1. (10分) 设$f(x)=\cos x,\ \ x\in\mathbb{R};\ \ g(x)=\arcsin(x),\ \ x\in[-1,1]$, 计算$g(f(x))$并画出草图.

$$
\begin{aligned}
&\arcsin(\cos x)=\arcsin(-\sin(x-\frac{\pi}{2}))=-\arcsin(\sin(x-\frac{\pi}{2}))\\
&=\begin{cases}-\arcsin(\sin(x-\frac{\pi}{2}-2k\pi)), & x-\frac{\pi}{2}\in[2k\pi-\frac{\pi}{2},2k\pi+\frac{\pi}{2}]\\ -\arcsin(-\sin(x-\frac{\pi}{2}-2k\pi-\pi)), & x-\frac{\pi}{2}\in[2k\pi+\frac{\pi}{2},2k\pi+\frac{3\pi}{2}]\end{cases}\\
&=\begin{cases}-(x-\frac{\pi}{2}-2k\pi), & x\in[2k\pi,2k\pi+\pi]\\ (x-\frac{\pi}{2}-2k\pi-\pi), & x\in[2k\pi+\pi,2(k+1)\pi]\end{cases}\\
&=\begin{cases}-x+(2k+\frac{1}{2})\pi), & x\in[2k\pi,2k\pi+\pi]\\ x-(2k+\frac{3}{2})\pi, & x\in[2k\pi+\pi,2(k+1)\pi]\end{cases}.
\end{aligned}
$$

2. (10分) 是否存在数列$\{a_n\}$使得$\lim\limits_{n\to\infty}(a_n+a_{n+1})=\alpha,\ \lim\limits_{n\to\infty}(a_n+a_{n+2})=\beta,\ \ \alpha\neq\beta$？若存在，给出一个例子; 若不存在，说明理由.

【设$\{a_n+a_{n+1}\}$ 和$\{a_n+a_{n+2}\}$都收敛, 证明$\{a_n\}$收敛.】

$a_n+a_{n+1}=x_n,\ a_n+a_{n+2}=y_n\ \ \Rightarrow\ \ 2a_n+a_{n+1}+a_{n+2}=x_n+y_n\ \ \Rightarrow\ \ 2a_n=x_n+y_n-x_{n+1}.$

$x_n,\ y_n$收敛 $\Rightarrow$ a_n收敛.

3. (10分) 设函数$f(x)$在$[a,b]$上只有第一类间断点, 证明$f(x)$在$[a,b]$上有界.

至多第一类间断 $\Rightarrow$ 各点左右极限存在 $\Rightarrow$ 局部有界 $\Rightarrow$ 有限覆盖 $\Rightarrow$ 整体有界.

4. (10分)设$f(x)\in C[a,b],\ f(a)f(b)<0$. 证明:

$\forall n=1,2,\cdots,\ \ \exists\{\xi_j\}_{j=1}^n\subset[a,b],\ \ i\neq j$时$,\xi_i\neq\xi_j,\ \ \ s.t.\ \sum\limits_{j=1}^n e^{f(\xi_j)}=n.$

$f(x)\in C[a,b],\ f(a)f(b)<0\ \ \Rightarrow\ \ \exists\xi_0\in(a,b)\ \ s.t.\ \ f(\xi_0)=0.$

不妨设$f(a)<0,f(b)>0$, 则

$f(x)\in C[a,b],f(\xi_0)=0\ \ \ \Rightarrow\ \ g(x)=e^{f(x)}\in C[a,b],\ \ g(\xi_0)=1,\ g(a)<1,\ \ g(b)>1.$

记$\delta=\min\{1-g(a),g(b)-1\}$, 则$g(x)\in C[a,b]$

$\Rightarrow\ \ \forall\eta\in(0,\delta],\ \ \exists\ \xi_1\in[a,\xi_0),\ \ s.t.\ \ g(\xi_1)=1-\eta;\ \ \exists\ \xi_2\in(\xi_0,b]\ \ s.t.\ \ g(\xi_2)=1+\eta.$

当$n=2k,\ k=1,2,3,\cdots$ 时, 取$\eta_j=\dfrac{j}{k}\delta,\ \ j=1,\cdots,k$

$\Rightarrow\ \ \exists\ \xi_{1,j}\in[a,\xi_0),\ \ s.t.\ \ g(\xi_{1,j})=1-\eta_j,\ \ j=1,\cdots,k;$

