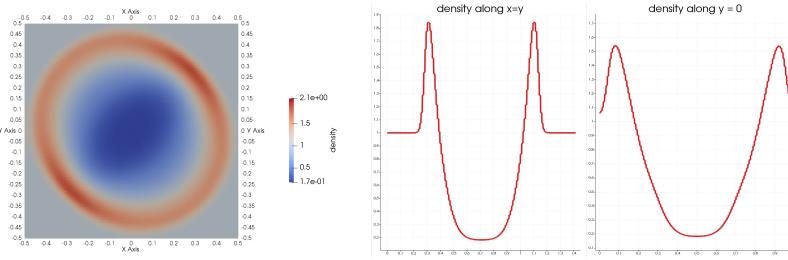


Figure 3.7: Density at time $t = 0.16$, Lax-Friedrichs flux with $\alpha = 1.5$

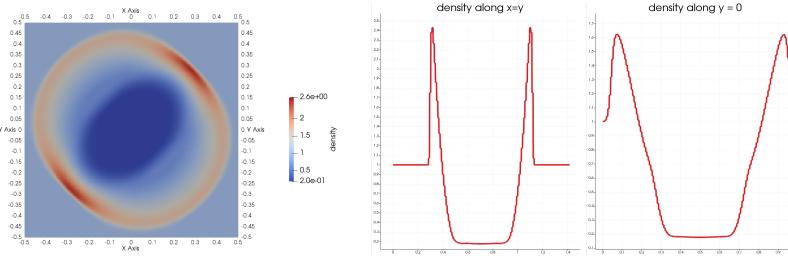


Figure 3.8: Density at time $t = 0.16$, HLLD flux

Discontinuous Galerkin method

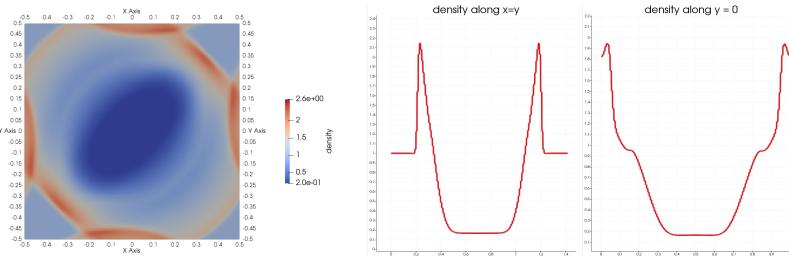


Figure 3.9: Density at time $t = 0.23$, Lax-Friedrichs flux with $\alpha = 0.5$

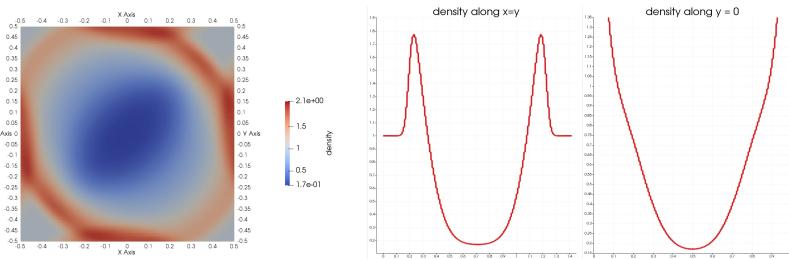


Figure 3.10: Density at time $t = 0.23$, Lax-Friedrichs flux with $\alpha = 1.5$

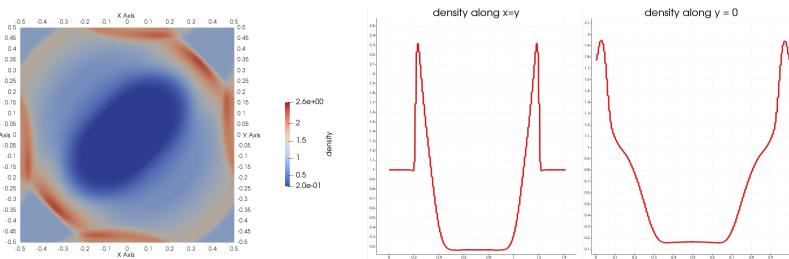


Figure 3.11: Density at time $t = 0.23$, HLLD flux

Discontinuous Galerkin method

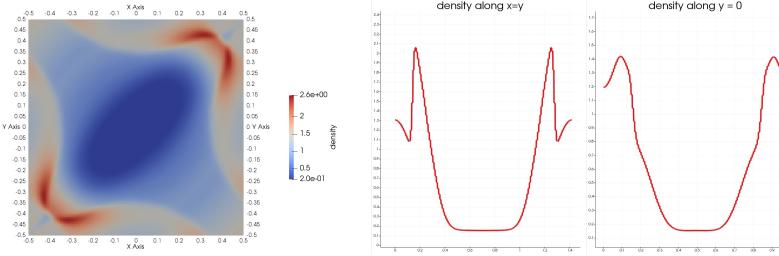


Figure 3.12: Density at time $t = 0.3$, Lax-Friedrichs flux with $\alpha = 0.5$

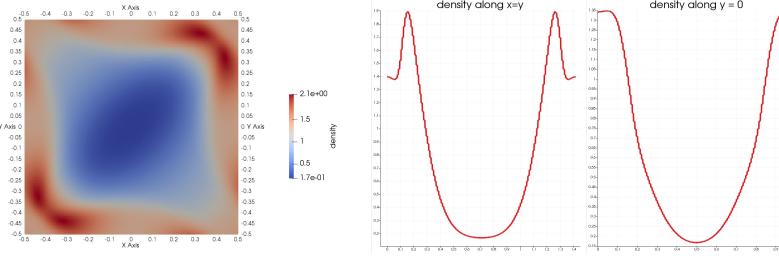


Figure 3.13: Density at time $t = 0.3$, Lax-Friedrichs flux with $\alpha = 1.5$

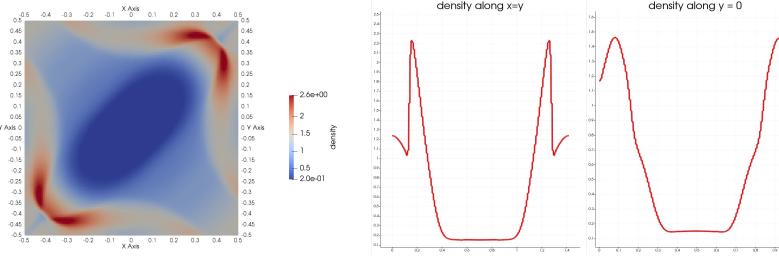


Figure 3.14: Density at time $t = 0.3$, HLLD flux

There are a few observations from all the triplets of Figures:

- The Lax-Friedrichs flux is very sensitive to the choice of α , for $\alpha = 1.5$, the solution is too diffusive,
- The HLLD flux is quantitatively and qualitatively very close to the 'ideal' Lax-Friedrichs flux where $\alpha = 0.5$.

The second observation could lead us to the conclusion that the Lax-Friedrichs flux, being simple to implement, is superior to HLLD which is quite complex. Unfortunately, as shown e.g. in [10], the Lax-Friedrichs flux is not stable enough to be used for arbitrary problem setup, also when handling solution with discontinuities, it is very dissipative [8].

Second reason why for practical usage of the developed code, the HLLD flux is strongly recommended is, that it is parameter-free, as opposed to Lax-Friedrichs

flux having the stabilization parameter α which makes it performing differently for different problems, as per the value of α .

3.2.4 Numerical handling of boundary conditions

In what follows, we are interested in using flux-induced inflow and outflow boundary conditions (see Section 2.3). Periodic boundary conditions are considered to be merely a special case of handling numerical fluxes across internal edges in Γ_I . To account for these boundary conditions, we need to investigate the term

$$\int_{\Gamma_{ij}} \mathbf{H}(\Psi_h|_{ij}, \Psi_h|_{ji}, \mathbf{n}_{ij}) \mathbf{v}_h$$

for $\Gamma_{ij} \in \Gamma_B$ (see Equation (3.3) on page 18). This term is used in Equation (3.18) for faces in Γ_I which are internal and always have 2 values connected to them - $\Psi_h|_{ij}$, $\Psi_h|_{ji}$ - which induces the notation. On a boundary face, the corresponding value to $\Psi_h|_{ij}$ can be defined in the same way as in the case of Γ_I , but $\Psi_h|_{ji}$ needs to be defined.

Inflow boundary condition

First, if we want to prescribe an inflow boundary condition (i.e. we know what values should the state vector Ψ_h have on $\Gamma_{ij} \in \Gamma_B$), we define

$$\overline{\Psi_h|_{ji}} \tag{3.37}$$

to be the prescribed value.

Outflow boundary condition

If we want to model an outflow boundary condition (i.e. do nothing condition), we may use the *consistency* of the numerical flux \mathbf{H} defined in Equation (3.16), and define

$$\overline{\Psi_h|_{ji}} = \Psi_h|_{ij}, \tag{3.38}$$

which is a suitable definition for the outflow boundary condition. It is important to mention, that setting the inflow boundary condition does not imply that solution values on this boundary equal to these prescribed value. This follows from the definition of broken Sobolev space (Equation (3.6)). Moreover the values of the solution on the boundary also depend on the numerical flux used, as the values on the boundary are merely one of the input parameters for the flux (See Equation (3.15)).

Periodic boundary conditions

Periodic boundary conditions come in the form described in Equation (2.42), for pair of points on the boundary (Equation (2.41)). The point mapping is the main

Slope limiting

complication encountered, namely in the distributed computations, and moreover if AMR is used in distributed computations (see Section 4.2). Otherwise, from the integral evaluation perspective, an edge on a periodic boundary is approached in the very same way as internal edges (we have values from both sides, and we evaluate the numerical flux in quadrature points).

DG full problem statement

Now, taking Equation (3.37) and Equation (3.38), we can enhance Equation (3.18) with an additional term, that will add the boundary conditions into the equation:

$$\sum_{\Gamma_{ij} \in \Gamma_B} \int_{\Gamma_{ij}} \mathbf{H}(\Psi_h|_{ij}, \overline{\Psi_h|_{ji}}, \mathbf{n}_{ij}) \mathbf{v}_h,$$

so that the complete semi-discrete problem reads:

$$\begin{aligned} & \int_{\Omega_t} \frac{\partial \Psi_h}{\partial t} \mathbf{v}_h - \sum_{K_i \in T_h} \int_{K_i} \mathbf{F}(\Psi_h) (\nabla \cdot \mathbf{v}_h) \\ & + \sum_{\Gamma_{ij} \in \Gamma_I} \int_{\Gamma_{ij}} \mathbf{H}(\Psi_h|_{ij}, \Psi_h|_{ji}, \mathbf{n}_{ij}) \mathbf{v}_h \\ & + \sum_{\Gamma_{ij} \in \Gamma_B} \int_{\Gamma_{ij}} \mathbf{H}(\Psi_h|_{ij}, \overline{\Psi_h|_{ji}}, \mathbf{n}_{ij}) \mathbf{v}_h \\ & = \int_{\Omega_t} \mathbf{S} \mathbf{v}_h. \end{aligned} \quad (3.39)$$

Now we can formulate the definition of the *semi-discrete solution* $\Psi_h = \Psi_h((t, \mathbf{x}))$ of MHD equations Equation (2.30) as

- A) $\Psi_h \in C^1((0, T), [V_h]^8)$,
- B) Equation (3.39) holds for all $t \in (0, T)$, and all $\mathbf{v} \in [V_h]^8$,
- C) $\Psi_h(0, \mathbf{x}) = \Pi_h \Psi^0(\mathbf{x})$,

where Π_h is a projection of the initial condition Ψ^0 onto $[V_h]^8$.

