COMPUTATIONAL FINANCE & RISK MANAGEMENT
UNIVERSITY *of* WASHINGTON
Department of Applied Mathematics

AMATH 460: Mathematical Methods for Quantitative Finance

1. Limits and Derivatives

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Outline

Outline

Course Information

- Instructors:
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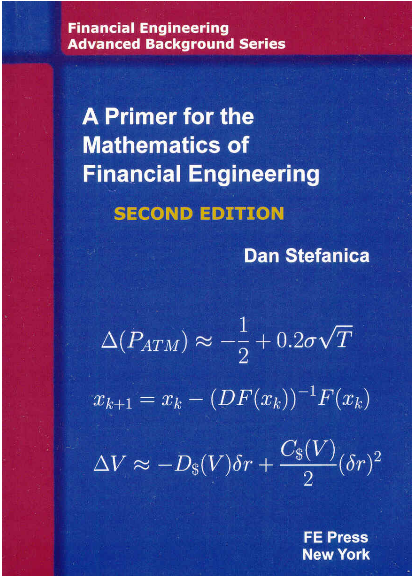
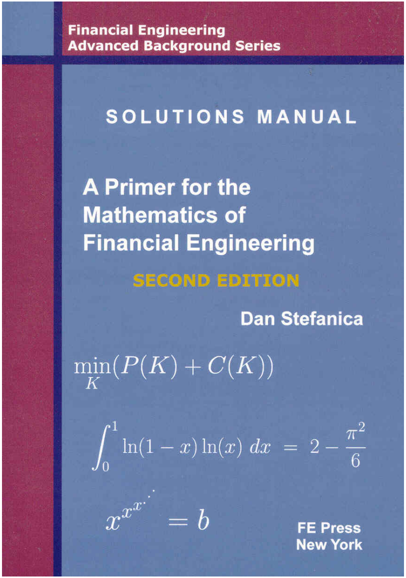
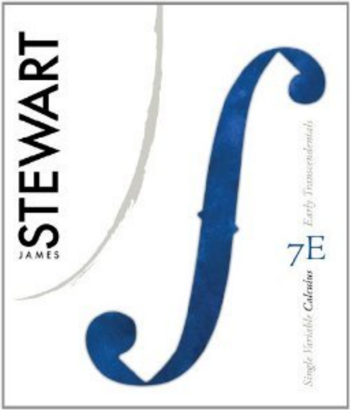
Recommended Texts





Recommended Texts



Calculus, Early Transcendentals
James Stewart
Any Edition
(Or equivalent big, fat calculus textbook)

Quarter Overview

Topics

1	Limits and Derivatives
2	Integration
3/4	Multivariable Calculus
5/6	Vectors, Matrices, Linear Algebra
7	Lagrange's Method, Taylor Series
8	Numerical Methods

Outline

Simple and Compound Interest Rules

Simple Interest

- Money accumulates interest proportional to the total time of the investment.

$$V = (1 + rn)\, A_0$$

Compound Interest

- Interest is paid regularly.

$$V = [\dots(1+r)[(1+r)\, A_0]] \quad = \quad [\dots(1+r)A_1] = (1+r)^n\, A_0$$

Compounding at Various Intervals

- Interest rate r is yearly but interest is paid more often (e.g., monthly).

$$V = [1 + (r/m)]^k\, A_0$$

$A_0 =$ Principal, $r =$ rate, $n = \#$ years, $m =$ periods/year, $k = \#$ periods

Present and Future Value

- Suppose the interest rate $r = 4\%$.
- In one year's time, the future value (FV) of \$100 will be \$104.
- Conversely, the present value (PV) of a \$104 payment in one year's time is \$100.
- Since

$$\mathrm{FV} = (1 + r)\,\mathrm{PV},$$

an expression for the present value is

$$\mathrm{PV} = \frac{1}{1 + r}\,\mathrm{FV} = d_1\,FV$$

where d_1 is the 1-year discount factor.

- More generally,

$$d_k = \frac{1}{[1 + (r/m)]^k}$$

the present value of a payment A_k received after k periods is $d_k\,A_k$.

Annuities

- An <u>annuity</u> is a contract that pays money regularly over a period of time.
- Question: suppose we would like to buy an annuity that pays \$100 at the end of the year for each of the next 10 years. At an interest rate of 4%, how much should we expect to pay?
- Solution: appropriately discount each of the 10 future payments and take the sum.
- The present value of the k^{th} payment is

$$\mathrm{PV}_k = d_k\, A_k = \frac{100}{(1+0.04)^k}$$

- The value of the annuity is then

$$V = \mathrm{PV}_1 + \mathrm{PV}_2 + \cdots + \mathrm{PV}_{10} = \sum_{k=1}^{10} \frac{100}{(1+0.04)^k}$$

Annuities (continued)

