How many digits can I rely on?

```
md"""
# How many digits can I rely on?
"""
```

For very big or very small inputs (x), how does the the function value $f(x)$ behave? Independent of doing the calculations for analysing the condition of f.

- **"Assumption"**: Subtraction is an arithmetic operation that erases information (because it is ill-conditioned).
- Function in its original-form and under the prev. assumption (ill-conditioned).

$$f(x) = \sqrt x - \sqrt{x+1}$$

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"""
```

```
x = 80000000
```

```
# (Big) Input
x = 80000000
```

```
f_x_original = -5.590169894276187e-5
```

```
f_x_original = sqrt(x) - sqrt(1+x)
```

With some algebra trick, we can transform the ill-conditioned problem into a better form (i.e. one that does not rely on the subtraction arith. operator, preferebly using $\{+, x, /, \sqrt{} \}$ since they are well-conditioned arith. operators: $\kappa(\cdot) = \{\frac{1}{2}, 1\} \quad \forall (\cdot) \in \{+, x, /, \sqrt{} \}$.

$$f(x) = \sqrt x - \sqrt{x+1} = (\sqrt x - \sqrt{x+1}) \cdot \frac{(\sqrt x + \sqrt{x+1})}{(\sqrt x + \sqrt{x+1})} = \frac{-1}{\sqrt x + \sqrt{x+1}}$$

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"""
```

```
f_x_well_cond = -5.590169926280193e-5
```

```
f_x_well_cond = (-1/(sqrt(x) + sqrt(x+1)))
```

```
diff = 3.2004006243107133e-13
```

```
diff = abs(f_x_original - f_x_well_cond)
```

Digit Analysis

Now, how many digits can I rely on for $f_{original}$?

- We are using IEEE double precision, so there are 16 decimal digits available.
- For the input $x = 80000000$, the computer sees it as $x_{computer} = 8 * 10^7$
- The ill-conditioned formulation of $f_{original-computer} = -5.590169894276187e - 5$, which for us humans is $f_{original} = -0.00005590169894276187$.

Now, out of the 16 available digits we observe that 5 digits have been erased (i.e. equal to zero, 1 before the comma and 4 after the comma), $16 - 5 = 11$, hence there are only 11 reliable digits.

Reasonable part of the output for the ill-conditioned formulation of f is -0.0000559016, the remaining digits are trash (i.e. $...9894276187$).

Now, how many digits can I rely on for $f_{well-cond.}$?

- The well-conditioned formulation gives $f_{well-cond.} = -5.590169926280193e-5$, which for us humans is $f_{well-cond.} = -0.00005590169926280193$.

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Reasonable part of the output for the well-conditioned formulation of f is -0.0000559016, the remaining digits are trash (i.e. $\ldots.9926280193$).

- Finally, observe that, $diff = |f_{original-computer} - f_{well-cond.}| = 3.2004006243107133e-13$, means that from digit 13th onwards that the f values differ.

Conclusion

The amount of digits I can rely on does not change for the input $x = 80000000$, yet we see some small differences appear across the different formulations of f.

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## Digit Analysis

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---

## Conclusion

The amount of digits I can rely on does not change for the input $x=80000000$, yet we see some small differences appear across the different formulations of $f$.
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Final Experiment

Let x be truely huge, say $x = 8 * 10^{23}$. Now, by comparing the ill- against the well-condition formulation of f, we observe that for very big values of x no single digit is reliable for $f_{original}$.

The last value of x that gives ($16 - 8 = 8$) reliable digits is $x_{last-power} = 8 * 10^{14}$. From the 15th power onwards, no single digit is reliable for the original formulation of f, contrary to $f_{well-cond.}$ which still returns a meaningful result, now with only $(16 - 8 = 8)$ reliable digits.

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## Final Experiment

Let $x$ be truely huge, say $x=8*10^{23}$. Now, by comparing the ill- against the well-condition formulation of $f$, we observe that for very big values of $x$ no single digit is reliable for $f_{original}$.

The last value of $x$ that gives ($16 - 8 = 8$) reliable digits is $x_{last-power}=8*10^{14}$. From the 15th power onwards, no single digit is reliable for the original formulation of $f$, contrary to $f_{well-cond.}$ which still returns a meaningful result, now with only $(16 - 8 = 8)$ reliable digits.
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Inputs

```
md""" #### Inputs """
```

x_bigger = 800000000000000000000000

```
x_bigger = 800000000000000000000000
```

x_last_power = 800000000000000

```
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```

1st Test

```
md""" #### 1st Test """
```

f_x_original_last_power = -1.862645149230957e-8

```
f_x_original_last_power = sqrt(x_last_power) - sqrt(1+x_last_power)
```

f_x_well_cond_last_power = -1.7677669529663683e-8

```
f_x_well_cond_last_power = (-1/(sqrt(x_last_power) + sqrt(x_last_power+1)))
```

diff_last_power = 0.0

```
diff_last_power = abs(f_x_original_last_power-f_x_original_last_power)
```

2nd Test

```
md""" #### 2nd Test """
```

f_x_original_bigger = 0.0

```
f_x_original_bigger = sqrt(x_bigger) - sqrt(1+x_bigger)
```

f_x_well_cond_bigger = -5.590169943749474e-13

```
f_x_well_cond_bigger = (-1/(sqrt(x_bigger) + sqrt(x_bigger+1)))
```

diff_bigger = 5.590169943749474e-13

```
diff_bigger = abs(f_x_original_bigger-f_x_well_cond_bigger)
```