- Notations & Units

Metric

time ← space →

$$g^{\mu\nu} = \eta^{\mu\nu} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \end{array}$$

e.g.

vectors

$$p_1 \cdot p_2 \equiv g^{\mu\nu} p_{1\mu} p_{2\nu} = p_1^0 p_2^0 - p_1^1 p_2^1 - p_1^2 p_2^2 - p_1^3 p_2^3$$

$$\partial^\mu = \eta^{\mu\nu} \partial_\nu \equiv (\partial_0, \vec{\partial}) \quad , \quad p_\mu \equiv (p^0, \vec{p})$$

$$\partial_\mu \equiv (\partial_0, -\vec{\partial})$$ ← different sign.

$$\Rightarrow \quad \partial^\mu j_\mu = g^{\mu\nu} \partial_\mu j_\nu = \partial_0 j_0 - (-\vec{\partial}) \cdot \vec{j} = \partial_0 j_0 + \vec{\partial} \cdot \vec{j}$$

$$\partial_\mu \partial^\mu \phi = \partial_0^2 \phi - \vec{\partial}^2 \phi$$

tensors

$$F^{\mu\nu} F_{\mu\nu} = g^{\alpha\mu} g^{\beta\nu} F_{\alpha\beta} F_{\mu\nu}$$

* Units : $c = 3 \times 10^8$ m/s

$\hbar = 1.055 \times 10^{-34}$ Joules · s

let's set $c = \hbar = 1$

e.g. $E^2 = p^2c^2 + m^2c^4 = p^2 + m^2$

$[\hat{x}, \hat{p}] = i\hbar = i$

From that, we can see that

$$[\text{Energy}] = [\text{momentum}] = [\text{mass}]$$

$$= [\text{length}]^{-1} = [\text{time}]^{-1}$$

We use $[\text{Energy}] = \text{GeV} = 10^9 \text{eV} = 1.602 \times 10^{-10}$ Joules

Some examples :

$$[\text{Area}] = [\text{length}]^{-2} = \text{GeV}^{-2}$$

$$[\text{velocity}] = \frac{[\text{length}]}{[\text{time}]} = [1] = [c] = \text{GeV}^0$$

$$\int d^3x = [\text{length}]^3 = \text{GeV}^{-3}$$

Some examples cont.

- $[\psi] = \mathrm{GeV}^{3/2}$ since $\int d^3x\, |\psi|^2 = 1$

non-relativistic wave function

GeV^{-3} GeV^0

- $[S] = \mathrm{GeV}^0$ since $\exp[i \frac{S}{\hbar}]$

action dimensionless

or $S \sim \int dt\, \frac{1}{2} m v^2$

GeV^{-1} GeV GeV^0

- then $[L] = \mathrm{GeV}$. since $S = \int dt\, L$

Lagrangian

and $[\mathcal{L}] = \mathrm{GeV}^4$

(1+3)-D lagrangian density $L = \int d^3x\, \mathcal{L}(x)$

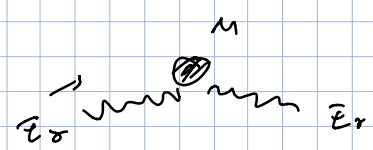
- $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, if $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$

$\Rightarrow [F^{\mu\nu}] = \mathrm{GeV}^2$

- Dimensional Analysis

- why is sky blue?

we study the elastic scattering of a photon with energy E_γ off an atom with mass M



we consider the case in which

$$E_\gamma \ll \Delta E \ll r_0^{-1} \ll M$$

excitation Energy Bohr radius mass

so it is elastic, in the sense that the electron can not be excited by the photon, since $E_\gamma \ll \Delta E$

the photon sees the atom as a whole, it can NOT resolve the atom.

To study the process, we want to write down the Lagrangian following some fundamental rules, out of several building blocks. The kinematic part of the Lagrangian should contain all physics of order E_r

- building blocks:

  $F_{\mu\nu}$ (E/B) $\rightsquigarrow$ create, annihilate a photon

  ϕ_ν $\sim a + a^\dagger$ $\rightsquigarrow$ create, annihilate a atom

  ∂_μ $\sim E_r, p_r$ $\rightsquigarrow$ Energy or momentum of photon

  v_μ $\sim$ 4-velocity of the atom $\rightsquigarrow$ almost at rest
  $v^\mu \sim (1, \vec{0})$

- Fundamental rules:

  * Atom number is conserved.
    - we do not create or destroy an atom in elastic scattering.

    $\Rightarrow$ $\phi_\nu^\dagger$ & ϕ_ν appear in pair!

    No $\phi_\nu^\dagger \phi_\nu \phi_\nu$ term in the Lagrangian is allowed!

  * Charge & Parity Conservation

    $\Rightarrow$ No $\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu}$

* Gauge Invariance, You will learn this in the future

but here All you need to know is that.

- a charge neutral object is Gauge invariant.

  so $\phi_v, \phi_v^\dagger$ is. (Atom is charge neutral)

- $F_{\mu\nu}$ is gauge invariant. (also anti-symmetric)

* We to construct out of the building blocks

  a scalar. Since $\mathcal{L}$ is a scalar

  $\Rightarrow$ all Lorentz indices have to be contracted.

So the Lagrangian we can write down is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$ ← kinematics of photon, not interacting

$$+ C_1 \underbrace{\phi_v^\dagger \phi_v}_{\text{physics} \sim E_\gamma} F_{\mu\nu} F^{\mu\nu} + C_2 \underbrace{\phi_v^\dagger \phi_v}_{\text{physics} \sim E_\gamma} V_\mu F^{\mu\nu} V^\alpha F_{\alpha\nu}$$

$$+ C_3 \underbrace{\phi_v^\dagger \phi_v}_{\text{physics} \sim E_\gamma} \partial_\mu F^{\mu\nu} F_{\alpha\nu} V^\alpha + \dots$$

interactions between photon and atom

$C_i = C_i(\Delta E_i, m, r_0^{-1})$ → called wilson coefficients.