3.3 Slope limiting

It is well known [7] that the Discontinuous Galerkin method exhibits nonphysical spurious oscillations in the vicinity of sharp discontinuities. Noteworthy is the fact, that with continuous Finite Element spaces, the situation is even worse, as the oscillations tend to propagate through the computational domain. With the DG method, the problem is localized to a single layer of elements bordering any sharp

Slope limiting

front. This behavior is not acceptable, and measures must be taken to eliminate such oscillations - methods aiming at solving this are usually labeled as flux limiters, slope limiters, or shock capturing schemes.

These methods can be categorized according to multiple aspects. Out of these, two are important from the perspective of this work. First categorization is whether the approach changes the equations by introducing additional term that 'smoothes' the solution near the sharp front (this may be understood as a form of *artificial viscosity / resistivity*) - such approach is proposed e.g. in [6].

In this work, such an approach is not preferred, as we aim at implementing a generally usable solver, where extensive analysis of the impact of a change in the governing equations for the particular problem is not possible.

Second categorization worth mentioning is whether the particular approach is suitable for multi-scale phenomena, where the solution can exhibit large jumps in all possible configurations with respect to the (non-uniformly refined) mesh. From a pragmatic perspective, a rather simple slope and robust limiting technique that does not change the governing equations is the Barth-Jespersen limiter [3]. The Barth-Jespersen limiter considers the piecewise-linear solution in the form

$$u_h(x) = u_c + \alpha_e (\nabla u)_c \cdot (x - x_c), \quad 0 \leq \alpha_e \leq 1, \quad (3.40)$$

where u_c is the cell average on the element e , and $(\nabla u)_c$ is the gradient of the solution on the element e .

The sought parameter α_e is the parameter (correction factor) that determines the maximum admissible slope and is defined as:

$$\alpha_e = \min_i \begin{cases} \min \left\{ 1, \frac{u_e^{max} - u_c}{u_i - u_c} \right\}, & \text{if } u_i - u_c > 0, \\ 1, & \text{if } u_i - u_c = 0, \\ \min \left\{ 1, \frac{u_e^{min} - u_c}{u_i - u_c} \right\}, & \text{if } u_i - u_c < 0, \end{cases} \quad (3.41)$$

where $u_i = u_c + (\nabla u)_c \cdot (x_i - x_c)$ is the unconstrained solution value at the vertex x_i , and u_e^{min}, u_e^{max} are the minimum and maximum cell averages of all elements sharing a common edge with the element e .

However, using this limiting technique, there are two suboptimal behavior features - the bounds for the limited solution are set on one hand too tight - solution values from elements meeting at a vertex but having no common edge/face are not taken into account (they might extend the interval for the admissible correction factor α_e , and on the other hand too loose - solution values on elements that share a common

Slope limiting

edge may extend the admissible interval for α_e based on the value at a vertex that does not belong to that particular common edge.

Both these two problems are solved in the slope limiting technique chosen for this work

3.3.1 Vertex-based limiter

Introduced by D. Kuzmin in [14], the Vertex-based limiter aims at being an improvement over the Barth-Jespersen limiter. It also considers the solution in the form Equation (3.40), but the definition of the correction factor α_e differs:

$$\alpha_e = \min_i \begin{cases} \min \left\{ 1, \frac{u_i^{max} - u_c}{u_i - u_c} \right\}, & \text{if } u_i - u_c > 0, \\ 1, & \text{if } u_i - u_c = 0, \\ \min \left\{ 1, \frac{u_i^{min} - u_c}{u_i - u_c} \right\}, & \text{if } u_i - u_c < 0, \end{cases} \quad (3.42)$$

where in this case u_i^{min}, u_i^{max} are defined in such a way that for each of the vertices they are initialized with a small and a large constant, respectively, and then in the loop over all elements that contain the i -th vertex, the values are updated as:

$$u_i^{max} = \max \{u_c, u_i^{max}\}, \quad u_i^{min} = \min \{u_c, u_i^{min}\}. \quad (3.43)$$

This slope limiting technique proves to have all the required attributes from the perspective of this work. The algorithm implementing this technique looks is presented

Slope limiting

in Algorithm 1 TODO algoritmus - tohle je jen placeholder

Algorithm 1: Assembling of the algebraic problem Equation (3.57)

```

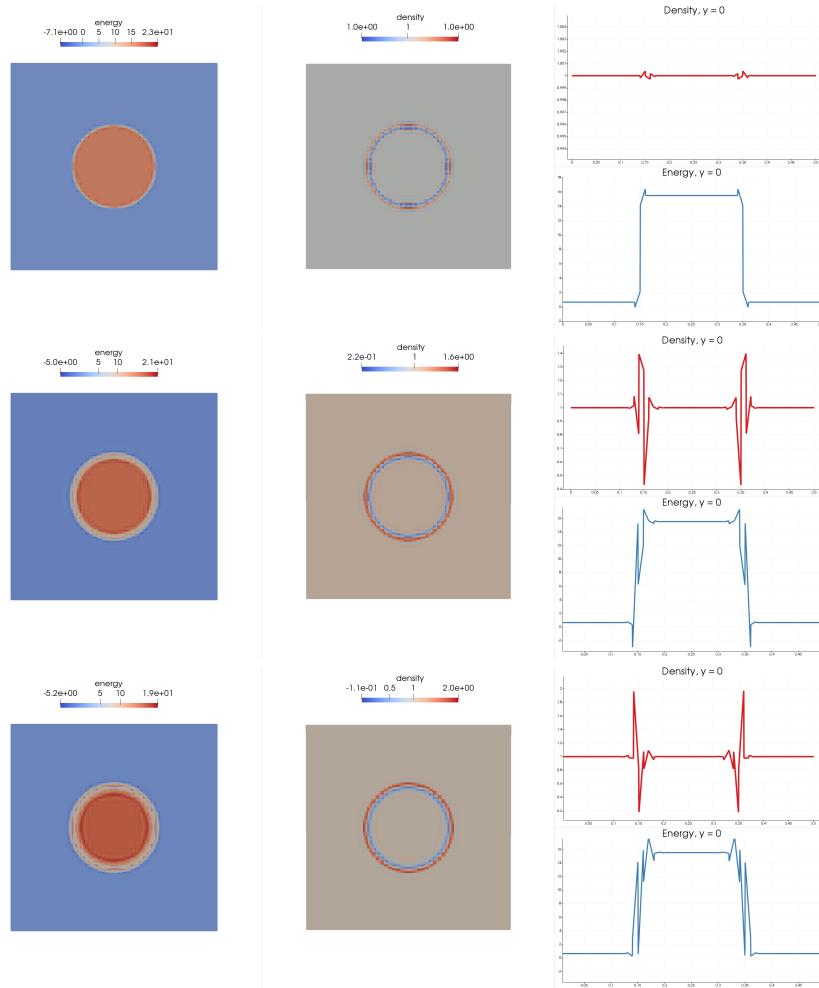
# 1 - Loop over elements
foreach  $K_i \in T_h$  do
    Data: Quadrature points  $\{x_1^i, \dots, x_n^i\}$ 
    Data: Jacobian of the mapping  $J_{K_i}$  mapping the reference element
        (unit cube) to the actual element
    Data: Quadrature weights  $\{w_1, \dots, w_n\}$ 
    # Loop over quadrature points
    foreach  $j \in \{1, \dots, n\}$  do
        Set:  $(JxW)_j = J_{K_i} \times w_j$ 
        # Loop over test functions
        foreach  $v \in v_h(K_i)$  do
            Data:  $l$  - index of  $v$  in the global system, i.e. row in
                Equation (3.54) - Equation (3.56)
            # Loop over basis functions
            foreach  $u \in v_h(K_i)$  do
                Data:  $m$  - index of  $u$  in the global system, i.e. column in
                    Equation (3.54)
                 $a_{lm} += (JxW)_j a_{lmi} j$ 
                 $b_l += (JxW)_j b_{li} j$ 
    # 2 - Loop over edges
    foreach  $\Gamma_{ij} \in T_h$  do
        Data: Quadrature points  $\{x_1^{ij}, \dots, x_{n_f}^{ij}\}$ 
        Data: Jacobian of the mapping  $J_{K^+} = J_{K^-}$  mapping the reference face
            (unit square) to the actual face, where  $K^+, K^-$  are elements
            adjacent to  $\Gamma_{ij}$  if this is an internal edge, or  $K^+ = K^-$  if this is a
            boundary edge.
        Data: Quadrature weights  $\{w_1, \dots, w_{n_f}\}$ 
        # Loop over quadrature points
        foreach  $j_f \in \{1, \dots, n_f\}$  do
            Set:  $(JxW)_{j_f} = J_{K^+} \times w_{j_f}$  # Here it does not matter if we
                choose  $J_{K^+}$  or  $J_{K^-}$ 
            # Loop over test functions
            foreach  $v \in v_h(K^+)$  do
                Data:  $l$  - index of  $v$  in the global system, i.e. row in
                    Equation (3.54) - Equation (3.56)
                 $b_l += (JxW)_{j_f} b'_{lij} j_f$ 

```

Slope limiting

Comparison of limited and unlimited solution

For illustration, the same problem as in Section 3.2.3 is considered, which is suitable for illustrating the undershoots and overshoots, as the problem contains a sharp front where this behavior is clearly visible. In the next Figures, first several snapshots of an unlimited solution, and then several snapshot of a limited solution are presented. Note that the unlimited solution can't progress beyond a certain point, as the undershoots and overshoots get so large that the density values get to be negative, therefore e.g. calculating its square root needed for the speed of sound evaluation is impossible. Even before that, we get to non-physical regime, and e.g. the energy values start growing beyond limit.



