COMPUTATIONAL FINANCE & RISK MANAGEMENT

UNIVERSITY *of* WASHINGTON

Department of Applied Mathematics

AMATH 460: Mathematical Methods for Quantitative Finance

2. Integrals

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Outline

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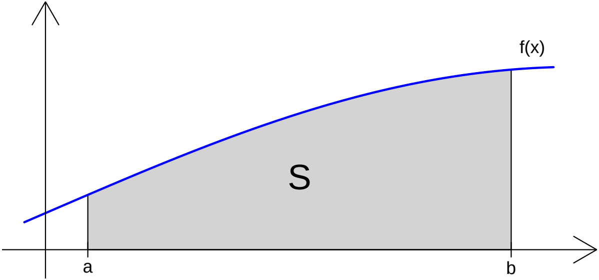
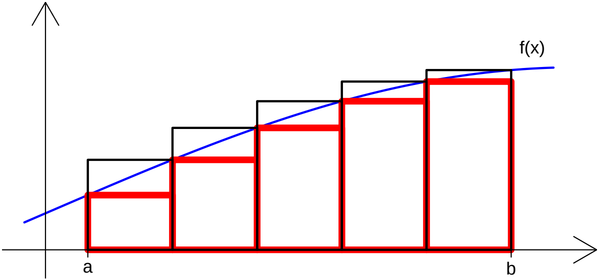
The Area Problem

- Want to find the area of the region S under $f(x)$ between a and b
  $f(x)$ continuous on $[a, b]$



The Area Problem



- Create a partition P_n of the interval $[a, b]$
- Use rectangles to compute upper and lower bounds for the area of S
- Consider $\lim_{n\to\infty} L(P_n, f(x))$ and $\lim_{n\to\infty} U(P_n, f(x))$

The Definite Integral

- If $\lim_{n\to\infty} L(P_n, f(x)) = \lim_{n\to\infty} U(P_n, f(x))$ then the definite integral of $f(x)$ from a to b is

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} L(P_n, f(x)) = A$$

- A is interpretable as the area of the region S
- $f(x)$ is integrable

Properties:

1. $$\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$$
2. $$\int_a^a f(x)\, dx = 0$$

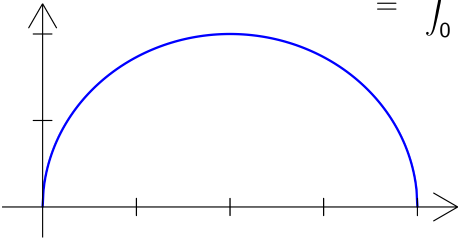
Properties (continued)

3. $$\int_a^b c\,dx = c(b-a) \qquad (c \text{ a real-valued constant})$$
4. $$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$
5. $$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx \qquad (c \text{ a real-valued constant})$$
6. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx \qquad (a < c < b)$$
7. $$f(x) \geq 0,\ x \in [a,b] \implies \int_a^b f(x)\,dx \geq 0$$
8. $$f(x) \geq g(x),\ x \in [a,b] \implies \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$
9. $$m \leq f(x) \leq M,\ x \in [a,b] \implies m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Example

- Evaluate $\int_0^4 \sqrt{4x - x^2}\, dx$

$$
\begin{aligned}
\int_0^4 \sqrt{4x - x^2}\, dx &= \int_0^4 \sqrt{-(x^2 - 4x)}\, dx \\
&= \int_0^4 \sqrt{4 - (x^2 - 4x + 4)}\, dx \\
&= \int_0^4 \sqrt{4 - (x^2 - 4x + 4)}\, dx \\
&= \int_0^4 \sqrt{4 - (x - 2)^2}\, dx
\end{aligned}
$$



$$\int_0^4 \sqrt{4x - x^2} = 2\pi$$

Equation of circle: $(x - x_0)^2 + (y - y_0)^2 = r^2$

Outline

Fundamental Theorem of Calculus

- Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. The antiderivative of $f(x)$ is a function $F(x)$ satisfying

$$F(x) = \int f(x)\,dx \quad \Longleftrightarrow \quad F'(x) = f(x)$$

Antiderivative not unique: $[F(x) + c]' = f(x)$

- Can use the Fundamental Theorem of Calculus to evaluate definite integrals (when a closed form of the antiderivative exists)

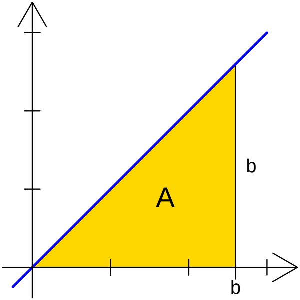
FToC: Let $f(x)$ be a continuous function on the interval $[a, b]$ and let $F(x)$ be the antiderivative of $f(x)$. Then,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example

