COMPUTATIONAL FINANCE & RISK MANAGEMENT
UNIVERSITY of WASHINGTON
Department of Applied Mathematics

AMATH 460: Mathematical Methods for Quantitative Finance

4. Multiple Integrals

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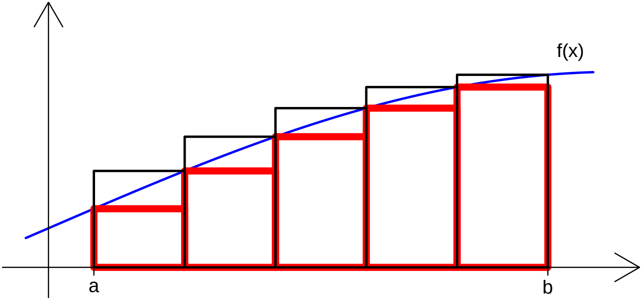
Outline

Outline

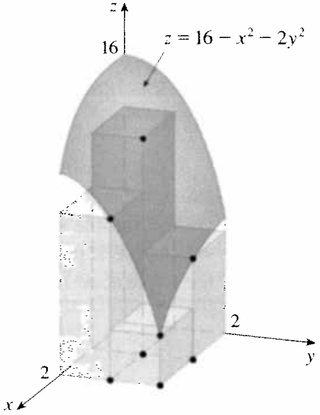
Double Integrals

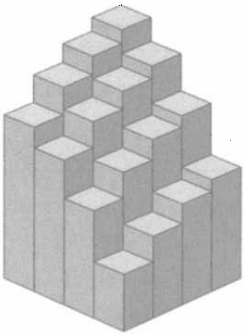
- Review of the definite integral of a single-variable function



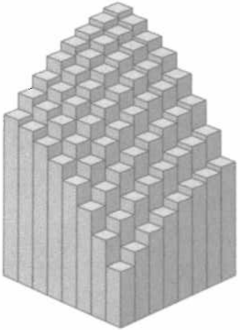
- $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(a+i\Delta x)\,\Delta x$ $\qquad \Delta x = \dfrac{b-a}{n}$

Double Integrals





(a) $m = n = 4$, $V \approx 41.5$



(b) $m = n = 8$, $V \approx 44.875$

$$\iint_{[0,2]\times[0,2]} f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(i\Delta x, j\Delta y)\,\Delta A$$

$$\Delta x = \frac{2-0}{m} \qquad \Delta y = \frac{2-0}{n} \qquad \Delta A = \Delta x\,\Delta y$$

Double Integrals

- In general, double integral over a rectangle $R = [a, b] \times [c, d]$

$$\iint_R f(x, y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(a + i\Delta x, c + j\Delta y)\, \Delta A$$

- If $f(x, y) \geq 0 \quad \forall (x, y) \in R$, then

$$V = \iint_R f(x, y)\, dA$$

is the volume of the region above R and below surface $z = f(x, y)$

Properties of Double Integrals

Linearity Properties:

- $$\iint_R [f(x,y) + g(x,y)]\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$$
- $$\iint_R cf(x,y)\, dA = c \iint_R f(x,y)\, dA$$

Comparison:

- If $f(x,y) \geq g(x,y)\ \forall (x,y) \in R$ then

$$\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA$$

Iterated Integrals

- Suppose $f(x, y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$
- Partial integration: fix x, integrate $f(x, y)$ as a function of y alone

$$A(x) = \int_c^d f(x, y)\, dy$$

- An iterated integral is the integral of $A(x)$ wrt x

$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

- Usually the brackets are omitted

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

- Iterating the other way

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

Double Integrals vs. Iterated Integrals

Big Question:

- What is the relationship between a double integral and an iterated integral?

$$\iint_R f(x,y)\,dA \quad \boxed{?} \quad \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Outline

Fubini's Theorem

- If $f(x, y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$ then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

- The order of iteration does not matter

Example

- Let $R = [1,3] \times [2,5]$ and $f(x,y) = 2y - 3x$. Compute $\iint_R f(x,y)\,dA$

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \int_2^5 \left[\int_1^3 (2y - 3x)\,dx\right] dy \\
&= \int_2^5 \left[\left(2xy - \frac{3}{2}x^2\right)\Big|_1^3\right] dy \\
&= \int_2^5 \left[\left(6y - \frac{27}{2}\right) - \left(2y - \frac{3}{2}\right)\right] dy \\
&= \int_2^5 \left[4y - 12\right] dy \\
&= \left[2y^2 - 12y\right]\Big|_2^5 \\
&= [50 - 60] - [8 - 24] = 6
\end{aligned}$$