Slope limiting

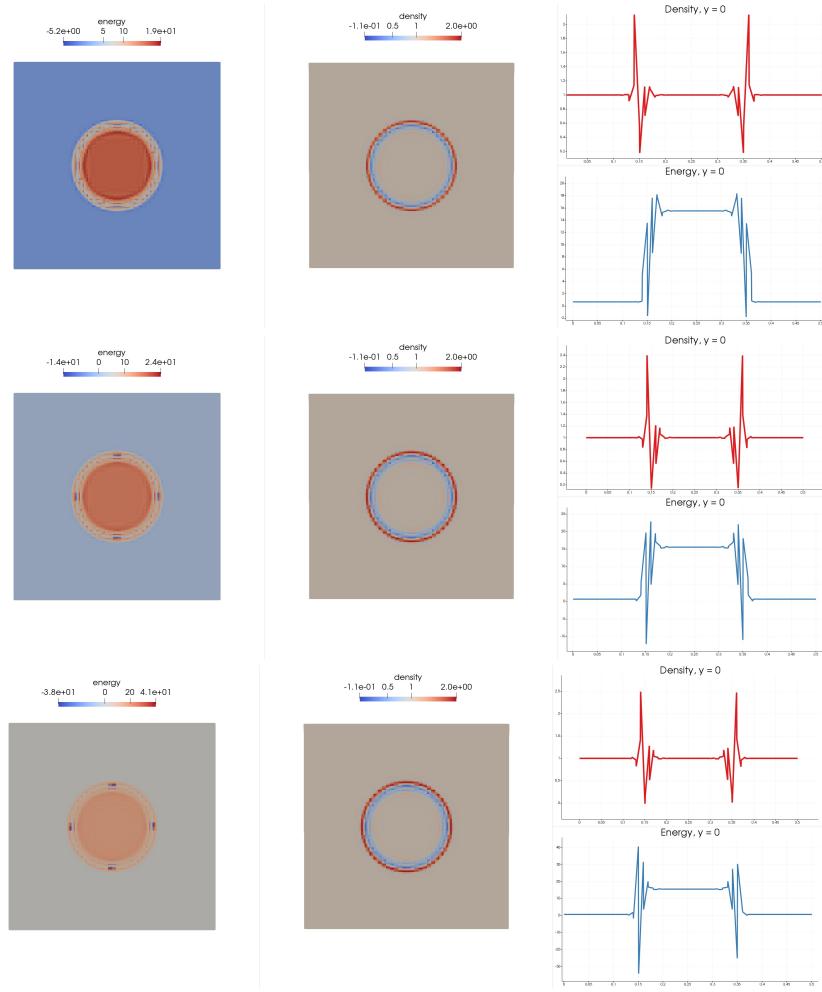
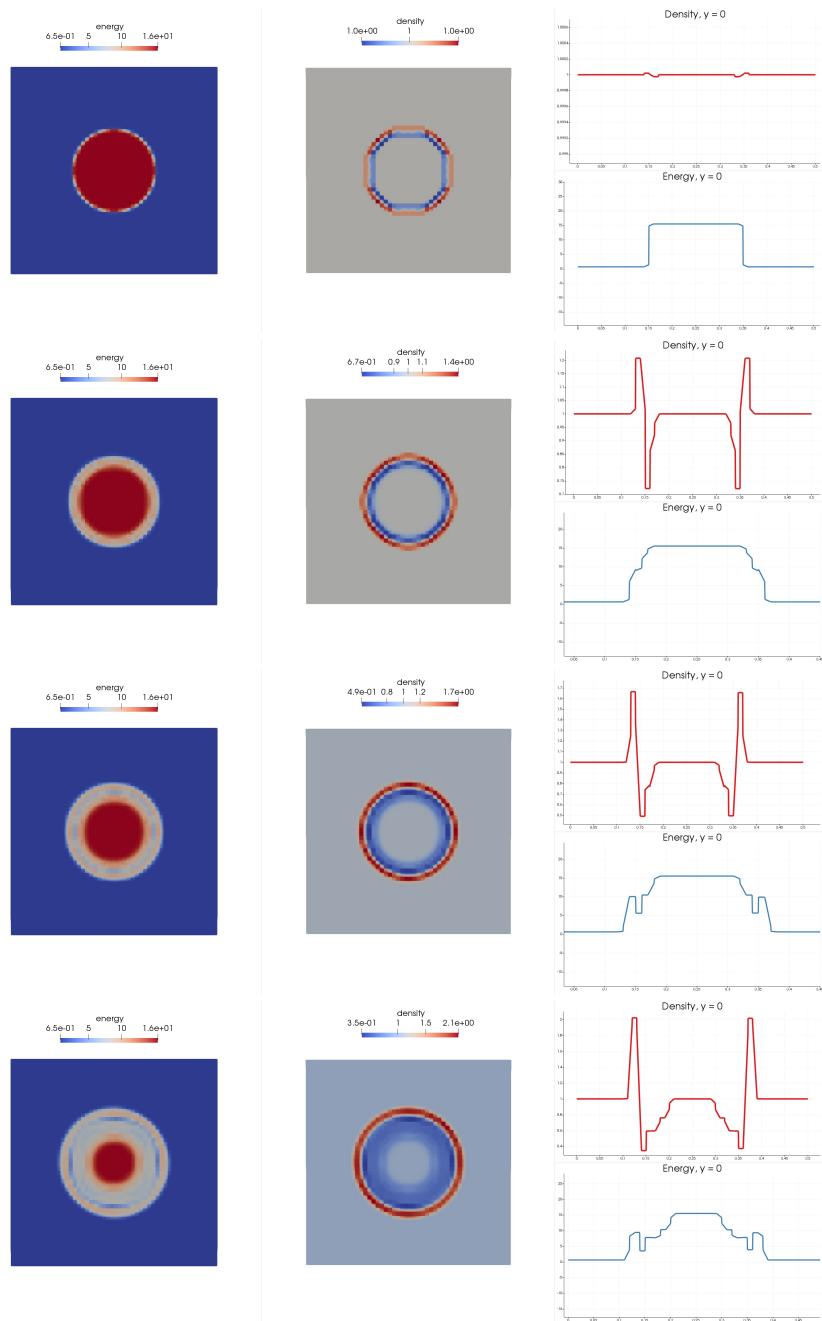


Figure 3.15: Unlimited solution - Energy, density, and their values over line $y = 0$, the solution cannot progress beyond the last snapshot, as the oscillations are orders of magnitude larger than the physical solution

Slope limiting



Divergence-free FE space

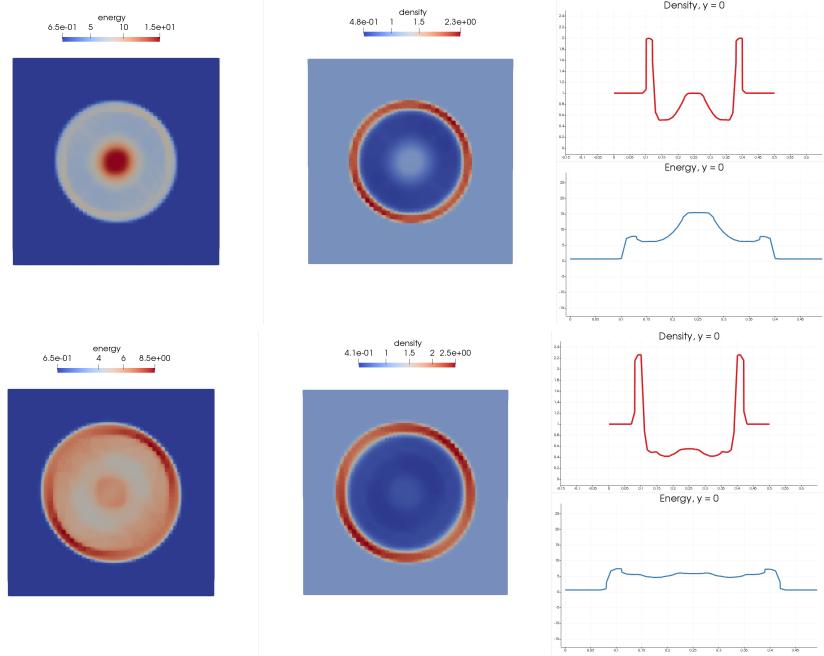


Figure 3.16: Solution limited with Vertex-based limiter - Energy, density, and their values over line $y = 0$

3.4 Divergence-free FE space

The divergence-free constraint of the magnetic field, $\mathbf{B} = 0$ (Gauss's law) is not enforced by the solution definition Table 2.1. Therefore, we need to perform additional work to be sure that we do not have a non-physical solution in the sense that the constraint is not satisfied.

There are two often used approaches to handle this problem - the Constraint-Transport (CT) method, and divergence cleaning. The first one is not suitable for this work, as it constraints the triangulation in such a way, that implementing Adaptive Mesh Refinement would be very complicated, if possible at all. The second approach, the divergence cleaning methods need additional postprocessing step which may be omitted for the sake of calculation efficiency. The approach taken in this work is to replace the standard FE space Equation (3.11) with basis functions Equation (3.12) for the magnetic field part (B) with a vector-valued (3-dimensional) space V_h^B of functions that have exactly

$$\nabla \cdot \mathbf{v}_h^B = 0, \quad \mathbf{v}_h^B \in V_h^B, \quad (3.44)$$

where these functions are as before discontinuous on interfaces Γ_{ij} . The basis of space V_h^B for piecewise-linear functions can be selected in several ways, in this work,

Divergence-free FE space

the following basis was selected:

$\begin{pmatrix} B_x(x, y, z) \\ B_y(x, y, z) \\ B_z(x, y, z) \end{pmatrix}$	Visualization	$\begin{pmatrix} B_x(x, y, z) \\ B_y(x, y, z) \\ B_z(x, y, z) \end{pmatrix}$	Visualization
--	---------------	--	---------------

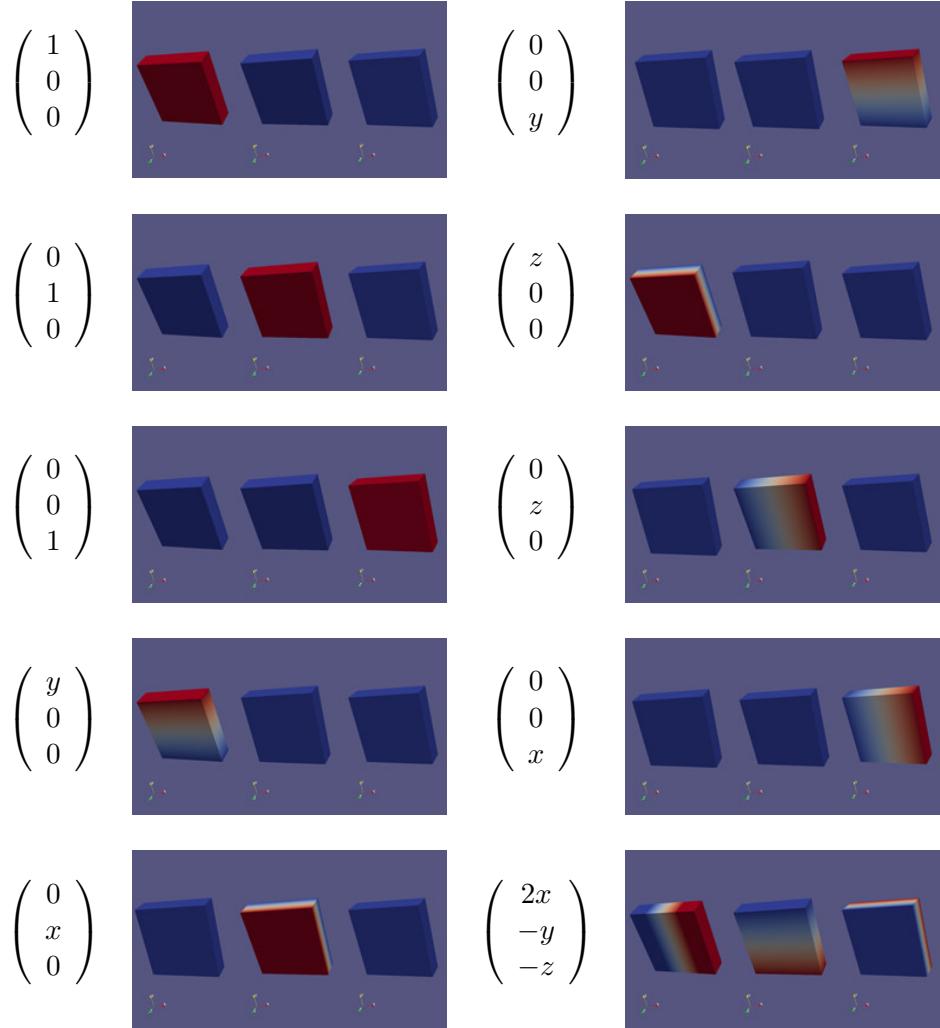


Table 3.1: Divergence-free space basis

In Table 3.1, it is obvious that only one function actually has more nonzero components.

Discretization in time

In what follows, the notation $[V_h]^8$ used before will be used for the space where the last three components are replaced by V_h^B . Note that there are some technicalities with respect to the usage of V_h^B needed to be handled in computation, e.g. in Equation (3.42), or later in Equation (3.60), Equation (3.62) that we do not explicitly attend to.

3.5 Discretization in time

Relations Equation (3.18) represent a system of ordinary differential equations which can be solved by a suitable numerical method. Since we are interested in applying the Rothe's method (as opposed to the method of lines, which switches the order of discretization in time and space), we now want to discretize the time derivative. In order to do so, we consider a partition $0 = t_0 < t_1 < t_2 < \dots$ of the time interval $(0, T)$ and set $\tau_k = t_{k+1} - t_k$. We use the notation \mathbf{w}_h^k for the approximation of $\mathbf{w}_h(t_k)$.