Recall: $V = \sum_{k=1}^{10} \frac{100}{(1+0.04)^k}$

$$
\begin{aligned}
(1+0.04)\,V &= 100 + \sum_{k=1}^{9} \frac{100}{(1+0.04)^k} \\
-\qquad V &= \sum_{k=1}^{9} \frac{100}{(1+0.04)^k} + \frac{100}{(1+0.04)^{10}} \\
\hline
0.04\,V &= 100 - \frac{100}{(1+0.04)^{10}}
\end{aligned}
$$

$$
\Longrightarrow \quad V = \frac{1}{0.04}\left(100 - \frac{100}{(1+0.04)^{10}}\right) = 811.09
$$

Perpetual Annuities

- A perpetual annuity or perpetuity pays a fixed amount periodically *forever*.
- Actually exist: instruments exist in the UK called consols.

$$
\begin{array}{rrcl}
 & (1+0.04)\,V & = & 100 + \displaystyle\sum_{k=1}^{\infty} \frac{100}{(1+0.04)^k} \\
- & V & = & \displaystyle\sum_{k=1}^{\infty} \frac{100}{(1+0.04)^k} \\
\hline
 & 0.04\,V & = & 100
\end{array}
$$

$$\implies \quad V = \frac{100}{0.04} = 2500$$

Summary: Present Value

- Discount factor d_k for a payment received after k periods, annual interest rate r, interest paid m times per year.

$$d_k = \frac{1}{[1 + (r/m)]^k}$$

- Value V of an annuity that pays an amount A annually, annual interest rate r, n total payments.

$$V = \frac{A}{r}\left[1 - \frac{1}{(1+r)^n}\right]$$

- Value V of a perpetual annuity that pays an amount A annually, annual interest rate r.

$$V = \frac{A}{r}$$

Outline

A Diverging Example

- Calculate
$$V = 1 - 1 + 1 - 1 + 1 \dots$$
- Rewrite as
$$V \stackrel{?}{=} \sum_{k=1}^{\infty} (+1 - 1) \stackrel{?}{=} \sum_{k=1}^{\infty} 0 = 0$$
- Also could rewrite as
$$V \stackrel{?}{=} \sum_{k=1}^{\infty} 1 - \sum_{k=1}^{\infty} 1 \stackrel{?}{=} 0$$
- But $\sum_{k=1}^{\infty} 1 \stackrel{?}{=} 1 + 1 + 1 + \sum_{k=4}^{\infty} 1 = 1 + 1 + 1 + \sum_{j=1}^{\infty} 1$
$$V \stackrel{?}{=} \sum_{k=1}^{\infty} 1 - \sum_{k=1}^{\infty} 1 \stackrel{?}{=} 1 + 1 + 1 + \sum_{j=1}^{\infty} 1 - \sum_{k=1}^{\infty} 1 \stackrel{?}{=} 3$$
- Can make V any integer value.

Notation

$\mathbb{R}$	the set of real numbers
$\mathbb{Z}$	the set of integers
$\in$	in: indicator of set membership
$\forall$	for all
$\exists$	there exists
$\rightarrow$	goes to
$g : \mathbb{R} \rightarrow \mathbb{R}$	a real-valued function with a real argument
$\lfloor x \rfloor$	floor: the largest integer less than or equal to x
$\lceil x \rceil$	ceiling: the smallest integer greater than or equal to x
$/$	not (e.g. *"not in"* would be $x \notin \mathbb{Z}$)

Definition of limit

- Let $g : \mathbb{R} \to \mathbb{R}$.
- The <u>limit</u> of $g(x)$ as $x \to x_0$ exists and is finite and equal to l if and only if for any $\epsilon > 0$ there exists $\delta > 0$ such that $|g(x) - l| < \epsilon$ for all $x \in (x_0 - \delta, x_0 + \delta)$.
- Alternatively,

$$\lim_{x \to x_0} g(x) = l \text{ iff } \forall\, \epsilon > 0,\ \exists\, \delta > 0 \text{ s.t. } |g(x) - l| < \epsilon, \forall\, |x - x_0| < \delta$$

- Similarly,

$$\begin{aligned}\lim_{x \to x_0} g(x) &= \infty \text{ iff } \forall\, C > 0,\ \exists\, \delta > 0 \text{ s.t. } g(x) > C, \\ &\qquad \forall\, |x - x_0| < \delta \\ \lim_{x \to x_0} g(x) &= -\infty \text{ iff } \forall\, C < 0,\ \exists\, \delta > 0 \text{ s.t. } g(x) < C, \\ &\qquad \forall\, |x - x_0| < \delta\end{aligned}$$

Modification for $x \to \infty$

- How to make sense of $|x - \infty| < \delta$?

$$\lim_{x \to \infty} g(x) = l \text{ iff } \forall\, \epsilon > 0,\ \exists\, b \text{ s.t. } |g(x) - l| < \epsilon,\ \forall\, x > b$$

- Example, recall pricing formulas:

$$\text{Annuity:} \qquad V_n = \frac{A}{r}\left[1 - \frac{1}{(1+r)^n}\right]$$

$$\text{Perpetuity:} \qquad V = \frac{A}{r}$$

Question: is $\lim_{n \to \infty} V_n$ equal to V?

Strategy: given $\epsilon > 0$, find N such that $|V_n - V| < \epsilon$ for $n > N$.