$\quad\ \exists\ \xi_{2,j}\in(\xi_0,b]\ \ s.t.\ \ g(\xi_{2,j})=1+\eta_j,\ \ j=1,\cdots,k.$

$\Rightarrow\ \ \sum\limits_{j=1}^k[g(\xi_{1,j})+g(\xi_{2,j})]=2k=n.$

当$n=2k+1,\ k=1,2,3,\cdots$ 时, $\sum\limits_{j=1}^k[g(\xi_{1,j})+g(\xi_{2,j})]+g(\xi_0)=2k+1=n.$

命题得证.

5. (15分)设$\lim\limits_{n\to\infty} a_n = a$, 试证:

$$\lim_{n\to\infty}\frac{p_1a_n+p_2a_{n-1}+\cdots+p_na_1}{p_1+p_2+\cdots+p_n}=a,$$

其中$p_k>0$而且$\lim\limits_{n\to\infty}\dfrac{p_n}{p_1+p_2+\cdots+p_n}=0.$

$\forall\varepsilon>0$

因为$\{a_n\}$收敛, 所以$\exists M>0\ \ s.t.\ \ |a_n-a|\le M,\ \ \forall n\in\mathbb{N}.$

且$a_n\to a\ (n\to\infty)\ \ \Rightarrow\ \ \exists N_1\in\mathbb{N}\ \ s.t.\ \ |a_n-a|<\dfrac{\varepsilon}{2},\ \ \forall n\ge N_1.$

$$\left|\frac{\sum\limits_{k=1}^{n}p_{n-k}a_k}{\sum\limits_{k=1}^{n}p_k}-a\right|=\left|\frac{\sum\limits_{k=1}^{n}p_{n-k}(a_k-a)}{\sum\limits_{k=1}^{n}p_k}\right|=\left|\frac{\sum\limits_{k=1}^{N_1}p_{n-k}(a_k-a)+\sum\limits_{k=N_1+1}^{n}p_{n-k}(a_k-a)}{\sum\limits_{k=1}^{n}p_k}\right|$$

$$\le\frac{\sum\limits_{k=1}^{N_1}p_{n-k}|a_k-a|}{\sum\limits_{k=1}^{n}p_k}+\frac{\sum\limits_{k=N_1+1}^{n}p_{n-k}|a_k-a|}{\sum\limits_{k=1}^{n}p_k}\le\frac{\sum\limits_{k=1}^{N_1}p_{n-k}M}{\sum\limits_{k=1}^{n}p_k}+\frac{\sum\limits_{k=N_1+1}^{n}p_{n-k}\frac{\varepsilon}{2}}{\sum\limits_{k=1}^{n}p_k}$$

$$\le M\sum_{k=1}^{N_1}\frac{p_{n-k}}{\sum\limits_{k=1}^{n}p_k}+\frac{\varepsilon}{2}\le M\sum_{k=1}^{N_1}\frac{p_{n-k}}{\sum\limits_{k=1}^{n-k}p_k}+\frac{\varepsilon}{2}.$$

因为$\lim\limits_{n\to\infty}\dfrac{p_n}{p_1+p_2+\cdots+p_n}=0$, 所以$\exists N_2\in\mathbb{N}\ \ s.t.\ \ 0<\dfrac{p_n}{\sum\limits_{k=1}^{n}p_k}<\dfrac{\varepsilon}{2MN_1},\ \ \ n>N_2.$

所以对上述的N_1, 在$n-N_1>N_2$, 即$n>N_1+N_2$时,

$0<\dfrac{p_{n-k}}{\sum\limits_{k=1}^{n-k}p_k}<\dfrac{\varepsilon}{2MN_1},\ \ \ k=1,2,\cdots,N_1.$

从而在$n>N_1+N_2$时, $\left|\dfrac{\sum\limits_{k=1}^{n}p_{n-k}a_k}{\sum\limits_{k=1}^{n}p_k}-a\right|\le M\sum\limits_{k=1}^{N_1}\dfrac{\varepsilon}{2MN_1}+\dfrac{\varepsilon}{2}=\varepsilon.$ 故$\lim\limits_{n\to\infty}\dfrac{\sum\limits_{k=1}^{n}p_{n-k}a_k}{\sum\limits_{k=1}^{n}p_k}=a.$