3.5.1 Discrete problem

Then we apply a time discretization scheme, for example, the simple explicit *Euler method* and our *fully discrete problem* reads: for each $k > 0$ find \mathbf{w}_h^{k+1} such that

A) $\Psi_h^{k+1} \in [V_h]^8$,

B) For all test functions $\mathbf{v}_h \in [V_h]^8$:

$$\begin{aligned} \int_{\Omega_t} \frac{\Psi_h^{k+1} - \Psi_h^k}{\tau} \mathbf{v}_h - \sum_{K_i \in T_h} \int_{K_i} \mathbf{F}(\Psi_h^k) (\nabla \cdot \mathbf{v}_h) \\ + \sum_{\Gamma_{ij} \in \Gamma_I} \int_{\Gamma_{ij}} \mathbf{H}(\Psi_h^k|_{ij}, \Psi_h^k|_{ji}, \mathbf{n}_{ij}) \mathbf{v}_h \\ + \sum_{\Gamma_{ij} \in \Gamma_B} \int_{\Gamma_{ij}} \mathbf{H}(\Psi_h^k|_{ij}, \overline{\Psi_h^k|_{ji}}, \mathbf{n}_{ij}) \mathbf{v}_h \\ = \int_{\Omega_t} \mathbf{S} \mathbf{v}_h, \end{aligned} \quad (3.45)$$

C) $\Psi_h^0(\mathbf{x}) = \Pi_h \Psi^0(\mathbf{x})$,

where Π_h is a projection of the initial condition Ψ^0 onto $[V_h]^8$.

3.5.2 Time step length

Time step length is an important attribute of the discretization. If it is too small, the calculation might be taking too long to finish, with unnecessary precision with respect to time. If it is too large, we may end up with unstable calculation and obtain

Algebraic formulation

results with nonphysical oscillations, or without a solution whatsoever. That is why we need to take extra care to derive the proper value. From the stability perspective, we have a condition for the upper bound of the time step - this condition is called the *Courant-Friedrichs-Lowy* condition. This condition is of the following form:

$$\tau_{max} = \min \left\{ \frac{\Delta_{x_{min}}}{c_{max}}, \frac{\Delta_{x_{min}}^2}{2\eta_{max}} \right\}, \quad (3.46)$$

where $\Delta_{x_{min}}$ is the smallest dimension of any element, η_{max} highest resistivity in the domain, and v_{max} highest velocity in the domain, where the following velocities are taken into account:

$$c_s = \sqrt{\frac{\gamma(\gamma-1)}{\rho}(U - \rho v^2 - U_B)}, \quad (3.47)$$

$$c_a = \sqrt{\frac{B^2}{\rho}}, \quad (3.48)$$

$$u, \quad (3.49)$$

where c_s is the speed of sound, c_a is the Alfvén speed, and u is the speed of plasma. We then take

$$c_{max} = \max \{c_s, c_a, u\}.$$

3.6 Algebraic formulation

Last step in the DG method discretization is to transform the system of equations ?? into a system of linear algebraic equations at every time step t_k and obtain the solution at this time step as the solution of this linear algebraic system.

First, we rearrange the system in the following manner:

$$\begin{aligned} \sum_{K_i \in T_h} \int_{K_i} \mathbf{v}_h \Psi_h^{k+1} &= \sum_{K_i \in T_h} \int_{K_i} \left[\Psi_h^k + \tau \mathbf{S} + \tau \mathbf{A}(\Psi_h^k) (\nabla \cdot \mathbf{v}_h) \right] \mathbf{v}_h \\ &\quad - \sum_{\Gamma_{ij} \in \Gamma_I} \int_{\Gamma_{ij}} \mathbf{H}(\Psi_h^k|_{ij}, \Psi_h^k|_{ji}, \mathbf{n}_{ij}) \mathbf{v}_h \\ &\quad - \sum_{\Gamma_{ij} \in \Gamma_B} \int_{\Gamma_{ij}} \mathbf{H}(\Psi_h^k|_{ij}, \overline{\Psi_h^k|_{ji}}, \mathbf{n}_{ij}) \mathbf{v}_h. \end{aligned} \quad (3.50)$$

We can see that the left hand side does not depend on the previous solution values, so there is no need to recalculate the matrix entries in every time step (unless we employ AMR, in which case the mesh and therefore the set of basis functions

Numerical integration

changes). Now

$$\Psi_h^{k+1} = \sum_{l=0}^{l=L} y_l \mathbf{v}_{hl}, L = \dim([V_h]^8) \quad (3.51)$$

for some (obviously finite) basis $\{v_{h1}, \dots, v_{hL}\}$ of $[V_h]^8$. Next, since Ψ_h^k , τ , S , \mathbf{A} (and the basis) are all known, we can define

$$a_{lm} = \sum_{K_i \in T_h} \int_{K_i} \mathbf{v}_{hl} \mathbf{v}_{hm}, \quad (3.52)$$

$$b_l = \sum_{K_i \in T_h} \int_{K_i} [\Psi_h^k + \tau \mathbf{S} + \tau \mathbf{A}(\Psi_h^k)(\nabla \cdot \mathbf{v}_{hl})] \mathbf{v}_{hl} \quad (3.53)$$

$$- \sum_{\Gamma_{ij} \in \Gamma_I} \int_{\Gamma_{ij}} \mathbf{H}(\Psi_h^k|_{ij}, \Psi_h^k|_{ji}, \mathbf{n}_{ij}) \mathbf{v}_{hl}$$

$$- \sum_{\Gamma_{ij} \in \Gamma_B} \int_{\Gamma_{ij}} \mathbf{H}(\Psi_h^k|_{ij}, \overline{\Psi_h^k|_{ji}}, \mathbf{n}_{ij}) \mathbf{v}_{hl},$$

$$A = \{a_{lm}\}_{l,m=1}^{l,m=L}, \quad (3.54)$$

$$b = \{b_l\}_{l=1}^{l=L}, \quad (3.55)$$

$$y = \{y_l\}_{l=1}^{l=L}, \quad (3.56)$$

and rewriting Equation (3.50) using Equation (3.52) - Equation (3.56), we come to the *fully discrete algebraic problem at time instance t_{k+1}* :

$$Ay = b, \quad (3.57)$$

whose well-posedness, and other attributes that allow for a successful solution of this system, come from the properties of the DG method. Now if we solve the system Equation (3.57), and obtain the solution vector y , we are able to reconstruct the discrete solution $\Psi_h^{k+1} \in [V_h]^8$ using the relation Equation (3.51).

In the implementation, we take the elements $K \in T_h$, of the triangulation T_h to be hexahedra.

3.7 Numerical integration

Evaluation of the integral values in Equation (3.52), Equation (3.53) is performed using the *Gaussian numerical quadrature*. A quadrature rule approximates the integral values by replacing the integral as a weighted sum of integrand values at specified points in the domain of integration. The Gaussian quadrature is constructed so that the approximation is exact for polynomials of degree $2n - 1$ (and less). This is acceptable, as our space V_h is constructed using polynomials - see section Section 3.2.1 on the page 20. We only need to take the value n to be corresponding to

Numerical integration

the value of p_m for the element K_m . The rule for both a 2-dimensional element face Γ , and a 3-dimensional cube K is derived from a one-dimensional approximation (where the interval $[-1, 1]$ is a convention):

$$\int_{-1}^1 f(x) dx = \sum_{i=1}^n w_i f(x_i),$$

where the numbers $w_i > 0$ are the *quadrature weights*, and the points (numbers in this case) x_i are the *quadrature points*, in the following way:

$$\begin{aligned} \int_{\Gamma} f(\mathbf{x}) d\mathbf{x} &= \int_{-1}^1 \int_{-1}^1 f(x_1, x_2) dx \approx \sum_{i=1}^n \sum_{j=1}^n w_i w_j f(x_{1i}, x_{2j}), \\ \int_K f(\mathbf{x}) d\mathbf{x} &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 f(x_1, x_2, x_3) dx \approx \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n w_i w_j w_k f(x_{1i}, x_{2j}, x_{3k}), \end{aligned}$$

and transformation to a generic rectangular hexahedron is performed using the transformation in one dimension:

$$\int_a^b f(x) dx \approx \frac{b-a}{2} \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right) dx.$$

Applying the Gaussian quadrature rule then results in the following one-dimensional approximation:

$$\int_a^b f(x) dx \approx \frac{b-a}{2} \sum_{i=1}^n w_i f\left(\frac{b-a}{2}x_i + \frac{a+b}{2}\right).$$

And the transformations in higher dimensions follow naturally. For $\Gamma = [a_1, b_1] \times [a_2, b_2]$ we have:

$$\int_{\Gamma} f(\mathbf{x}) d\mathbf{x} \approx \frac{b_2 - a_2}{2} \frac{b_1 - a_1}{2} \sum_{i=1}^n \sum_{j=1}^n w_i w_j f\left(\frac{b_1 - a_1}{2}x_i + \frac{a_1 + b_1}{2}, \frac{b_2 - a_2}{2}x_j + \frac{a_2 + b_2}{2}\right).$$

Numerical integration

Taking now e.g. Equation (3.52), and notation for quadrature points and weights, we can write (omitting the operand $\mathbf{x} = (x_1, x_2, x_3)$):

$$a_{lm} = \sum_{K_i \in T_h} \int_{K_i} \mathbf{v}_{hl} \mathbf{v}_{hm}, \quad (3.58)$$

$$a_{lm} := \sum_{K_i \in T_h} \int_{K_i} f(\mathbf{v}_{hl}, \mathbf{v}_{hm}), \quad (3.59)$$

$$a_{lm} \approx \sum_{K_i \in T_h} \sum_{\vec{j}=\overrightarrow{1}}^{\vec{n}} f(\mathbf{v}_{hl}(\mathbf{x}_{\vec{j}}^i), \mathbf{v}_{hm}(\mathbf{x}_{\vec{j}}^i)) w_{\vec{j}}, \quad (3.60)$$

where \vec{j} is a multi-index used in sum over (volumetric) quadrature points $\mathbf{x}_{\vec{j}}^i \in K_i$. Similarly for the right-hand side (Equation (3.53)):

$$\begin{aligned} b_l &= \sum_{K_i \in T_h} \int_{K_i} [\Psi_h^k + \tau \mathbf{S} + \tau \mathbf{A}(\Psi_h^k) (\nabla \cdot \mathbf{v}_{hl})] \mathbf{v}_{hl} \\ &\quad - \sum_{\Gamma_{ij} \in \Gamma_I} \int_{\Gamma_{ij}} \mathbf{H}(\Psi_h^k|_{ij}, \Psi_h^k|_{ji}, \mathbf{n}_{ij}) \mathbf{v}_{hl} \\ &\quad - \sum_{\Gamma_{ij} \in \Gamma_B} \int_{\Gamma_{ij}} \mathbf{H}(\Psi_h^k|_{ij}, \overline{\Psi_h^k|_{ji}}, \mathbf{n}_{ij}) \mathbf{v}_{hl}, \end{aligned} \quad (3.61)$$

$$b_l := \sum_{K_i \in T_h} \int_{K_i} g(\mathbf{v}_{hl}) - \sum_{\Gamma_{ij} \in \Gamma_I} \int_{\Gamma_{ij}} g'(\mathbf{v}_{hl}) - \sum_{\Gamma_{ij} \in \Gamma_B} \int_{\Gamma_{ij}} g''(\mathbf{v}_{hl}) \quad (3.62)$$

$$\begin{aligned} b_l &\approx \sum_{K_i \in T_h} \sum_{\vec{j}=\overrightarrow{1}}^{\vec{n}} g(\mathbf{v}_{hl}(\mathbf{x}_{\vec{j}}^i)) w_{\vec{j}} \\ &\quad - \sum_{\Gamma_{ij} \in \Gamma_I} \sum_{\vec{j}_f=\overrightarrow{1}}^{\vec{n}_f} g'(\mathbf{v}_{hl}(\mathbf{x}_{\vec{j}_f}^{ij})) w_{\vec{j}_f} \\ &\quad - \sum_{\Gamma_{ij} \in \Gamma_B} \sum_{\vec{j}_f=\overrightarrow{1}}^{\vec{n}_f} g''(\mathbf{v}_{hl}(\mathbf{x}_{\vec{j}_f}^{ij})) w_{\vec{j}_f}, \end{aligned} \quad (3.63)$$

where in addition to \mathbf{j} as explained before, \mathbf{j}_f is a multi-index used sum over face quadrature points $\mathbf{x}_{\mathbf{j}_f}^{ij} \in \Gamma_{ij}$. Based on this, we can define

$$a_{lmi}\mathbf{j} = f\left(\mathbf{v}_{hl}\left(\mathbf{x}_{\mathbf{j}}^i\right), \mathbf{v}_{hm}\left(\mathbf{x}_{\mathbf{j}}^i\right)\right) w_{\mathbf{j}}, \quad (3.64)$$

$$b_{li}\mathbf{j} = g\left(\mathbf{v}_{hl}\left(\mathbf{x}_{\mathbf{j}}^i\right)\right) w_{\mathbf{j}}, \quad (3.65)$$

$$b'_{lij}\mathbf{j}_f = \begin{cases} g'\left(\mathbf{v}_{hl}\left(\mathbf{x}_{\mathbf{j}_f}^{ij}\right)\right) w_{\mathbf{j}} & \text{if } \Gamma_{ij} \in \Gamma_I \\ g''\left(\mathbf{v}_{hl}\left(\mathbf{x}_{\mathbf{j}_f}^{ij}\right)\right) w_{\mathbf{j}} & \text{if } \Gamma_{ij} \in \Gamma_B \end{cases}. \quad (3.66)$$