Example

- Given $\epsilon > 0$, find N such that $|V_n - V| < \epsilon$ for $n > N$

$$\begin{aligned} |V_n - V| &= \left| \frac{A}{r} \left[1 - \frac{1}{(1+r)^n} \right] - \frac{A}{r} \right| \\ &= \left| \frac{A}{r} - \frac{A}{r(1+r)^n} - \frac{A}{r} \right| \\ &= \frac{A}{r(1+r)^n} \end{aligned}$$

- Next, solve for N such that

$$\frac{A}{r(1+r)^n} \le \frac{\epsilon}{2}$$

for $n > N$.

Example (continued)

$$\begin{aligned} \frac{A}{r(1+r)^n} &= \frac{\epsilon}{2} \\ (1+r)^n &= \frac{2A}{r\epsilon} \\ n\log(1+r) &= \log\left(\frac{2A}{r\epsilon}\right) \\ n &= \frac{\log(2A) - \log(r\epsilon)}{\log(1+r)} \end{aligned}$$

- Choose $N = \left\lceil \frac{\log(2A) - \log(r\epsilon)}{\log(1+r)} \right\rceil$ then for $n > N$

$$|V_n - V| \leq \frac{A}{r(1+r)^n} \leq \frac{\epsilon}{2} < \epsilon$$

Summary: Limits

- Given $\epsilon > 0$, can find N such that $|V_n - V| < \epsilon$ for $n > N$, thus

$$\lim_{n \to \infty} V_n = V$$

- Definition of a limit

$$\lim_{x \to x_0} g(x) = l \text{ iff } \forall\, \epsilon > 0,\ \exists\, \delta > 0 \text{ s.t. } |g(x) - l| < \epsilon,\ \forall\, |x - x_0| < \delta$$

Outline

1. Course Organization
2. Present Value
3. Limits
4. **Evaluating Limits**
5. Continuity and Asymptotes
6. Differentiation
7. Product Rule and Chain Rule
8. Higher Derivatives
9. Bond Duration
10. l'Hôpital's Rule

Evaluating Limits

- Observation: working with the definition = not fun!
- Suppose that c is a real constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

(i) $$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

(ii) $$\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$$

(iii) $$\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

(iv) $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \qquad \text{if} \quad \lim_{x \to a} g(x) \neq 0$$

(v) $$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$$

Evaluating Limits

- If $P(x)$ and $Q(x)$ are polynomials and $c > 1$ is a real constant then

(a) $$\lim_{x\to\infty} \frac{P(x)}{c^x} = \lim_{x\to\infty} P(x)\, c^{-x} = 0$$

(b) $$\lim_{x\to\infty} \frac{\log |Q(x)|}{P(x)} = 0$$

- Examples

(a) $$\lim_{x\to\infty} x^2 e^{-x} = 0$$

(b) $$\lim_{x\to\infty} \frac{\log(x^3)}{x} = 0$$

Evaluating Limits

- Notation: $x \searrow 0$ means $x \to 0$ with $x > 0$; $k! = k \cdot (k-1) \cdots 2 \cdot 1$
- Let $c > 0$ be a positive constant, then

  (a) $\lim_{x \to \infty} c^{\frac{1}{x}} = 1$

  (b) $\lim_{x \to \infty} x^{\frac{1}{x}} = 1$

  (c) $\lim_{x \searrow 0} x^x = 1$

- If k is a positive integer and if $c > 0$ is a positive constant, then

  (a) $\lim_{k \to \infty} k^{\frac{1}{k}} = 1$

  (b) $\lim_{k \to \infty} c^{\frac{1}{k}} = 1$

  (c) $\lim_{k \to \infty} \frac{c^k}{k!} = 0$

Example

$$\lim_{x\to\infty} \frac{1}{\sqrt{x^2-4x+1}-x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2-4x+1}-x} \; \frac{\sqrt{x^2-4x+1}+x}{\sqrt{x^2-4x+1}+x}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2-4x+1}+x}{x^2-4x+1-x^2}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2-4x+1}+x}{1-4x} \; \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}+1}{\frac{1}{x}-4}$$

$$= \frac{\sqrt{\lim_{x\to\infty}\left(1-\frac{4}{x}+\frac{1}{x^2}\right)}+1}{\lim_{x\to\infty}\left(\frac{1}{x}-4\right)} = \frac{1+1}{-4} = -\frac{1}{2}$$