Example (continued)

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \int_1^3 \left[\int_2^5 (2y-3x)\,dy\right]\,dx \\
&= \int_1^3 \left[\left(y^2-3xy\right)\Big|_2^5\right]\,dx \\
&= \int_1^3 \left[(25-15x)-(4-6x)\right]\,dx \\
&= \int_1^3 \left[21-9x\right]\,dx \\
&= \left[21x-\frac{9}{2}x^2\right]\Big|_1^3 \\
&= \left[63-\frac{81}{2}\right]-\left[21-\frac{9}{2}\right] = 42-36 = 6
\end{aligned}$$

Double Integrals Non-Rectangular Regions

- If $f(x,y)$ is continuous on a region D that can be described

$$D = \{(x,y) : a \le x \le b, \;\; g_1(x) \le y \le g_2(x)\ \}$$

then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

- If $f(x,y)$ is continuous on a region D that can be described

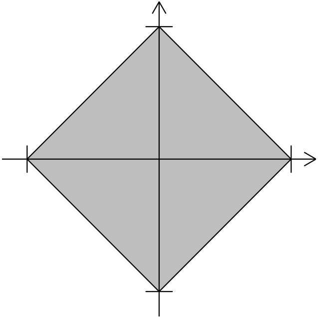
$$D = \{(x,y) : c \le y \le d, \;\; h_1(y) \le x \le h_2(y)\ \}$$

then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dx\,dy$$

Example

- Let $D = \{(x, y) : |x| + |y| \leq 1\}$
  diamond w/ corners at $(0, \pm 1)$ and $(\pm 1, 0)$
- Compute the integral of $f(x, y) = 1$ over D



$$\iint_D 1 \, dA$$

$$= \int_{-1}^{0} \left[\int_{-1-x}^{1+x} 1 \, dy \right] dx + \int_{0}^{1} \left[\int_{x-1}^{1-x} 1 \, dy \right] dx$$

$$= \int_{-1}^{0} \left[\; y \; \Big|_{-1-x}^{1+x} \right] dx + \int_{0}^{1} \left[\; y \; \Big|_{x-1}^{1-x} \right] dx$$

Example (continued)

$$
\begin{aligned}
&= \int_{-1}^{0} \Big[[1+x] - [-1-x] \Big]\, dx + \int_{0}^{1} \Big[[1-x] - [x-1] \Big]\, dx \\
&= \int_{-1}^{0} [2+2x]\, dx + \int_{0}^{1} [2-2x]\, dx \\
&= [2x+x^2]\Big|_{-1}^{0} + [2x-x^2]\Big|_{0}^{1} \\
&= \Big[[0+0] - [-2+1] \Big] + \Big[[2-1] - [0-0] \Big] \\
&= 1+1 \\
&= 2
\end{aligned}
$$

Outline

Change of Variables: Single Variable Case

- Let $f(x)$ be a continuous function
- Let $g(s)$ be a continuously differentiable and invertible function
  - Implies $g(s)$ either strictly increasing or strictly decreasing
- $g(s)$ maps the interval $[c, d]$ into the interval $[a, b]$, i.e.,

$$s \in [c, d] \rightarrow x = g(s) \in [a, b]$$

- Integration by substitution says:

$$\int_{x=a}^{x=b} f(x)\, dx = \int_{s=g^{-1}(a)}^{s=g^{-1}(b)} f(g(s))\, g'(s)\, ds$$

- Let $f(x, y)$ be a continuous function
- Want to compute: $\iint_D f(x, y)\, dA$
- Let Ω be a domain such that the mapping
  - $x = x(s, t)$
  - $y = y(s, t)$

  of a point $(s, t) \in \Omega$ to a point $(x, y) \in D$ is one-to-one and onto
  - $x(s, t)$ and $y(s, t)$ continuously differentiable
- That is,

$$(s, t) \in \Omega \longleftrightarrow (x, y) = (x(s, t), y(s, t)) \in D$$

- Want to find a function $h(s, t)$ such that

$$\iint_D f(x, y)\, dx\, dy = \iint_\Omega h(s, t)\, ds\, dt$$

Change of Variables

- Get started

$$f(x,y) = f(x(s,t), y(s,t))$$

- In the single variable case, if $x = g(s)$ then

$$dx = g'(s)\, ds$$

- In the 2-variable case, $(x,y) = (x(s,t), y(s,t))$ is a vector-valued function of 2 variables
- The gradient of $(x(s,t), y(s,t))$ is the 2×2 array

$$D(x(s,t), y(s,t)) = \begin{bmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{bmatrix}$$