3.8 Assembling the algebraic problem

Now we have a clear expression how to evaluate the integral values Equation (3.52), Equation (3.53) using Equation (3.64), Equation (3.65), Equation (3.66), but we need to construct the matrix A (Equation (3.54)), and the right-hand-side vector b (Equation (3.55)) in an effective manner. This is generally achieved through a *element-wise* assembling of these structures. Key to this is to create a data structure that identifies for a particular element K_i all the test functions \mathbf{v}_{hl} that make sense to be evaluated (have non-empty support) on K_i , i.e. we are looking for the set

$$\mathbf{v}_h(K_i) = \{\mathbf{v}_h \in V_h : \text{supp}(\mathbf{v}_h) \cap K_i \neq \emptyset\}, \quad (3.67)$$

and do the same for the faces Γ_i (both boundary, and internal):

$$\mathbf{v}_h(\Gamma_i) = \{\mathbf{v}_h \in V_h : \text{supp}(\mathbf{v}_h) \cap \Gamma_i \neq \emptyset\}. \quad (3.68)$$

Assembling the algebraic problem

Now the assembling procedure looks like this:

Algorithm 2: Assembling of the algebraic problem Equation (3.57)

```

# 1 - Loop over elements
foreach  $K_i \in T_h$  do
    Data: Quadrature points  $\{x_1^i, \dots, x_n^i\}$ 
    Data: Jacobian of the mapping  $J_{K_i}$  mapping the reference element
        (unit cube) to the actual element
    Data: Quadrature weights  $\{w_1, \dots, w_n\}$ 
    # Loop over quadrature points
    foreach  $j \in \{1, \dots, n\}$  do
        Set:  $(JxW)_j = J_{K_i} \times w_j$ 
        # Loop over test functions
        foreach  $v \in v_h(K_i)$  do
            Data:  $l$  - index of  $v$  in the global system, i.e. row in
                Equation (3.54) - Equation (3.56)
            # Loop over basis functions
            foreach  $u \in v_h(K_i)$  do
                Data:  $m$  - index of  $u$  in the global system, i.e. column in
                    Equation (3.54)
                 $a_{lm} += (JxW)_j a_{lmi} j$ 
                 $b_l += (JxW)_j b_{lij}$ 

# 2 - Loop over edges
foreach  $\Gamma_{ij} \in T_h$  do
    Data: Quadrature points  $\{x_1^{ij}, \dots, x_{n_f}^{ij}\}$ 
    Data: Jacobian of the mapping  $J_{K^+} = J_{K^-}$  mapping the reference face
        (unit square) to the actual face, where  $K^+, K^-$  are elements
        adjacent to  $\Gamma_{ij}$  if this is an internal edge, or  $K^+ = K^-$  if this is a
        boundary edge.
    Data: Quadrature weights  $\{w_1, \dots, w_{n_f}\}$ 
    # Loop over quadrature points
    foreach  $j_f \in \{1, \dots, n_f\}$  do
        Set:  $(JxW)_{j_f} = J_{K^+} \times w_{j_f}$  # Here it does not matter if we
            choose  $J_{K^+}$  or  $J_{K^-}$ 
        # Loop over test functions
        foreach  $v \in v_h(K^+)$  do
            Data:  $l$  - index of  $v$  in the global system, i.e. row in
                Equation (3.54) - Equation (3.56)
             $b_l += (JxW)_{j_f} b'_{lij} j_f$ 

```

An important remark needs to be added here, with respect to the fact, that we aim for a distributed computation. As stated before, the domain decomposition approach leads to each of the processors $P_i \in \{P_0, \dots, P_N\}$ of the total number of pro-

cessors employed for the computation owning only a subset of elements $\{K \in T\}_i$. If the processor P_i did not have any data about other elements, we would not be able to perform the evaluation involving b' defined in Equation (3.66), or more precisely g' defined in Equation (3.62) as the values $\Psi_h^k|_{ji}$ for edges Γ_{ij} of elements from $\{K \in T\}_i$ will not always be there.

To amend this situation, in the distributed triangulation, each processor holds not only the mesh element data it owns, but also data for mesh elements it needs - in Discontinuous Galerkin method, for exactly the purposes of internal face integral evaluation described in the previous paragraph. This data is called *ghost elements*, or *ghost cells*, as these are only copies provided by other processors which own the particular elements.

TODO obrazek domain decomposition s ghost cell layer

Note that in the described algorithm, and also with respect to the remarks in the previous paragraphs, periodic boundary conditions are not specifically handled, as they are only a technically more complex case of internal edges (both from the perspective of necessity of utilizing ghost cells, and from the perspective of evaluating b_l in Algorithm 2).

3.9 Time-stepping and linearization

3.9.1 Time-stepping

The simple time discretization scheme that we use in Section 3.5.1 on the page 40 allows us to simply implement the time-stepping in the following fashion:

Algorithm 3: Time-stepping procedure

```

Set:  $y_0$  = (initial solution)
Set:  $ts = 1$  # initial time step
Set:  $t = 0.00$  # initial time
# Loop over time steps
for ;  $t < T$ ;  $t = t + \tau$ ,  $ts = ts + 1$  do
    Data: Solution from the previous time step  $y^{ts-1}$ 
    1. Call procedure Algorithm 2 to obtain  $A, b(y^{ts-1})$ 
    2. Solve the problem  $Ay^{ts} = b(y^{ts-1})$  # See note below
    3. (If necessary) postprocess  $y^{ts}$ 
    4. Calculate updated value of  $\tau$ 
```

Note The step 2 (solving the algebraic problem) is of course a key point in the overall process. Because of its importance, the aim of this work is not to describe, or even implement an algebraic solver for this purpose. Many scientific teams have spent many years on publicly available open-source solvers that are usable by the

Performance considerations

software we develop for the purpose of solving the MHD phenomena. We use the existing solvers.

3.10 Performance considerations

3.10.1 Parallelization

To be able to perform any large scale calculations, we need to be able to utilize the performance of hardware at maximum. Being able to use modern, multi-core computers is an absolute must to achieve good performance, as the execution time when using parallel execution can decrease by a factor of corresponding to the number of cores - and modern machines have tens of cores available.

The parallelization is possible at several places in the overall algorithm Algorithm 2. But it makes most sense to parallelize the outer-most loop over elements, and over edges.

Another point for parallelization is the algebraic solver. As explained in Section 3.9.1, we rely on existing software packages for finding the solution of the algebraic system Equation (3.57) - all the used solvers support and heavily utilize parallelization.

3.10.2 Vectorization

Similarly as in section Section 3.10.1, the goal here is to be able to solve the discretized problem in the most efficient (fastest) way possible. One of the features that (modern) hardware offers is to employ vectorization instructions - i.e. unary or binary instructions that operate on $N, N > 1$ values (in case of unary instructions), or $N, N > 1$ pairs of values (in case of binary instructions) at the same time - the number N depends on the capabilities of the CPU, and on the precision (single or double). On the hardware that was available to perform calculations for this thesis (and based on always-used double precision), $N = \{4, 8\}$, using such instructions requires both using them in the code (one has to specify that these instructions shall be generated explicitly), and having the compiler aware of these, and able to utilize them in the generated machine code - both compilers used for work on this thesis (GNU gcc, and Microsoft Visual Studio) support vectorization instructions (SSE, SSE2, AVX, AVX2).

Vectorization helps heavily with respect to CPU time spent on calculation. In the algorithm Algorithm 2, vectorized instructions are used in:

- evaluation of the expressions $a_{lm}+ = JxW_j a_{lmKj}$
- evaluation of expressions $b_l+ = JxW_j b_{lKj}|_{edge|volumetric}$
- calculation of JxW_j

Performance considerations

3.10.3 Distribution

- AZ DO PLNE PRACE

4 Adaptive Mesh Refinement

As the Adaptive Mesh Refinement (AMR) is a very important algorithm in the overall numerical solution, handling the multi-scale aspect of the studied problems, in this chapter, a description of what needs to be taken care of for the DG method, in the distributed settings, and what concrete decisions were made during this work, and justification of these in light of the requirements on the numerical solution.

Note that in what follows, the symbol T or $T_{i \in \{0,1,2,\dots\}}$ shall denote a general computational mesh and a particular mesh in the series of refined meshes respectively. We shall drop the subscript h from T_h referring to the dimension (maximum of diameters of all elements) of the mesh where possible.

4.1 Overview of the AMR

The general schema of any Adaptive Mesh Refinement algorithm is described here:

Algorithm 4: Generic AMR algorithm

Data: Mesh T_0

Result: A mesh T_n and a solution \mathbf{y}_n on this mesh satisfying the solution acceptance criteria

$i = 0$

while *true* **do**

 obtain solution \mathbf{y}_i on T_i

 evaluate solution \mathbf{y}_i acceptance criteria

if *solution acceptance criteria satisfied* **then**

 | return

else

 | identify subset T_i^r of all elements $K \in T_i$ to be refined, $T_i^r \subseteq T_i$

 | obtain T_{i+1} by refining (at least) all $K \in T_i^r$

 | $i = i + 1$

In Line 4, *solution acceptance criteria* is usually either spatial error estimate threshold, or number of elements, etc. In the same description, the note that there might be other elements $K \notin T_i^r$ refined in order to maintain some desired attributes of the mesh, such as 1-irregularity (the refinement level difference of two elements sharing a common edge / face is at most one), smoothness of mesh (there is e.g. no unrefined elements for which all, or a majority of neighboring elements would be refined), etc.

Overview of the AMR

Since we are dealing with evolution equations, it is necessary to specify how the AMR algorithm relates with the non-AMR solution algorithm Algorithm 3. There are several points we need to take into considerations:

- slope limiting as a postprocessing step after the solution must not be omitted in case of higher-order (e.g. linear) basis functions
- the solution needs to be transferred to the refined mesh in order to be able to assemble the matrix and the right-hand side on the refined mesh in the next adaptivity iteration
- since the solution evolves and the refinements that contribute to error reduction at time step n do not contribute to error reduction at time step $n + m$ (the solution was e.g. oscillating or potentially discontinuous at time step n , but is smooth at time step $n + m$), we also want to revert such refinements as the simulation time progresses, we call this process *coarsening* of elements. For this we shall define a set T_i^c of all elements to be coarsened.