6. (15分)证明：对任何有界正数列$a_n>0,\ n=1,2,\cdots$, 均有$\varlimsup\limits_{n\to\infty}\left(\dfrac{a_1+a_{n+1}}{a_n}\right)^n=+\infty.$

【证明：对任何正数列$a_n>0,\ n=1,2,\cdots$, 均有$\varlimsup\limits_{n\to\infty}\left(\dfrac{a_1+a_{n+1}}{a_n}\right)^n\ge e.$ 】

【反正法一】

如果$\varlimsup\limits_{n\to\infty}\left(\dfrac{a_1+a_{n+1}}{a_n}\right)^n<e$, 则$\exists N\in\mathbb{N}\ \ s.t.\ \ \dfrac{a_1+a_{n+1}}{a_n}<1+\dfrac{1}{n},\ \ \ n\ge N.$

$\Rightarrow\ \ \ a_1+a_{n+1}<\dfrac{n+1}{n}a_n\ \ \ \Rightarrow\ \ \ \dfrac{a_{n+1}}{n+1}<\dfrac{a_n}{n}-\dfrac{a_1}{n+1},\ \ \ n\ge N.$

$\Rightarrow\ \dfrac{a_{n+1}}{n+1}<\dfrac{a_N}{N}-a_1\sum\limits_{k=N+1}^{n}\dfrac{1}{k},\ \ n>N,$ 　但是$\lim\limits_{n\to\infty}\sum\limits_{k=N+1}^{n}\dfrac{1}{k}=+\infty,$

所以$\exists N_1\in\mathbb{N}\ \ s.t.\ \dfrac{a_{n+1}}{n+1}<0,\ \ n>N_1$. 这与已知条件矛盾.

【反正法二】【在$\varliminf\limits_{n\to\infty}a_n<+\infty$的条件下】

如果$\varlimsup\limits_{n\to\infty}\left(\dfrac{a_1+a_{n+1}}{a_n}\right)^n<e,$

则$\forall\varepsilon>0,\ \ \exists N\in\mathbb{N}\ \ s.t.\ \left(\dfrac{a_1+a_{n+1}}{a_n}\right)^n<e+\varepsilon,\ \ n>N.$

$\Rightarrow\ \ \dfrac{a_1+a_{n+1}}{a_n}<(e+\varepsilon)^{\frac{1}{n}},\ \ n>N.$

$\Rightarrow\ \ a_1+a_{n+1}<(e+\varepsilon)^{\frac{1}{n}}a_n,\ \ n>N.$

$\Rightarrow\ \ a_1+\varliminf\limits_{n\to\infty}a_{n+1}\le\varliminf\limits_{n\to\infty}a_n,\ \ \Rightarrow\ \ a_1=0$. 这与已知条件矛盾.

！！！！！！在$\varliminf\limits_{n\to\infty}a_n<+\infty$的条件下, 上述$e$换成任何正数都导致相同结果, 所以$\varlimsup\limits_{n\to\infty}\left(\dfrac{a_1+a_{n+1}}{a_n}\right)^n=+\infty.$

7. (15分) 讨论$f(x)=\dfrac{x}{1+x\cos^2x}$在$[0,+\infty)$上的一致连续.

令$x_n=2n\pi+\dfrac{\pi}{2},\ \ n\in\mathbb{N}$, 则$f(x_n)=x_n$.

令$x_n'=2n\pi+\dfrac{\pi}{2}+\alpha_n,\ \ n\in\mathbb{N}$,其中$\lim\limits_{n\to\infty}\alpha_n=0$.

则$f(x_n')=\dfrac{x_n'}{1+x_n'\sin^2\alpha_n}\le\dfrac{1}{\sin^2\alpha_n}\le\dfrac{\pi^2}{4}\left(\dfrac{1}{\alpha_n}\right)^2.$

这里使用了不等式 $\dfrac{2}{\pi}x<\sin x<x,\ \ x\in(0,\dfrac{\pi}{2})$.