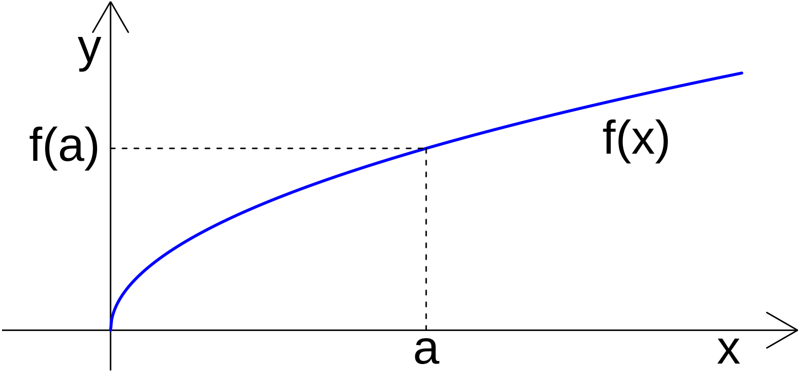
Example

$$
\begin{aligned}
&\lim_{x\to\infty} \frac{1}{\sqrt{x^2-4x+1}-x+2} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{x^2-4x+1}-(x-2)} \; \frac{\sqrt{x^2-4x+1}+(x-2)}{\sqrt{x^2-4x+1}+(x-2)} \\
&= \lim_{x\to\infty} \frac{\sqrt{x^2-4x+1}+x-2}{x^2-4x+1-(x-2)^2} \\
&= \lim_{x\to\infty} \frac{\sqrt{x^2-4x+1}+x-2}{x^2-4x+1-[x^2-4x+4]} \\
&= \lim_{x\to\infty} \frac{\sqrt{x^2-4x+1}+x-2}{-3} \\
&\to -\infty
\end{aligned}
$$

Outline

Continuity

- Most of the time: $\lim_{x \to a} f(x) = f(a)$



Continuity Definitions

- A function f is <u>continuous</u> at a if

$$\lim_{x \to a} f(x) = f(a)$$

- A function is <u>right-continuous</u> at a if

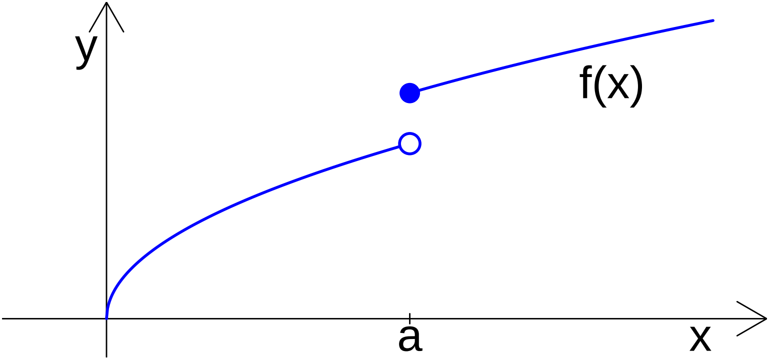
$$\lim_{x \searrow a} f(x) = f(a)$$

- A function is <u>left-continuous</u> at a if

$$\lim_{x \nearrow a} f(x) = f(a)$$

- A function is <u>continuous</u> on an interval (c, d) if it is continuous at every number $a \in (c, d)$.

(Dis)continuity



- Is $f(x)$ continuous at a?
- Is $f(x)$ right-continuous at a?
- Is $f(x)$ left-continuous at a?

Continuity Theorems

- Let f and g be continuous at a and c be a real-valued constant. Then

  1. $f + g$ 2. $f - g$ 3. cf
  4. fg 5. $\frac{f}{g}$ $(g(a) \neq 0)$

  are continuous at a.

- Any polynomial is continuous on $\mathbb{R}$.

- The following functions are continuous on their domains:
  - polynomials
  - rational functions
  - root functions
  - logarithmic functions
  - trigonometric functions
  - inverse trigonometric functions
  - exponential functions

- If g is continuous at a, and f is continuous at $g(a)$, then $f \circ g(x) = f\left(g(x)\right)$ is continuous at a.

Example

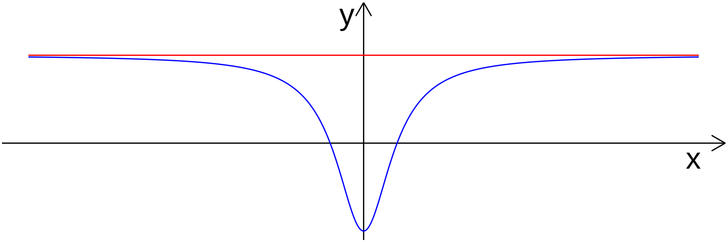
- Where is the following function continuous?

$$f(x) = \frac{\log(x) + \tan^{-1}(x)}{x^2 - 1}$$

- $\tan^{-1}(x)$ means $\arctan(x)$, not $\frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)} = \cot(x)$
- $\log(x)$ is continuous for $x > 0$ and $\arctan(x)$ for $x \in \mathbb{R}$
  - thus $\log(x) + \arctan(x)$ is continuous for $x \in (0, \infty)$
- $x^2 - 1$ is a polynomial $\implies$ continuous everywhere
- $f(x)$ is continuous on $(0, \infty)$ except where $x^2 - 1 = 0$
- $\implies f(x)$ is continuous on the intervals $(0, 1)$ and $(1, \infty)$

Horizontal Asymptotes

- What happens to $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ as x becomes large?