The algorithm looks like this:

Algorithm 5: AMR for time-discretized problems

```

Set:  $y_0 =$  (initial solution)
Set:  $ts = 1$  # initial time step
Set:  $t = 0.00$  # initial time
# Loop over time steps
for ;  $t < T$ ;  $t = t + \tau$ ,  $ts = ts + 1$  do
    Data: Solution from the previous time step  $y^{ts-1}$  expressed on the mesh
     $T_0^{ts}$ 
     $i = 0$ 
    while true do
        call procedure Algorithm 2 to obtain  $A_i, b_i (y_i^{ts-1})$  on the mesh  $T_i^{ts}$ 
        solve the problem  $A_i y_i^{ts} = b_i (y_i^{ts-1})$ 
        postprocess the solution  $y_i^{ts}$ 
        evaluate solution  $y_i^{ts}$  acceptance criteria
        if solution acceptance criteria satisfied then
             $y^{ts} = y_i^{ts}$ 
            break While loop
        else
            identify subset  $T_i^r$  of all elements  $K \in T_i^{ts}$  to be refined,  $T_i^r \subseteq T_i^{ts}$ 
            identify subset  $T_i^c$  of all elements  $K \in T_i^{ts}$  to be coarsened,
             $T_i^c \subseteq T_i^{ts}$ 
            obtain  $T_{i+1}^{ts}$  by refining (at least) all  $K \in T_i^r$  and coarsening a
            subset of  $T_i^c$ 
            transfer the solution  $y_i^{ts}$  onto  $T_{i+1}^{ts}$ 
             $i = i + 1$ 
        calculate updated value of  $\tau$ 

```

The critical points of the algorithm are

- solution acceptance criteria evaluation,
- identification of subset T_i^r ,
- identification of subset T_i^c .

For the last two points, we shall consider a function r

$$r : T_i^{ts} \rightarrow [0, +\infty), \quad (4.1)$$

which shall be called *refinement indicator*, and the set $T_i^r = \{K^r\}$ shall be then defined as a set of all such elements for which one of these criteria are satisfied:

- $r(K^r) > \alpha \cdot \max \{r(K) \mid K \in T_i^{ts}\}$, or
- $r(K^r) > \beta \cdot \sum \{r(K) \mid K \in T_i^{ts}\}$.

The set $T_i^c = \{K^c\}$ shall be defined similarly as

- $r(K^c) < \gamma \cdot \max \{r(K) \mid K \in T_i^{ts}\}$, or
- $r(K^c) < \delta \cdot \sum \{r(K) \mid K \in T_i^{ts}\}$.

Note that the parameters $0 < \gamma \leq \alpha < 1$, $0 < \delta \leq \beta < 1$ are artificial, and do not affect the overall solution quality (as the solution acceptance criteria has to be met independently of choices of their values). Nevertheless, the choices may affect performance - e.g. for a high α, β , many steps are needed in order to satisfy the solution acceptance criteria, and on the other hand, if values are too low, there is an unnecessary high number of degrees of freedom that do not contribute substantially to the reduction of the solution error.

Please also note, that description of edge cases, and limitations are not given here for brevity. These cases include e.g. what happens if an element is both selected for coarsening and refinement, or how we maintain a 'minimal' refinement level so that we do not coarsen beyond a rational limit.

4.2 Adaptive-mesh refinement and DG

With the Discontinuous Galerkin method, using AMR brings additional complexity into the evaluation of face integrals described in Equation (3.62), Equation (3.66). To describe the process in detail, definition of the numerical flux Equation (3.15) needs to be taken into account. What is evaluated during the assembling procedure Algorithm 2 is the term Equation (3.66). In order to evaluate

TODO obrázky možných případů sousedících elementů, jak zassemblovat

TODO zmínka o peridických podmírkách, že to je zase o tom samém, pridat nekam

Reference solution approach

4.2.1 Relationship with slope limiters

TODO tady hlavne ze ten relativne drahý napocet sousedu ve vrcholu (je drahý proto, ze na to klasické struktury nejsou delané) musim tedy delat rozumne, pak ze samozrejme musim respektovat refinement a zajimaji me jen aktivni elementy

4.3 Reference solution approach

As the aim of this work is to prepare a universally usable solver for MHD problems, the refinement indicator Equation (4.1) must ideally not be dependent on any attributes of the solved problem data (initial condition, boundary conditions, physical quantities such as γ , etc.).

5 Results

In this section, results from the computation using the implemented software are presented. There are two classical benchmarks for 3-dimensional MHD equations, namely the MHD Blast [15], [1], and the Orszag-Tang vortex [20]. And then the main section of this work contains results from the flux tube eruption model on and above the Sun's surface.

5.1 Benchmarks

The benchmarks presented in this Section do not have an exact analytical solution, but the formation of waves and discontinuities is well studied, and benchmarking is usually performed on the basis of comparing the structure and presence of non-physical attributes.

5.1.1 MHD Blast

MHD Blast - original version

This benchmark has been used for decades - [29], [15], [1] - in a variety of configurations and as a benchmark in software - e.g. [25]. The setup as described in [29], [15] is defined (although in [15] with interchanged x - and y - coordinates) by the initial conditions:

$$\begin{aligned} \gamma &= 5/3 & (5.1) \\ p_0(\mathbf{x}, t) &= 100 \text{ for } |\mathbf{x}| < 0.1 \\ p_0(\mathbf{x}, t) &= 1 \text{ for } |\mathbf{x}| \geq 0.1 \\ \rho(\mathbf{x}, t = 0) &= 1, \\ p(\mathbf{x}, t = 0) &= p_0(\mathbf{x}, t), \\ \mathbf{u}_1(\mathbf{x}, t = 0) &= 0, \\ \mathbf{u}_2(\mathbf{x}, t = 0) &= 0, \\ \mathbf{u}_3(\mathbf{x}, t = 0) &= 0, \\ \mathbf{B}_1(\mathbf{x}, t = 0) &= 0, \\ \mathbf{B}_2(\mathbf{x}, t = 0) &= 100, \\ \mathbf{B}_3(\mathbf{x}, t = 0) &= 0. \end{aligned}$$

Benchmarks

Total energy is calculated using Equation (1.1), Equation (1.2), Equation (1.3). The domain Ω is a square, with scaling quite arbitrarily used in the papers. In the case of [15], $\Omega = [0, 1] \times [0, 1]$, in this work it is $\Omega = [-0.25, 0.25] \times [-0.25, 0.25]$.

It is obvious from the initial setup, that the example is true to its name, and it is in fact a blast of the over-pressured area $|\mathbf{x}| < 0.1$, where pressure p is $100\times$ larger than elsewhere in the domain. This setup is completed with simple outflow boundary condition Equation (2.40). In the next figures, the solution as presented in [15] is compared to the solution obtained with the approach described in this work. Note that the solution from [15] needed to have the axes transformed ($x = y$) with respect to the original paper. The figure Figure 5.1 is taken from the article [15] from the page 33. Unfortunately the paper does not specify the precise time at which the snapshots are taken.

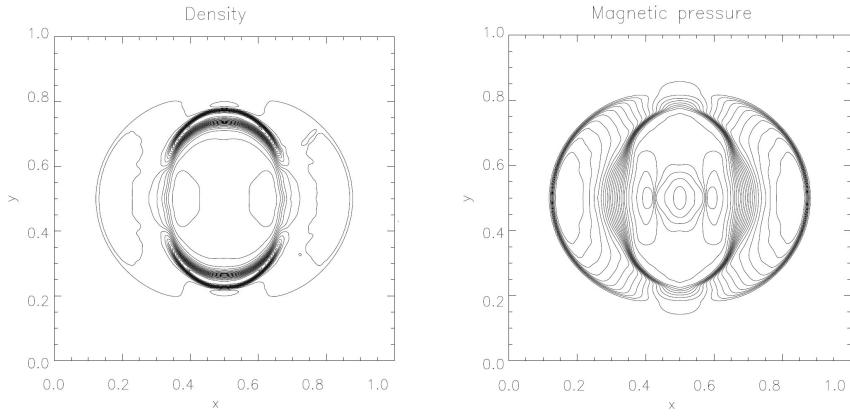


Figure 5.1: Results from [15], density(left), magnetic energy(right)

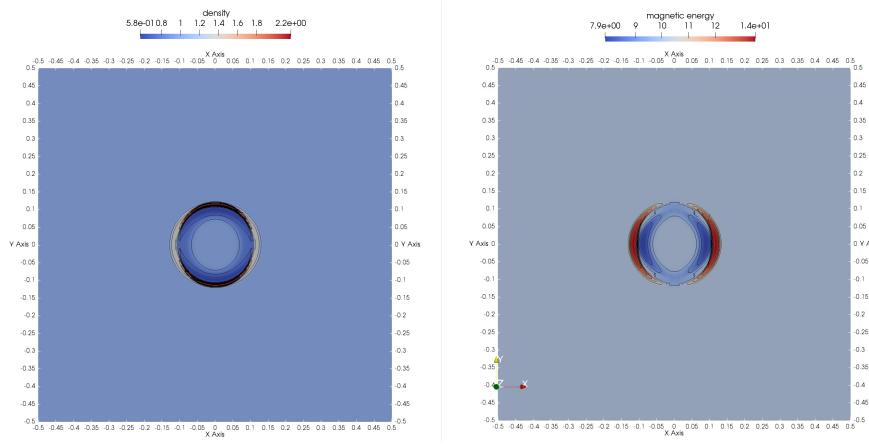


Figure 5.2: Obtained results, $t = 1 \times 10^{-3}$, density(left), magnetic energy(right)

Benchmarks

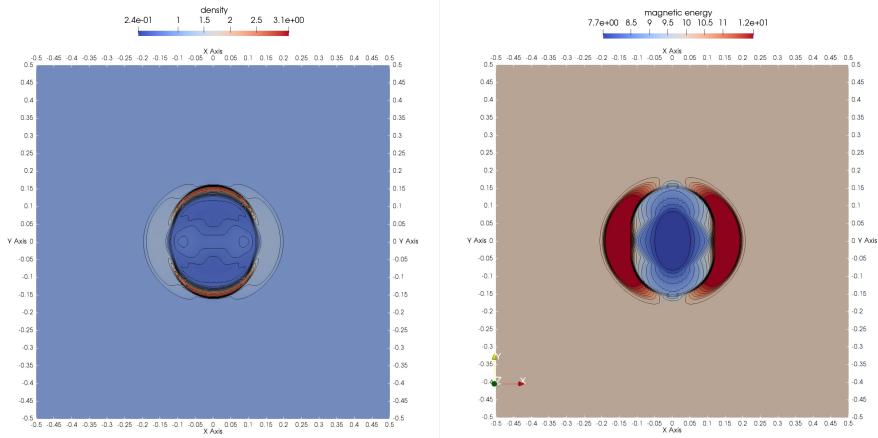


Figure 5.3: Obtained results, $t = 6 \times 10^{-3}$, density(left), magnetic energy(right)

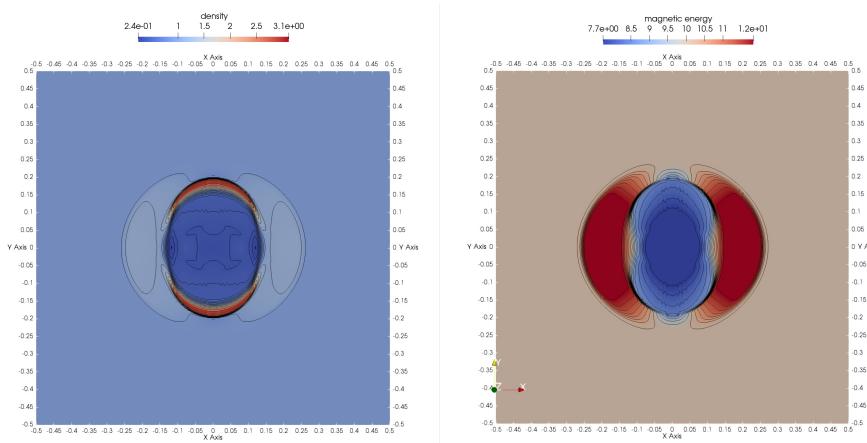


Figure 5.4: Obtained results, $t = 11 \times 10^{-3}$, density(left), magnetic energy(right)