Jacobian

- The 2-variable equivalent of $dx = g'(s)\,ds$ is

$$dx\,dy = \left|\left[\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right]\right|\,ds\,dt$$

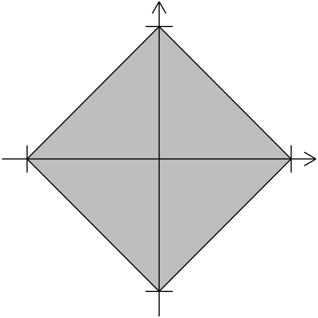
  and the quantity in the square brackets is called the Jacobian

- 2 dimensional change of variables formula:

$$\iint_D f(x,y)\,dx\,dy = \iint_\Omega f(x(s,t),y(s,t))\,\left|\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right|\,ds\,dt$$

Example

- Example from previous section
- Let $D = \{(x, y) : |x| + |y| \leq 1\}$
  diamond w/ corners at $(0, \pm 1)$ and $(\pm 1, 0)$
- Compute the integral of $f(x, y) = 1$ over D



$$\iint_D 1 \, dA = \int_{-1}^{0} \left[\int_{-1-x}^{1+x} 1 \, dy \right] dx + \int_{0}^{1} \left[\int_{x-1}^{1-x} 1 \, dy \right] dx$$

Example (continued)

- Consider the change of variables:

$$s = x + y, \qquad t = x - y$$

- Solve for x and y in terms of s and t:

$$x = \frac{s+t}{2}, \qquad y = \frac{s-t}{2}$$

- Compute the partial derivatives of the change of variables:

$$\frac{\partial x}{\partial s} = \frac{1}{2}, \qquad \frac{\partial x}{\partial t} = \frac{1}{2}, \qquad \frac{\partial y}{\partial s} = \frac{1}{2}, \qquad \frac{\partial y}{\partial t} = -\frac{1}{2}$$

- Compute the Jacobian:

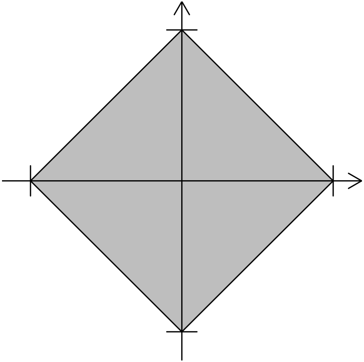
$$\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} = \frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2}\frac{1}{2} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

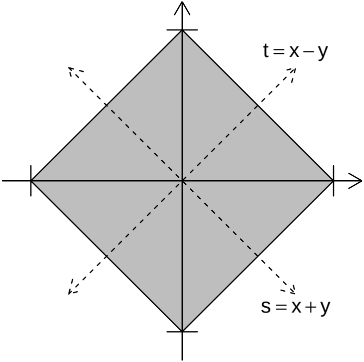
Example (continued)

- Multivariate change of variables formula:

$$\iint_D 1\,dx\,dy = \iint_\Omega 1 \left| \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \right| ds\,dt = \iint_\Omega \frac{1}{2}\,ds\,dt$$

- Finally, domain of integration (Ω):





Example (continued)

- Domain of integration: $\Omega = [-1, 1] \times [-1, 1]$

$$
\begin{aligned}
\iint_D 1 \, dA &= \int_{-1}^{0} \left[\int_{-1-x}^{1+x} 1 \, dy \right] dx + \int_{0}^{1} \left[\int_{x-1}^{1-x} 1 \, dy \right] dx \\
&= \iint_\Omega \frac{1}{2} \, ds \, dt \\
&= \int_{t=-1}^{t=1} \left[\int_{s=-1}^{s=1} \frac{1}{2} \, ds \right] dt \\
&= \int_{t=-1}^{t=1} \left[\frac{s}{2} \Big|_{s=-1}^{s=1} \right] dt \\
&= \int_{t=-1}^{t=1} 1 \, dt \\
&= 2
\end{aligned}
$$

Outline

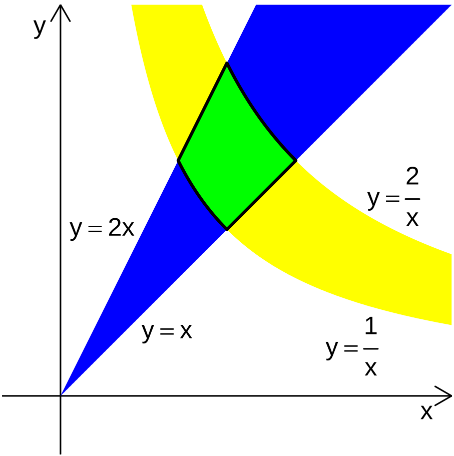
Change of Variables Example

- Evaluate $\displaystyle \iint_D x\,dx\,dy$ where

$$D = \Big\{(x, y) \in \mathbb{R}^2 : x \geq 0,\quad 1 \leq xy \leq 2,\quad 1 \leq \frac{y}{x} \leq 2\Big\}$$