- Suppose $F(x) = \frac{1}{2}x^2$
- Then $f(x) = F'(x) = x$
- **FToC** says area under the curve $f(x) = x$ from 0 to b is

$$\int_0^b f(x)\,dx = F(b) - F(0) = \frac{1}{2}b^2 - \frac{1}{2}0^2 = \frac{1}{2}b^2$$



- Area of a triangle

$$A = \frac{1}{2}\,\text{base} \times \text{height}$$

- $\displaystyle\int_0^b f(x)\,dx = \frac{1}{2}b^2$

Evaluating Definite Integrals

- Evaluating definite integrals boils down to finding antiderivatives
- Not as straight-forward as differentiation
- Anti-Power Rule: $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c$
- Dictionary method:

$f(x)$	$F(x)$		$f(x)$	$F(x)$
0	0		c	cx
$\sin(x)$	$-\cos(x)$		$\cos(x)$	$\sin(x)$
e^x	e^x		$\frac{1}{x}$	$\log(x)$
$\vdots$	$\vdots$		$\vdots$	$\vdots$

Examples

- $$\int_2^5 3x^2 + 2x + 1\,dx = \int_2^5 3x^2\,dx + \int_2^5 2x\,dx + \int_2^5 dx$$
$$= x^3\Big|_2^5 + x^2\Big|_2^5 + x\Big|_2^5$$
$$= [5^3 - 2^3] + [5^2 - 2^2] + [5 - 2] = 141$$

- $$\int_0^\pi 2\sin(x)\,dx = 2\int_0^\pi \sin(x)\,dx$$
$$= 2\left[-\cos(x)\right]\Big|_0^\pi$$
$$= -2\left[\cos(\pi) - \cos(0)\right]$$
$$= -2\left[-1 - 1\right] = 4$$

Examples

- $$\int_{-1}^{1} e^x \, dx = e^x \Big|_{-1}^{1} = e^1 - e^{-1} = e - \frac{1}{e} \approx 2.350402$$

- $$\int_1^4 \frac{dx}{x} = \int_1^4 \frac{1}{x} \, dx = \log(x) \Big|_1^4 = \log(4) - \log(1) = \log(4) \approx 1.386294$$

Outline

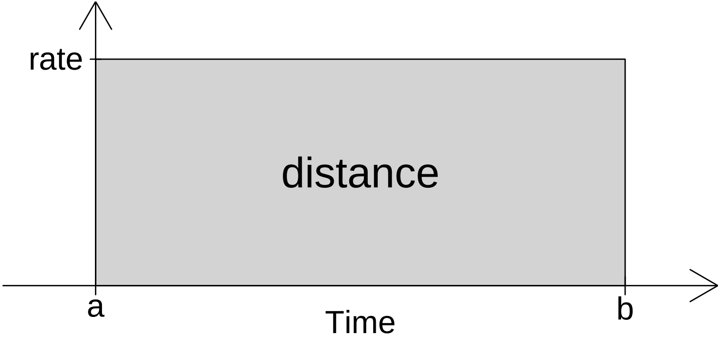
$$\text{distance} = \text{rate} \times \text{time}$$

The Distance Formula

- Works well for distances.

$$100 \text{ miles} = 50 \text{ mph} \times 2 \text{ hours}$$

- Intuition for many problems. What are we really doing?



The Job Problem

- If Lenny can do a job in 6 hours and Carl can do the same job in 4 hours how long does the job take if they work together?

- It's the rates that are additive
  - Lenny can do $\frac{1}{6}$ of the job per hour
  - Carl can do $\frac{1}{4}$ of the job per hour
  - Together, they can do $\frac{1}{6} + \frac{1}{4} = \frac{5}{12}$ of the job per hour

- Here, the job is *distance*, so have to solve for time

$$\text{time} = \frac{\text{distance}}{\text{rate}} = \frac{1}{\frac{5}{12}} = \frac{12}{5} = 2\,\text{hours}\;\; 24\,\text{minutes}$$

The Job Problem (continued)

- Solution assumes that Lenny and Carl work at a constant rate.
- Suppose instead that
  - Lenny works at a rate $L(x)$
  - Carl works at a rate $C(x)$
- Expressed as definite integrals

$$\int_0^6 L(x)\,dx = 1 \qquad \text{and} \qquad \int_0^4 C(x)\,dx = 1$$

- The problem is to find T such that

$$\int_0^T [L(x) + C(x)]\,dx = 1$$

The Job Problem (continued)

- Suppose $L(x) = \dfrac{12 - x}{54}$ and $C(x) = \dfrac{8 - x}{24}$
- How long does the job take if Lenny and Carl work together?
- Want to find T such that

$$
\begin{aligned}
\int_0^T [L(x) + C(x)]\, dx &= 1 \\
216 \cdot \int_0^T \left[\frac{12 - x}{54} + \frac{8 - x}{24}\right] dx &= 216 \cdot 1 \\
\int_0^T [(48 - 4x) + (72 - 9x)]\, dx &= 216 \\
\int_0^T [120 - 13x]\, dx &= 216
\end{aligned}
$$

