Combined Adaptive Multimesh hp-FEM/hp-DG for Multiphysics Coupled Problems of Compressible Flow

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Motivation

In multiphysics problems, physical field exhibit qualitative differences:

- singularities
- boundary layers
- smooth areas

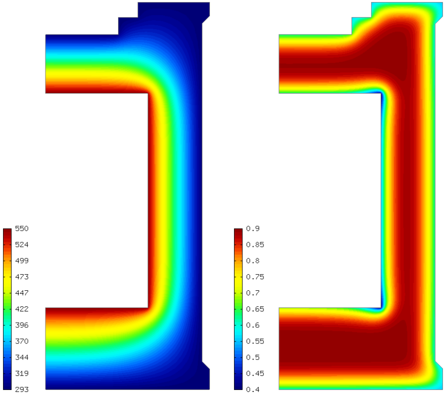
These phenomena require specific treatment.

Our goal: approximate every physical quantity in an optimal way.

- appropriate mesh (for each component of the system)
- appropriate Sobolev space ($H^1, L^2, H_{curl}, H_{div}$, ..)
- appropriate method (continuous FEM, DG method) of higher order

Component specific meshes example

Coupled system of linear second-order PDE
Heat and moisture transfer in concrete



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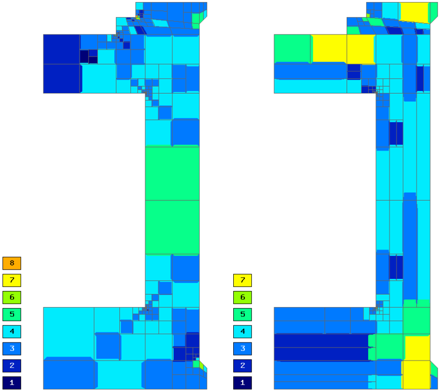
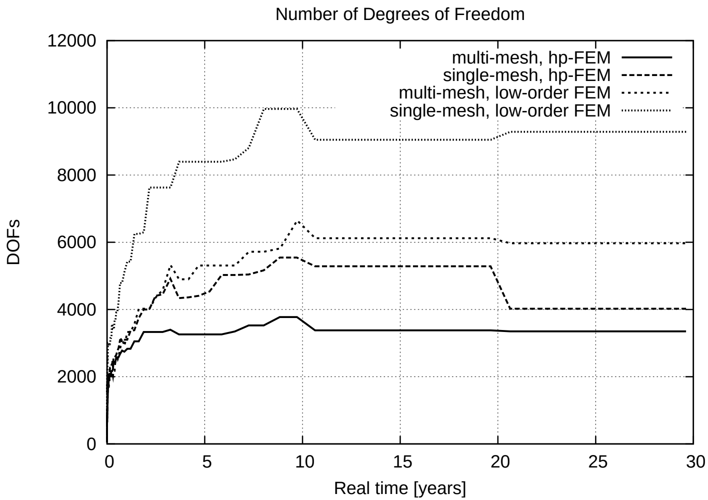


Component specific meshes example - DOFs



Monolithic Multimesh Adaptive FEM

- meshes come from one **common coarse mesh** (can be one element) that we call master mesh
- every mesh in the system is obtained from this one via an **arbitrary sequence of** elementary **refinements**
- adaptivity puts all elements of all meshes into one array, error estimate as a difference between the coarse and reference mesh approximations
- elements sorted according to the error, elements with the largest errors are **refined as in the standard hp-FEM**
  - well suited for time multiscale transient problems

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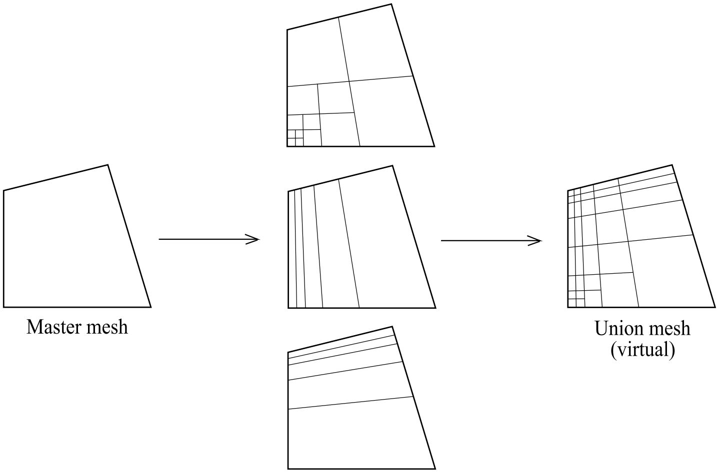

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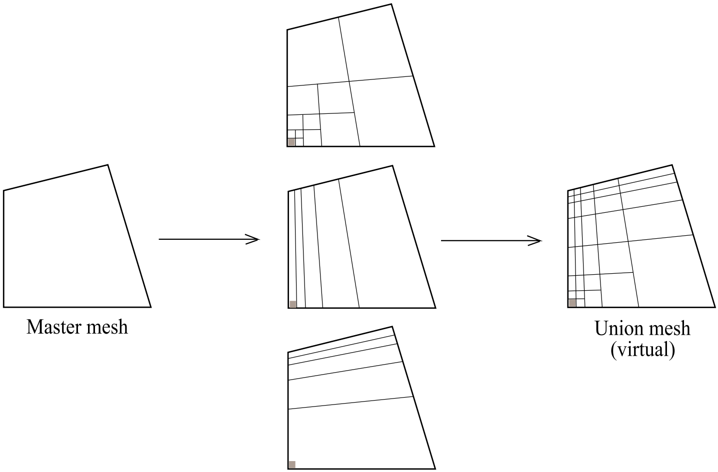

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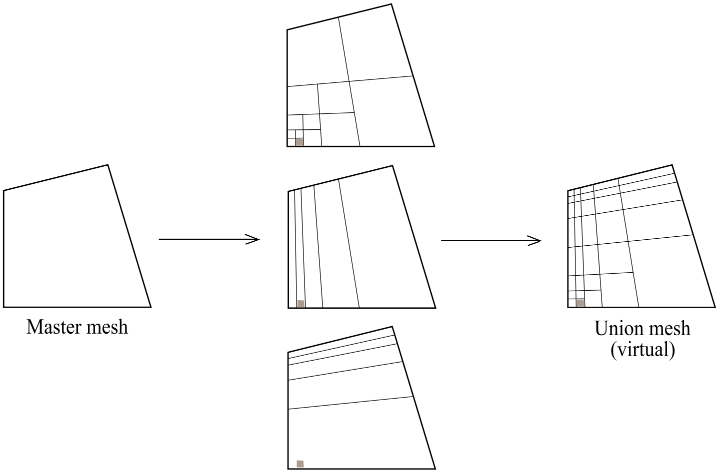

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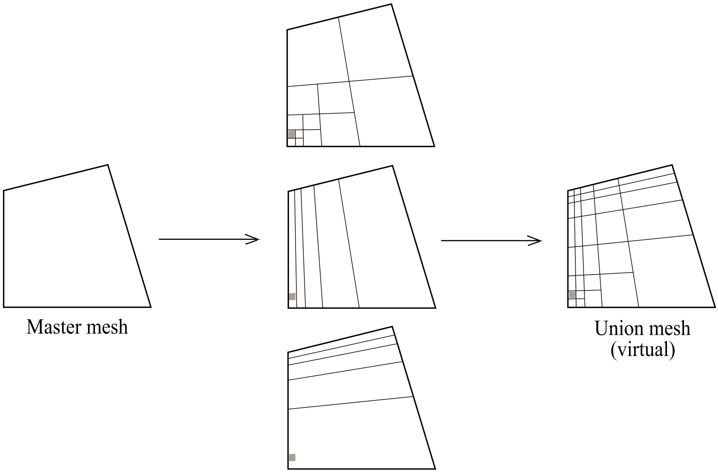

