COMPUTATIONAL FINANCE & RISK MANAGEMENT
UNIVERSITY *of* WASHINGTON
Department of Applied Mathematics

AMATH 460: Mathematical Methods for Quantitative Finance

3. Partial Derivatives

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Outline

Outline

Scalar Valued Functions

- A function of several variables that takes values in $\mathbb{R}$ is called a <u>scalar valued</u> function.
- Notation: $f : \mathbb{R}^n \to \mathbb{R}$

$$y = f(x_1, x_2, \ldots, x_n) \qquad y \in \mathbb{R}, \quad x_j \in \mathbb{R} \;\text{ for }\; j = 1, \ldots, n$$

- Example: Black-Scholes Formula for a European Call Option Price

Inputs:				
	S	asset price	σ	asset volatility
	K	strike price	r	risk-free interest rate
	T	maturity	q	asset continuous dividend rate
	t	time		

$$C(S, t; \cdot) = \\ S\, e^{-q(T-t)}\, \Phi(d_+(S, t; \cdot)) \;-\; K\, e^{-r(T-t)}\, \Phi(d_-(S, t; \cdot))$$

$$d_+(S, t; \cdot) = \frac{\log\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\ldots}, \quad d_-(S, t; \cdot) = d_+(S, t; \cdot) - \sigma\sqrt{T-t}$$

Partial Derivatives

- Let $f : \mathbb{R}^n \to \mathbb{R}$
- The partial derivative of f with respect to x_j is denoted by $\frac{\partial f}{\partial x_j}(x_1, \ldots, n_n)$ and is defined as

$$\frac{\partial f}{\partial x_j}(\cdot) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

  if the limit exists and is finite.

- In practice, to compute $\frac{\partial f}{\partial x_j}$
  - fix x_k for $k \neq j$
  - differentiate f as a function of one variable x_j

Example

- $f(x,y) = x^2 y + e^{-xy^3}$

$$
\begin{aligned}
\frac{\partial}{\partial x} f(x,y) &= \frac{\partial}{\partial x}\left[y\, x^2\right] + \frac{\partial}{\partial x}\left[e^{-xy^3}\right] && \text{let } u = -xy^3 \\
&= y \frac{\partial}{\partial x}\left[x^2\right] + \frac{\partial}{\partial x}\left[e^u\right] \\
&= 2xy + e^u \frac{\partial u}{\partial x} && \frac{\partial u}{\partial x} = -y^3 \\
&= 2xy - y^3 e^{-xy^3}
\end{aligned}
$$

Example (continued)

- $f(x,y) = x^2 y + e^{-xy^3}$

$$
\begin{aligned}
\frac{\partial}{\partial y} f(x,y) &= \frac{\partial}{\partial y}\left[y\,x^2\right] + \frac{\partial}{\partial y}\left[e^{-xy^3}\right] && \text{let } u = -xy^3 \\
&= x^2 \frac{\partial}{\partial y}[y] + \frac{\partial}{\partial y}\left[e^u\right] \\
&= x^2 + e^u \frac{\partial u}{\partial y} && \frac{\partial u}{\partial y} = -3xy^2 \\
&= x^2 - 3xy^2 e^{-xy^3}
\end{aligned}
$$

The Gradient

- Let $f(x) = f(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$. The <u>gradient</u> of $f(x)$ is denoted by $D\,f(x)$ and is defined to be the following $1 \times n$ array of partial derivatives.

$$D\,f(x) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \cdots & \dfrac{\partial f}{\partial x_n} \end{bmatrix}$$

Vector Valued Functions

- A function of one or several variables that takes values in a multidimensional space is called a vector valued function.

- Notation: $f : \mathbb{R}^n \to \mathbb{R}^m$

$$f(x_1, \ldots, x_n) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ f_2(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{bmatrix}$$

- Partial derivatives have the form

$$\frac{\partial f_i}{\partial x_j}(x_1, \ldots, x_n)$$

- There are $n \times m$ first-order partial derivatives in total. Yikes!

Outline

Higher Order Partial Derivatives

- For functions of a single variable:

$$\frac{d^2}{dx^2} f(x) = \frac{d}{dx}\left[\frac{d}{dx} f(x)\right] = \frac{d}{dx} f'(x) = f''(x)$$

- For functions of several variables:

$$\frac{\partial^2}{\partial x^2} e^{xy} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} e^{xy}\right] = \frac{\partial}{\partial x}\left[y e^{xy}\right] = y \frac{\partial}{\partial x} e^{xy} = y^2 e^{xy}$$

$$\frac{\partial^2}{\partial y^2} e^{xy} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y} e^{xy}\right] = \frac{\partial}{\partial y}\left[x e^{xy}\right] = x \frac{\partial}{\partial y} e^{xy} = x^2 e^{xy}$$

- For functions of several variables also have mixed partial derivatives:

$$\frac{\partial^2}{\partial x \partial y} e^{xy} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y} e^{xy}\right] = \frac{\partial}{\partial x}\left[x e^{xy}\right] = e^{xy} + x \frac{\partial}{\partial x} e^{xy} = e^{xy} + xy e^{xy}$$

Mixed Partial Derivatives

- What is the relationship between $\dfrac{\partial^2}{\partial x \partial y}$ and $\dfrac{\partial^2}{\partial y \partial x}$?