Since

$$[\phi_v] = \text{GeV}^{3/2}, \quad [F^{\mu\nu}] = \text{GeV}^2, \quad [\mathcal{L}] = \text{GeV}^4$$

$$\Rightarrow \quad [C_1] = [C_2] = \text{GeV}^{-3}, \qquad [C_3] = \text{GeV}^{-5}$$

$\Rightarrow \quad c_1 \sim \left(\frac{1}{\Delta E}\right)^3. \quad c_2 \sim \left(\frac{1}{\Delta E}\right)^3. \quad [c_3] \sim \left(\frac{1}{\Delta E}\right)^5$

$$\Rightarrow \quad c_1 \phi_v^\dagger \phi_v F_{\mu\nu} F^{\mu\nu} \sim \frac{E_\gamma^7}{\Delta E^3} \sim \left(\frac{E_\gamma}{\Delta E}\right)^3 E_\gamma^4$$

$$c_2 \phi_v^\dagger \phi_v v_\mu F^{\mu\nu} F_{\alpha\nu} v^\alpha \sim \frac{E_\gamma^7}{\Delta E^3} \sim \left(\frac{E_\gamma}{\Delta E}\right)^3 E_\gamma^4$$

$$c_3 \phi_v^\dagger \phi_v \partial_\mu F^{\mu\nu} \partial^\alpha F_{\alpha\nu} \sim \frac{E_\gamma^9}{\Delta E^5} \sim \left(\frac{E_\gamma}{\Delta E}\right)^5 E_\gamma^4$$

since $E_\gamma \ll \Delta E$. $\Rightarrow$ c_3 term can be negligible when compared with c_1 & c_2 term.

the cross section is proportional to the transition amplitude^2

$$\underset{\text{cross-section}}{\sigma} = |\langle \underset{\text{atom}}{A}\, \underset{\text{photon}}{\gamma} | \mathcal{L}_{int} | A\gamma \rangle|^2 \times \text{phase space factor of photon}$$

$$= |\langle A\gamma | c_1 \cdots + c_2 \cdots | A\gamma \rangle|^2 \times \text{phase space factor of photon}$$

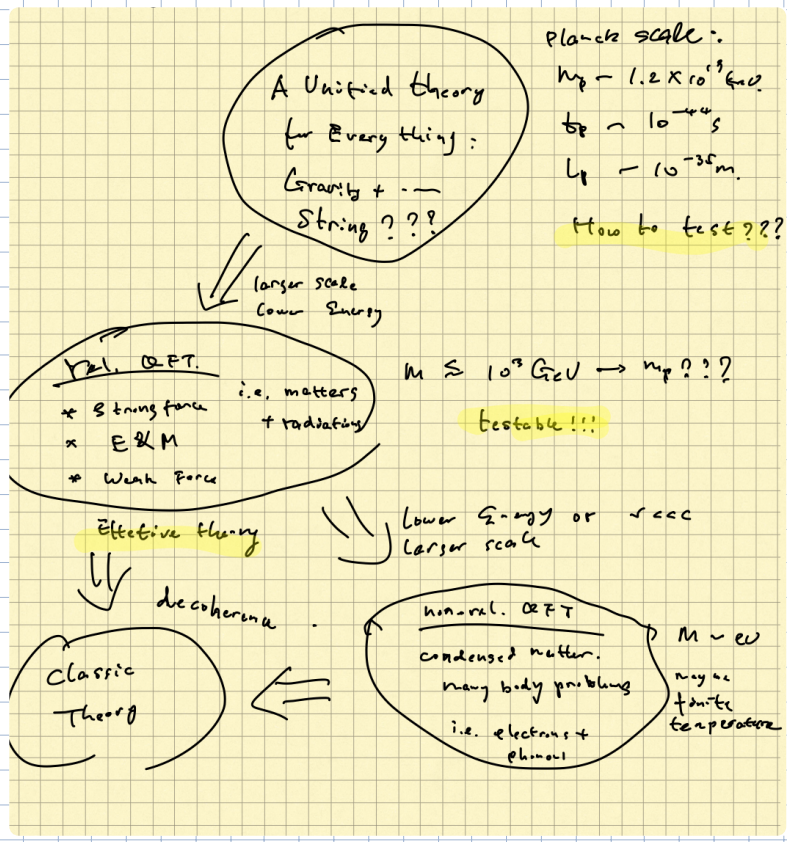
$$= \Big[\underbrace{|c_1|^2}_{\left(\frac{1}{\Delta E}\right)^6} + \underbrace{|c_2|^2}_{\left(\frac{1}{\Delta E}\right)^6} + \underbrace{(c_1^* c_2 + c.c.)}_{\left(\frac{1}{\Delta E}\right)^6} \Big] \times \text{phase space factor of photon}$$

since $[\sigma] = [\text{length}]^{-2} = \text{GeV}^{-2}$

$$\Rightarrow \quad \sigma = \frac{E_\gamma^4}{\Delta E^6} \quad \text{to make the dimension correct!}$$

$$\Rightarrow \quad \sigma \propto \frac{1}{\lambda^4} \leftarrow \text{Rayleigh law w/o doing any calculation!!}$$

- why we study QFT ?



- QFT can NOT be valid to arbitrarily high Energy Scale.
- It is an Effective theory for $E \lesssim 10^3$ TeV or even higher.
- QFT is NOT necessarily relativistic
- relativity + QM $\Rightarrow$ Many body $\Rightarrow$ Field Theory.