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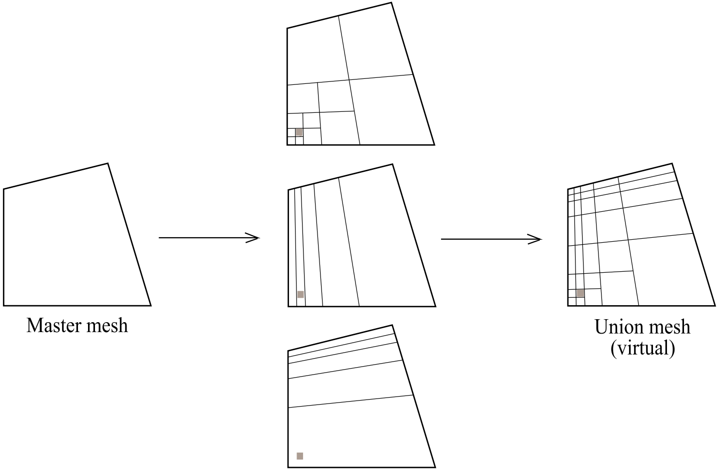

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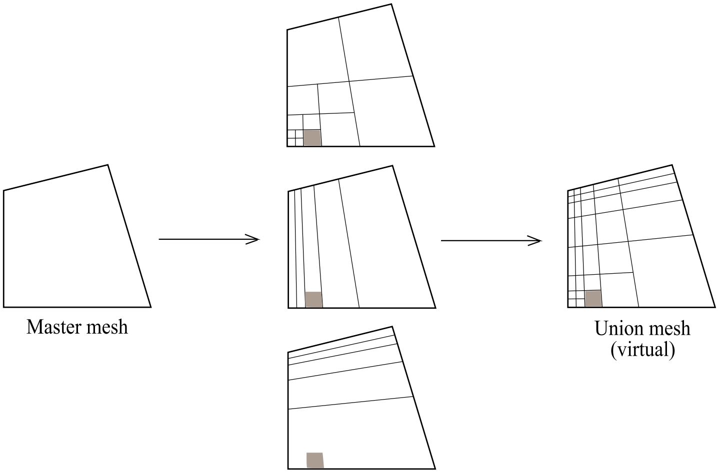

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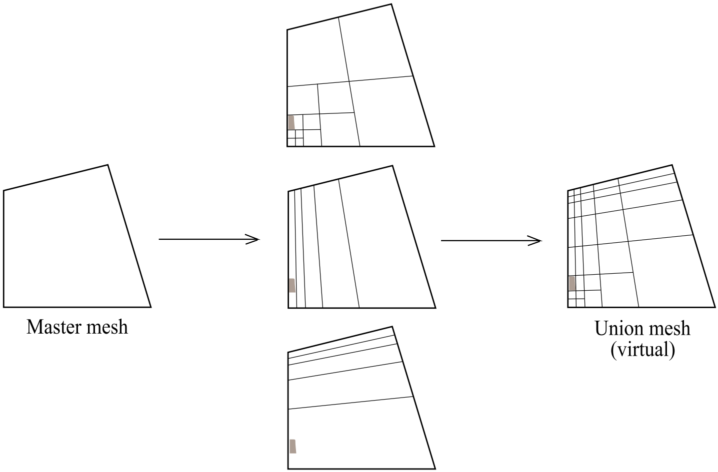

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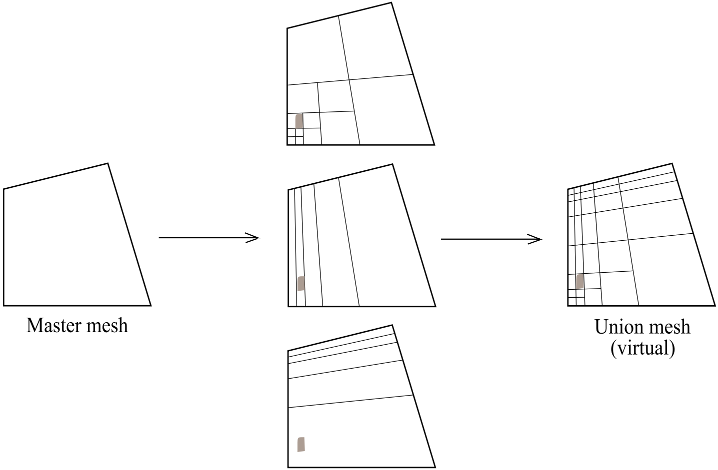

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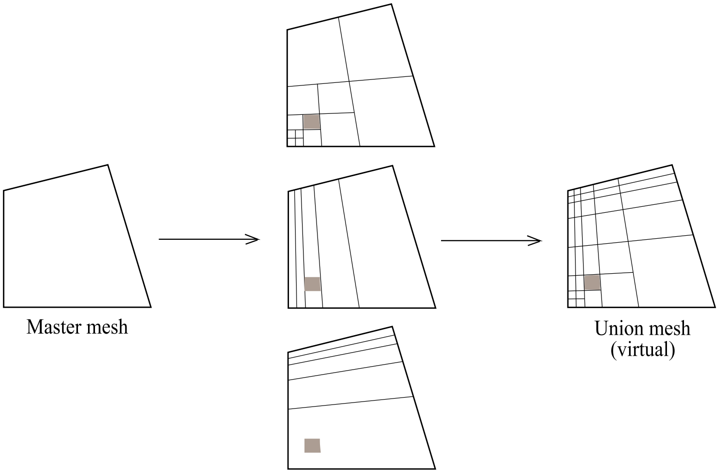

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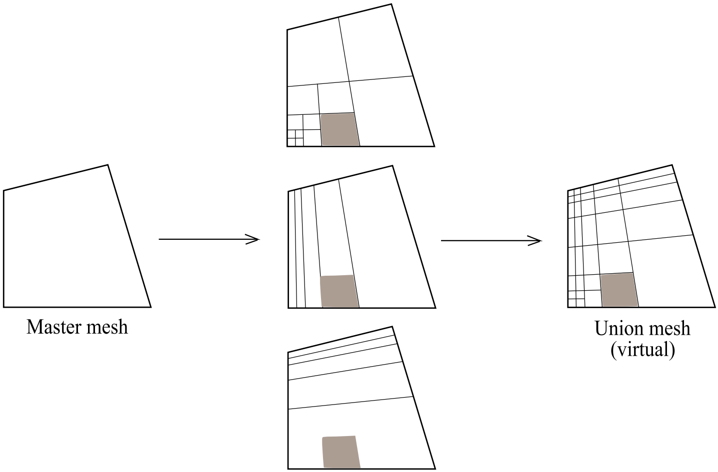

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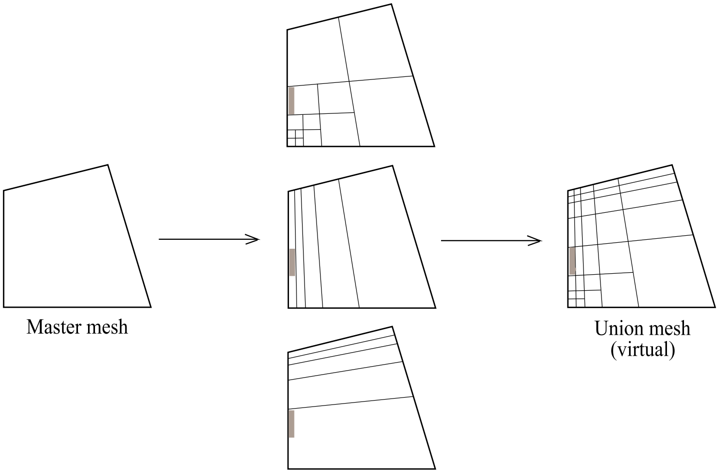