The Job Problem (continued)

$$\begin{aligned}
(120x - \frac{13}{2}x^2)\Big|_0^T &= 216 \\
(120T - \frac{13}{2}T^2) - (120 \cdot 0 - \frac{13}{2} \cdot 0^2) &= 216 \\
13T^2 - 240T + 432 &= 0
\end{aligned}$$

- Solve using quadratic formula: just under 2 hours and 2 minutes

Probability Density Functions

- Let $f_X(x)$ be the probability density function of a continuous random variable X, then

$$P(X \in [a, b]) = \int_a^b f(x)\, dx$$

- The support of a random variable is the set of points S such that $f_X(x) > 0$ for $x \in S$

- Properties: if $f_X(x)$ is a pdf then

  (i) $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

  (ii) $\int_S f_X(x)\, dx = 1$

Example

- The pdf of a Beta$(2,2)$ distribution is

$$f_X(x) = \begin{cases} 6x(1-x) & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

- Verify $f_X(x) \geq 0$

- Verify that the definite integral of $f(x)$ over its support is 1

$$\begin{aligned} \int_S f_X(x)\,dx = \int_0^1 6x(1-x)x\,dx &= 6\int_0^1 [x - x^2]\,dx \\ &= 6\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]\Big|_0^1 \\ &= 6\left[\left(\frac{1}{2} - \frac{1}{3}\right) - (0-0)\right] \\ &= 6 \cdot \frac{1}{6} = 1 \end{aligned}$$

Outline

Integration by Parts

- Compute an antiderivative using the product rule backwards
- Let $f(x)$ and $g(x)$ be continuous, integrable functions

$$\begin{aligned} \frac{d}{dx}\left[F(x)\,G(x)\right] &= f(x)\,G(x)\,dx + F(x)\,g(x)\,dx \\ F(x)\,G(x) &= \int f(x)\,G(x)\,dx + \int F(x)\,g(x)\,dx \end{aligned}$$

- Rearrange terms to get integration by parts formula:

$$\int F(x)\,g(x)\,dx = F(x)\,G(x) - \int f(x)\,G(x)\,dx$$

- Idea: choose $F(x)$ and $g(x)$ so that $\int f(x)G(X)\,dx$ is easy to compute.

Integration by Parts: Example 1

- $\int \log(1+x)\,dx$
- Let $F(x) = \log(1+x)$, $G(x) = (1+x)$, and $g(x) = 1$

$$
\begin{aligned}
\int F(x)\,g(x)\,dx &= F(x)\,G(x) - \int f(x)\,G(x)\,dx \\
\int \log(1+x)\,dx &= (1+x)\log(1+x) - \int \frac{(1+x)}{(1+x)} dx \\
&= (1+x)\,\log(1+x) - \int dx \\
&= (1+x)\,\log(1+x) - x + C
\end{aligned}
$$

Integration by Parts: Example 2

- $\int x^2 \log(x)\, dx$
- Let $F(x) = \log(x)$, $G(x) = \frac{1}{3}x^3$, and $g(x) = x^2$

$$\begin{aligned}
\int F(x)\, g(x)\, dx &= F(x)\, G(x) - \int f(x)\, G(x)\, dx \\
\int x^2 \log(x)\, dx &= \frac{1}{3}x^3 \log(x) - \int \frac{1}{x}\, \frac{1}{3}x^3 dx \\
&= \frac{1}{3}x^3 \log(x) - \int \frac{1}{3}x^2\, dx \\
&= \frac{1}{3}x^3 \log(x) - \frac{1}{9}x^3 + C
\end{aligned}$$

- For definite integrals

$$\begin{aligned}
\int_a^b \left(f(x)\,G(x) + F(x)\,g(x)\right) dx &= \int_a^b \frac{d}{dx}\left[F(x)\,G(x)\right] \\
&= \left[F(x)\,G(x)\right]\Big|_a^b \\
&= F(b)\,G(b) - F(a)\,G(a)
\end{aligned}$$

- Rearrange terms to obtain

$$\int_a^b f(x)\,G(x)\,dx = F(b)\,G(b) - F(a)\,G(a) - \int_a^b F(x)\,g(x)\,dx$$

Integration by Parts: Example 3

- $\displaystyle\int_1^3 x\, e^x\, dx$
- Let $F(x) = x$, $G(x) = e^x$, and $g(x) = e^x$

$$
\begin{aligned}
\int_1^3 F(x)\, g(x)\, dx &= F(x)\, G(x)\Big|_1^3 - \int_1^3 f(x)\, G(x)\, dx \\
\int_1^3 x\, e^x\, dx &= x\, e^x\Big|_1^3 - \int_1^3 e^x\, dx \\
&= x\, e^x\Big|_1^3 - e^x\Big|_1^3 \\
&= 3e^3 - e - [e^3 - e] \\
&= 2e^3
\end{aligned}
$$