- Can assume $x > 0$

$$D = \begin{cases} \frac{1}{x} \leq y \leq \frac{2}{x} \\ x \leq y \leq 2x \end{cases}$$



$$\iint_D x\,dx\,dy = \int_{\frac{\sqrt{2}}{2}}^{1} \int_{\frac{1}{x}}^{2x} x\,dy\,dx + \int_{1}^{\sqrt{2}} \int_{x}^{\frac{2}{x}} x\,dy\,dx$$

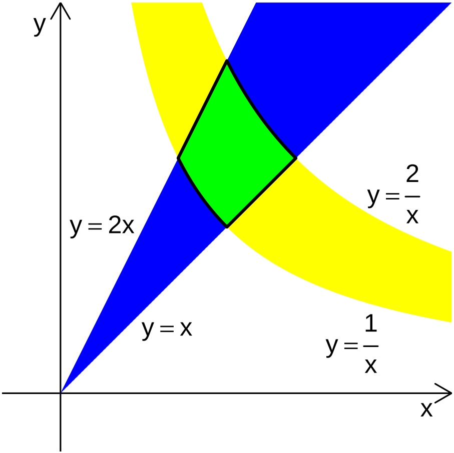
$$
\begin{aligned}
&= \int_{\frac{\sqrt{2}}{2}}^{1} \left[\int_{\frac{1}{x}}^{2x} x\, dy \right] dx + \int_{1}^{\sqrt{2}} \left[\int_{x}^{\frac{2}{x}} x\, dy \right] dx \\
&= \int_{\frac{\sqrt{2}}{2}}^{1} \left[xy \,\Big|_{y=\frac{1}{x}}^{y=2x} \right] dx + \int_{1}^{\sqrt{2}} \left[xy \,\Big|_{y=x}^{y=\frac{2}{x}} \right] dx \\
&= \int_{\frac{\sqrt{2}}{2}}^{1} [2x^2 - 1]\, dx + \int_{1}^{\sqrt{2}} [2 - x^2]\, dx \\
&= [\frac{2}{3}x^3 - x] \Big|_{\frac{\sqrt{2}}{2}}^{1} + [2x - \frac{1}{3}x^3] \Big|_{1}^{\sqrt{2}} \\
&= \left[\left(\frac{2}{3} - 1 \right) - \left(\frac{2}{3} \frac{(\sqrt{2})^3}{2^3} - \frac{\sqrt{2}}{2} \right) \right] + \left[\left(2\sqrt{2} - \frac{1}{3}(\sqrt{2})^3 \right) - \left(2 - \frac{1}{3} \right) \right] \\
&= -\frac{1}{3} - \frac{\sqrt{2}}{6} + \frac{3\sqrt{2}}{6} + \frac{12\sqrt{2}}{6} - \frac{4\sqrt{2}}{6} - \frac{5}{3} = \frac{-12 + 10\sqrt{2}}{6} = \frac{-6 + 5\sqrt{2}}{3}
\end{aligned}
$$

- Consider the change of variables

$$s = xy, \qquad t = \frac{y}{x}$$

$$D = \Big\{(x, y) \in \mathbb{R}^2 : x \geq 0,$$
$$1 \leq xy \leq 2,$$
$$1 \leq \frac{y}{x} \leq 2\Big\}$$

- $\Omega = [1, 2] \times [1, 2]$



$$\iint_D x\, dx\, dy = \int_{t=1}^{t=2} \int_{s=1}^{s=2} x\, dx\, dy$$

Again, Using a Change of Variables

- First, solve for functions $x = x(s,t)$ and $y = y(s,t)$:

$$x = \sqrt{\frac{s}{t}}, \qquad y = \sqrt{st}$$

- Partial derivatives for the change of variables are:

$$\begin{aligned}
\frac{\partial x}{\partial s} &= \frac{\partial}{\partial s}\left[s^{\frac{1}{2}} t^{-\frac{1}{2}}\right] \\
&= \frac{1}{2} s^{-\frac{1}{2}} t^{-\frac{1}{2}} = \frac{1}{2\sqrt{st}}
\end{aligned}
\qquad
\begin{aligned}
\frac{\partial x}{\partial t} &= \frac{\partial}{\partial t}\left[s^{\frac{1}{2}} t^{-\frac{1}{2}}\right] \\
&= -\frac{1}{2} s^{\frac{1}{2}} t^{-\frac{3}{2}} = -\frac{\sqrt{s}}{2t\sqrt{t}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial y}{\partial s} &= \frac{\partial}{\partial s}\left[s^{\frac{1}{2}} t^{\frac{1}{2}}\right] \\
&= \frac{1}{2} s^{-\frac{1}{2}} t^{\frac{1}{2}} = \frac{\sqrt{t}}{2\sqrt{s}}
\end{aligned}
\qquad
\begin{aligned}
\frac{\partial y}{\partial t} &= \frac{\partial}{\partial t}\left[s^{\frac{1}{2}} t^{\frac{1}{2}}\right] \\
&= \frac{1}{2} s^{\frac{1}{2}} t^{-\frac{1}{2}} = \frac{\sqrt{s}}{2\sqrt{t}}
\end{aligned}$$

Again, Using a Change of Variables

- Jacobian for the change of variables:

$$\begin{aligned}
\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} &= \frac{1}{2\sqrt{st}}\frac{\sqrt{s}}{2\sqrt{t}} - \left(-\frac{\sqrt{s}}{2t\sqrt{t}}\right)\frac{\sqrt{t}}{2\sqrt{s}} \\
&= \frac{1}{4t} + \frac{1}{4t} \\
&= \frac{1}{2t}
\end{aligned}$$

- Change of variables formula:

$$\iint_D x\, dx\, dy = \int_{t=1}^{t=2}\int_{s=1}^{s=2}\sqrt{\frac{s}{t}}\,\frac{1}{2t}\, ds\, dt$$

$$
\begin{aligned}
\int_{t=1}^{t=2}\left[\int_{s=1}^{s=2}\sqrt{\frac{s}{t}}\,\frac{1}{2t}\,ds\right]dt &= \frac{1}{2}\int_{t=1}^{t=2}\left[\int_{s=1}^{s=2} s^{\frac{1}{2}}t^{-\frac{3}{2}}\,ds\right]dt \\
&= \frac{1}{2}\int_{t=1}^{t=2}\left[\frac{2}{3}s^{\frac{3}{2}}t^{-\frac{3}{2}}\,\Big|_{s=1}^{s=2}\right]dt \\
&= \frac{1}{3}\int_{t=1}^{t=2}\left[2\sqrt{2}t^{-\frac{3}{2}} - t^{-\frac{3}{2}}\right]dt \\
&= \frac{1}{3}\int_{t=1}^{t=2}(2\sqrt{2}-1)\,t^{-\frac{3}{2}}\,dt \\
&= \frac{2\sqrt{2}-1}{3}\left[-2t^{-\frac{1}{2}}\,\Big|_{t=1}^{t=2}\right] \\
&= \frac{2\sqrt{2}-1}{3}(2-\sqrt{2}) \\
&= \frac{5\sqrt{2}-6}{3}
\end{aligned}
$$

Outline

Separable Functions

- Take another look at the example in the last section

$$\int_{t=1}^{t=2}\left[\int_{s=1}^{s=2}\sqrt{\frac{s}{t}}\,\frac{1}{2t}\,ds\right]dt \;=\; \frac{1}{2}\int_{t=1}^{t=2}\left[\int_{s=1}^{s=2}s^{\frac{1}{2}}\,t^{-\frac{3}{2}}\,ds\right]dt$$

- Since t (and any function of t) is constant while integrating wrt s

$$\int_{t=1}^{t=2}\left[\int_{s=1}^{s=2}\sqrt{\frac{s}{t}}\,\frac{1}{2t}\,ds\right]dt \;=\; \frac{1}{2}\int_{t=1}^{t=2}t^{-\frac{3}{2}}\left[\int_{s=1}^{s=2}s^{\frac{1}{2}}\,ds\right]dt$$

- The double integral is the product of 2 single-variable definite integrals

$$\int_{t=1}^{t=2}\left[\int_{s=1}^{s=2}\sqrt{\frac{s}{t}}\,\frac{1}{2t}\,ds\right]dt \;=\; \frac{1}{2}\left[\int_{s=1}^{s=2}s^{\frac{1}{2}}\,ds\right]\left[\int_{t=1}^{t=2}t^{-\frac{3}{2}}\,dt\right]$$