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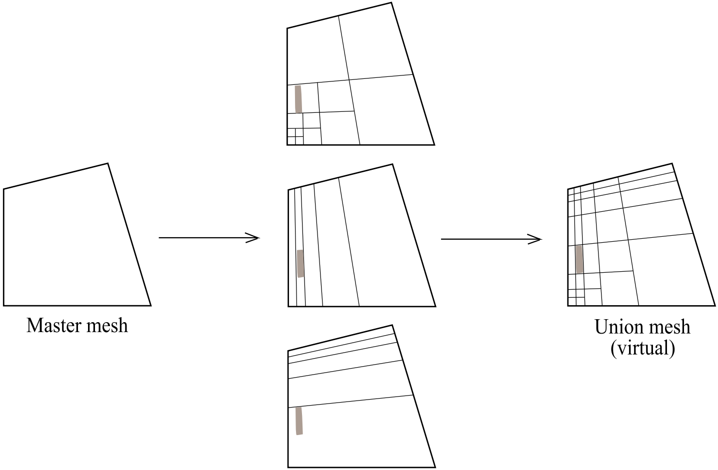

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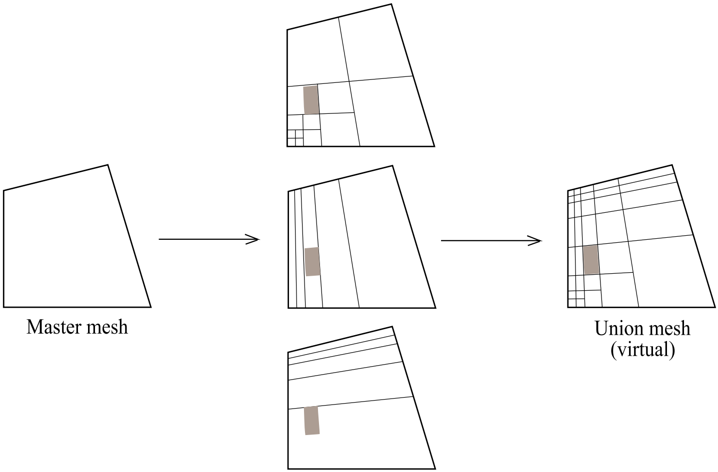

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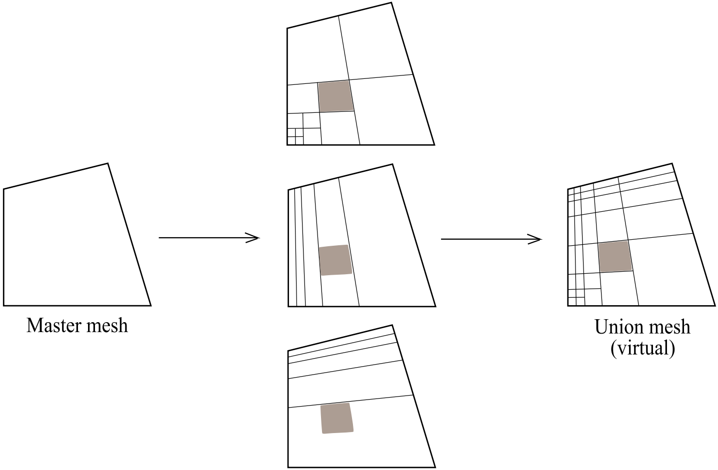

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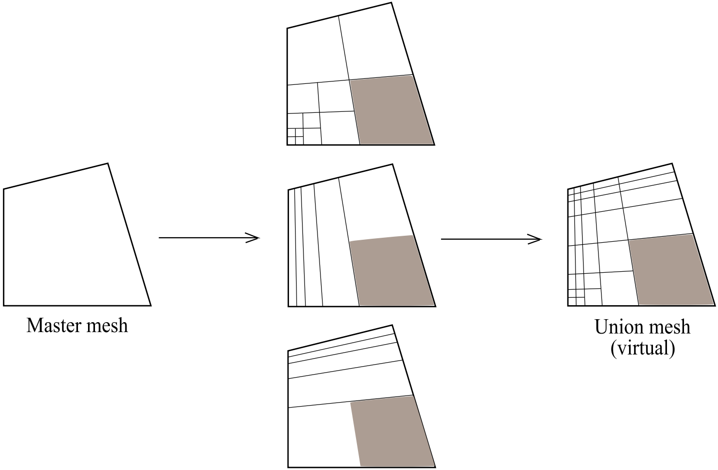

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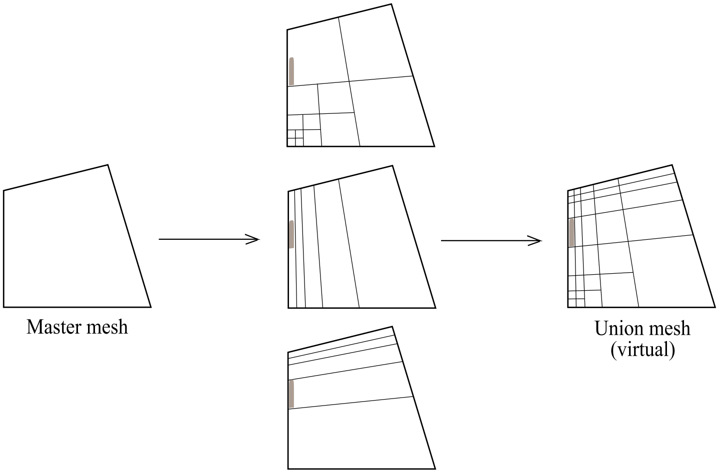

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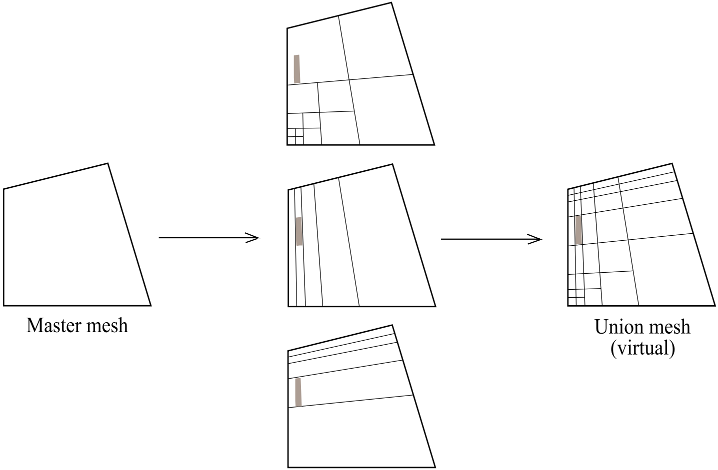

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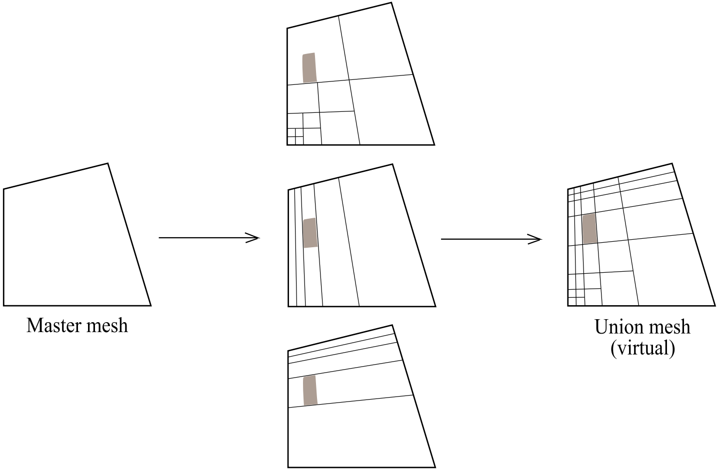

