

1 LTI CT DE

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$
$$\sum_{k=0}^N a_k x^k = a_N \prod_{i=1}^p (x - x_i)^{n_i}$$
$$y_h(t) = \sum_{i=1}^p (\sum_{j=0}^{n_i-1} A_{i,j} t^j) e^{x_i t}$$

If $x(t) = \sum_{k=0}^m C_k t^k e^{x_i t}$

$$y_p(t) = (\sum_{k=0}^m B_k t^k) t^{n_i} e^{x_i t}, n_i \geq 0$$

Determine constants. $y(t_0), \frac{dy(t_0)}{dy}, \dots, \frac{d^{N-1} y(t_0)}{dy^{N-1}}$

DT DE

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$
$$\sum_{k=0}^N a_{N-k} x^k = a_0 \prod_{i=1}^p (x - x_i)^{n_i}$$
$$y_h[n] = \sum_{i=1}^p (\sum_{j=0}^{n_i-1} A_{i,j} n^j) x_i^n$$

if $x[n] = \sum_{i=0}^m C_i n^i s^n$

$$y_p[n] = (\sum_{k=0}^m B_k n^k) n^{n_i} x_i^n, n_i \geq 0$$

Special: $x_i = 1$

Unit Impulse Resp

$$h_p(t) = 0$$

Matching $\delta^{(k)}(t), \delta^{(k-1)}(t), \delta^{(k-2)}(t) \dots$ (Only singularity part)

\int_{0-}^{0+} to determine coefficient

$$y(t) = \int_{\tau} x(\tau) h(t - \tau)$$

2 Fourier Analysis CT FS

$$x(t) = x(t + T), \omega_0 = \frac{2\pi}{T}$$

$$c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t} = \begin{cases} x(t) \\ \frac{x(t+) + x(t-)}{2} \end{cases}$$

Real Signals

When $c_k^* = c_{-k}$

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t))$$
$$= A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k)$$

$$|c_k|^2 = |c_{-k}|^2 = \frac{a_k^2 + b_k^2}{4}$$

Examples

$$x(t) = a[u(t + \frac{\tau}{2}) - u(t - \frac{\tau}{2})]$$

$$c_0 = a \frac{\tau}{T}$$

$$c_k = \frac{a\tau}{T} Sa(\frac{k\omega_0 \tau}{2})$$

$$Sa(t) = \begin{cases} \frac{\sin(t)}{t}, t \neq 0 \\ 1, t = 0 \end{cases}$$

Parseval Theorem

$$\frac{1}{T} \int_T |x(t)|^2 dt = \sum_k |c_k|^2 = A_0^2 + \frac{1}{2} \sum_{k \geq 1} A_k^2$$

Properties

$$\mathcal{FS}\{x(t)\} = c_k, \mathcal{FS}\{x(t - t_0)\} = c_k e^{-jk\omega_0 t_0}$$

$$\mathcal{FS}\{x(t)\} = c_k, \mathcal{FS}\{x^*(t)\} = c_{-k}^*$$

$$x(t) = \sum \frac{c_k + c_{-k}^*}{2} e^{jk\omega_0 t} + j \sum \frac{c_k - c_{-k}^*}{2j} e^{jk\omega_0 t}$$

$$\mathcal{FS}\{x(t)y(t)\} = \sum c_l d_{k-l} = c_k * d_k$$

$$\mathcal{FS}\{x'(t)\} = jk\omega_0 c_k$$

$$\mathcal{FS}\{y(t)\} = \int_0^t x(\tau) d\tau, (0 < t < T), y(t) = y(t + T)$$

$$y(t) = \sum_{k \neq 0} \frac{c_k}{jk\omega_0} (e^{jk\omega_0 t} - 1) + c_0 t$$

2.1 DT FS

$$x[n] = x[n + N]$$

$$x[n] = \sum_{k=0}^{N-1} c_k e^{-\frac{j2\pi kn}{N}}, 0 \leq n < N$$

$$c_k = \frac{1}{N} \sum_{l=0}^{N-1} x[l] e^{-\frac{j2\pi kl}{N}} = c_{k+N}$$

Examples

$$x[n] = \begin{cases} 1, |n| \leq N_1 \\ 0, N_1 < |n| \leq \frac{N}{2} \end{cases},$$

$$\mathcal{FS}\{x[n]\} = \begin{cases} \frac{\sin(\frac{2\pi k(N_1 + \frac{1}{2})}{N})}{N \sin(\frac{2\pi k}{2N})}, k \neq tN \\ \frac{2N_1 + 1}{N}, k = tN \end{cases}$$

$$\mathcal{FS}\{e^{j\omega_0 n}\} = \begin{cases} 1, k = m + tN \\ 0, otherwise \end{cases}, \omega_0 = \frac{2\pi m}{N}$$