Benchmarks

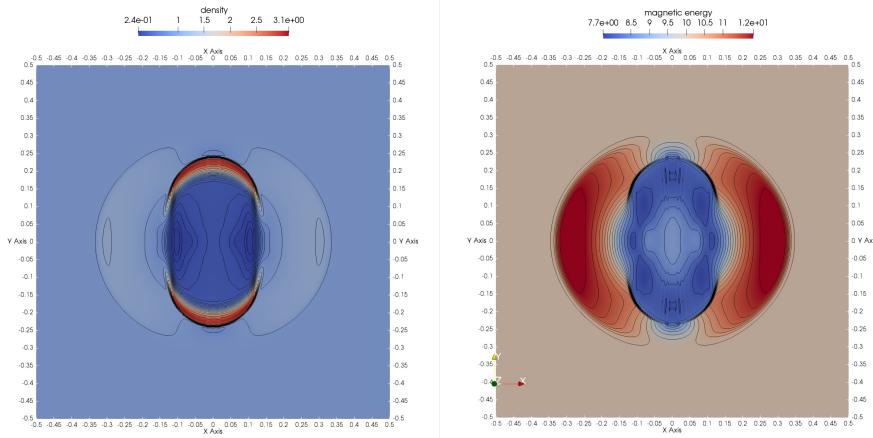


Figure 5.5: Obtained results, $t = 16 \times 10^{-3}$, density(left), magnetic energy(right)

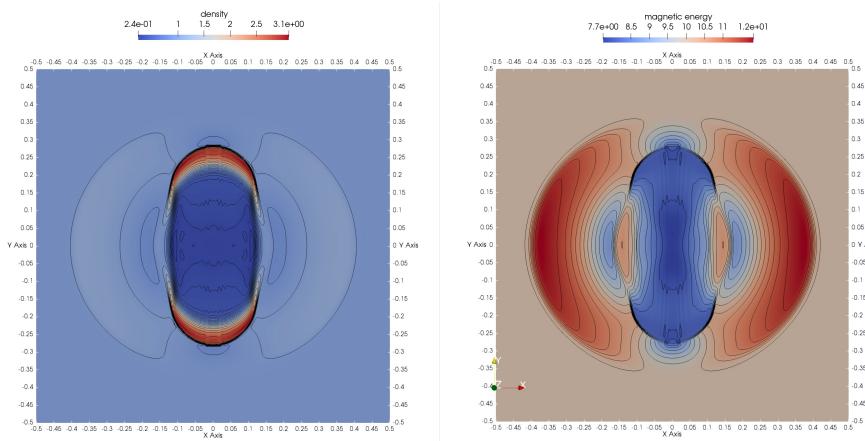


Figure 5.6: Obtained results, $t = 23 \times 10^{-3}$, density(left), magnetic energy(right)

The solution in Figure 5.5, Figure 5.6 is apparently almost identical to that in Figure 5.1. To demonstrate the distributed nature of the computation, in Figure 5.7, color-mapping of elements $K \in T$ to processors owning the particular element is presented (see Section 3.1.1 for details). Note that there were 48 processors used for the computation.

Benchmarks

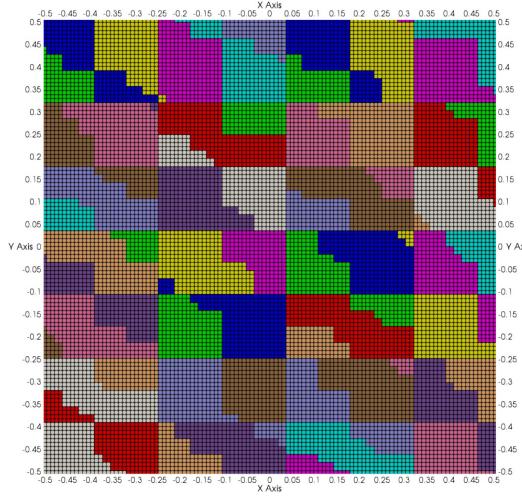


Figure 5.7: Color-mapping of elements to processors for the computation of Figures 5.2 to 5.6

This however, was a static calculation with 200 mesh elements in the x - and y -dimensions. In order to see how the AMR (see ??) performs, the same computation was performed in adaptive setup. Starting from a very coarse mesh of 10 elements in both dimensions, the evolving mesh and its distribution to processors are shown in ??-?? - the distribution to processors on the left, the mesh elements on the right.

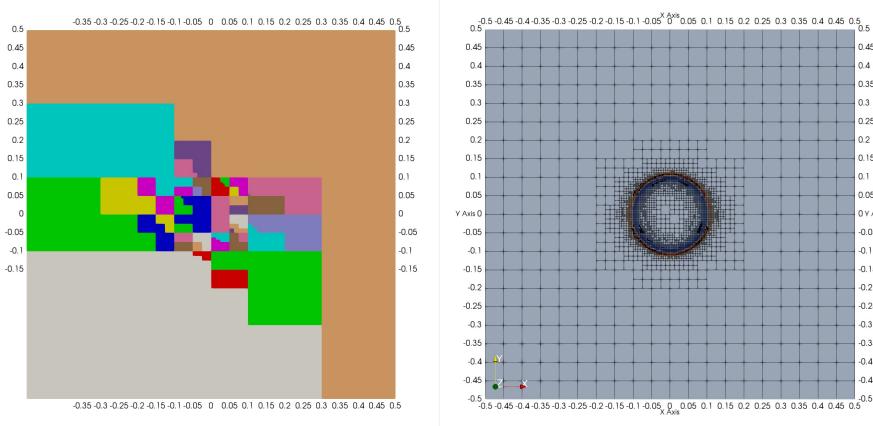


Figure 5.8: Obtained results, initial state, distribution of elements (right) to processors (left)

Benchmarks

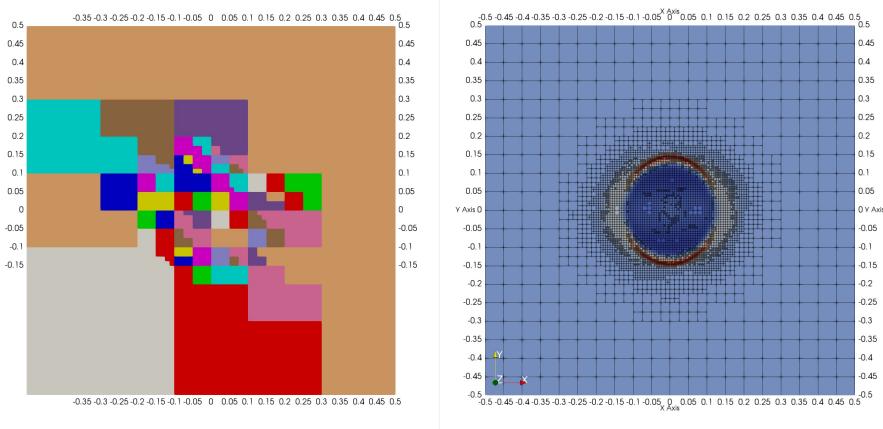


Figure 5.9: Obtained results, $t = 5 \times 10^{-3}$, distribution of elements (right) to processors (left)

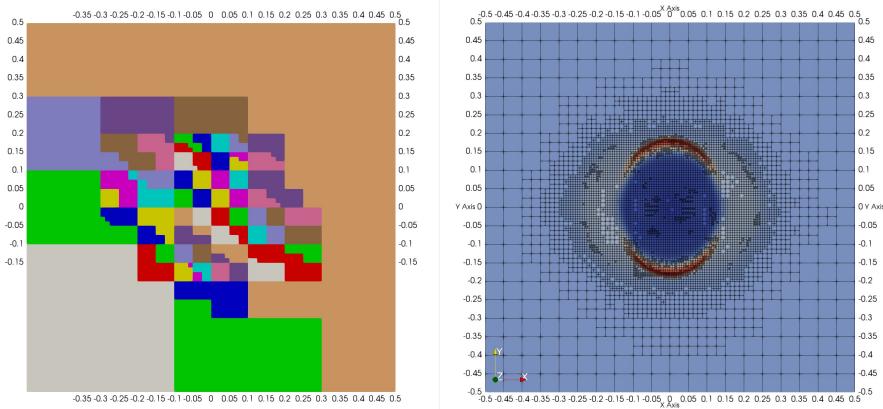


Figure 5.10: Obtained results, $t = 10 \times 10^{-3}$, distribution of elements (right) to processors (left)

Benchmarks

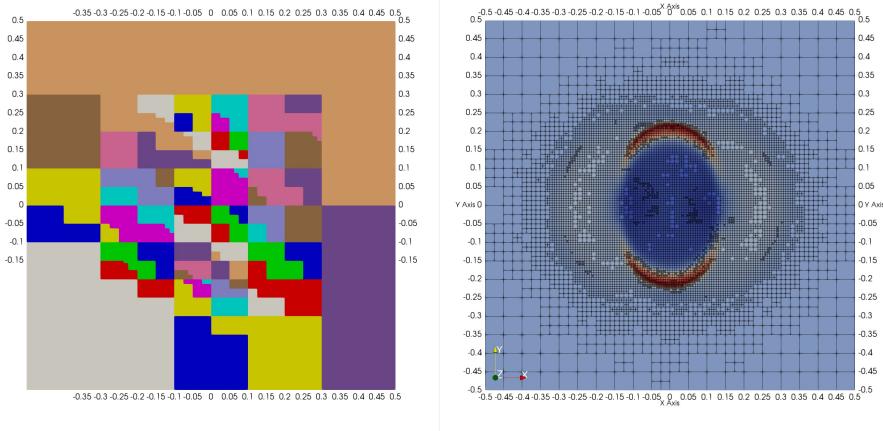


Figure 5.11: Obtained results, $t = 15 \times 10^{-3}$, distribution of elements (right) to processors (left)

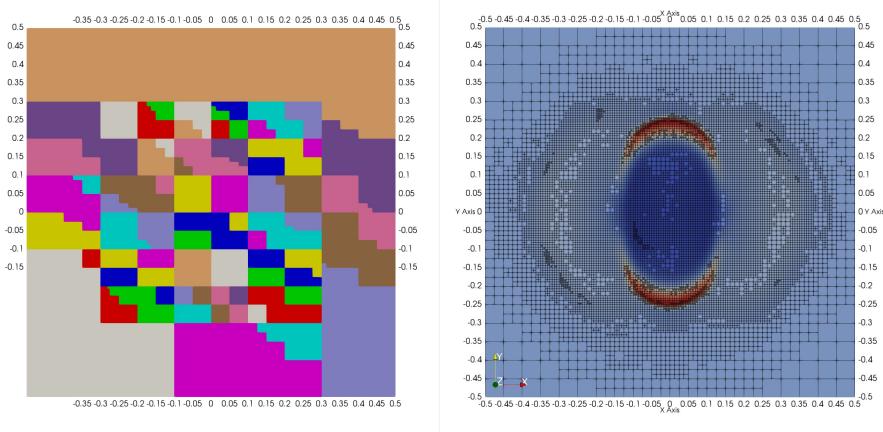


Figure 5.12: Obtained results, $t = 20 \times 10^{-3}$, distribution of elements (right) to processors (left)

Benchmarks

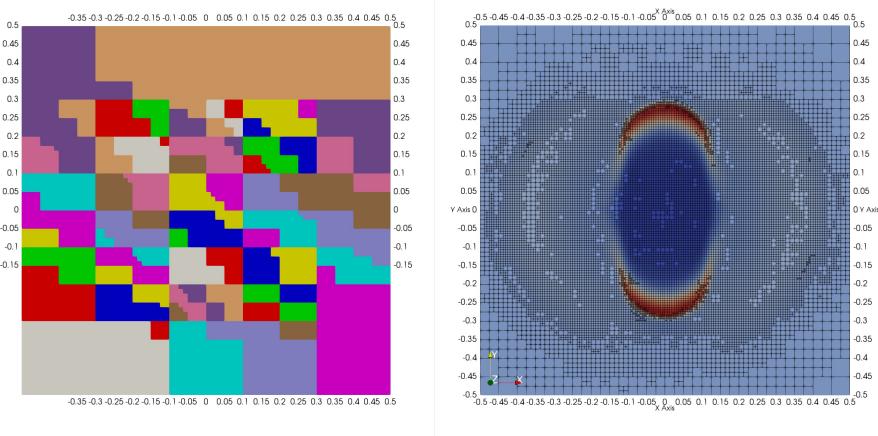


Figure 5.13: Obtained results, $t = 25 \times 10^{-3}$, distribution of elements (right) to processors (left)