Outline

Integration by Substitution

- Integration by parts = product rule backward
- Integration by substitution = chain rule backward

- Let $F(x) = \int f(x)\,dx$ and $x = g(u)$. By the chain rule

$$\frac{d}{du}\left[F(g(u))\right] = F'(g(u))\,g'(u) = f(g(u))\,g'(u)$$

- Integrate both sides with respect to u

$$F(g(u)) = \int f(g(u))\,g'(u)\,du$$

- **Integration by substitution** Let $f(x)$ be integrable and $g(x)$ invertible and continuously differentiable. The substitution $x = g(u)$ changes the integration variable from x to u as follows:

$$\int f(x)\,dx = \int f(g(u))\,g'(u)\,du$$

Integration by Substitution: Example 1

- $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$
- Let $u = \sqrt{x}$, then $x = u^2$ and $dx = 2u\,du$

$$\begin{aligned}
\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx &= \int \frac{e^u}{u}\,2u\,du \\
&= 2\int e^u\,du \\
&= 2\,e^u + C \\
&= 2e^{\sqrt{x}} + C
\end{aligned}$$

($x > 0$ assumed)

Integration by Substitution: Example 2

- $\displaystyle\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$

- Let $u = e^x - e^{-x}$, then $du = (e^x + e^{-x})\,dx$

$$\begin{aligned}
\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx &= \int \frac{1}{(e^x - e^{-x})}\,(e^x + e^{-x})\,dx \\
&= \int \frac{1}{u}\,du \\
&= \log(u) + C \\
&= \log(e^x - e^{-x}) + C
\end{aligned}$$

Integration by Substitution: Definite Integrals

- Using the Fundamental Theorem of Calculus:

$$\begin{aligned}
\int_{x=a}^{x=b} f(x)\,dx &= \int_{u=g^{-1}(a)}^{u=g^{-1}(b)} f(g(u))\,g'(u)\,du \\
&= \int_{u=g^{-1}(a)}^{u=g^{-1}(b)} \frac{d}{du}\big[F(g(u))\big]\,du \\
&= F(g(u))\Big|_{u=g^{-1}(a)}^{u=g^{-1}(b)} \\
&= F(g(g^{-1}(b))) - F(g(g^{-1}(a))) \\
&= F(b) - F(a)
\end{aligned}$$

- Definite integral by substitution rule:

$$\int_{x=a}^{x=b} f(x)\,dx = \int_{u=g^{-1}(a)}^{u=g^{-1}(b)} f(g(u))\,g'(u)\,du$$

Integration by Substitution: Example 3

- $\int_{-1}^{0} x^2(x^3-1)^4\,dx$

- Let $u = x^3 - 1$, then $du = 3x^2\,dx$

$$
\begin{aligned}
\int_{-1}^{0} x^2(x^3-1)^4\,dx &= \frac{1}{3}\int_{-1}^{0}(x^3-1)^4\,3x^2\,dx \\
&= \frac{1}{3}\int_{u=-2}^{u=-1} u^4\,du \\
&= \frac{1}{3}\cdot\frac{1}{5}u^5\Big|_{u=-2}^{u=-1} \\
&= \frac{1}{15}(-1)^5 - \frac{1}{15}(-2)^5 \\
&= \frac{31}{15}
\end{aligned}
$$

Outline

Quadratic Formula

- A quadratic function has the form

$$f(x) = ax^2 + bx + c$$

where a, b and c are real-valued constants and $a \neq 0$

- The well-known quadratic formula tells us the solutions to $f(x) \overset{\text{set}}{=} 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Derivation of the quadratic formula relies on a technique called completing the square

Quadratic Formula (continued)

$$\begin{aligned} ax^2 + bx + c &\overset{\text{set}}{=} 0 \\ x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\ x^2 + 2\frac{b}{2a}x + \underline{\qquad} + \frac{c}{a} - \underline{\qquad} &= 0 \end{aligned}$$

$$\left(x + \frac{b}{2a}\right)^2 = x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2}$$

$$\begin{aligned} x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2} &= 0 \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{4ac}{a^2} \end{aligned}$$

Quadratic Formula (continued)

$$
\begin{aligned}
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x &= \frac{-b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

Completing the Square

- Compute $\displaystyle\int_0^\infty e^{-\frac{(\log(y)-\mu)^2}{2\sigma^2}}\,dy$ $\qquad$ μ, σ real-valued constants