注意到$f(x)$在$x=x_n$点取到极大值, 所以$|f(x_n)-f(x_n')|=f(x_n)-f(x_n')>x_n-\dfrac{\pi^2}{4}\left(\dfrac{1}{\alpha_n}\right)^2.$

若取$\alpha_n=n^{-\frac{1}{4}}$, 则$\lim\limits_{n\to\infty}|f(x_n)-f(x_n')|=+\infty$. 所以$f(x)$在$[0,+\infty)$ 上不一致连续.

【只要$x_n-\dfrac{\pi^2}{4}\left(\dfrac{1}{\alpha_n}\right)^2>\varepsilon_0>0$, 并注意$f(x_n)$是极大值.】

8. (15分)设$f(x)\in C[a,b],\ \ f([a,b])\subset[a,b],\ \ x_0\in[a,b],\ \ \ x_{n+1}=f(x_n),\ \ n=0,1,2,\cdots$. 证明：$\{x_n\}$收敛的充分必要条件是$\lim\limits_{n\to\infty}(x_{n+1}-x_n)=0$.

【必要性】(1分)x_n收敛时, 当然$\lim\limits_{n\to\infty}(x_{n+1}-x_n)=0$.

【充分性】:

【方法一】:

【结论一】(7分):设序列$\{x_n\}$有界, 且$\lim\limits_{n\to\infty}(x_{n+1}-x_n)=0$. 再设$l=\varliminf\limits_{n\to\infty}x_n$和$L=\varlimsup\limits_{n\to\infty}x_n$. 则$[l,\ L]$中的任意一个数都是此序列的一个子列的极限.

【反证法】如果存在$\eta\in(\ell,L)$使得η不是x_n的任何子列的极限,

则$\exists\delta>0,\ N\in\mathbb{N}_{\not{r}},\ \ s.t.\ \ V(\eta,\delta)\cap\{x_n\}_{n=N_0}^{\infty}=\emptyset.$

同时, $\lim\limits_{n\to\infty}(x_{n+1}-x_n)=0\ \ \Rightarrow\ \ \exists N\in\mathbb{N}\ (N>N_0)\ \ s.t.\ |x_{n+1}-x_n|<\delta,\ \forall n>N.$

这说明, x_n在$n>N$之后将无法跨越纵轴上的分离带$V(\eta,\delta)$,

从而要么$x_n < \eta - \delta,\ \forall n > N$, 要么$x_n > \eta + \delta,\ \forall n > N$.
$x_n < \eta - \delta,\ \forall n > N \quad \Rightarrow \quad \overline{\lim_{n\to\infty}}\, x_n \le \eta - \delta$, 与$L = \overline{\lim_{n\to\infty}}\, x_n$矛盾.
$x_n > \eta + \delta,\ \forall n > N \quad \Rightarrow \quad \underline{\lim_{n\to\infty}}\, x_n \ge \eta + \delta$, 与$\ell = \underline{\lim_{n\to\infty}}\, x_n$矛盾.

【结论二】(5分)设$f(x) \in C[a,b]$, $f([a,b]) \subset [a,b]$, $x_0 \in [a,b]$, $x_{n+1} = f(x_n)$, $n = 0,1,2,\cdots$, $\lim\limits_{n\to\infty}(x_{n+1} - x_n) = 0$. 记$\overline{\lim_{n\to\infty}}\, x_n = h$, $\underline{\lim_{n\to\infty}}\, x_n = \ell$. 如果$\ell < h$, 则$\forall \xi \in (\ell, h)$, $f(\xi) = \xi$.

【证明】:$\forall \xi \in (\ell,h)$, $\exists\ \{x_{n_k}\}_{k+1}^{\infty} \subset \{x_n\}\ \ s.t.\ \lim\limits_{k\to\infty} x_{n_k} = \xi$.
$f(x) \in C[a,b] \quad \Rightarrow \quad \lim\limits_{k\to\infty} f(x_{n_k}) = f(\xi) \quad \Rightarrow \quad \lim\limits_{k\to\infty} x_{n_k+1} = f(\xi)$.
$\lim\limits_{n\to\infty}(x_{n+1} - x_n) = 0 \quad \Rightarrow \quad \lim\limits_{k\to\infty}(x_{n_k+1} - x_{n_k}) = 0 \quad \Rightarrow \quad f(\xi) = \xi$.