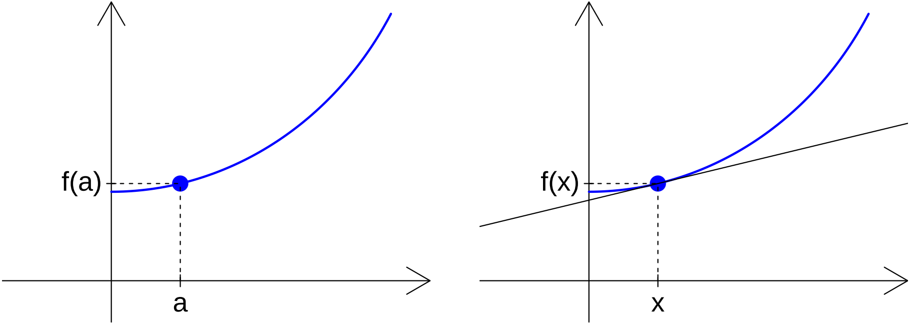
- The line $y = L$ is called a horizontal asymptote if

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

Outline

Tangent Line

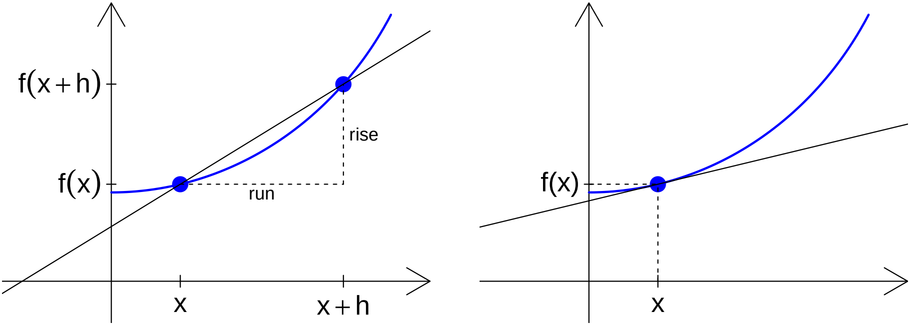
- Goal: find the slope of the line tangent to $y = f(x)$ at a point a.



- A line $y = l(x)$ is tangent to the curve $y = f(x)$ at a point a if there is $\delta > 0$ such that
  - $f(x) > l(x)$ on $(a - \delta, a)$ and $(a, a + \delta)$ or
  - $f(x) < l(x)$ on $(a - \delta, a)$ and $(a, a + \delta)$

  and $f(a) = l(a)$.

Strategy



- Use the slope of a line connecting a nearby point as an estimate of the slope of the tangent line.

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(x+h) - f(x)}{h}$$

- Get successively better estimates by letting $h \to 0$

Definition of Derivative

- A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a point $x \in \mathbb{R}$ if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists.

- When this limit exists, the derivative of $f(x)$, denoted by $f'(x)$, is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- A function $f(x)$ is differentiable on an open interval (a, b) if it is differentiable at every point $x \in (a, b)$.

Example

- Compute the derivative of $f(x) = 3x^2 + 5x + 1$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{3(x+h)^2 + 5(x+h) + 1 - [3x^2 + 5x + 1]}{h} \\
&= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 + 5x + 5h + 1 - 3x^2 - 5x - 1}{h} \\
&= \lim_{h\to 0} \frac{6xh + 3h^2 + 5h}{h} \\
&= \lim_{h\to 0} \left[6x + 3h + 5\right] \\
&= 6x + 5
\end{aligned}
$$

Properties of Derivatives

- The derivative of a constant is 0.
- Linearity: let $f(x)$ and $g(x)$ be differentiable functions and let a and b be real-valued constants. The derivative of

  $$l(x) = a\, f(x) + b\, g(x)$$

  is

  $$l'(x) = a\, f'(x) + b\, g'(x)$$

- Power Rule: Let n be a real-valued constant. The derivative of

  $$f(x) = x^n$$

  is

  $$f'(x) = n\, x^{n-1}$$

Summary

- Definition: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$
- Derivative of a constant: if $f(x) = c$ then $f'(x) = 0$.
- Linearity: $\big(a\,f(x) + b\,g(x)\big)' = a\,f'(x) + b\,g'(x)$
- Power Rule: $(x^c)' = c\,x^{c-1} \quad$ for $c \neq 0$.

Outline

Product Rule

- Suppose $f(x)$ and $g(x)$ are differentiable functions.
- Let $p(x) = f(x)\,g(x)$
- Then $p(x)$ is differentiable, and

$$p'(x) = \left(f(x)\,g(x)\right)' = f'(x)\,g(x) + f(x)\,g'(x)$$

Examples

Recall: $p'(x) = \left(f(x)\,g(x)\right)' = f'(x)\,g(x) + f(x)\,g'(x)$

- $p(x) = x\,e^x$
$$\begin{aligned} f(x) &= x, \quad g(x) = e^x \\ p'(x) &= 1 \cdot e^x + x\,e^x \\ &= (x+1)e^x \end{aligned}$$

- $p(t) = \sqrt{t}(1-t)$
$$\begin{aligned} f(t) &= \sqrt{t} = t^{\frac{1}{2}}, \quad g(t) = (1-t) \\ p'(x) &= \frac{1}{2}t^{-\frac{1}{2}} \cdot (1-t) + t^{\frac{1}{2}} \cdot (-1) \\ &= \frac{(1-t)}{2\sqrt{t}} - \sqrt{t} = \frac{1-3t}{2\sqrt{t}} \end{aligned}$$