- Let $u = \log(y) - \mu$, then $y = e^{u+\mu} = e^u e^\mu$, and $dy = y\,du$

$$
\begin{aligned}
\int_0^\infty e^{-\frac{(\log(y)-\mu)^2}{2\sigma^2}}\,dy &= \int_{u=-\infty}^{u=\infty} y e^{-\frac{u^2}{2\sigma^2}}\,du \\
&= \int_{-\infty}^{\infty} e^u e^\mu e^{-\frac{u^2}{2\sigma^2}}\,du \\
&= e^\mu \int_{-\infty}^{\infty} e^{-\frac{u^2}{2\sigma^2}} e^u\,du
\end{aligned}
$$

Completing the Square (continued)

- Take a closer look at the integrand:

$$\begin{aligned} e^{-u^2/2\sigma^2} e^u &= e^{\frac{-u^2+2\sigma^2 u}{2\sigma^2}} = e^{\frac{-u^2+2\sigma^2 u-\sigma^4+\sigma^4}{2\sigma^2}} \\ &= e^{\frac{-(u^2-2\sigma^2 u+\sigma^4)+\sigma^4}{2\sigma^2}} = e^{\frac{-(u-\sigma^2)^2+\sigma^4}{2\sigma^2}} = e^{\frac{-(u-\sigma^2)^2}{2\sigma^2}} e^{\frac{\sigma^2}{2}} \end{aligned}$$

$$\begin{aligned} \int_0^\infty e^{-\frac{(\log(y)-\mu)^2}{2\sigma^2}}\, dy &= e^\mu \int_{-\infty}^\infty e^{\frac{-(u-\sigma^2)^2}{2\sigma^2}} e^{\frac{\sigma^2}{2}}\, du \\ &= e^\mu e^{\frac{\sigma^2}{2}} \int_{-\infty}^\infty e^{\frac{-(u-\sigma^2)^2}{2\sigma^2}}\, du \end{aligned}$$

Hint: $\int_{-\infty}^\infty e^{\frac{-(u-\sigma^2)^2}{2\sigma^2}}\, du = \sqrt{2\pi}\sigma$

$$= \sqrt{2\pi}\sigma\, e^{\mu+\frac{\sigma^2}{2}}$$

Outline

Differentiating Definite Integrals

- The definite integral $\int_a^b f(x)\,dx$ is a real number
- If the limits are functions of another variable (say t) then the result is a function of t

$$g(t) = \int_{a(t)}^{b(t)} f(x)\,dx$$

- More generally, the integrand may be a function of x and t

$$g(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$$

(more on this in lesson 3)

- How to calculate $\dfrac{d}{dt}\left[\int_{a(t)}^{b(t)} f(x)\,dx\right]$

Differentiating Definite Integrals

- Let $F(x) = \int f(x)\,dx$ and

$$g(t) = \int_{a(t)}^{b(t)} f(x)\,dx$$

- By the Fundamental Theorem of Calculus

$$g(t) = F(b(t)) - F(a(t))$$

- By the chain rule

$$g'(t) = F'(b(t))\,b'(t) - F'(a(t))\,a'(t) = f(b(t))\,b'(t) - f(a(t))\,a'(t)$$

- Result: let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} f(x)\,dx\right] = f(b(t))\,b'(t) - f(a(t))\,a'(t)$$

  where $a(t)$ and $b(t)$ are differentiable functions

Example

- Let

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_0^{b(x)} e^{-\frac{t^2}{2}}\, dt$$

$$b(x) = 5\left(\log(x) + 0.04 + \frac{(0.20)^2}{2}\right) = 5\left(\log(x) + 0.06\right)$$

Compute $g'(x)$

- Recall: $\displaystyle \frac{d}{dt}\left[\int_{a(t)}^{b(t)} f(x)\, dx\right] = f(b(t))\, b'(t) - f(a(t))\, a'(t)$

- Since $a = 0$: $\displaystyle \frac{d}{dt}\left[\int_{0}^{b(t)} f(x)\, dx\right] = f(b(t))\, b'(t)$

Example (continued)

- Putting things together:

$$g'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{25\,(\log(x)+0.06)^2}{2}}\ b'(x)$$

- Need to compute $b'(x)$

$$\begin{aligned} b(x) &= 5\,(\log(x) + 0.06) \\ b'(x) &= \frac{5}{x} \end{aligned}$$