【结论三】(2分)设$f(x) \in C[a,b]$, $f([a,b]) \subset [a,b]$, $x_0 \in [a,b]$, $x_{n+1} = f(x_n)$, $n = 0,1,2,\cdots$, $\lim\limits_{n\to\infty}(x_{n+1} - x_n) = 0$. 则$\{x_n\}$收敛.

【证明】根据结论二, 如果$\underline{\lim_{n\to\infty}}\, x_n = \ell < h = \overline{\lim_{n\to\infty}}\, x_n = h$, 则$(\ell,h)$ 内的所有点都是$f(x)$的不动点. 同时$\exists n_0 \in \mathbb{N}\ \ s.t.\ x_{n_0} \in (\ell,h) \quad \Rightarrow \quad x_{n+1} = f(x_n) \equiv x_{n_0},\ \ n \ge n_0 \quad \Rightarrow \quad \lim\limits_{n\to\infty} x_n = x_{n_0}$. 这一方面说明$x_n$收敛, 另一方面也与$\ell < h$矛盾. 所以$\ell = h$.
综合以上三条结论, 充分性得证.

【方法二】

【结论一】(12分)设$f(x) \in C[a,b]$, $f([a,b]) \subset [a,b]$, $x_0 \in [a,b]$, $x_{n+1} = f(x_n)$, $n = 0,1,2,\cdots$, $\lim\limits_{n\to\infty}(x_{n+1} - x_n) = 0$. 记$\overline{\lim_{n\to\infty}}\, x_n = h$, $\underline{\lim_{n\to\infty}}\, x_n = \ell$. 如果$\ell < h$, 则$\forall \xi \in (\ell, h)$, $f(\xi) = \xi$.

【证明】如果$\exists \xi \in (\ell,h)\ \ s.t.\ f(\xi) > \xi\ \ (f(\xi) < \xi$ 时, 同理可证).
则$f(x) \in C[a,b] \quad \Rightarrow \quad \exists \delta > 0\ \ s.t.\ f(x) - x > \varepsilon_0 = \frac{f(\xi)-\xi}{2},\ \ \forall x \in V(\xi,\delta)$.
取$\delta_0 = \min\{\delta, \varepsilon_0\}$, 则$\forall x \in V(\xi,\delta_0) \quad \Rightarrow \quad f(x) - x > \delta_0$.
这说明, x_n一旦进入$V(\xi,\delta_0)$, 必将导致$x_{n+1} - x_n = f(x_n) - x_n > \delta_0$.
但$\lim\limits_{n\to\infty}(x_{n+1} - x_n) = 0 \quad \Rightarrow \quad \exists N \in \mathbb{N}\ \ s.t.\ |x_{n+1} - x_n| < \dfrac{\delta_0}{2},\ \ \forall n > N$.
从而$n > N$之后, x_n将无法进入$V(\xi,\delta_0)$无法跨越$V(\xi,\delta_0) \subset (\ell,h)$.
此与$\overline{\lim_{n\to\infty}}\, x_n = h$, $\underline{\lim_{n\to\infty}}\, x_n = \ell$. 矛盾.

【结论二】(2分)设$f(x) \in C[a,b]$, $f([a,b]) \subset [a,b]$, $x_0 \in [a,b]$, $x_{n+1} = f(x_n)$, $n = 0,1,2,\cdots$, $\lim\limits_{n\to\infty}(x_{n+1} - x_n) = 0$. 则$\{x_n\}$收敛.

【证明】根据结论二, 如果$\underline{\lim_{n\to\infty}}\, x_n = \ell < h = \overline{\lim_{n\to\infty}}\, x_n = h$, 则$(\ell,h)$ 内的所有点都是$f(x)$的不动点. 同时$\exists n_0 \in \mathbb{N}\ \ s.t.\ x_{n_0} \in (\ell,h) \quad \Rightarrow \quad x_{n+1} = f(x_n) \equiv x_{n_0},\ \ n \ge n_0 \quad \Rightarrow \quad \lim\limits_{n\to\infty} x_n = x_{n_0}$. 这一方面说明$x_n$收敛, 另一方面也与$\ell < h$矛盾. 所以$\ell = h$.
综合以上两条结论, 充分性得证.