Separable Functions

$$\begin{aligned}
\frac{1}{2}\left[\int_{s=1}^{s=2} s^{\frac{1}{2}}\, ds\right]\left[\int_{t=1}^{t=2} t^{-\frac{3}{2}}\, dt\right] &= \frac{1}{2}\left[\frac{2}{3}s^{\frac{3}{2}}\Big|_{s=1}^{s=2}\right]\left[-2t^{-\frac{1}{2}}\Big|_{t=1}^{t=2}\right] \\
&= \frac{-2}{3}\left[s^{\frac{3}{2}}\Big|_{s=1}^{s=2}\right]\left[t^{-\frac{1}{2}}\Big|_{t=1}^{t=2}\right] \\
&= -\frac{2}{3}\left[2\sqrt{2}-1\right]\left[\frac{1}{\sqrt{2}}-1\right] \\
&= -\frac{2}{3}\left[2-2\sqrt{2}-\frac{1}{\sqrt{2}}+1\right] \\
&= -\frac{1}{3}\left[4-4\sqrt{2}-\sqrt{2}+2\right] \\
&= \frac{5\sqrt{2}-6}{3}
\end{aligned}$$

Separable Functions

- Let $R = [a, b] \times [c, d]$ be a rectangle
- Let $f(x, y)$ be a continuous, separable function

$$f(x, y) = g(x)\, h(y)$$

$g(x)$ and $h(x)$ continuous

- Double integral of f over R is the product of single-variable integrals

$$\begin{aligned}
\iint_R f(x, y)\, dx\, dy &= \int_c^d \int_a^b g(x)\, h(y)\, dx\, dy \\
&= \int_c^d h(y) \left[\int_a^b g(x)\, dx \right] dy \\
&= \left[\int_a^b g(x)\, dx \right] \left[\int_c^d h(y)\, dy \right]
\end{aligned}$$

Outline

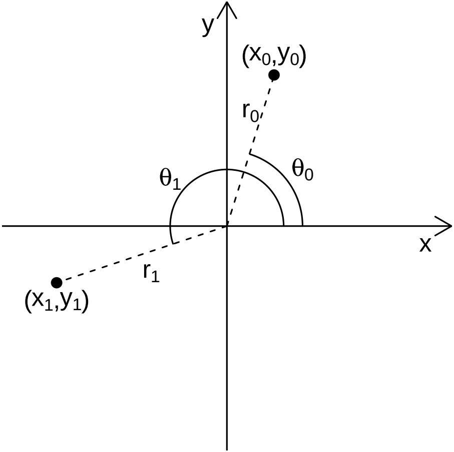
Polar Coordinates

- Describe points in $\mathbb{R}^2$ using
  - $r \in [0, \infty)$
  - $\theta \in [0, 2\pi)$
- $r =$ distance from origin
- $\theta =$ angle counter clockwise from positive x-axis
- $(x, y) \longleftrightarrow (r, \theta)$

$$x(r,\theta) = r\cos(\theta)$$

$$y(r,\theta) = r\sin(\theta)$$

- Can simplify integration problems



Change of Variables to Polar Coordinates

- The change of variables is:

$$x(r,\theta) = r\,\cos(\theta) \qquad y(r,\theta) = r\,\sin(\theta)$$

- The partial derivatives of the change of variables are:

$$\frac{\partial x}{\partial r} = \cos(\theta) \quad \frac{\partial x}{\partial \theta} = -r\,\sin(\theta) \quad \frac{\partial y}{\partial r} = \sin(\theta) \quad \frac{\partial y}{\partial \theta} = r\,\cos(\theta)$$

- The Jacobian is:

$$\begin{aligned}
\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} &= \cos(\theta)\cdot r\,\cos(\theta) - (-r\,\sin(\theta))\cdot\sin(\theta) \\
&= r\left[\cos^2(\theta) + \sin^2(\theta)\right] \\
&= r
\end{aligned}$$

Change of Variables to Polar Coordinates

- The change of variable formula to polar coordinates:

$$\iint_D f(x,y)\,dx\,dy = \iint_D f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta$$

- Integrate $f(x,y)$ over a disk of radius R centered at the origin:

$$\iint_{D(0,R)} f(x,y)\,dx\,dy = \int_0^{2\pi}\left[\int_0^R f(r\cos(\theta), r\sin(\theta))\,r\,dr\right]d\theta$$

- Integrate $f(x,y)$ over the plane $\mathbb{R}^2$:

$$\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = \int_0^{2\pi}\left[\int_0^\infty f(r\cos(\theta), r\sin(\theta))\,r\,dr\right]d\theta$$