- The derivative of $g(x)$ is

$$g'(x) = \frac{5}{x\sqrt{2\pi}} e^{-\frac{25\,(\log(x)+0.06)^2}{2}}$$

Outline

Improper Integrals

- Two types: infinite intervals and unbounded functions
- First type: integrate a function $f(x)$ over either $(-\infty, b]$ or $[a, \infty)$
- The improper integral of $f(x)$ over $[a, \infty)$ exists iff

$$\lim_{t \to \infty} \int_a^t f(x)\, dx$$

exists and is finite

- Similarly, the improper integral of $f(x)$ over $[a, \infty)$ exists iff

$$\lim_{t \to -\infty} \int_t^b f(x)\, dx$$

exists and is finite

- When the limits exist and are finite

$$\int_a^{\infty} f(x)\, dx = \lim_{t \to \infty} \int_a^t f(x)\, dx$$ and $$\int_{-\infty}^{b} f(x)\, dx = \lim_{t \to -\infty} \int_t^b f(x)\, dx$$

Adding and Subtracting Improper Integrals

- Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function over the interval $[a, \infty)$. If $b > a$, then $f(x)$ is integrable over the interval $[b, \infty)$ and

$$\int_a^\infty f(x)\,dx - \int_b^\infty f(x)\,dx = \int_a^b f(x)\,dx$$

- Let $f(x)$ be an integrable function over the interval $(-\infty, b)$. If $a < b$, then $f(x)$ is integrable over the interval $(-\infty, a]$ and

$$\int_{-\infty}^b f(x)\,dx - \int_{-\infty}^a f(x)\,dx = \int_a^b f(x)\,dx$$

Improper Integrals

- The integral $\int_{-\infty}^{\infty} f(x)\,dx$ exists iff there is a point a such that both

$$\int_{-\infty}^{a} f(x)\,dx \quad \text{and} \quad \int_{a}^{\infty} f(x)\,dx$$

  exist

- If $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ exists

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx \\
&= \lim_{t\to-\infty} \int_{t}^{a} f(x)\,dx + \lim_{t\to\infty} \int_{a}^{t} f(x)\,dx
\end{aligned}$$

Improper Integrals

- Why not $$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_{-t}^{t} f(x)\,dx \;\; ?$$

  Since
  $$\int_{-t}^{t} x\,dx = \frac{x^2}{2}\bigg|_{-t}^{t} = \frac{1}{2}\left[t^2 - (-t)^2\right] = 0, \quad \forall t > 0$$

  Thus
  $$\lim_{t\to\infty} \int_{-t}^{t} x\,dx = 0$$

- Does not have the property that
  $$\int_{-\infty}^{a} x\,dx + \int_{a}^{\infty} x\,dx = \int_{-\infty}^{\infty} x\,dx$$

- Taking $a = 0$ gives $-\infty + \infty$ which is not defined

Improper Integrals

- However, if we already know that $\int_{-\infty}^{\infty} f(x)\,dx$ exists, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_{-t}^{t} f(x)\,dx$$

- Formula can be used to evaluate improper integrals

Example

- Show that for integer $\alpha \geq 0$,
$$\int_0^\infty x^\alpha e^{-x^2}\, dx$$
exists

- Need to show that
$$\lim_{t\to\infty} \int_0^t x^\alpha e^{-x^2}\, dx < \infty$$

- Recall that $\displaystyle\lim_{x\to\infty} \frac{P(x)}{e^x} = 0$, in particular, $\displaystyle\lim_{x\to\infty} \frac{x^{\alpha+2}}{e^{x^2}} = 0$

Example (continued)

- Since $\lim_{x\to\infty} \dfrac{x^{\alpha+2}}{e^{x^2}} = 0$ there is $M > 0$ such that

$$x^{\alpha+2}e^{-x^2} < 1 \iff x^{\alpha}e^{-x^2} < \frac{1}{x^2} \qquad \forall\, x \geq M$$

- For $t > M$

$$\begin{aligned}
\int_0^t x^{\alpha}e^{-x^2}\,dx &= \int_0^M x^{\alpha}e^{-x^2}\,dx + \int_M^t x^{\alpha}e^{-x^2}\,dx \\
&< c + \int_M^t \frac{dx}{x^2} \\
&< c - \frac{1}{x}\bigg|_M^t \\
&< c - \left[\frac{1}{t} - \frac{1}{M}\right]
\end{aligned}$$

Example (continued)

$$\begin{aligned} \int_0^t x^\alpha e^{-x^2}\, dx &< c - \left[\frac{1}{t} - \frac{1}{M}\right] \\ &< c + \frac{1}{M} - \frac{1}{t} \\ &< c + \frac{1}{M} \end{aligned}$$

- Thus $\displaystyle \lim_{t\to\infty} \int_0^t x^\alpha e^{-x^2}\, dx \;<\; c + \frac{1}{M} \;<\; \infty$
- Finally, $\displaystyle \int_0^\infty x^\alpha e^{-x^2}\, dx$ exists

Outline

Improper Integrals

- Second type: the integrand $f(x)$ is unbounded at either a or b, i.e.,

$$\lim_{x \searrow a} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \nearrow b} f(x) = \pm\infty$$

