HW 2: QFT warm-ups

March 19, 2019

will be officially posted on March 21 and due on April 4. Before March 21, any modifications to the problems are possible.

1 Problem 1: Elastic Rod

We extend our studies from the simple harmonic oscillator to the quantization of an elastic rod undergoing a small longitudinal vibration, which you will soon see is a $1+1$ dimensional field system. The problem can be modeled first by N-coupled harmonic oscillators and then taking the continuous limit.

We model the elastic rod by N particles with the same mass m, attached by massless springs with elastic constant k to their two closest neighbours. When in equilibrium, the distance between two closest particles is a. If the displacement of the i-th particle from its equilibrium position is measured by the quantity q_i, then the Lagrangian of this system is given by

$$L = T - V = \sum_{i=1}^{N} \frac{1}{2} m\, \dot{q}_i^2 - \frac{1}{2} k \sum_{i=0}^{N} (q_i - q_{i+1})^2 \,, \tag{1}$$

where $q_0 = q_{N+1} = 0$, which means that the two ends of the rod are fixed.

- Show that if we make N large and thus a small, the Lagrangian can be written as

$$L = \int \mathrm{d}x \left[\frac{1}{2} \rho\, \dot{q}(t,x)^2 - \frac{1}{2} Y \left(\partial_x q(t,x) \right)^2 \right] \equiv \int \mathrm{d}x\, \mathcal{L} \tag{2}$$

  where $\rho = \lim_{a \to 0} \frac{m}{a}$ is the mass density and $Y = \lim_{a \to 0} k\, a$ is the Youngs modulus. Here we introduced the Lagrangian density $\mathcal{L}$. For simplicity, from now on, we set $\rho = 1$ and $Y = 1$.

- Find and write out the Euler-Lagrangian equation for the canonic position $q(t,x)$. And show that

$$q(t,x) = \int_{-\infty}^{\infty} \mathrm{d}k \frac{1}{\sqrt{2\omega(k)}} \left(e^{i\omega t} e^{-ikx} a^\dagger(k) + e^{-i\omega t} e^{ikx} a(k) \right) \,, \tag{3}$$

with $\omega > 0$ is a solution to the Euler-Lagrangian equation, if $\omega(k) = |k|$. We note that one main difference between this solution and the simple harmonic is that in the latter case, the ω takes one definite given value, but here the ω can take infinite possible values. Here we somehow assumed that the length of the rod l is long enough ($l \to \infty$) and we can ignore the effect of the boundaries. The particles in the bulk of the rod are therefore"free".

- Use the definition $p(t,x) = \frac{\delta \mathcal{L}(\dot{q},q)}{\delta \dot{q}(t,x)}$ to find the canonic momentum $p(t,x)$ and the Hamiltonian density $\mathcal{H}$, defined via the Hamiltonian H

$$H = \int \mathrm{d}x\, \mathcal{H}(p,q)\,. \tag{4}$$

  Note that please write p explicitly in terms of $a(k)$ and $a^\dagger(k)$ but only need to write $\mathcal{H}$ in terms of $p(t,x)$ and $q(t,x)$.

- Now we are able to quantize the system by imposing the (equal time) quantization condition $[q(t,x_1), p(t,x_2)] = i\delta(x_1 - x_2)$, in other words, if x_1 and x_2 are different points, then $q(x_1)$ and $p(x_2)$ commute, otherwise they do not. This assumption is reasonable, since from the discrete model at the very beginning, different points on the rod relate to different particles and it is nature that we can measure particle i_1's position q_{i_1} and simultaneously measure particle i_2's momentum p_{i_2}.