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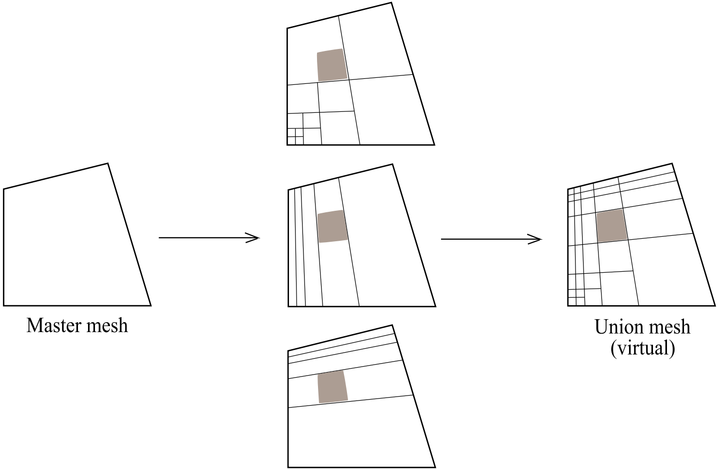

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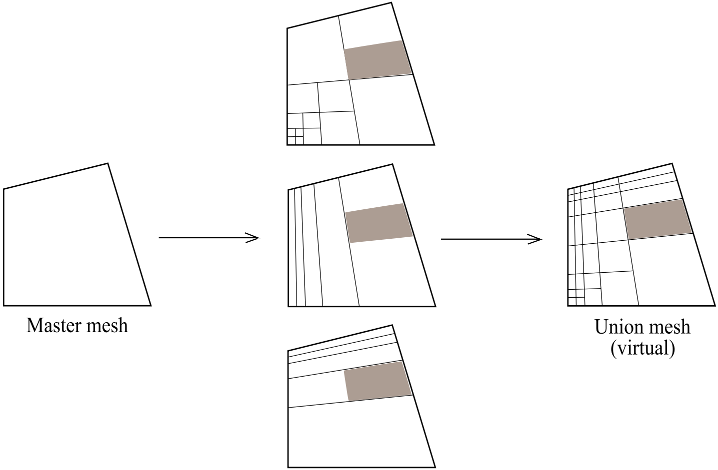

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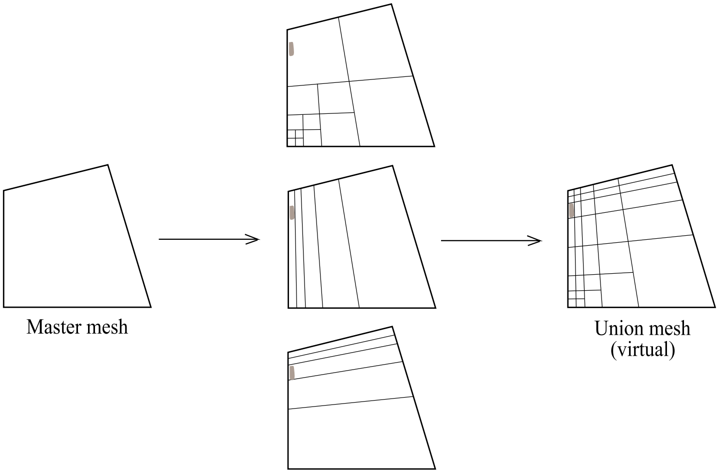

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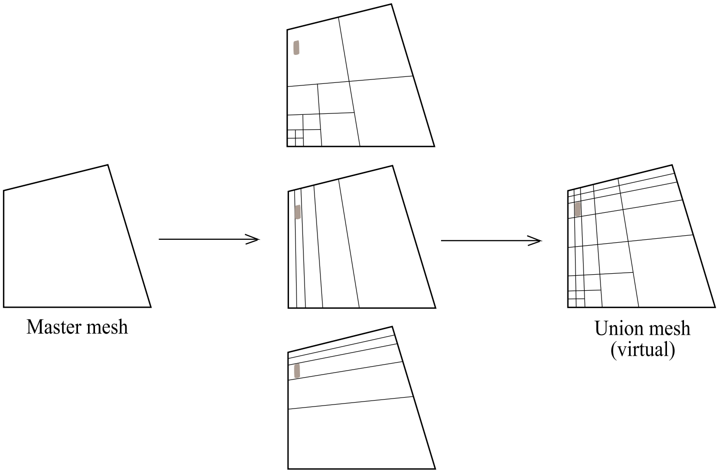

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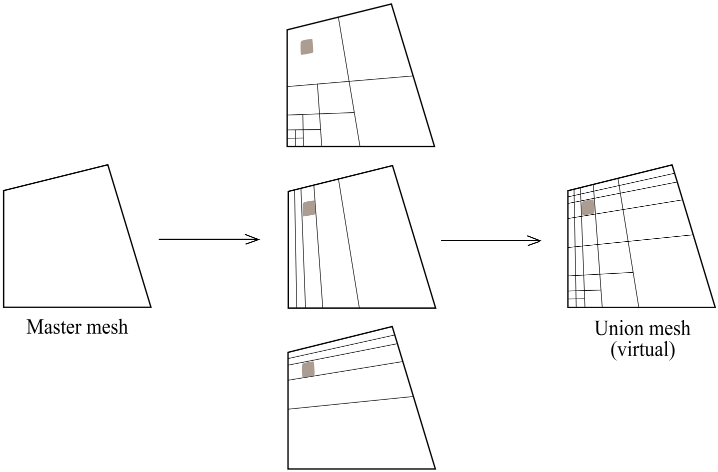

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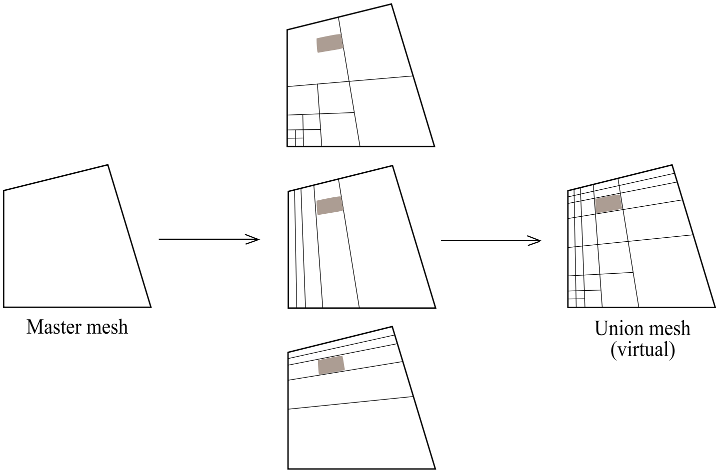

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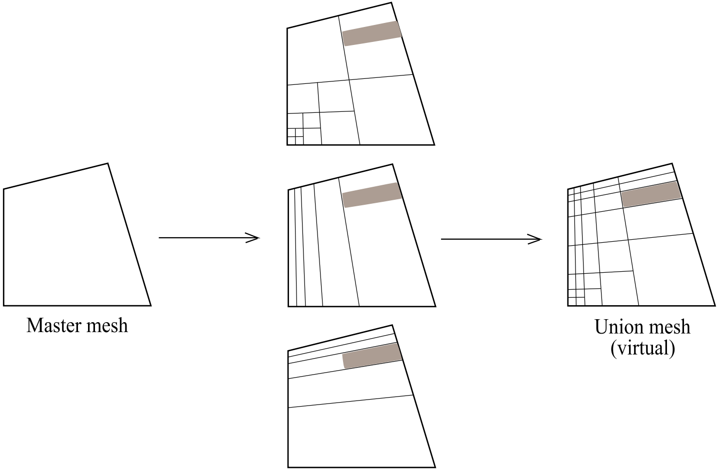