MHD Blast - extended version

An extended version of the benchmark has been used in [9], [26], and a similar problem was used also in [28] - the description of the benchmark is also available at: <http://www.astro.princeton.edu/~jstone/Athena/tests/blast/blast.html>. In this version, the domain dimensions are set as a rectangle: $\Omega = [-0.5, 0.5] \times [-0.75, 0.75]$. The initial conditions are a little different than in the case of Equation (5.1), and read:

$$\begin{aligned} \gamma &= 5/3 & (5.2) \\ p_0(\mathbf{x}, t) &= 10 \text{ for } |\mathbf{x}| < 0.1 \\ p_0(\mathbf{x}, t) &= 0.1 \text{ for } |\mathbf{x}| \geq 0.1 \\ \rho(\mathbf{x}, t=0) &= 1, \\ p(\mathbf{x}, t=0) &= p_0(\mathbf{x}, t), \\ \mathbf{u}_1(\mathbf{x}, t=0) &= 0, \\ \mathbf{u}_2(\mathbf{x}, t=0) &= 0, \\ \mathbf{u}_3(\mathbf{x}, t=0) &= 0, \\ \mathbf{B}_1(\mathbf{x}, t=0) &= \frac{1}{\sqrt{2}}, \\ \mathbf{B}_2(\mathbf{x}, t=0) &= \frac{1}{\sqrt{2}}, \\ \mathbf{B}_3(\mathbf{x}, t=0) &= 0, \end{aligned}$$

with periodic boundary conditions on the top-bottom, and left-right parts of the boundary. That is, with respect to Equation (2.41), the two pairs $\Gamma_1, \Gamma_2, \Gamma'_1, \Gamma'_2$ are

Benchmarks

specified as follows:

$$\Gamma_1 = \{-0.5\} \times [-0.75, 0.75], \quad (5.3)$$

$$\Gamma_2 = \{0.5\} \times [-0.75, 0.75], \quad (5.4)$$

and

$$\Gamma'_1 = [-0.5, 0.5] \times \{-0.75\}, \quad (5.5)$$

$$\Gamma'_2 = [-0.5, 0.5] \times \{0.75\}. \quad (5.6)$$

Calculation specification

A mesh T_h formed by rectangular hexahedra was used. The value of the time step τ used was set according to the CFL condition (Section 3.5.2). Illustration of the obtained results follows below - these results were obtained using one node of the department's computational cluster with these parameters:

- CPU: Intel(R) Xeon(R) CPU E5-2680 v3 @ 2.50GHz
- # of Cores: 48 per node (nodes 1, 2), 16 per node (nodes 3, 4)
- Vectorization support: AVX2
- Parallelization implemented: Intel TBB
- Distributed calculation implemented: OpenMPI
- RAM: 512 GB
- OS: Ubuntu 15.10

Results

The results show distribution of plasma - ρ in the domain.

TODO - moje vysledky Blastu

TODO - moje vysledky Blastu s adaptivitou

5.1.2 Orszag-Tang vortex

This problem was first described in [20] and has been extensively used as a benchmark for 2- and 3- dimensional MHD code ([29], [15], [22], [?], and many others). It is a simple model of the evolution of MHD turbulence including interactions between the several shock waves that appear. The Orszag-Tang system is defined by

Benchmarks

the initial conditions:

$$\begin{aligned}
\rho_0 &= \frac{25}{36\pi} & (5.7) \\
p_0 &= \frac{5}{12\pi} \\
B_0 &= \sqrt{\frac{1}{4\pi}} \\
\gamma &= 5/3 \\
\rho(\mathbf{x}, t=0) &= \rho_0, \\
p(\mathbf{x}, t=0) &= p_0, \\
\mathbf{u}_1(\mathbf{x}, t=0) &= -\sin(2\pi y), \\
\mathbf{u}_2(\mathbf{x}, t=0) &= \sin(2\pi x), \\
\mathbf{u}_3(\mathbf{x}, t=0) &= 0, \\
\mathbf{B}_1(\mathbf{x}, t=0) &= -B_0 \sin(2\pi y), \\
\mathbf{B}_2(\mathbf{x}, t=0) &= B_0 \sin(4\pi x), \\
\mathbf{B}_3(\mathbf{x}, t=0) &= 0, \\
U(\mathbf{x}, t=0) &= \frac{p_0}{\gamma - 1} + U_m(\mathbf{x}, t) + U_k(\mathbf{x}, t),
\end{aligned}$$

where the last term is an application of Equation (1.1), Equation (1.2), Equation (1.3). The domain Ω is set as $\Omega = [0, 1] \times [0, 1]$ and the problem is equipped with periodic boundary conditions on the top-bottom, and left-right parts of the boundary, similarly as in Section 5.1.1. This configuration is strongly unstable, leading to a wide spectrum of propagating MHD modes and shock waves.

As before, in order to compare with reference papers, figures are presented from these papers - see Figure 5.8. The images are taken from pages 30 ([15]), and 20/282 ([29]) respectively. All are taken at $t = 0.5$.

Flux tube eruption model

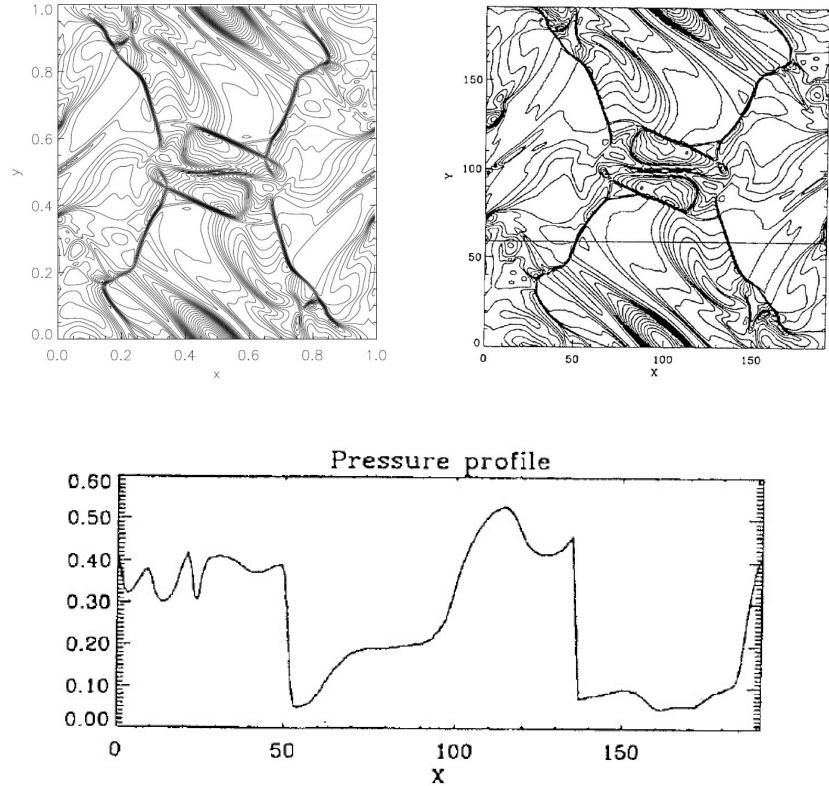


Figure 5.14: p isolines from [15] (top left), p isolines from [29] (top right), p along $y = 0.3125$ from [29] (bottom).

TODO - vysledky z cite otnew
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5.2 Flux tube eruption model

This model is based on the original Titov-Demoulin model from [27], as used in [13].

The model parameters are as follows. Note that k_B is the Boltzmann constant $k_B = 1.38064852 \times 10^{-23} \frac{\text{J}}{\text{K}}$, m_p is the plasma mass, and g gravitational acceleration.

$$\beta = 0.05 \quad \dots \text{Plasma beta} \quad (5.8)$$

$$L_G = 2 k_B \frac{T_{ext}}{(m_p g)} = 1.2 \times 10^8 \text{ [m]} \quad (5.9)$$

$$L_G = 20 \quad \dots \text{Coronal height scale in dimension-less units} \quad (5.10)$$

$$!!! \quad (5.11)$$

TODO - Pr

$$(5.12)$$

$$(5.13)$$

$$N_t = -3 \quad \dots \text{Torus winding number} \quad (5.14)$$

$$R = 4 \quad \dots \text{Torus major radius} \quad (5.15)$$

$$2L = 4 \quad \dots \text{Magnetic charge separation distance} \quad (5.16)$$

$$d = 2 \quad \dots \text{Geometrical factor} \quad (5.17)$$

$$q_{mag} = \left| \frac{\ln(8e^{-5/4}R)}{4} N_t \left(\frac{L}{R} \right)^2 \left[1 + \left(\frac{R}{L} \right)^2 \right]^{3/2} \right| \quad (5.18)$$

$$q_{mag} \approx \text{Normalised magnetic charge correspond} \quad (5.19)$$

$$q_{mag} = \frac{\ln(8R) - \frac{5}{4}}{4} |N_t| \frac{\left(1 + \left(\frac{L}{R} \right)^2 \right) \sqrt{1 + \left(\frac{L}{R} \right)^2}}{\frac{L}{R}} \quad (5.20)$$

$$(5.21)$$

TODO - Ner

$$(5.22)$$

$$(5.23)$$

$$H = 2 \frac{N_t^2}{R^2} \quad \dots \text{"Helicity" factor inside the loop} \quad (5.24)$$

$$\frac{T_{ext}}{T_{in}} = 10 \quad \dots \text{Coronal/prominence temperature ratio} \quad (5.25)$$

$$(5.26)$$

TODO - Pre

$$(5.27)$$

$$(5.28)$$

$$(5.29)$$

6 Conclusion, outlook

Further work will focus on real-world industrial and astrophysical problems, mainly study of the magnetic field reconnection - [4], and other phenomena occurring both in solar physics and in industrial applications of plasma flow. Before the implemented software package is ready to be used for solution of such problems, it needs to be improved in several ways - first, the benchmark shown in this work clearly showed, that a more stable numerical scheme needs to be implemented. Second, the performance needs to be tweaked by the use of distributed computations. This will need to employ some of the following techniques:

- Second-order scheme for the discretization of the time derivative
- Adaptive algorithm for the discretization of the time derivative
- Shock-capturing for high-order DG scheme to prevent non-physical oscillations
- Adaptive algorithm for the discretization of the space derivatives communication
- A specific shaperset of basis and test functions based on Taylor expansions.

After the implementation is stable for the benchmark presented here, and as well as additional benchmarks (such as Orszag-Tang Vortex - [20]), next steps are distributed calculation of larger real-world problems and further work on performance improvements. Performance improvements will include some of the following:

- Caching of values that are necessary in multiple spaces of the algorithm (utilizing the RAM)
- Further use of vectorization for evaluation of integral quantities
- Distributed computation using the Message Passing Interface
- Optimal selection of both solver, and preconditioner for the solution of the resulting algebraic problems

Step after that is adding of additional physics into the implementation, mainly in the form of magnetic reconnection models. From the implementation point of view, additional physics usually implies extending the matrix assembly routines by evaluating more physical quantities, possibly in a different setup (e.g. point values, particle-based quantities, etc.), or either post-processing or iterating the results in order to add some new rules and relations that must hold for the solution.

Plan of work:

6 Conclusion, outlook

Quarter	Activities
Q2/2016	Stabilize the algorithm, achieve both stable, reliable, and fast calculation of 2 - 3 benchmarks
Q3/2016	Implementing more numerical fluxes, performance improvements
Q4/2016	Adding physics (e.g. magnetic reconnection), start with real-world problems
Q1+Q2/2017	Distributed computations, adaptive algorithms

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