- Already saw that $\dfrac{\partial^2}{\partial x \partial y} e^{xy} = e^{xy} + xy e^{xy}$

$$\frac{\partial^2}{\partial y \partial x} e^{xy} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x} e^{xy}\right] = \frac{\partial}{\partial y}\left[y e^{xy}\right] = e^{xy} + y \frac{\partial}{\partial y} e^{xy} = e^{xy} + xy e^{xy}$$

- For $f(x, y) = e^{xy}$ have

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y} e^{xy} = e^{xy} + xy e^{xy} = \frac{\partial^2}{\partial y \partial x} e^{xy} = \frac{\partial^2 f}{\partial y \partial x}$$

- But, $f(x, y) = e^{xy}$ has a certain *"symmetry"* wrt differentiation
- Let's see what happens when $f(x, y) = x^2 y + e^{-xy^3}$

Mixed Partial Derivatives

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(x^2 y + e^{-xy^3}\right)\right] && \text{let } u = -xy^3 \\
&= \frac{\partial}{\partial x}\left[x^2 + \frac{\partial}{\partial y} e^u\right] \\
&= \frac{\partial}{\partial x}\left[x^2 + e^u \frac{\partial u}{\partial y}\right] && \frac{\partial u}{\partial y} = -3xy^2 \\
&= \frac{\partial}{\partial x}\left[x^2 - 3xy^2 e^{-xy^3}\right] \\
&= 2x - \left[3y^2 e^{-xy^3} + 3xy^2 \frac{\partial}{\partial x} e^u\right] \\
&= 2x - 3y^2 e^{-xy^3} - 3xy^2 e^u \frac{\partial u}{\partial x} && \frac{\partial u}{\partial x} = -y^3 \\
&= 2x - 3y^2 e^{-xy^3} + 3xy^5 e^{-xy^3}
\end{aligned}
$$

Mixed Partial Derivatives

$$
\begin{aligned}
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(x^2 y + e^{-xy^3}\right)\right] && \text{let } u = -xy^3 \\
&= \frac{\partial}{\partial y}\left[2xy + \frac{\partial}{\partial x} e^u\right] \\
&= \frac{\partial}{\partial y}\left[2xy + e^u \frac{\partial u}{\partial x}\right] && \frac{\partial u}{\partial x} = -y^3 \\
&= \frac{\partial}{\partial y}\left[2xy - y^3 e^u\right] \\
&= 2x - \left[3y^2 e^{-xy^3} + y^3 \frac{\partial}{\partial y} e^u\right] \\
&= 2x - 3y^2 e^{-xy^3} - y^3 e^u \frac{\partial u}{\partial y} && \frac{\partial u}{\partial y} = -3xy^2 \\
&= 2x - 3y^2 e^{-xy^3} + 3xy^5 e^{-xy^3}
\end{aligned}
$$

Mixed Partial Derivatives

- When $f(x, y) = x^2y + e^{-xy^3}$ have

$$\frac{\partial^2 f}{\partial x \partial y} = 2x - 3y^2 e^{-xy^3} + 3xy^5 e^{-xy^3} = \frac{\partial^2 f}{\partial y \partial x}$$

- **Theorem** If all of the partial derivatives of order k of the function $f(x)$ exist and are continuous, then the order in which partial derivatives of $f(x)$ of order at most k are computed does not matter.

The Hessian

- Let $f(x) = f(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$. The Hessian of $f(x)$ is denoted by $D^2 f(x)$ and is defined to be the following $n \times n$ array of (mixed) partial derivatives.

$$D^2 f(x) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \dfrac{\partial^2 f}{\partial x_n \partial x_1} \\[2ex] \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_n \partial x_n} \\[2ex] \vdots & \vdots & \ddots & \vdots \\[2ex] \dfrac{\partial^2 f}{\partial x_1 \partial x_n} & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Outline

Functions of Two Variables

- Let $f = f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be a scalar-valued function
- The partial derivatives of f are also functions of x and y:

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \\ \frac{\partial f}{\partial y}(x, y) &= \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h} \end{aligned}$$

- The gradient of $f(x, y)$ is

$$D\, f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x}(x, y) & \frac{\partial f}{\partial y}(x, y) \end{bmatrix}$$

- The Hessian of $f(x, y)$ is

$$D^2 f(x,y) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial^2 x}(x,y) & \dfrac{\partial^2 f}{\partial y \partial x}(x,y) \\ \\ \dfrac{\partial^2 f}{\partial x \partial y}(x,y) & \dfrac{\partial^2 f}{\partial^2 y}(x,y) \end{bmatrix}$$

Example

- Let $f(x,y) = x^2y^3$. Evaluate Df and D^2f at the point $(1,2)$

$$
\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= 2xy^3 \\
\frac{\partial f}{\partial y}(x,y) &= 3x^2y^2 \\
\frac{\partial^2 f}{\partial x^2}(x,y) &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}(x,y)\right] = \frac{\partial}{\partial x}\left[2xy^3\right] = 2y^3 \\
\frac{\partial^2 f}{\partial x \partial y}(x,y) &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}(x,y)\right] = \frac{\partial}{\partial x}\left[3x^2y^2\right] = 6xy^2 = \frac{\partial^2 f}{\partial y \partial x}(x,y) \\
\frac{\partial^2 f}{\partial y^2}(x,y) &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}(x,y)\right] = \frac{\partial}{\partial y}\left[3x^2y^2\right] = 6x^2y
\end{aligned}
$$