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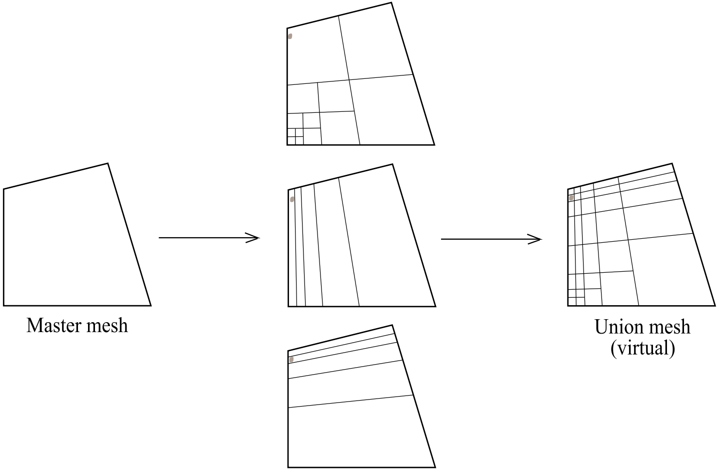

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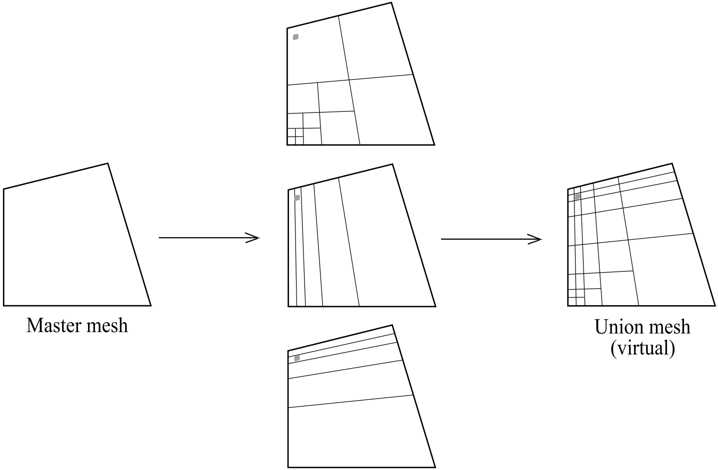

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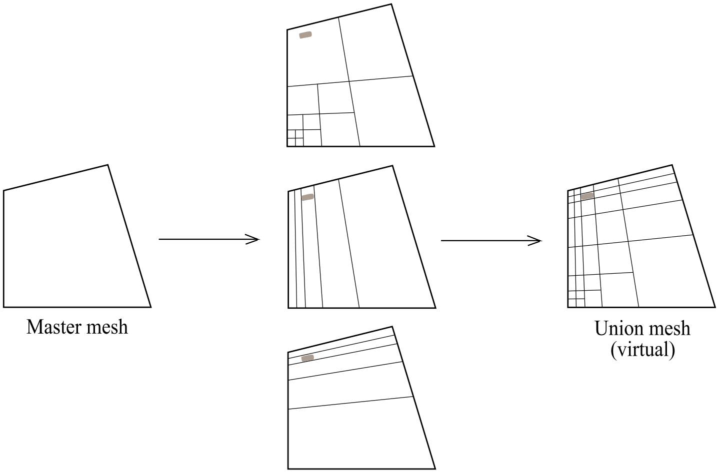

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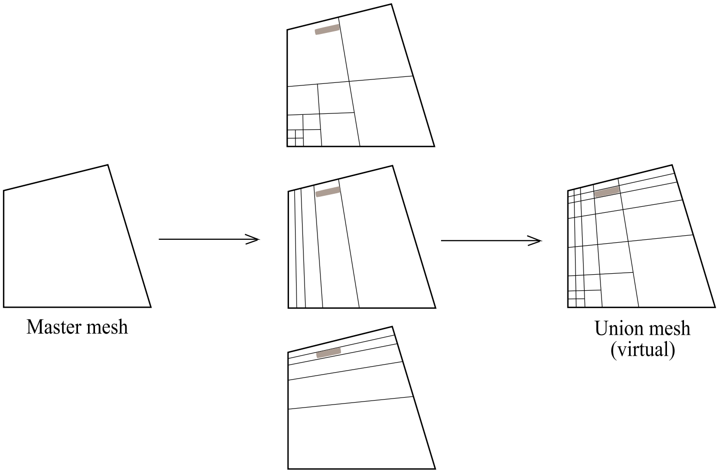

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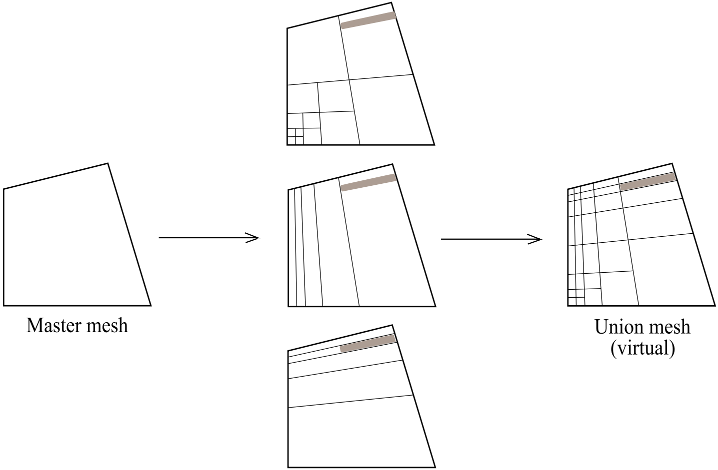

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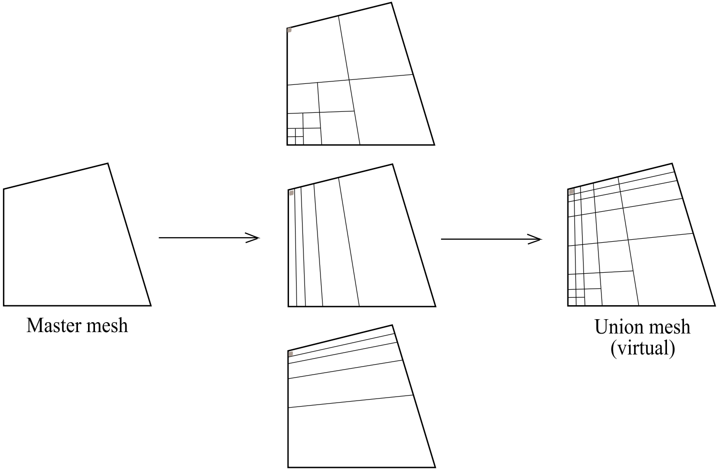

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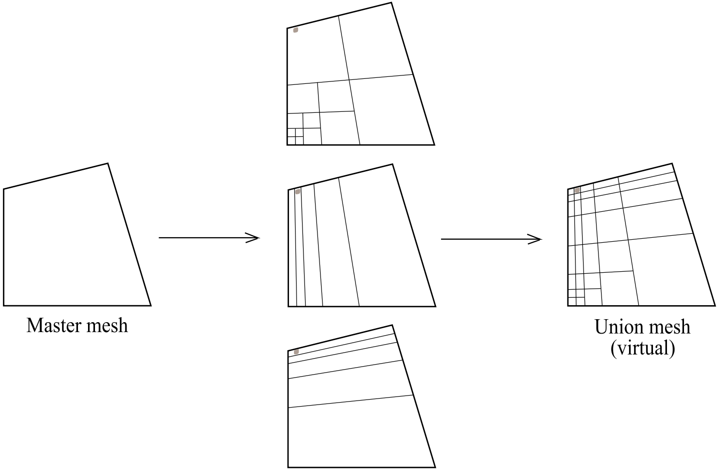