2.2 CT FT

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Examples

$$\mathcal{FS}\{x(t) = a[u(t + \frac{\tau}{2}) - u(t - \frac{\tau}{2})]\} = a\tau Sa(\frac{\omega\tau}{2})$$

$$\mathcal{FS}\{x(t) = e^{-\alpha t} u(t), (\alpha > 0)\} = \frac{1}{\alpha + j\omega}$$

$$\mathcal{FS}\{\delta(t)\} = 1, \mathcal{FS}\{1\} = 2\pi\delta(\omega)$$

$$\mathcal{FS}\{e^{j\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$$

$$\mathcal{FS}\{\text{sgn}(t)\} = \frac{2}{j\omega}$$

Periodic Signals

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}$$

$$\mathcal{FS}\{x(t)\} = \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t} e^{-j\omega t} dt$$

$$= 2\pi \sum_{k=-\infty}^{+\infty} c_k \delta(\omega - k\omega_0)$$

Parseval Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Properties

$$\mathcal{FS}\{x(t)\} = X(j\omega), \mathcal{FS}\{X(jt)\} = 2\pi x(j^2\omega)$$

$$\mathcal{FS}\{x(t - \tau)\} = X(j\omega) e^{-j\omega\tau}$$

$$\mathcal{FS}\{x(t)e^{j\omega_0 t}\} = X(j(\omega - \omega_0))$$

$$\mathcal{FS}\{x(\alpha t)\} = \frac{1}{|\alpha|} X(\frac{j\omega}{\alpha})$$

$$\mathcal{FS}\{x^*(t)\} = X^*(-j\omega)$$

$$\mathcal{FS}\{x'(t)\} = j\omega X(j\omega), \mathcal{FS}\{tx(t)\} = j \frac{dX(j\omega)}{d\omega}$$

$$\mathcal{FS}\{\int_{-\infty}^t x(\tau) d\tau\} = \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

$$\mathcal{FS}\{-\frac{x(t)}{it} + \pi x(0)\delta(t)\} = \int_{-\infty}^{\omega} X(j\eta) d\eta$$

$$\mathcal{F}\{x_1(t) * x_2(t)\} = X_1(j\omega)X_2(j\omega),$$

$$\mathcal{F}\{x_1(t)x_2(t)\} = \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$$

2.3 DT FT

$$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n} = X(e^{j(\omega+2\pi)}) (*)$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Examples

$$\mathcal{F}\{|[n] \leq N_1|\} = \frac{\sin\omega(N_1 + \frac{1}{2})}{\sin(\frac{\omega}{2})}$$

$$\mathcal{F}\{\frac{\sin(Wn)}{\pi n}\} = [u(\omega + W) - u(\omega - W)](\text{Periodic}, T = 2\pi)$$

$$x[n] = \begin{cases} 1, n \geq 0 \\ -1, n < 0 \end{cases}$$

$$\mathcal{F}\{x[n]\} = \frac{2}{1 - e^{-j\omega}}$$

$$\mathcal{F}\{u[n]\} = \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 2k\pi)$$

$$\mathcal{F}\{e^{j\omega_0 n}\} = \sum_{n=-\infty}^{+\infty} e^{-j(\omega - \omega_0)n} = 2\pi \sum \delta(\omega - \omega_0 - 2k\pi)$$

$$\mathcal{F}\{(\frac{n+r-1}{n}) a^n u[n]\} = (1 - ae^{-j\omega})^{-r}, |a| < 1$$

Periodic Signals

$$x[n] = x[n + N], \text{Fourier Series } x[n] = \sum_{k=0}^{N-1} c_k e^{-\frac{j2\pi kn}{N}}$$

$$c_k = \frac{1}{N} \sum x[l] e^{-\frac{j2\pi kl}{N}} = c_{k+N}$$

$$\mathcal{F}\{x(t)\} = \sum_{k=0}^{N-1} c_k \mathcal{F}\{e^{-\frac{j2\pi kn}{N}}\} = \sum_{k=-\infty}^{+\infty} 2\pi c_k \delta(\omega - \frac{2k\pi}{N})$$