Example (continued)

$$D\,f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \end{bmatrix} = \begin{bmatrix} 2xy^3 & 3x^2y^2 \end{bmatrix}$$

$$D\,f(1,2) = \begin{bmatrix} 2 \times 1 \times 2^3 & 3 \times 1^2 \times 2^2 \end{bmatrix} = \begin{bmatrix} 16 & 12 \end{bmatrix}$$

$$D^2 f(x,y) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial y \partial x} \\ \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y \end{bmatrix}$$

$$D^2 f(1,2) = \begin{bmatrix} 2 \times 2^3 & 6 \times 1 \times 2^2 \\ 6 \times 1 \times 2^2 & 6 \times 1^2 \times 2 \end{bmatrix} = \begin{bmatrix} 16 & 24 \\ 24 & 12 \end{bmatrix}$$

Outline

Chain Rule for Functions of a Single Variable

- Let $f(x)$ be a differentiable function
- Let $x = g(t)$ where $g(t)$ is a differentiable function
- $f(x)$ can be thought of as a function of t: $f(x) = f(g(t))$, and

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

- Example: evaluate $\dfrac{d}{dt}\log(\cos(t))$

  Let $f(x) = \log(x)$, $x = \cos(t)$

$$\frac{df}{dt} = \frac{d}{dt}f(x) = \frac{df}{dx}\frac{dx}{dt} = \frac{1}{x}\left[-\sin(t)\right] = -\frac{1}{\cos(t)}\sin(t) = -\tan(t)$$

Chain Rule for Functions of 2 Variables

- Let $f(x, y)$ be a differentiable function
- Let $x = g(t)$ and $y = h(t)$ where g and h are differentiable functions

- $f(x, y) = f\left(g(t), h(t)\right)$ is a function of t and

$$\frac{df}{dt}(g(t), h(t)) = \frac{\partial f}{\partial x}(g(t), h(t))\ g'(t) + \frac{\partial f}{\partial y}(g(t), h(t))\ h'(t)$$

- Using Leibniz notation:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example

- Let $f(x, y) = x^2 + y + xy^3, \quad x = e^{2t}, \quad \text{and} \quad y = t^2$
- First, by direct computation

$$
\begin{aligned}
f(t) &= e^{4t} + t^2 + t^6 e^{2t} \\
\frac{d}{dt} f(t) &= 4e^{4t} + 2t + \left[6t^5 e^{2t} + t^6\, 2e^{2t}\right] \\
\frac{df}{dt} &= 2t^6 e^{2t} + 6t^5 e^{2t} + 2t + 4e^{4t}
\end{aligned}
$$

Example (continued)

- Again, using the chain rule

$$
\begin{aligned}
f(x,y) &= x^2 + y + xy^3 \\
\frac{\partial f}{\partial x} &= 2x + y^3 \qquad\qquad \frac{\partial f}{\partial y} = 1 + 3xy^2 \\
\frac{dx}{dt} &= 2e^{2t} \qquad\qquad\qquad \frac{dy}{dt} = 2t
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt} f(x,y) &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (2x + y^3)\,2e^{2t} + (1 + 3xy^2)\,2t \\
&= (2e^{2t} + t^6)\,2e^{2t} + (1 + 3t^4 e^{2t})\,2t \\
&= 2t^6 e^{2t} + 6t^5 e^{2t} + 2t + 4e^{4t}
\end{aligned}
$$

Chain Rule for Functions of 2 Variables

- Let $f(x, y)$ be a differentiable function
- Let $x = g(s, t)$ and $y = h(s, t)$ where g and h are differentiable
- $f(x, y) = f(g(s, t), h(s, t))$ is a function of s and t

$$
\begin{aligned}
\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\
&= \frac{\partial f}{\partial x}\frac{\partial g}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial s}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\
&= \frac{\partial f}{\partial x}\frac{\partial g}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial t}
\end{aligned}
$$

Chain Rule for Functions of n Variables

- In general, let $f = f(x_1, \ldots, x_n)$ be a function of n variables
- For $i = 1, \ldots, n$, let $x_i = x_i(t_1, \ldots, t_m)$ be functions of m variables
- The partial derivative of f wrt t_j is

$$\frac{\partial f}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial t_j}$$

Example

- $f(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_1x_3 + 2x_3^2$
  $x_1(t_1, t_2) = t_1^2 - t_2^2 + 1, \;\; x_2(t_1, t_2) = t_2^2 + t_1 + 1, \;\; x_3(t_1, t_2) = -t_1^2 - 1$
- Compute $\frac{\partial f}{\partial t_1}$

$$\begin{aligned}
\frac{\partial f}{\partial t_1} &= \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial t_1} \\
&= (2x_1 + x_2 + x_3)(2t_1) + (x_1)(1) + (x_1 + 4x_3)(-2t_1) \\
&= (2t_1^2 - 2t_2^2 + 2 + t_2^2 + t_1 + 1 - t_1^2 - 1)(2t_1) \\
&\quad + (t_1^2 - t_2^2 + 1) + (t_1^2 - t_2^2 + 1 - 4t_1^2 - 4)(-2t_1) \\
&= (t_1^2 - t_2^2 + t_1 + 2)(2t_1) + (t_1^2 - t_2^2 + 1) + (3t_1^2 + t_2^2 + 3)(2t_1) \\
&= 8t_1^3 + 3t_1^2 + 10t_1 - t_2^2 + 1
\end{aligned}$$