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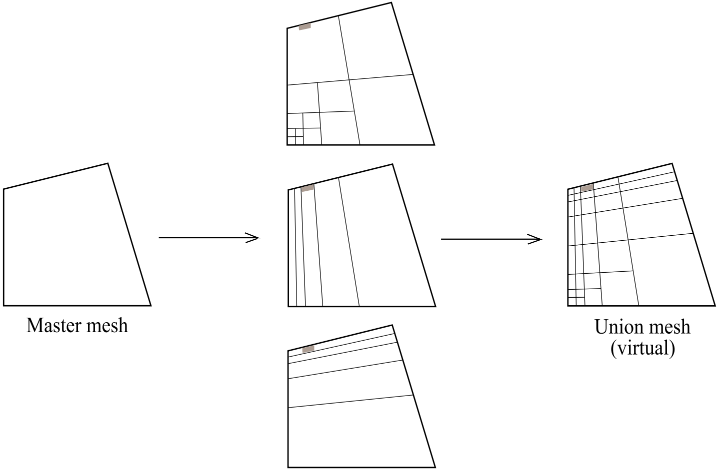

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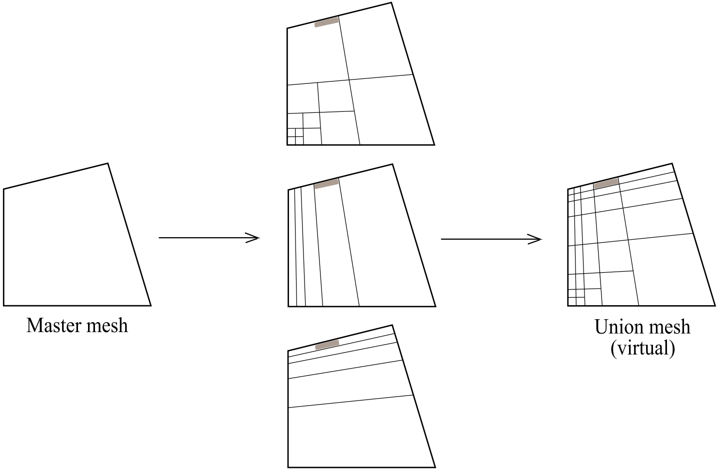

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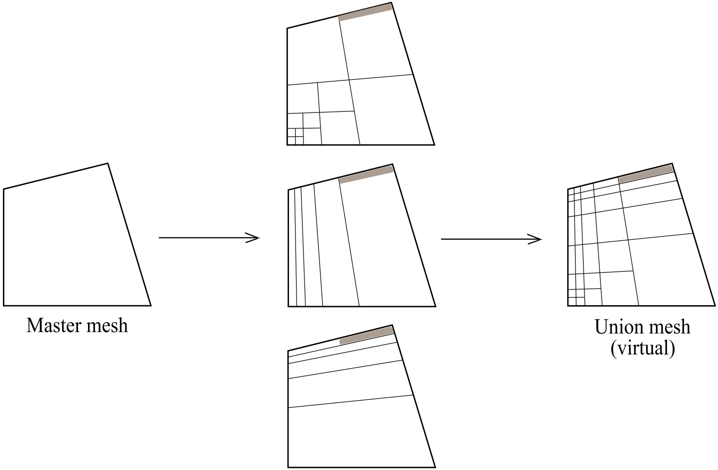
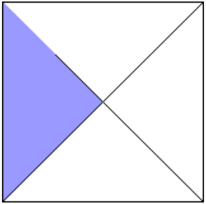


Hanging nodes

Hanging nodes

- Regular mesh



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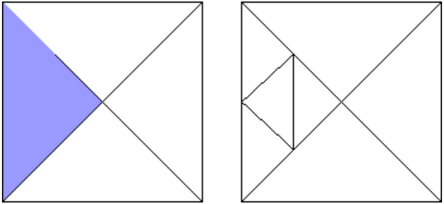

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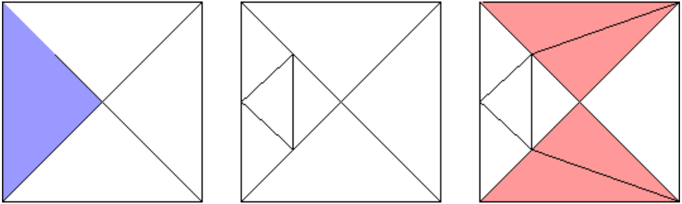

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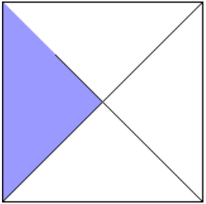
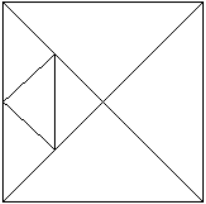
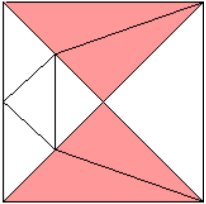
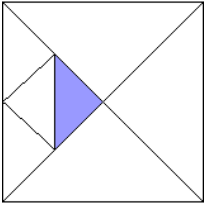






- One-level hanging nodes (1-irregular mesh)



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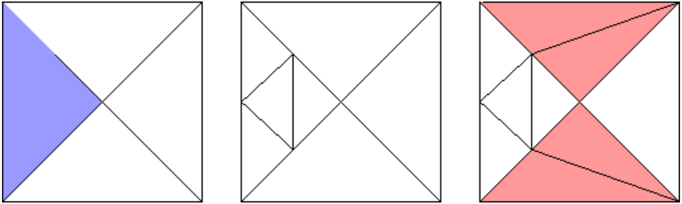

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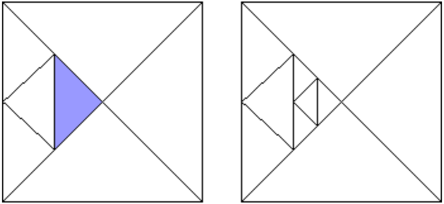

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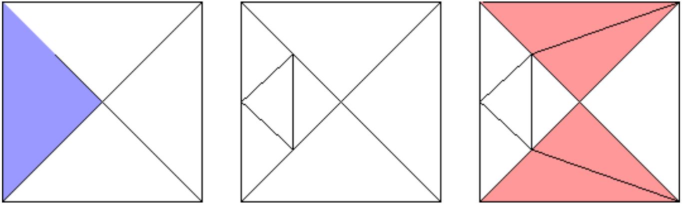

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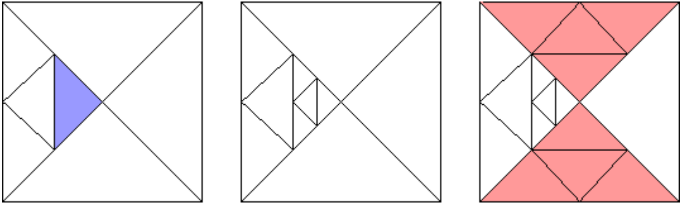
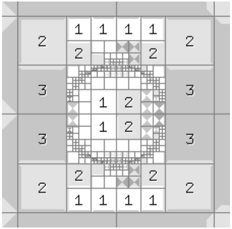


Hanging nodes

- Forced refinements
  - introduce unnecessary DOF
  - spoil element shapes
  - have recursive nature
  - cause incompatible refinements in the multi-mesh *hp*-FEM
- Arbitrary-level hanging nodes:



P.S., J. Cerveny, I. Dolezel: Arbitrary-Level Hanging Nodes and Automatic Adaptivity in the hp-FEM, Math. Comput. Simul. 77 (2008), 117 - 132.