- $\displaystyle\int_a^b f(x)\,dx$ exists iff

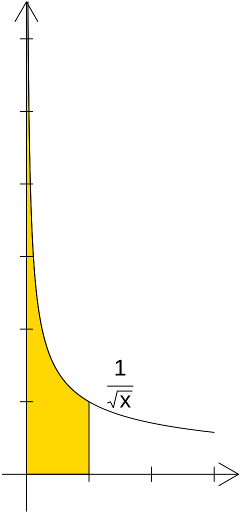
$$\lim_{t \searrow a} \int_t^b f(x)\,dx \qquad \text{or} \qquad \lim_{t \nearrow b} \int_a^t f(x)\,dx$$

  exists and is finite

- If both $\lim_{x \searrow a} f(x) = \pm\infty$ and $\lim_{x \nearrow b} f(x) = \pm\infty$, choose $c \in (a, b)$ such that $f(c) < \infty$ and use above results

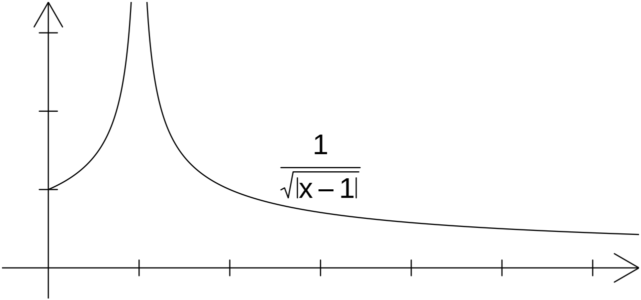
Example

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{x}} &= \lim_{t \searrow 0} \int_t^1 x^{-\frac{1}{2}}\, dx \\ &= \lim_{t \searrow 0} \left[\; 2x^{\frac{1}{2}} \Big|_t^1 \;\right] \\ &= \lim_{t \searrow 0} \left[2 - 2\sqrt{t}\right] \\ &= 2\end{aligned}$$



Example

- Can combine both types of improper integrals:
- Compute $\displaystyle\int_0^\infty \frac{dx}{\sqrt{|x-1|}}$



Example (continued)

$$\int_0^\infty \frac{dx}{\sqrt{|x-1|}} = \lim_{t \nearrow 1} \int_0^t \frac{dx}{\sqrt{1-x}} + \lim_{t \searrow 1} \int_t^2 \frac{dx}{\sqrt{x-1}} + \lim_{t \to \infty} \int_2^t \frac{dx}{\sqrt{x-1}}$$

$$\lim_{t \nearrow 1} \int_0^t \frac{dx}{\sqrt{1-x}} = \lim_{t \nearrow 1} \left[-2\sqrt{1-x} \Big|_0^t \right] = \lim_{t \nearrow 1} \left[-2\sqrt{1-t} + 2 \right] = 2$$

$$\lim_{t \searrow 1} \int_t^2 \frac{dx}{\sqrt{x-1}} = \lim_{t \searrow 1} \left[2\sqrt{x-1} \Big|_t^2 \right] = \lim_{t \searrow 1} \left[2 - 2\sqrt{t-1} \right] = 2$$

$$\lim_{t \to \infty} \int_2^t \frac{dx}{\sqrt{x-1}} = \lim_{t \to \infty} \left[2\sqrt{x-1} \Big|_2^t \right] = \lim_{t \to \infty} \left[2\sqrt{t-1} - 2 \right] = \infty$$

Example (continued)

- Conclusion:

$$\int_0^{\infty} \frac{dx}{\sqrt{|x-1|}}$$

does not exist because

$$\lim_{t\to\infty} \int_2^{t} \frac{dx}{\sqrt{x-1}}$$

is not finite

Outline

Differentiating Improper Integrals

- Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and suppose that $\int_{-\infty}^{\infty} f(x)\,dx$ exists
- Let $b(t)$ be a differentiable function and define

$$g(t) = \int_{-\infty}^{b(t)} f(x)\,dx$$

- To compute $g'(t)$, rewrite as

$$g(t) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{b(t)} f(x)\,dx$$

- Since the first term is a constant, its derivative wrt to t is 0, thus

$$\frac{d}{dt}\left[g(t)\right] = \frac{d}{dt}\left[\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{b(t)} f(x)\,dx\right] = \frac{d}{dt}\left[\int_{0}^{b(t)} f(x)\,dx\right]$$

Differentiating Improper Integrals

- $g'(t) = \dfrac{d}{dt}\left[F(b(t)) - F(0)\right] = f(b(t))\, b'(t)$

- Similarly, let $a(t)$ be a differentiable function and define

$$h(t) = \int_{a(t)}^{\infty} f(x)\, dx$$

- By the same argument,

$$\frac{d}{dt}\left[h(t)\right] = \frac{d}{dt}\left[\int_{a(t)}^{0} f(x)\, dx + \int_{0}^{\infty} f(x)\, dx\right] = \frac{d}{dt}\left[\int_{a(t)}^{0} f(x)\, dx\right]$$