Outline

Implicit Functions

- So far, functions have expressed one variable in terms of another (or others), e.g.,

$$y = f(x) = \sqrt{1 - x^2} \qquad \text{or} \qquad y = \frac{\sin(x)}{x}$$

- An implicit function is defined by a more general relation between the variables, e.g.,

$$x^2 + y^2 = 1 \qquad \text{or} \qquad x^3 + y^3 = 6xy$$

- In some cases, possible to solve for one variable as an explicit function (or functions) of the others.

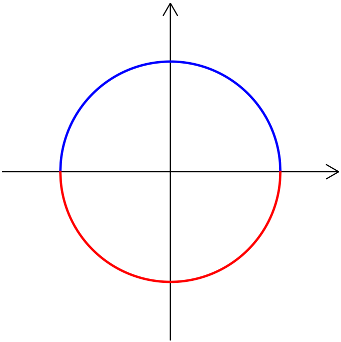
- Terminology: let $F(x, y, z)$ be a function of 3 variables. The set of points that satisfy

$$F(x, y, z) = 0$$

  is called the locus defined by F.

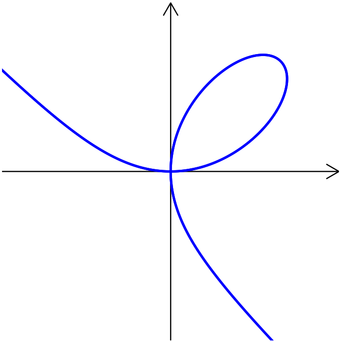
Implicit Functions

Circle



$$x^2 + y^2 = 1$$

Folium



$$x^3 + y^3 = 6xy$$

- blue curve: $y = \sqrt{1 - x^2}$
- red curve: $y = -\sqrt{1 - x^2}$
- blue curve: more difficult ...

Circle

Slope of tangent line to the unit circle

- Case 1: blue curve $y \geq 0$

$$\begin{aligned} \frac{d}{dx} y &= \frac{d}{dx}\sqrt{1-x^2} \\ \frac{dy}{dx} &= \frac{d}{dx}(1-x^2)^{\frac{1}{2}} \\ &= \frac{1}{2}(1-x^2)^{-\frac{1}{2}}\,(-2x) \\ &= \frac{-x}{\sqrt{1-x^2}} \end{aligned}$$

- Case 2: red curve $y < 0$

$$\begin{aligned} \frac{d}{dx} y &= \frac{d}{dx}\left[-\sqrt{1-x^2}\right] \\ \frac{dy}{dx} &= \frac{d}{dx}\left[-(1-x^2)^{\frac{1}{2}}\right] \\ &= -\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\,(-2x) \\ &= \frac{-x}{-\sqrt{1-x^2}} \end{aligned}$$

Derivatives of Implicit Functions Using the Chain Rule

- Recall that if y is a function of x then the chain rule says

$$\frac{d}{dx}y^2 = \frac{d}{dy}\left[y^2\right]\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$\begin{aligned}
x^2 + y^2 &= 1 \\
\frac{d}{dx}\left[x^2 + y^2\right] &= \frac{d}{dx}1 \\
\frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= 0 \\
2x + 2y\frac{dy}{dx} &= 0
\end{aligned}$$

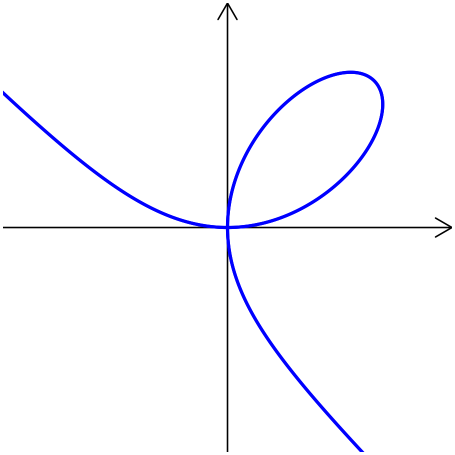
$$\begin{aligned}
2y\frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= \frac{-x}{y} \\
&= \frac{-x}{\sqrt{1-x^2}} \qquad (y \geq 0) \\
&= \frac{-x}{-\sqrt{1-x^2}} \qquad (y < 0)
\end{aligned}$$

Example

- Compute $\frac{dy}{dx}$ for the Folium

$$\begin{aligned}
x^3 + y^3 &= 6xy \\
\frac{d}{dx}\left[x^3 + y^3\right] &= \frac{d}{dx}\left[6xy\right] \\
\frac{d}{dx}\left[x^3\right] + \frac{d}{dx}\left[y^3\right] &= 6y + 6x\frac{d}{dx}\left[y\right] \\
3x^2 + 3y^2\frac{dy}{dx} &= 6y + 6x\frac{dy}{dx} \\
(3y^2 - 6x)\frac{dy}{dx} &= 6y - 3x^2 \\
\frac{dy}{dx} &= \frac{2y - x^2}{y^2 - 2x}
\end{aligned}$$

