6.630 Solution to Problem Set 1

Solution P1.1

(a) $\overline{E}_1$ and $\overline{E}_3$ satisfy the wave equation with $k = \omega(\mu_0\epsilon_0)^{1/2}$,

(b) By Faraday's law

$$\frac{\partial}{\partial t}\overline{H} = -\frac{1}{\mu_0}\nabla \times \overline{E}$$

$$\overline{H}_1 = 0$$

$$\overline{H}_2 = \begin{cases} \hat{y}\dfrac{\sqrt{2}k}{\omega\mu_0}\cos\left[\omega t + k(x-z)/\sqrt{2}\right] & , \quad x - z > 0 \\ \hat{y}\dfrac{-\sqrt{2}k}{\omega\mu_0}\cos\left[\omega t + k(z-x)/\sqrt{2}\right] & , \quad x - z < 0 \end{cases}$$

$$\overline{H}_3 = (\hat{x} - \hat{z})\frac{-k}{\omega\mu_0}\cos(\omega t + ky)$$

(c) $\overline{E}_3$ qualifies as electromagnetic waves.
$\overline{E}_1$ violates Gauss' Theorem $\nabla \cdot \overline{E} = 0$.
The derivative of $\overline{E}_2$ and $\overline{H}_2$ do not exit at $x = z$, so they violate all of the Maxwell equations.
The $\overline{k}$-vector for $\overline{E}_3$ is $\overline{k}_3 = -\hat{y}k$. $\overline{E}_3$ and $\overline{H}_3$ are all perpendicular to the direction of propagation as $\overline{k}_3 \cdot \overline{E}_3 = 0$ and $\overline{k}_3 \cdot \overline{H}_3 = 0$.

Solution P1.2

(a-i) 60 Hz: $\lambda = c/f = 5 \times 10^6 (m)$
(a-ii) AM radio (535–1605 kHz): $\lambda = 186.9 \sim 560.8(m)$
(a-iii) FM radio (88–108 MHz): $\lambda = 2.778 \sim 3.409(m)$
(a-iv) C-band (4–6 GHz): $\lambda = 0.05 \sim 0.075(m)$
(a-v) Visible light ($\sim 10^{14}$ Hz): $\lambda =\sim 3 \times 10^{-6}(m)$
(a-vi) X-rays ($\sim 10^{18}$ Hz): $\lambda =\sim 3 \times 10^{-10}(m)$

(b-i) 1 km: $f = c/\lambda = 3 \times 10^5 (Hz)$
(b-ii) 1 m: $f = 3 \times 10^8 (Hz)$
(b-iii) 1 mm: $f = 3 \times 10^{11} (Hz)$
(b-iv) 1 μm: $f = 3 \times 10^{14} (Hz)$
(b-v) 1 Å: $f = 3 \times 10^{18} (Hz)$

(c-i) 1 km: $k = 2\pi/\lambda = K_o/\lambda = 10^{-3}K_o$
(c-ii) 1 m: $k = 1K_o$
(c-iii) 1 mm: $k = 10^3 K_o$
(c-iv) 1 μm: $k = 10^6 K_o$
(c-v) 1 Å: $k = 10^{10} K_o$

Solution P1.3

$$\begin{aligned}
\nabla \times \left(\nabla \times \overline{E}\right) &= \nabla \times \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{bmatrix} \\
&= \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) & \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) & \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \end{bmatrix} \\
&= \left[\frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial y} + \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) E_x\right] \hat{x} \\
&+ \left[\frac{\partial}{\partial y}\left(\frac{\partial E_y}{\partial y} + \frac{\partial E_x}{\partial y} + \frac{\partial E_z}{\partial z}\right) - \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) E_y\right] \hat{y} \\
&+ \left[\frac{\partial}{\partial z}\left(\frac{\partial E_y}{\partial y} + \frac{\partial E_x}{\partial y} + \frac{\partial E_z}{\partial z}\right) - \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) E_z\right] \hat{z} \\
&= \nabla\left(\nabla \cdot \overline{E}\right) - \nabla^2 \overline{E}
\end{aligned}$$

$$\begin{aligned}
&\nabla \cdot \left(\overline{E} \times \overline{H}\right) = \nabla \cdot \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_x & E_y & E_z \\ H_x & H_y & H_z \end{bmatrix} \\
&= \frac{\partial}{\partial x}\left(E_y H_z - E_z H_y\right) + \frac{\partial}{\partial y}\left(E_z H_x - E_x H_z\right) + \frac{\partial}{\partial z}\left(E_x H_y - E_y H_x\right) \\
&= H_x\left(\frac{\partial}{\partial y} E_z - \frac{\partial}{\partial z} E_y\right) + H_z\left(\frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x\right) + H_y\left(\frac{\partial}{\partial z} E_x - \frac{\partial}{\partial x} E_z\right) \\
&- E_x\left(\frac{\partial}{\partial y} H_z - \frac{\partial}{\partial z} H_y\right) - E_y\left(\frac{\partial}{\partial z} H_x - \frac{\partial}{\partial x} H_z\right) - E_z\left(\frac{\partial}{\partial x} H_y - \frac{\partial}{\partial y} H_x\right) \\
&= \overline{H} \cdot \left(\nabla \times \overline{E}\right) - \overline{E} \cdot \left(\nabla \times \overline{H}\right)
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot \left(\nabla \times \overline{A}\right) &= \nabla \cdot \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{bmatrix} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\nabla \times (\nabla \phi) &= \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{bmatrix} \\
&= \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z}\right) \hat{x} + \left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z}\right) \hat{y} + \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y}\right) \hat{z} \\
&= 0
\end{aligned}$$

Solution P1.4

The phase ϕ is a function of both space and time. To get the wave velocity, one needs to fix a phase and trace this constant phase in space and time. In other words, $\Delta\phi = 0$ in both space and time coordinates. Therefore, we can take derivative with respect to time or space to get the wave velocity.

If k is increased, the slope tends to be flat, meaning the wave velocity decreased. In addition, we can see that the wave becomes denser in space.