- $h'(t) = \dfrac{d}{dt}\left[F(0) - F(a(t))\right] = -f(a(t))\, a'(t)$

Example: Delta of a European Call Option

- Suppose the risk-free rate is 0.04
- The Black-Scholes price for a European call option on a non-dividend paying asset with strike price of 1, volatility of 20%, and 3 months to maturity is

$$C(S) = S\Phi(d_1) - e^{-0.01}\Phi(d_2)$$

where

$$\begin{aligned} d_1(S) &= 10\big[\log(S) + 0.015\big] \\ d_2(S) &= d_1(S) - 0.1 \\ \Phi(z) &= \int_{-\infty}^{z} \phi(x)\, dx = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \end{aligned}$$

- Want to compute $\Delta(C) = \dfrac{d}{dS} C(S)$

Example (continued)

- Derivative of the first term:

$$\begin{aligned}
\frac{d}{dS}\big[S\,\Phi(d_1(S))\big] &= \Phi(d_1(S)) + S\,\frac{d}{dS}\left[\int_{-\infty}^{d_1(S)} \phi(x)\,dx\right] \\
&= \Phi(d_1(S)) + S\,\phi(d_1(S))\,\frac{d}{dS}\big[d_1(S)\big]
\end{aligned}$$

Recall: $d_1(S) = 10\big[\log(S) + 0.015\big]$ so $\frac{d}{dS}\big[d_1(S)\big] = \frac{10}{S}$

$$\begin{aligned}
&= \Phi(d_1(S)) + S\,\phi(d_1(S))\,\frac{10}{S} \\
&= \Phi(d_1(S)) + 10\,\phi(d_1(S))
\end{aligned}$$

Example (continued)

- Derivative of the second term:

$$\begin{aligned} \frac{d}{dS}\left[e^{-0.01}\,\Phi(d_2(S))\right] &= e^{-0.01}\,\frac{d}{dS}\left[\int_{-\infty}^{d_2(S)} \phi(x)\,dx\right] \\ &= e^{-0.01}\phi(d_2(S))\,\frac{d}{dS}\left[d_2(S)\right] \end{aligned}$$

Recall: $d_2(S) = d_1(S) - 0.1$ so $\frac{d}{dS}\left[d_2(S)\right] = \frac{d}{dS}\left[d_1(S)\right] = \frac{10}{S}$

$$= e^{-0.01}\phi(d_2(S))\,\frac{10}{S}$$

Example (simplification)

- Putting the two terms together:

$$\Delta(C) = \Phi(d_1) + 10\,\phi(d_1(S)) - \frac{10}{S}\,e^{-0.01}\phi(d_2(S))$$

- But we're not finished yet
- Recall: $\phi(x) = \dfrac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}$
- Write third term in terms of $d_1(S)$:

$$\begin{aligned}
\frac{10}{S}\,e^{-0.01}\phi(d_2(S)) &= \frac{10}{S}\,e^{-0.01}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{d_2^2(S)}{2}} \\
&= \frac{10}{S}\,e^{-0.01}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{(d_1(S)-0.1)^2}{2}}
\end{aligned}$$

Example (simplification)

$$
\begin{aligned}
&= \frac{10}{S}\, e^{-0.01}\, \frac{1}{\sqrt{2\pi}}\, e^{-\frac{d_1^2(S) - 0.2 d_1(S) + 0.01}{2}} \\
&= \frac{10}{S}\, e^{-0.01}\, \frac{1}{\sqrt{2\pi}}\, e^{-\frac{d_1^2(S) - 0.2 d_1(S) + 0.01}{2}} \\
&= \frac{10}{S}\, e^{-0.015}\, \frac{1}{\sqrt{2\pi}}\, e^{-\frac{d_1^2(S)}{2}}\, e^{0.1 d_1(S)} \\
&= \frac{10}{S}\, e^{-0.015} \phi(d_1(S)) e^{0.1 d_1(S)} \\
&= \frac{10}{S}\, e^{-0.015} \phi(d_1(S)) e^{\log(S) + 0.015} \\
&= 10\, \phi(d_1(S))
\end{aligned}
$$

Example (simplification)

- Putting it all together:

$$\begin{aligned}\Delta(C) &= \Phi(d_1) + 10\,\phi(d_1(S)) - \frac{10}{S}\,e^{-0.01}\phi(d_2(S)) \\ &= \Phi(d_1) + 10\,\phi(d_1(S)) - 10\,\phi(d_1(S)) \\ &= \Phi(d_1)\end{aligned}$$

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