  From this, show that (again, here we assume the length of the rod l is large enough so to take the $l \to \infty$ limit.)

$$[a(k), a^\dagger(k')] = \delta(k-k')\,, \quad [a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0\,. \tag{5}$$

- Now express the Hamiltonian H in terms of $a^\dagger(k)$ and $a(k)$. Suppose we denote the ground state as $|0\rangle$ which satisfies $a(k)|0\rangle = 0$, show that both the ground state energy $E = \langle 0|H|0\rangle$ and the energy density $\mathcal{E} = E/l$ of the system are divergent (infinite)!! But in the energy E, we have both the ultraviolet (UV) and infrared (IR) divergences, while in the density $\mathcal{E}$, we only have the IR divergence. By "UV divergence", we mean the divergence due to $\omega \to \infty$ and by "IR divergence", we mean the divergence due to $\omega \to 0$ or $l \to \infty$.

- Calculate the Feynman propagator $\Delta_F(t_1 - t_2, x_1 - x_2) = \langle 0|T[q(t_1,x_1)q(t_2,x_2)]|0\rangle$.

- Now we disturb the system by adding a potential term

$$V = \int \mathrm{d}x\, \frac{\lambda}{4!}[q(x,t)]^4\,, \tag{6}$$

  find out the λ correction to the energy density $\mathcal{E}$ by calculating the transition amplitude with $t \to -i\tau$. Note that the $\mathcal{O}(\lambda)$ correction itself is infrared divergent and thus again infinite!!

Hint: here everything is almost the same as the harmonic oscillator, for instance the Wick theorem still applies. The only differences are that the feynman propagator should be replaced by the one you derived in the previous problem. The vertex now involves not only $\int_0^T \mathrm{d}\tau$ but also $\int \mathrm{d}x$ due to the form of the potential V. Also now the ω is not a fixed value but being integrated over.

- In the correction you obtained above, you should encounter an integral of the form

$$\int_0^\infty \frac{\mathrm{d}\omega}{\omega^2} \tag{7}$$

the divergence is due to the poor behavior of the integrand when $\omega \to 0$. But we note that if the rod has a finite length l, the ω can never been 0 but takes the values of $\frac{n\pi}{l}$ with $n = 1, 2, 3, 4, \ldots$, which can be seen by demanding the solution of $q(t, x)$ in question#2 vanishes at the ends of the rod. The only reason here we have $\omega \to 0$ is soley due to the fact we have taken the $l \to \infty$ limit. Now suppose we recover the finite l and show that we can replace the integral by

$$\int_0^\infty \frac{\mathrm{d}\omega}{\omega^2} \to \sum_{n=1}^{\infty} \frac{l^2}{n^2\pi^2} \frac{\pi}{l}\,. \tag{8}$$

Perform the summation over n explicitly and use the result to find the finite $\mathcal{O}(\lambda)$ corrections to the energy density $\mathcal{E}$.

So from this problem you can see the differences between the "rod QFT" and the "simple harmonic QFT" are: 1. now the canonic variables (e.g. q or p) depend on both t and x (or label i. here x is a label! just like t. It is not an operator!) and therefore they are $1 + 1$ dimensional objects, while in the simple harmonics they only depend on the time t and therefore $1 + 0$ dimensional. 2. the tiny difference leads to the fact that $1 + 1$ dimensional QFT can be thought of an infinite sum of harmonic oscillators ($N \to \infty$). For instance, $H \sim \int \mathrm{d}\omega \frac{1}{2}\omega$ is a summation over an infinite numbers of harmonic oscillators with different ω's, all the way to $\omega = \infty$ (but for finite N, ω of the normal modes can only take finite numbers of values). 3. and this infinite-harmonic-oscillator-picture (or to say infinite degrees of freedom) leads to the UV divergence in the QFT byeond $1 + 0$-dimension. 4. If we approximate the size of the system, say the length of the rod here, as infinity, we may encounter in our theory the IR divergence. Usually the lower the dimension, the stronger the IR divergence is.

Now we are finished with the canonic quantization of the 1-dimensional rod. Then how about quantizing a mattress, which is a 2-dimensional surface? and how about we put our universe in a box, disturb some fields of it and try to quantize them?