Component specific function spaces

Component specific function spaces

Mathematically, solutions of different equations lie in different function spaces, e.g.

- Temperature, displacement, etc. - H^1 $(H^2, ..)$.
- Electromagnetics - H_{curl} (discontinuous vector fields with continuous tangential components along all mesh edges $\sim$ gradients of H^1 functions).
- Magnetism - H_{div} (discontinuous vector fields with continuous normal components along all mesh edges $\sim$ divergences of H^1 functions).
- Pressure (in Navier-Stokes equations) - L^2.

What about compressible flow? H^1 too small, L^2 too big.

Discontinuous Galerkin method

- discontinuous piecewise polynomial approximation
- local character
- well suited for conservation laws
- the non-uniqueness of the solution on mesh edges is handled as in FVM by introducing a suitable numerical flux

DG methods

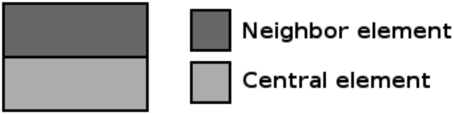
- Low order ($p = 0$) $\sim$ Finite Volume Method
  - easy to implement
  - robust
  - need for higher order approximation
- Higher order ($p > 0$, hp-adaptivity)
  - good properties from Finite Volume Method
  - possible need of handling oscillations (undershoots/overshoots)

Arbitrary-level hanging nodes with DG

- values from both sides of the edge
- correct pairing of integration points

Arbitrary-level hanging nodes with DG

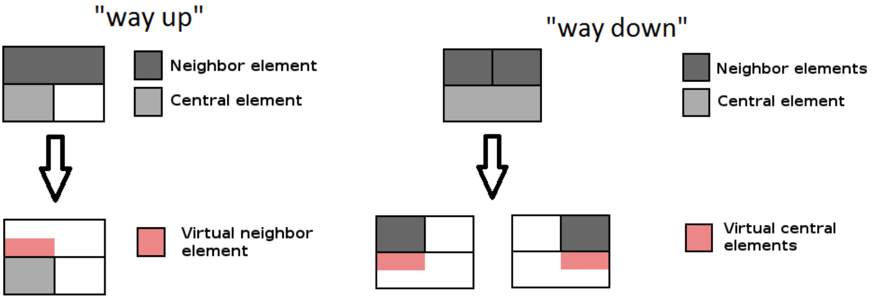
Simple case



- equal integration order on the edge from both sides $\Rightarrow$ identical integration points

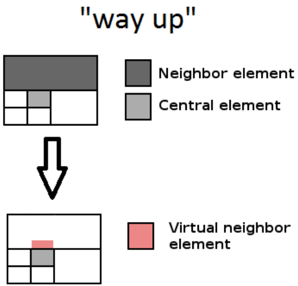
Arbitrary-level hanging nodes with DG

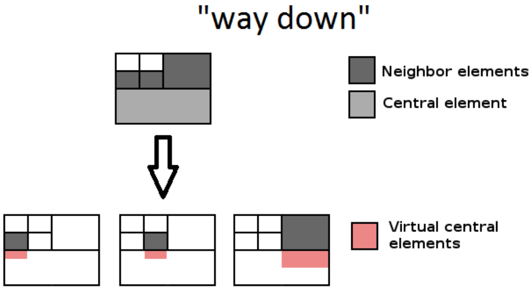
1-level hanging nodes
use of virtual sub-elements



Arbitrary-level hanging nodes with DG

2-level hanging nodes
use of virtual sub-elements





Model problem

Compressible Euler equations and the advection-diffusion equation (concentration)

- compressible Euler equations
  - first order system of hyperbolic laws
- linear advection-diffusion equation
  - second order equation

Euler equations

Non-conservative form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (1)$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}^T) + \nabla p = 0 \quad (2)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{u}(E + p)) = 0, \quad (3)$$

where $E = \rho e + \frac{1}{2}\rho u^2$.

Conservative form for DG discretization

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{f}_x}{\partial x} + \frac{\partial \mathbf{f}_y}{\partial y} = 0, \tag{4}$$

where

$$\mathbf{w} = \begin{pmatrix} \varrho \\ \rho u_1 \\ \rho u_2 \\ E \end{pmatrix} = \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix} \mathbf{f}_x = \begin{pmatrix} \rho u_1 \\ \rho u_1^2 + p \\ \rho u_1 u_2 \\ u_1(E+p) \end{pmatrix} = \begin{pmatrix} w_1 \\ \frac{w_1^2}{w_0} + p \\ \frac{w_1 w_2}{w_0} \\ \frac{w_1}{w_0}(w_3 + p) \end{pmatrix}$$

$$\mathbf{f}_y = \begin{pmatrix} \rho u_2 \\ \rho u_2 u_1 \\ \rho u_2^2 + p \\ u_2(E+p) \end{pmatrix} = \begin{pmatrix} w_2 \\ \frac{w_2 w_1}{w_0} \\ \frac{w_2^2}{w_0} + p \\ \frac{w_2}{w_0}(w_3 + p) \end{pmatrix} \tag{5}$$

$$p = \frac{R}{c_v}\left(E - \frac{1}{2}\rho\left(u_1^2 + u_2^2\right)\right) = \frac{R}{c_v}\left(w_3 - \frac{w_1^2 + w_2^2}{2 w_0}\right). \tag{6}$$

Advection-diffusion equation

$$\frac{\partial c}{\partial t} + \epsilon \Delta c + (\mathbf{u} \cdot \nabla) c = 0 \tag{7}$$

$$\frac{\partial c}{\partial t} + \epsilon \Delta c + \left(\frac{w_1}{w_0} \frac{\partial c}{\partial x} + \frac{w_2}{w_0} \frac{\partial c}{\partial y} \right) = 0. \tag{8}$$

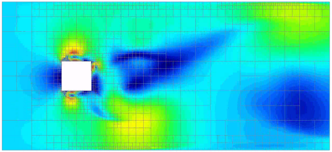
Optional variational multiscale stabilization (adding a stabilization form)

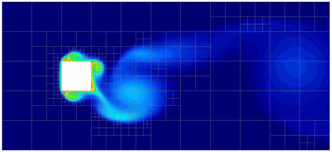
$$S(u, v) = \int_K \beta \left(-\frac{w_1}{w_0} \frac{\partial v}{\partial x} - \frac{w_2}{w_0} \frac{\partial v}{\partial y} + \epsilon \Delta v \right) * \left(-\frac{w_1}{w_0} \frac{\partial u}{\partial x} - \frac{w_2}{w_0} \frac{\partial u}{\partial y} + \epsilon \Delta u \right), \tag{9}$$

where $\beta = \frac{1}{\sqrt{9\left(\frac{4\epsilon}{|K|^2}\right)^2 + 4\left(\frac{|u|}{|K|}\right)^2}}$ and ϵ is diffusivity.

