Homework 3.

Problem 1:

a. $x-\tilde{x} = \begin{pmatrix} 0.2 \\ 0 \\ -0.4 \end{pmatrix}$

$\therefore \|x-\tilde{x}\|_\infty = 0.4$

$A\tilde{x}-b = \begin{pmatrix} 0 \\ -0.3 \\ -0.2 \end{pmatrix}$

$\therefore \|A\tilde{x}-b\|_\infty = 0.3$

b. $x-\tilde{x} = \begin{pmatrix} 0.33 \\ 0.9 \\ -0.8 \end{pmatrix}$

$\therefore \|x-\tilde{x}\|_\infty = 0.9$

$A\tilde{x}-b = \begin{pmatrix} 0.27 \\ -0.16 \\ 0.21 \end{pmatrix}$

$\|A\tilde{x}-b\|_\infty = 0.27.$

Problem 2:

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sqrt{\lambda_{max}(A^TA)} = \sqrt{\max_{1\le i\le n} |\lambda_i|}$$

λ_i为A^TA的特征值.

$\because A^T = A$

$\therefore \|A\|_2 = \max_{1\le i\le n} |\lambda'_i|$, λ'_i为A的特征值

$\because \rho(A) = \max\{|\lambda_i|, i=1,2,\cdots,n\}$

$\therefore \|A\|_2 = \rho(A)$

Problem 3:

a. $\max\{|a_{i1}|\} = 5.31$

$$\therefore \begin{bmatrix} 5.31 & -6.1 \\ 0.03 & 58.9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 47 \\ 59.2 \end{bmatrix}$$

$$\therefore \left[\begin{array}{cc:c} 5.31 & -6.1 & 47 \\ 0 & 58.9 & 58.9 \end{array}\right].$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$ same as actual solution

b. $\max\{|a_{i1}|\} = 6.11$

$$\therefore \left[\begin{array}{ccc:c} 6.11 & -14.2 & 21 & -139 \\ 3.03 & -12.1 & 14 & -119 \\ -3.03 & 12.1 & -7 & 120 \end{array}\right].$$

$$\therefore \left[\begin{array}{ccc:c} 6.11 & -14.2 & 21 & -139 \\ 3.03 & -12.1 & 14 & -119 \\ 0 & 0 & 7 & 1 \end{array}\right] \quad \therefore \left[\begin{array}{ccc:c} 6.11 & -14.2 & 21 & -139 \\ 0 & -5.06 & 3.59 & -50. \\ 0 & 0 & 7 & 1 \end{array}\right]$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.05 \\ 9.98 \\ \frac{1}{7} \end{bmatrix}$$

Problem 4:

a. $A = \begin{bmatrix} 4 & 1 & -1 \\ -1 & 3 & 1 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$= D - L - U$

$\therefore \cancel{A}\ x^{(k)} = D^{-1}[L+U]x^{(k-1)} + D^{-1}b.$

$\therefore \cancel{x^{(1)}=}\ D^{-1} = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \quad L+U = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix}$

$\therefore x^{(1)} = D^{-1}[L+U]x^{(0)} + D^{-1}b = \begin{bmatrix} 1.25 \\ -\frac{4}{3} \\ 0.2 \end{bmatrix}$

$x^{(2)} = D^{-1}[L+U]x^{(1)} + D^{-1}b = [1.63 \quad -0.98 \quad 0.23]^T$

$x^{(3)} = D^{-1}(L+U)x^{(2)} + D^{-1}b = [1.55 \quad -0.87 \quad -0.06]^T$

b. $A = \begin{bmatrix} -2 & 1 & \frac{1}{2} \\ 1 & -2 & -\frac{1}{2} \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore D^{-1} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \quad L+U = \begin{bmatrix} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ 0 & -1 & 0 \end{bmatrix}$

$x^{(1)} = D^{-1}(L+U)x^{(0)} + D^{-1}b = [-2 \quad 2 \quad 0]^T$

$x^{(2)} = D^{-1}(L+U)x^{(1)} + D^{-1}b = [-1 \quad 1 \quad -1]^T$

$x^{(3)} = D^{-1}(L+U)x^{(2)} + D^{-1}b = [-1.75 \quad 1.75 \quad -0.5]^T$

Problem 5: $x_{(0)} = [0 \quad 0 \quad 0]^T$

a. $D=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ $L=\begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ -3 & -3 & 0 \end{bmatrix}$ $U=\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

Jacobi:

$X = [0.035 \quad -0.237 \quad 0.657]^T$, iteration: 8 times

Gauss-Seidel:

$X = [0.0361 \quad -0.2366 \quad 0.6573]^T$, iteration: 5 times

b. $D=\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$ $L=\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ $U=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Jacobi:

$X = [0.9956 \quad 0.9572 \quad 0.7911]^T$, iteration: 5 times

Gauss-Seidel:

$X = [0.9950 \quad 0.9575 \quad 0.7915]^T$, iteration: 3 times

Problem 6:

$A x_k = \rho_k x_k$

assume that: $\exists\, c_1 \cdots c_k \neq 0$, st $c_1 x_1 + \cdots + c_k x_k = 0$

$\therefore c_1 \rho_1 x_1 + c_2 \rho_2 x_2 + \cdots + c_k \rho_k x_k = 0$

$c_1 \rho_k x_1 + c_2 \rho_k x_2 + \cdots + c_k \rho_k x_k = 0$

$\therefore c_1 (\rho_1 - \rho_k) x_1 + c_2 (\rho_2 - \rho_k) x_2 + \cdots + c_{n-1} (\rho_{k-1} - \rho_k) x_{k-1} = 0$

repeat the operation for k-2 times:

$m_1 (\rho_1 - \rho_3) x_1 + m_2 (\rho_2 - \rho_3) x_2 = 0$

$\therefore n_1 (\rho_1 - \rho_2) x_1 = 0$

$\therefore n_1 = 0 \Rightarrow n_2 = 0 \Rightarrow \cdots \Rightarrow c_i = 0,\ i = 1 \cdots k$

$\therefore x_1 \cdots x_k$ is a linearly independent set

Problem 7:

$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|$

assume that A is not invertible

$\det(A) = 0$

$X = (x_1, \cdots, x_n)^t$, $|x_k| = \max\{|x_i|\}$

$\therefore \sum_{j=1}^{n} a_{kj} a_j = 0$, $\therefore |a_{kk}|\,|x_k| = |\sum_{j\neq k} a_{kj} x_j|$

$\because |a_{kk}|\,|x_k| \geq |x_k| \sum_{j\neq k} |a_{kj}| \geq \sum_{j\neq k} |a_{kj} x_j|$

$\therefore$ contradiction

$\therefore$ A is invertible