Chain Rule

- Suppose $f(x)$ and $g(x)$ are differentiable functions.
- The composite function $(g \circ f)(x) = g(f(x))$ is differentiable, and

$$((g \circ f)(x))' = g'(f(x))\ f'(x)$$

- Alternative notation: let $u = f(x)$ and $g = g(u) = g(f(x))$, then

$$\frac{dg}{dx} = \frac{dg}{du}\ \frac{du}{dx}$$

(Leibniz notation)

Examples

Recall: $((g \circ f)(x))' = g'(f(x))\, f'(x)$

- $(g \circ f)(x) = \sqrt{x^2 + 1} = (x^2 + 1)^{\frac{1}{2}}$

$$
\begin{aligned}
((g \circ f)(x))' &= \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \cdot 2x \\
&= \frac{x}{\sqrt{x^2 + 1}}
\end{aligned}
$$

- $(g \circ f)(x) = (x^3 - 1)^{100}$

$$
\begin{aligned}
((g \circ f)(x))' &= 100(x^3 - 1)^{99} \cdot 3x^2 \\
&= 300x^2\,(x^3 - 1)^{99}
\end{aligned}
$$

Substitution Method

Recall Leibniz notation: $\dfrac{dg}{dx} = \dfrac{dg}{du}\,\dfrac{du}{dx}$

- Compute $\frac{d}{dx}\sin(x^2)$; let $u = x^2$, $g(u) = \sin(u)$ and $\frac{du}{dx} = 2x$

$$\begin{aligned} \frac{d}{dx}\sin(u) &= \frac{d}{du}\sin(u) \cdot \frac{du}{dx} \\ &= \cos(u) \cdot 2x \\ &= 2x\cos(x^2) \end{aligned}$$

- Compute $\frac{d}{d\theta}e^{\sin(\theta)}$; let $u = \sin(\theta)$, $g(u) = e^u$ and $\frac{du}{d\theta} = \cos(\theta)$

$$\begin{aligned} \frac{d}{dx}e^{\sin(\theta)} &= \frac{d}{du}e^u \cdot \frac{du}{d\theta} \\ &= e^u \cdot \cos(\theta) \\ &= \cos(\theta)\, e^{\sin(\theta)} \end{aligned}$$

Derivative of the Inverse Function

- The inverse function of $f(x)$ is the function such that $f^{-1}(f(x)) = x$
- Let $f : [a, b] \to [c, d]$ be a differentiable function
- Let $f^{-1} : [c, d] \to [a, b]$ be the inverse function of $f(x)$
- $f^{-1}(x)$ is differentiable for $x \in [c, d]$ where $f'(f^{-1}(x)) \neq 0$ and

$$\left(f^{-1}(x)\right)' = \frac{1}{f'(f^{-1}(x))}$$

Example

- $\log(e^x) = x \quad \Longrightarrow \quad f(x) = e^x$, $f'(x) = e^x$, and $f^{-1}(x) = \log(x)$

$$\left(\log(x)\right)' = \left(f^{-1}(x)\right)' = \frac{1}{e^{\log(x)}} = \frac{1}{x}$$

Summary: Product Rule and Chain Rule

Product Rule

$$\big(f(x)\,g(x)\big)' = f'(x)\,g(x) + f(x)\,g'(x)$$

Chain Rule

$$\big((g \circ f)(x)\big)' = g'(f(x))\; f'(x)$$

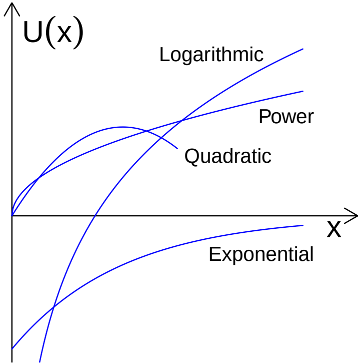
$$\frac{dg}{dx} = \frac{dg}{du}\;\frac{du}{dx}$$

Outline

Utility Functions

- A utility function is a real-valued function $U(x)$ defined on the real numbers.
- Used to compare wealth levels: if $U(x) > U(y)$ then $U(x)$ is preferred.

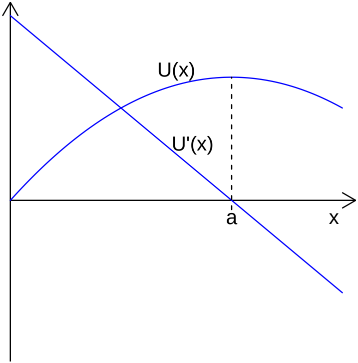
Exponential	$U(x) = -e^{-ax}$ $(a > 0)$
Logarithmic	$U(x) = \log(x)$
Power	$U(x) = bx^b$ $b \in (0, 1]$
Quadratic	$U(x) = x - bx^2$ $(b > 0)$

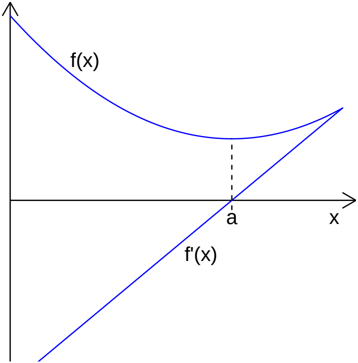


- What is the derivative telling us?