CT FS and DT FT

$$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n} = X(e^{j(\omega+2\pi)})$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}$$

$$x[n] = c_{-n} \rightarrow x(t) = X(e^{j\omega_0 t})$$

Properties

$$x_k[n] = \begin{cases} x[n/k], n/k \in \mathbb{Z} \\ 0, otherwise \end{cases}$$

$$\mathcal{F}\{x_k[n]\} = X(e^{jk\omega})$$

$$\mathcal{F}\{x[n] - x[n - 1]\} = (1 - e^{-j\omega})X(e^{jk\omega})$$

$$\mathcal{F}\{nx[n]\} = j \frac{dX(e^{j\omega})}{d\omega}$$

$$\mathcal{F}\{\sum_{k \leq n} x[k]\} = \frac{X(e^{j\omega})}{(1 - e^{-j\omega})} + \pi X(e^{j0}) \sum \delta(\omega - 2k\pi)$$

$$\mathcal{F}\{x[n]y[n]\} = \frac{1}{2\pi} \int_{2\pi} X(e^{j\theta}) Y(e^{j(\omega - \theta)}) d\theta$$

2.4 DFT

$$X[k] = X(e^{-\frac{j2\pi k}{N}}) \text{ (sampling DT FT)}$$

Distortionless

$$Y(j\omega) = X(j\omega)H(j\omega) = AX(j\omega)e^{-j\omega\tau}$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = AX(e^{j\omega})e^{-j\omega n_0}$$

Group Delay

$$\tau(\omega) = -\frac{d}{d\omega} \phi(H(j\omega))$$

2.5 Sampling

Impulse

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s), \omega_s = \frac{2\pi}{T}$$

$$x_p(t) = x(t)p(t), X_p(j\omega) = \frac{1}{T} \sum_k X(j(\omega - k\omega_s)) \text{ (copy and paste)}$$

If $X(j\omega) = 0(|\omega| > \omega_M)$, No overlap in freq domain if $\omega_s > 2\omega_M$

lowpass filter $\omega_c \in (\omega_M, \omega_s - \omega_M)$ to restore signal

Zero-Order Holding

$$x_0(t) = \sum_{k=-\infty}^{+\infty} x(kT)p_0(t - kT) = x(t)p(t) * p_0(t)$$

$$p_0(t) = u(t) - u(t - T)$$

$$P(j\omega) = T e^{-\frac{j\omega T}{2}} Sa(\frac{\omega T}{2})$$

$$X_0(j\omega) = \sum_{k=-\infty}^{+\infty} e^{-\frac{j\omega T}{2}} Sa(\frac{\omega T}{2}) X(j(\omega - k\omega_s))$$

$$H_r(j\omega) = \frac{e^{-\frac{j\omega T}{2}}}{Sa(\frac{\omega T}{2})}, |\omega| < |\omega_c|$$

Linear Interpolation

$$x_1(t) = \sum_{n=-\infty}^{+\infty} x(kT)p_1(t - kT), p_1(t) = (1 - \frac{|t|}{T})[u(t + T) - u(t - T)]$$

$$P_1(j\omega) = T Sa^2(\frac{\omega T}{2})$$

$$X_1(j\omega) = \sum_{k=-\infty}^{+\infty} Sa^2(\frac{\omega T}{2}) X(j(\omega - k\omega_s))$$

$$H_r(j\omega) = Sa^{-2}(\frac{\omega T}{2}), |\omega| < |\omega_c|$$

Processing

$$y(t) = x(t) * h(t), y_d[n] = x_d[n] * h_d[n], x_d[n] = x(nT), y_d[n] = y(nT)$$

$$Y_d(e^{j\Omega}) = X_d(e^{j\Omega})H_d(e^{j\Omega}) \text{ (also band limited)}$$

$$X_d(e^{j\Omega}) = X_p(\frac{j\Omega}{T}) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(j\frac{j(\Omega - 2k\pi)}{T})$$

$$H_d(e^{j\Omega}) = H(\frac{j\Omega}{T}), -\pi \leq \Omega < \pi$$

If $H(j\omega)$ also band limited (for $|\omega| > \frac{\omega_s}{2}$), $h_d[n] = T h(nT)$