- The latter can be useful for integrating probability density functions

Example

- Let $D = \{(x, y) : x^2 + y^2 \leq 1\}$ (disk of radius 1 centered at origin)
- Compute the integral

$$\iint_D (1 - x^2 - y^2)\, dx\, dy$$

- Try once using xy-coordinates
- Then try once using polar coordinates

$$
\begin{aligned}
\iint_D f(x,y)\,dy\,dx &= \int_{-1}^{1}\left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1-x^2-y^2)\,dy\right]dx \\
&= \int_{-1}^{1}\left[\left((1-x^2)y-\frac{y^3}{3}\right)\Bigg|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\right]dx \\
&= \int_{-1}^{1}\left[2(1-x^2)\sqrt{1-x^2}-\frac{2(\sqrt{1-x^2})^3}{3}\right]dx \\
&= \int_{-1}^{1}\left[\frac{6(\sqrt{1-x^2})^3-2(\sqrt{1-x^2})^3}{3}\right]dx \\
&= \frac{4}{3}\int_{-1}^{1}\left[(\sqrt{1-x^2})^3\right]dx \\
&\;\vdots \qquad\qquad\qquad \vdots \\
&= \frac{1}{6}\left(x\sqrt{1-x^2}(5-2x^2)+3\arcsin(x)\right)\Bigg|_{-1}^{1}
\end{aligned}
$$

Example (in polar coordinates)

$$
\begin{aligned}
\iint_D f(x,y)\,dy\,dx &= \int_0^{2\pi}\int_0^1 \left[1 - r^2\cos^2(\theta) - r^2\sin^2(\theta)\right] r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \left[1 - r^2\left(\cos^2(\theta) + \sin^2(\theta)\right)\right] r\,dr\,d\theta \\
&= \int_0^{2\pi}\left[\int_0^1 \left[r - r^3\right] dr\right] d\theta \\
&= \int_0^{2\pi}\left[\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1\right] d\theta \\
&= \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{\pi}{2}
\end{aligned}
$$

Outline

The Standard Normal Density

- The standard normal density

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

- The function Φ used in the Black-Scholes formula

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)\, dt$$

- Raises $2\frac{1}{2}$ questions:

1 Where does the $\frac{1}{\sqrt{2\pi}}$ come from?

2 Why not use a closed-form expression for Φ?

$2\frac{1}{2}$ Where does the $\frac{1}{\sqrt{2\pi}}$ come from?

The Standard Normal Density

- Since ϕ is a probability density function

$$\int_{-\infty}^{\infty} \phi(x)\, dx = \int_{-\infty}^{\infty} \phi(x)\, dx = 1$$

- Implies that

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

- Change of variables

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

- The problem is e^{-x^2} does not have an antiderivative

- Let

$$M = \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

- Want to show that $M = \sqrt{\pi}$

- Can also express M as

$$M = \int_{-\infty}^{\infty} e^{-y^2}\, dy$$

- Now want to show that $M^2 = \pi$

$$M^2 = \left[\int_{-\infty}^{\infty} e^{-x^2}\, dx\right] \left[\int_{-\infty}^{\infty} e^{-y^2}\, dy\right]$$

- This is the double integral of a separable function

Change to Polar Coordinates

- Unseparate the double integral

$$
\begin{aligned}
M^2 &= \left[\int_{-\infty}^{\infty} e^{-x^2}\,dx\right]\left[\int_{-\infty}^{\infty} e^{-y^2}\,dy\right] \\
&= \int_{-\infty}^{\infty} e^{-x^2}\left[\int_{-\infty}^{\infty} e^{-y^2}\,dy\right] dx \\
&= \int_{-\infty}^{\infty} e^{-x^2} \int_{-\infty}^{\infty} e^{-y^2}\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dy\,dx \\
&= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dy\,dx
\end{aligned}
$$

- Change to polar coordinates

$$
= \int_0^{2\pi}\int_0^{\infty} e^{-[r\cos(\theta)]^2+[r\sin(\theta)]^2}\,r\,dr\,d\theta
$$

Change to Polar Coordinates

$$\begin{aligned}
M^2 &= \int_0^{2\pi}\int_0^{\infty} e^{-[r\cos(\theta)]^2+[r\sin(\theta)]^2}\, r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-r^2[\cos^2(\theta)+\sin^2(\theta)]}\, r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta & \text{let } u = -r^2 \\
&= \left[\int_0^{2\pi} d\theta\right]\left[-\frac{1}{2}\int_{u=0}^{u=-\infty} e^{u}\,(-2r\,dr)\right] & du = -2r\,dr \\
&= \left[\ \theta\ \Big|_0^{2\pi}\ \right]\left[-\frac{1}{2}e^{u}\ \Big|_0^{-\infty}\ \right] \\
&= [2\pi - 0]\left[-\frac{1}{2}\left(\lim_{t\to-\infty} e^{t} - 1\right)\right] = 2\pi \cdot \frac{1}{2} = \pi
\end{aligned}$$