Critical Points

- A critical point of a function $f(x)$ is a number a in the domain of $f(x)$ where either $f'(a) = 0$ or $f'(a)$ does not exist.
- If f has a local min or max at a, then a is a critical point.





- local max if $f'(x)$ decreasing at a, local min if increasing

Higher Derivatives

- If $f(x)$ is a differentiable function, $f'(x)$ is also a function.
- If $f'(x)$ is also differentiable, its derivative is denoted by

$$f''(x) = \big(f'(x)\big)'$$

- $f''(x)$ is called the <u>second derivative</u> of $f(x)$
- In Leibniz notation:

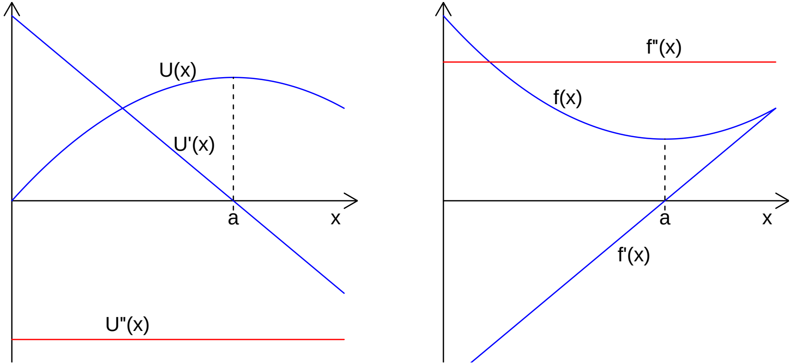
$$\frac{d}{dx}\left[\frac{d}{dx}f(x)\right] = \frac{d^2}{dx^2}f(x) = \frac{d^2f}{dx^2}$$

- Alternative notation: $f''(x) = D^2f(x)$
- No reason to stop at 2:

$$f^{(n)}(x) = \frac{d^n}{dx^n}f(x) = \frac{d^nf}{dx^n} = D^nf(x)$$

provided $f^{(n-1)}(x)$ differentiable

First and Second Order Conditions

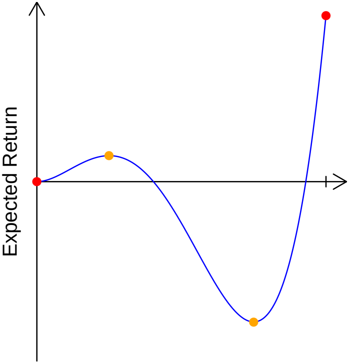


- $f'(x) = 0$ and $f''(x) < 0 \implies$ local maximum
- $f'(x) = 0$ and $f''(x) > 0 \implies$ local minimum

Absolute Minimum and Maximum Values

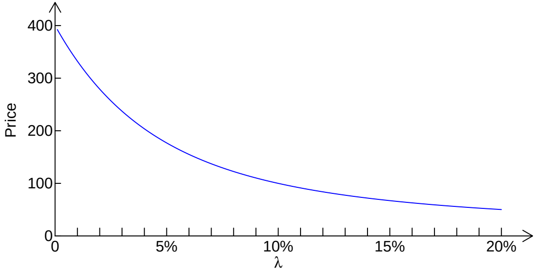
- Want to invest a fixed sum in 2 assets to maximize expected return.
- Find the absolute maximum value of $f(x)$ on the interval $[0, 1]$
  More generally, on a closed interval $[a, b]$

1. Evaluate $f(x)$ at the critical points in (a, b)
2. Evaluate $f(x)$ at a and b
3. The global max is the max of the values in steps 1 & 2.
4. The global min is the min of the values in steps 1 & 2.



Outline

Bond Pricing Formula



- The price P of a bond is

$$P = \frac{F}{[1+\lambda]^n} + \sum_{k=1}^{n} \frac{C}{1+\lambda^k} = \frac{F}{[1+\lambda]^n} + \frac{C}{\lambda}\left[1 - \frac{1}{[1+\lambda]^n}\right]$$

where: $C =$ coupon payment $\quad n = \#$ coupon periods remaining

$F =$ face value $\quad \lambda =$ yield to maturity

Duration and Sensitivity

- Let $PV_k = \frac{c_k}{[1+\lambda]^k}$, $\qquad P = \sum_{k=1}^{n} PV_k = \sum_{k=1}^{n} \frac{c_k}{[1+\lambda]^k}$

- The duration of a bond is a weighted average of times that payments are made.

$$D = \sum_{k=1}^{n} \frac{PV_k}{P} k \qquad \text{or} \qquad D\,P = \sum_{k=1}^{n} k\, PV_k$$

- Sensitivity: how is price affected by a change in yield?

$$\begin{aligned} PV_k &= \frac{c_k}{[1+\lambda]^k} = c_k\,[1+\lambda]^{-k} \\ \frac{d\,PV_k}{d\lambda} &= -\,k\,c_k\,[1+\lambda]^{-k-1} = -\frac{k\,c_k}{[1+\lambda]^{k+1}} = -\frac{k}{1+\lambda} PV_k \end{aligned}$$