Example (continued)



$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

Outline

The Greeks

- Let V be the value of a portfolio of derivative securities based on one underlying asset
- The rates of change of the value V wrt pricing parameters (e.g., asset price, volatility, etc.) useful for hedging
- These rates of change are called <u>the Greeks</u> of the portfolio
- Consider a portfolio containing a single European call option
- Price using Black-Scholes

Inputs:				
	S	asset price	σ	asset volatility
	K	strike price	r	(continuous) risk-free interest rate
	T	maturity	q	(continuous) asset dividend rate
	t	time		

The Greeks

- Black-Scholes formula for a European call option:

$$C(S,t) = Se^{-q(T-t)}\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$

where

$$\begin{aligned}
\Phi(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\, dx \\
d_+ &= \frac{\log\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\
d_- &= d_+ - \sigma\sqrt{T-t} = \frac{\log\left(\frac{S}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}$$

Put-Call Parity

- $C(t)$ and $P(t)$ prices of European call and put options on same asset
- Same maturity T and strike price K
- Put-Call parity states that

$$P(t) + S(t)e^{-q(T-t)} - C(t) = Ke^{-r(T-t)}$$

The Greeks

- Delta (Δ): rate of change of C wrt S $\qquad \Delta = \dfrac{\partial C}{\partial S}$
- Gamma (Γ): rate of change of Δ wrt S $\qquad \Gamma = \dfrac{\partial \Delta}{\partial S} = \dfrac{\partial^2 C}{\partial S^2}$
- Theta (Θ): rate of change of C wrt t $\qquad \Theta = \dfrac{\partial C}{\partial t}$
- Rho (ρ): rate of change of C wrt r $\qquad \rho = \dfrac{\partial C}{\partial r}$
- Vega: rate of change of C wrt σ $\qquad \text{vega} = \dfrac{\partial C}{\partial \sigma}$

Outline

Delta

- The Delta (Δ) of a European call option is the rate of change of $C(S,t)$ wrt the asset price S

$$
\begin{aligned}
\Delta &= \frac{\partial}{\partial S} C(S,t) \\
&= \frac{\partial}{\partial S}\left[S e^{-q(T-t)} \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-) \right] \\
&= e^{-q(T-t)} \frac{\partial}{\partial S}\left[S \Phi(d_+) \right] - K e^{-r(T-t)} \frac{\partial}{\partial S}\left[\Phi(d_-) \right]
\end{aligned}
$$

- By the product rule:

$$
= e^{-q(T-t)} \left[\Phi(d_+) + S \frac{\partial}{\partial S} \Phi(d_+) \right] - K e^{-r(T-t)} \frac{\partial}{\partial S}\left[\Phi(d_-) \right]
$$

Delta (continued)

- Consider the partial derivatives

$$\frac{\partial}{\partial S}\left[\Phi(d_\pm)\right]$$

- The chain rule says

$$\frac{\partial}{\partial S}\left[\Phi(d_\pm)\right] = \frac{\partial}{\partial d_\pm}\left[\Phi(d_\pm)\right]\frac{\partial d_\pm}{\partial S}$$

- Recall definition of Φ

$$\begin{aligned}
\Phi(d_\pm) &= \int_{-\infty}^{d_\pm} \phi(x)\,dx = \int_{-\infty}^{d_\pm} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \\
\frac{\partial}{\partial d_\pm}\Phi(d_\pm) &= \frac{\partial}{\partial d_\pm}\int_{-\infty}^{d_\pm} \phi(x)\,dx = \frac{\partial}{\partial d_\pm}\int_{-\infty}^{d_\pm} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \\
\frac{\partial}{\partial d_\pm}\Phi(d_\pm) &= \phi(d_\pm) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(d_\pm)^2}{2}}
\end{aligned}$$

Delta (continued)

- Results so far

$$\Delta = e^{-q(T-t)}\left[\Phi(d_+) + S\frac{\partial}{\partial S}\Phi(d_+)\right] - Ke^{-r(T-t)}\frac{\partial}{\partial S}\left[\Phi(d_-)\right]$$

$$\frac{\partial}{\partial S}\left[\Phi(d_\pm)\right] = \frac{\partial}{\partial d_\pm}\left[\Phi(d_\pm)\right]\frac{\partial d_\pm}{\partial S}$$

$$\frac{\partial}{\partial d_\pm}\Phi(d_\pm) = \phi(d_\pm) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(d_\pm)^2}{2}}$$

- Substituting

$$\Delta = e^{-q(T-t)}\left[\Phi(d_+) + S\,\phi(d_+)\frac{\partial d_+}{\partial S}\right] - Ke^{-r(T-t)}\,\phi(d_-)\frac{\partial d_-}{\partial S}$$

Delta (continued)

- Finally, need to compute partial derivatives of d_+ and d_- wrt to S

$$
\begin{aligned}
d_\pm &= \frac{\log\left(\frac{S}{K}\right) + \left(r - q \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\
\frac{\partial}{\partial S} d_\pm &= \frac{1}{\sigma\sqrt{T-t}} \frac{\partial}{\partial S}\left[\log\left(\frac{S}{K}\right)\right] + \frac{\partial}{\partial S}\left[\frac{\left(r - q \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right]
\end{aligned}
$$