In summary:

- Started out with
$$M = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$
- Showed that $M^2 = \pi$ and thus that $M = \sqrt{\pi}$
- Can conclude that
$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Outline

Marginal Density of a Bivariate Normal Distribution

Bivariate normal density function: $f_{X,Y}(x, y; \mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$

$$\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{\dfrac{(x-\mu_x)^2}{\sigma_x^2} - \dfrac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \dfrac{(y-\mu_y)^2}{\sigma_y^2}}{2(1-\rho^2)}\right]$$

The marginal density is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$$

- Let X and Y be returns on an asset in consecutive periods
- Assume that $\mu_x = \mu_y = \mu$ and $\sigma_x = \sigma_y = \sigma$

Marginal Density

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[-\frac{(x-\mu)^2 - 2\rho(x-\mu)(y-\mu) + (y-\mu)^2}{2\sigma^2(1-\rho^2)}\right] dy$$

Guess that $f_X(x)$ is normal with mean μ and variance σ^2

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$\times \int_{-\infty}^{\infty} \mathcal{C} \exp\left[\frac{(x-\mu)^2}{2\sigma^2} - \frac{(x-\mu)^2 - 2\rho(x-\mu)(y-\mu) + (y-\mu)^2}{2\sigma^2(1-\rho^2)}\right] dy$$

where $\mathcal{C} = \dfrac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}}$

Marginal Density

Lets look at the quantity in the square brackets

$$\left[\frac{(x-\mu)^2}{2\sigma^2} - \frac{(x-\mu)^2 - 2\rho(x-\mu)(y-\mu) + (y-\mu)^2}{2\sigma^2(1-\rho^2)}\right]$$

$$= \left[\frac{(1-\rho^2)(x-\mu)^2 - (x-\mu)^2 + 2\rho(x-\mu)(y-\mu) - (y-\mu)^2}{2\sigma^2(1-\rho^2)}\right]$$

$$= \left[\frac{-\rho^2(x-\mu)^2 + 2\rho(x-\mu)(y-\mu) - (y-\mu)^2}{2\sigma^2(1-\rho^2)}\right]$$

$$= \left[-\frac{\rho^2(x-\mu)^2 - 2\rho(x-\mu)(y-\mu) + (y-\mu)^2}{2\sigma^2(1-\rho^2)}\right]$$

$$= \left[-\frac{\left[\rho(x-\mu) - (y-\mu)\right]^2}{2\sigma^2(1-\rho^2)}\right] = \left[-\frac{\left[y - (\mu + \rho(x-\mu))\right]^2}{2(\sigma\sqrt{1-\rho^2})^2}\right]$$

Marginal Density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$
$$\times \int_{-\infty}^{\infty} \mathcal{C} \exp\left[\frac{(x-\mu)^2}{2\sigma^2} - \frac{(x-\mu)^2 - 2\rho(x-\mu)(y-\mu) + (y-\mu)^2}{2\sigma^2(1-\rho^2)}\right] dy$$

Lets look just at the integrand

$$\frac{1}{\sqrt{2\pi}(\sigma\sqrt{1-\rho^2})} \exp\left[-\frac{\left[y - (\mu + \rho(x-\mu))\right]^2}{2(\sigma\sqrt{1-\rho^2})^2}\right]$$

Let $m = (\mu + \rho(x-\mu))$ and $s = (\sigma\sqrt{1-\rho^2})$, the integrand becomes

$$\frac{1}{\sqrt{2\pi}s} \exp\left[-\frac{(y-m)^2}{2s^2}\right]$$

Marginal Density

- The integral becomes

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}s} \exp\left[-\frac{(y-m)^2}{2s^2}\right] dy$$

- Integral of a normal density over the real line, thus equal to 1
- The guess for the marginal density was correct

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}s} \exp\left[-\frac{(y-m)^2}{2s^2}\right] dy \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \end{aligned}$$

Bonus: Conditional Density Function

- The function that we integrated out is the conditional density
- Denoted by $f_{Y|X}(y|x)$

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}(\sigma\sqrt{1-\rho^2})} \exp\left[-\frac{\left[y - (\mu + \rho(x-\mu))\right]^2}{2(\sigma\sqrt{1-\rho^2})^2}\right]$$

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