Discretization of time dependent problems

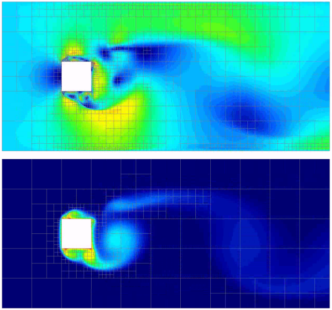
- Rothe's method
  - discretization in time first
  - discretized problem at each time level solved by adaptive hp-FEM
- dynamical meshes
  - meshes evolve in time
  - independent of each other





Discretization of time dependent problems

- Rothe's method
  - discretization in time first
  - discretized problem at each time level solved by adaptive hp-FEM
- dynamical meshes
  - meshes evolve in time
  - independent of each other



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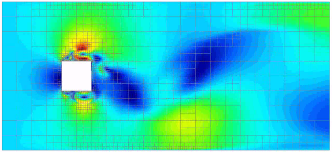

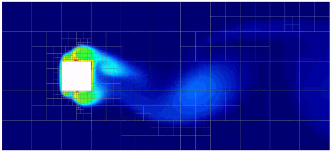

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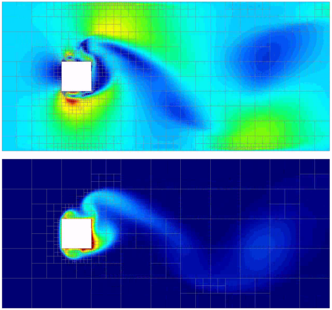

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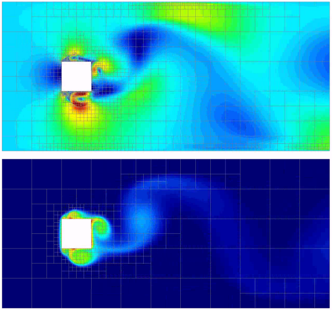


Discretization in time

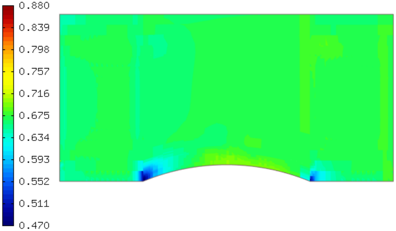
Semi-implicit discretization

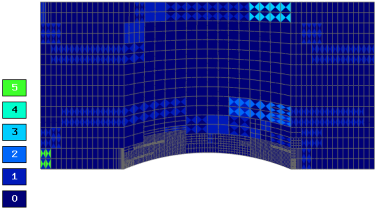
- linear terms discretized implicitly
- nonlinear terms explicitly
- possible increase of time step

Discretization in space

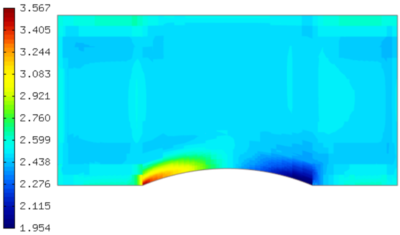
- adaptive hp-DG for the Euler equations
  - Steger-Warming numerical flux (easy linearization)
  - polynomial orders up to 10 (orders used in practice are lower)
  - Shock capturing
  - Flux limiting
- adaptive hp-FEM for the advection-diffusion equation
  - polynomial orders up to 10

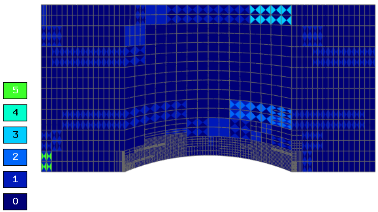
Evolution of Mach number, $\epsilon = 0.01$



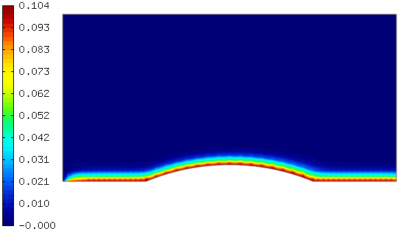


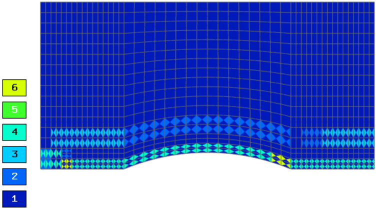
Evolution of pressure, $\epsilon = 0.01$





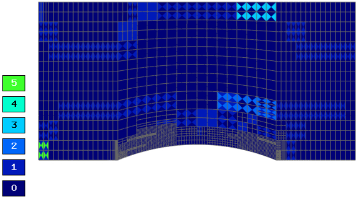
Evolution of Concentration, $\epsilon = 0.01$

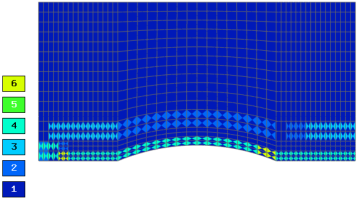




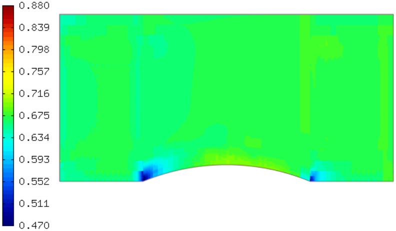
Comparison of meshes, $\epsilon = 0.01$

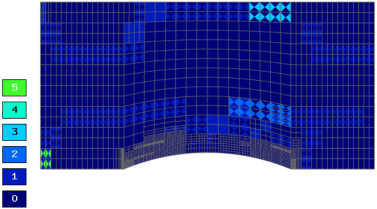
Mesh for the flow (top), mesh for the concentration (bottom)



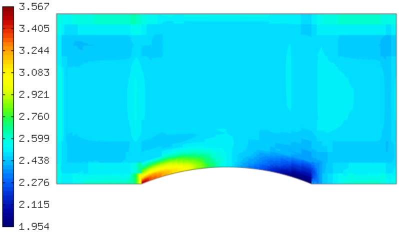


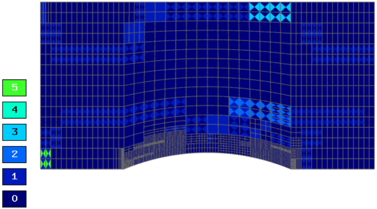
Evolution of Mach number, $\epsilon = 0.001$



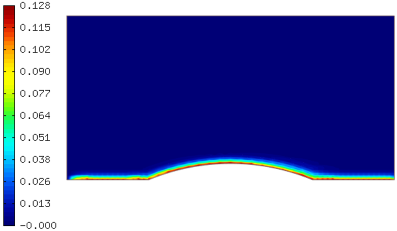


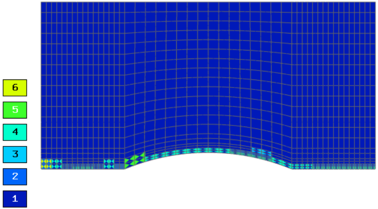
Evolution of pressure, $\epsilon = 0.001$





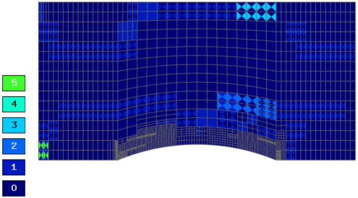
Evolution of Concentration, $\epsilon = 0.001$

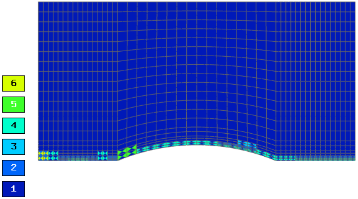




Comparison of meshes, $\epsilon = 0.001$

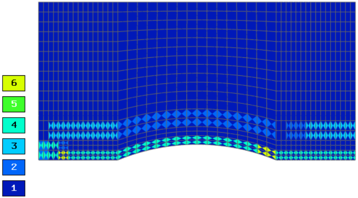
Mesh for the flow (top), mesh for the concentration (bottom)

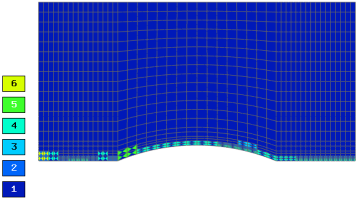




Comparison of meshes

Concentration meshes for $\epsilon = 0.01$ (top) and $\epsilon = 0.001$ (bottom)





Outlook

- \- Use the fully implicit scheme
- \- Use JFNK method (NOX package from the Trilinos library)
- \- Use the "full" Newton's method

Thank you for your attention.