Duration and Sensitivity

$$
\begin{aligned}
P &= \sum_{k=1}^{n} PV_k \\
\frac{d}{d\lambda} P &= \frac{d}{d\lambda} \sum_{k=1}^{n} PV_k = \sum_{k=1}^{n} \frac{d\, PV_k}{d\lambda} \\
&= -\sum_{k=1}^{n} \frac{k}{1+\lambda} PV_k \\
&= -\frac{1}{1+\lambda} \sum_{k=1}^{n} k\, PV_k \\
&= -\frac{1}{1+\lambda} D\, P \equiv -D_M\, P
\end{aligned}
$$

- D_M is called the modified duration

Duration

- Price Sensitivity Formula

$$\frac{dP}{d\lambda} = -D_M P$$

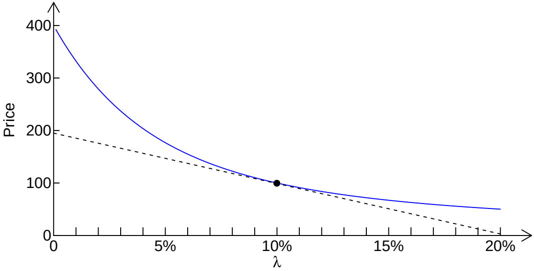
- Since

$$\frac{1}{P}\frac{dP}{d\lambda} = -D_M$$

  D_M measures the relative change in price as λ changes.

- Duration measures interest rate sensitivity.

Linear Approximation



- The tangent line can be used to approximate the price.
- The approximation can be improved by adding a quadratic term based on the convexity of the price-yield curve.
  More on this later

Summary: Bond Duration

Duration

$$D = \sum_{k=1}^{n} \frac{PV_k}{P} k \qquad \text{or} \qquad D\,P = \sum_{k=1}^{n} k\,PV_k$$

Modified Duration

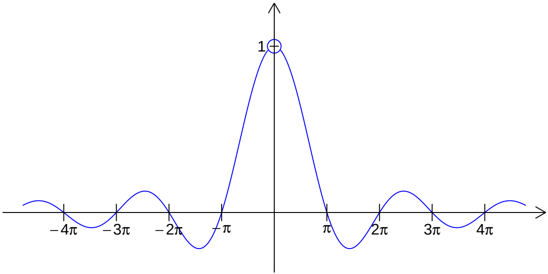
$$D_M = \frac{1}{1+\lambda} D$$

Price Sensitivity Formula

$$\frac{dP}{d\lambda} = -D_M P$$

Outline

- Problem: how to evaluate $\displaystyle \lim_{x \to 0} \frac{\sin(x)}{x}$?



- $\displaystyle \lim_{x \to 0} \frac{\sin(x)}{x} = \frac{\lim_{x \to 0} \sin(x)}{\lim_{x \to 0} x} = \frac{0}{0}$

- Not allowed since $\displaystyle \lim_{x \to 0}$ denominator $= 0$

l'Hôpital's Rule

- Let x_0 be a real number (including $\pm\infty$) and let $f(x)$ and $g(x)$ be differentiable functions.
  (i) Suppose $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) = 0$. If $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists and there is an interval (a, b) containing x_0 such that $g'(x) \neq 0$ for all $x \in (a, b) \setminus 0$, then $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ exists and

  $$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

  (ii) Suppose $\lim_{x\to x_0} f(x) = \pm\infty$ and $\lim_{x\to x_0} g(x) = \pm\infty$. If $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists and there is an interval (a, b) containing x_0 such that $g'(x) \neq 0$ for all $x \in (a, b) \setminus 0$, then $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ exists and

  $$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

  When $x_0 = \pm\infty$, intervals are of the form $(-\infty, b)$ and (a, ∞).

Examples

- $\displaystyle \lim_{x\to 0} \frac{\sin(x)}{x}$ since $\displaystyle \lim_{x\to 0} \sin(x) = 0$ and $\displaystyle \lim_{x\to 0} x = 0$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}\sin(x)}{\frac{d}{dx}x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

- $\displaystyle \lim_{x\to\infty} \frac{x^2-1}{x^2+1}$ since $\displaystyle \lim_{x\to\infty} x^2 - 1 = \infty$ and $\displaystyle \lim_{x\to\infty} x^2 + 1 = \infty$

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = \lim_{x\to\infty} \frac{\frac{d}{dx}x^2-1}{\frac{d}{dx}x^2+1} = \lim_{x\to\infty} \frac{2x}{2x} =?$$

Since $\displaystyle \lim_{x\to\infty} 2x = \infty$ have to use l'Hôpital's rule again:

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = \lim_{x\to\infty} \frac{\frac{d}{dx}x^2-1}{\frac{d}{dx}x^2+1} = \lim_{x\to\infty} \frac{2x}{2x} = \lim_{x\to\infty} \frac{\frac{d}{dx}2x}{\frac{d}{dx}2x} = \lim_{x\to\infty} \frac{2}{2} = 1$$

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