- Let $u = \frac{S}{K}$, then by the chain rule

$$
\begin{aligned}
\frac{\partial}{\partial S} d_\pm &= \frac{1}{\sigma\sqrt{T-t}} \frac{\partial}{\partial S}\left[\log(u)\right] = \frac{1}{\sigma\sqrt{T-t}} \frac{1}{u}\frac{\partial u}{\partial S} = \frac{1}{\sigma\sqrt{T-t}} \frac{K}{S}\frac{1}{K} \\
&= \frac{1}{S\,\sigma\sqrt{T-t}}
\end{aligned}
$$

Delta (continued)

- Putting it all together ...

$$\Delta = e^{-q(T-t)}\left[\Phi(d_+) + S\,\phi(d_+)\frac{\partial d_+}{\partial S}\right] - Ke^{-r(T-t)}\,\phi(d_-)\frac{\partial d_-}{\partial S}$$

$$\frac{\partial}{\partial S}d_\pm = \frac{1}{S\,\sigma\sqrt{T-t}}$$

- Yields the following expression for Δ

$$\Delta = e^{-q(T-t)}\Phi(d_+) + \frac{e^{-q(T-t)}\phi(d_+)}{\sigma\sqrt{T-t}} - \frac{Ke^{-r(T-t)}\phi(d_-)}{S\,\sigma\sqrt{T-t}}$$

Outline

Gamma (Γ)

- Gamma (Γ) is the rate of change of Delta (Δ)
- Hence: Gamma (Γ) is the second partial derivative of C wrt S

$$\Gamma = \frac{\partial}{\partial S}\Delta = \frac{\partial}{\partial S}\,\frac{\partial C}{\partial S} = \frac{\partial^2 C}{\partial S^2}$$

- So, all we have to do is . . .

$$\Gamma = \frac{\partial}{\partial S}\Delta = \frac{\partial}{\partial S}\left[e^{-q(T-t)}\Phi(d_+) + \frac{e^{-q(T-t)}\phi(d_+)}{\sigma\sqrt{T-t}} - \frac{Ke^{-r(T-t)}\phi(d_-)}{S\,\sigma\sqrt{T-t}}\right]$$

- Luckily, there is a shortcut

Simplifying the Expression for Delta

- The textbook says

$$\Delta = e^{-q(T-t)}\Phi(d_+) + \frac{e^{-q(T-t)}\phi(d_+)}{\sigma\sqrt{T-t}} - \frac{Ke^{-r(T-t)}\phi(d_-)}{S\,\sigma\sqrt{T-t}}$$

- Finding an expression for Γ much easier if

$$\frac{e^{-q(T-t)}\phi(d_+)}{\sigma\sqrt{T-t}} - \frac{Ke^{-r(T-t)}\phi(d_-)}{S\,\sigma\sqrt{T-t}} \stackrel{?}{=} 0$$

- Strategy: manipulate $\phi(d_-)$ into $\phi(d_+)$ and see what falls out

$$\phi(d_-) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{d_-^2}{2}\right) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\left(d_+ - \sigma\sqrt{T-t}\,\right)^2}{2}\right)$$

Simplifying the Expression for Delta

$$
\begin{aligned}
\phi(d_-) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\left(d_+ - \sigma\sqrt{T-t}\right)^2}{2}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d_+^2 - 2d_+\sigma\sqrt{T-t} + \sigma^2(T-t)}{2}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d_+^2}{2}\right) \exp\left(-\frac{-2d_+\sigma\sqrt{T-t}}{2}\right) \exp\left(-\frac{\sigma^2(T-t)}{2}\right) \\
&= \phi(d_+) \exp\left(d_+\sigma\sqrt{T-t}\right) \exp\left(-\sigma^2(T-t)/2\right)
\end{aligned}
$$

Reminder: $d_+ = \dfrac{\log\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$

$$
= \phi(d_+) \exp\left[\log\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)\right] \exp\left[-\frac{\sigma^2(T-t)}{2}\right]
$$

Simplifying the Expression for Delta

$$\begin{aligned}\phi(d_-) &= \phi(d_+)\,\frac{S}{K}\exp\left[\left(r-q+\frac{\sigma^2}{2}\right)(T-t)-\frac{\sigma^2}{2}(T-t)\right]\\ &= \phi(d_+)\,\frac{S}{K}e^{r(T-t)}\,e^{-q(T-t)}\end{aligned}$$

- Substituting

$$\frac{e^{-q(T-t)}\phi(d_+)}{\sigma\sqrt{T-t}}-\frac{Ke^{-r(T-t)}\phi(d_-)}{S\,\sigma\sqrt{T-t}}$$

$$\begin{aligned}&= \frac{e^{-q(T-t)}\phi(d_+)}{\sigma\sqrt{T-t}}-\frac{Ke^{-r(T-t)}\phi(d_+)\,\frac{S}{K}\,e^{r(T-t)}\,e^{-q(T-t)}}{S\,\sigma\sqrt{T-t}}\\ &= \frac{e^{-q(T-t)}\phi(d_+)}{\sigma\sqrt{T-t}}-\frac{e^{-q(T-t)}\phi(d_+)}{\sigma\sqrt{T-t}}=0\end{aligned}$$

Gamma (Γ)

$$\Gamma = \frac{\partial}{\partial S}\Delta = \frac{\partial}{\partial S}\left[e^{-q(T-t)}\Phi(d_+) + \frac{e^{-q(T-t)}\phi(d_+)}{\sigma\sqrt{T-t}} - \frac{Ke^{-r(T-t)}\phi(d_-)}{S\,\sigma\sqrt{T-t}}\right]$$

$$= \frac{\partial}{\partial S}\left[e^{-q(T-t)}\Phi(d_+)\right]$$

$$= e^{-q(T-t)}\frac{\partial}{\partial S}\left[\Phi(d_+)\right]$$

$$= e^{-q(T-t)}\frac{\partial}{\partial d_+}\Phi(d_+)\,\frac{\partial d_+}{\partial S} \qquad \frac{\partial d_\pm}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

$$= \frac{e^{-q(T-t)}}{S\sigma\sqrt{T-t}}\,\phi(d_+)$$

$$\Gamma = \frac{\partial}{\partial S}\Delta = \frac{e^{-q(T-t)}}{S\sigma\sqrt{T-t}}\,\frac{1}{\sqrt{2\pi}}e^{-\frac{d_+^2}{2}}$$

Outline

Rho (ρ)

- Rho (ρ) is the rate of change of the value of the portfolio wrt the risk-free interest rate r
- Black-Scholes formula

$$C(S,t) = Se^{-q(T-t)}\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$

- Derivative wrt r:

$$\rho \;=\; \frac{\partial}{\partial r}\left[Se^{-q(T-t)}\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)\right]$$

- Product rule:

$$\rho = Se^{-q(T-t)}\,\frac{\partial}{\partial r}\Phi(d_+)$$

$$-\left[-K(T-t)e^{-r(T-t)}\,\Phi(d_-) + Ke^{-r(T-t)}\frac{\partial}{\partial r}\Phi(d_-)\right]$$

Rho (ρ)

- Chain rule:

$$\rho = Se^{-q(T-t)}\phi(d_+)\,\frac{\partial d_+}{\partial r} - \left[-K(T-t)e^{-r(T-t)}\,\Phi(d_-) + Ke^{-r(T-t)}\phi(d_-)\,\frac{\partial d_-}{\partial r}\right]$$

- Need partial derivatives of d_+ and d_- wrt r:

$$d_\pm = \frac{\log\left(\frac{S}{K}\right) + \left(r - q \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = \frac{r\sqrt{T-t}}{\sigma} + \mathcal{C}$$

$$\frac{\partial}{\partial r}d_\pm = \frac{\sqrt{T-t}}{\sigma}$$

Rho (ρ)

- Substituting $\frac{\partial}{\partial r} d_{\pm} = \frac{\sqrt{T-t}}{\sigma}$ ...

$$\rho = K(T-t)e^{-r(T-t)}\,\Phi(d_-)$$

$$+ Se^{-q(T-t)}\phi(d_+)\,\frac{\sqrt{T-t}}{\sigma} + Ke^{-r(T-t)}\phi(d_-)\,\frac{\sqrt{T-t}}{\sigma}$$

- Rewrite as ...

$$\rho = K(T-t)e^{-r(T-t)}\,\Phi(d_-)$$

$$+ S(T-t)\left[\frac{e^{-q(T-t)}\phi(d_+)}{\sigma\sqrt{T-t}} + \frac{Ke^{-r(T-t)}\phi(d_-)}{S\,\sigma\sqrt{T-t}}\right]$$

$$\rho = K(T-t)e^{-r(T-t)}\,\Phi(d_-)$$

Vega

- Vega is the rate of change of the value of the portfolio wrt the volatility σ
- Black-Scholes formula

$$C(S,t) = Se^{-q(T-t)}\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$

- Derivative wrt σ:

$$\text{vega} = \frac{\partial}{\partial \sigma}\left[Se^{-q(T-t)}\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)\right]$$

- Chain rule:

$$\text{vega} = Se^{-q(T-t)}\phi(d_+)\,\frac{\partial d_+}{\partial \sigma} - Ke^{-r(T-t)}\phi(d_-)\,\frac{\partial d_-}{\partial \sigma}$$

Vega

- Partial derivatives of d_+ and d_- wrt σ:

$$\begin{aligned}
d_\pm &= \frac{\log\left(\frac{S}{K}\right) + \left(r - q \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\
&= \frac{\log\left(\frac{S}{K}\right) + (r-q)(T-t)}{\sqrt{T-t}}\sigma^{-1} \pm \frac{(T-t)}{2\sqrt{T-t}}\sigma \\
\frac{\partial d_\pm}{\partial\sigma} &= -\frac{\log\left(\frac{S}{K}\right) + (r-q)(T-t)}{\sqrt{T-t}}\sigma^{-2} \pm \frac{(T-t)}{2\sqrt{T-t}} \\
&= -\frac{\log\left(\frac{S}{K}\right) + (r-q)(T-t)}{\sigma^2\sqrt{T-t}} \pm \frac{\frac{\sigma^2}{2}(T-t)}{\sigma^2\sqrt{T-t}} \\
&= -\frac{\log\left(\frac{S}{K}\right) + (r - q \mp \frac{\sigma^2}{2})(T-t)}{\sigma^2\sqrt{T-t}}
\end{aligned}$$

Bag of Tricks

- Substitute

$$\frac{\partial}{\partial \sigma} d_{\pm} = -\frac{\log\left(\frac{S}{K}\right) + (r - q \mp \frac{\sigma^2}{2})(T-t)}{\sigma^2\sqrt{T-t}}$$

into

$$\text{vega} = Se^{-q(T-t)}\phi(d_+)\,\frac{\partial d_+}{\partial \sigma} - Ke^{-r(T-t)}\phi(d_-)\,\frac{\partial d_-}{\partial \sigma}$$

- Looks like it's going to get worse before it gets better
- Another approach ...
- Already have

$$\frac{e^{-q(T-t)}\phi(d_+)}{\sigma\sqrt{T-t}} - \frac{Ke^{-r(T-t)}\phi(d_-)}{S\,\sigma\sqrt{T-t}} = 0$$

- Rewrite as

$$Se^{-q(T-t)}\phi(d_+) = Ke^{-r(T-t)}\phi(d_-)$$

Vega

- The expression for vega becomes

$$\begin{aligned}
\text{vega} &= Se^{-q(T-t)}\phi(d_+)\,\frac{\partial d_+}{\partial \sigma} - Ke^{-r(T-t)}\phi(d_-)\,\frac{\partial d_-}{\partial \sigma} \\
&= Se^{-q(T-t)}\phi(d_+)\left(\frac{\partial d_+}{\partial \sigma} - \frac{\partial d_-}{\partial \sigma}\right) \\
&= Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial \sigma}(d_+ - d_-)
\end{aligned}$$

- Recall: $d_- = d_+ - \sigma\sqrt{T-t}$ so that

$$\begin{aligned}
d_+ - d_- &= d_+ - (d_+ - \sigma\sqrt{T-t}) \\
\frac{\partial}{\partial \sigma}(d_+ - d_-) &= \frac{\partial}{\partial \sigma}\sigma\sqrt{T-t} \\
&= \sqrt{T-t}
\end{aligned}$$

Vega

- Substuting . . .

$$\text{vega} = Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial\sigma}(d_+ - d_-)$$

$$\text{vega} = S\sqrt{T-t}\,e^{-q(T-t)}\phi(d_+)$$

Outline

Theta (Θ)

- Theta is the rate of change of the value of the portfolio wrt the time t
- Black-Scholes formula

$$C(S,t) = Se^{-q(T-t)}\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$

- Derivative wrt t:

$$\Theta = \frac{\partial}{\partial t}\left[Se^{-q(T-t)}\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)\right]$$

- Product rule twice:

$$\Theta = qSe^{-q(T-t)}\Phi(d_+) + Se^{-q(T-t)}\frac{\partial}{\partial t}\Phi(d_+)$$

$$- \left[rKe^{-r(T-t)}\Phi(d_-) + Ke^{-r(T-t)}\frac{\partial}{\partial t}\Phi(d_-)\right]$$

Theta (Θ)

$$
\begin{aligned}
\Theta &= qSe^{-q(T-t)}\Phi(d_+) - rKe^{-r(T-t)}\Phi(d_-) \\
&\qquad + \left[Se^{-q(T-t)}\frac{\partial}{\partial t}\Phi(d_+) - Ke^{-r(T-t)}\frac{\partial}{\partial t}\Phi(d_-)\right] \\
&= qSe^{-q(T-t)}\Phi(d_+) - rKe^{-r(T-t)}\Phi(d_-) \\
&\qquad + \left[Se^{-q(T-t)}\phi(d_+)\frac{\partial d_+}{\partial t} - Ke^{-r(T-t)}\phi(d_-)\frac{\partial d_-}{\partial t}\right] \\
&= qSe^{-q(T-t)}\Phi(d_+) - rKe^{-r(T-t)}\Phi(d_-) \\
&\qquad + Se^{-q(T-t)}\phi(d_+)\left(\frac{\partial d_+}{\partial t} - \frac{\partial d_-}{\partial t}\right)
\end{aligned}
$$

Theta (Θ)

- Again, rewrite the quantity

$$\begin{aligned}
\left(\frac{\partial d_+}{\partial t} - \frac{\partial d_-}{\partial t}\right) &= \frac{\partial}{\partial t}(d_+ - d_-) \\
&= \frac{\partial}{\partial t}\sigma\sqrt{T-t} \quad = \sigma\frac{\partial}{\partial t}(T-t)^{\frac{1}{2}} \\
&= \frac{-\sigma}{2\sqrt{T-t}}
\end{aligned}$$

- Substitute into ...

$$\begin{aligned}
\Theta = qSe^{-q(T-t)}\Phi(d_+) - rKe^{-r(T-t)}\Phi(d_-) \\
+ \; Se^{-q(T-t)}\phi(d_+)\left(\frac{\partial d_+}{\partial t} - \frac{\partial d_-}{\partial t}\right)
\end{aligned}$$

Theta (Θ)

- Finally . . .

$$\Theta = -\frac{\sigma S e^{-q(T-t)}}{2\sqrt{T-t}}\phi(d_+) + qSe^{-q(T-t)}\Phi(d_+) - rKe^{-r(T-t)}\Phi(d_-)$$

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