# Non-Linear Optimization Assignment

**SC42055 Optimization in Systems and Control**

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$E_1$, $E_2$ and $E_3$ are parameters changing from 0 to 9 for each group according to the mean of the last three numbers of their Student IDs:

$E_1 = \frac{D_{a,1}+D_{b,1}}{2}$, $E_2 = \frac{D_{a,2}+D_{b,2}}{2}$, $E_3 = \frac{D_{a,3}+D_{b,3}}{2}$,

where $D_{a,3}$ is the right-most digit of one student and $D_{b,3}$ is the right-most digit of the other student.

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Traffic congestion on freeways is a critical problem due to its negative impact on the environment and many other important consequences (higher delays, waste of fuel, higher accident risk probability, etc.). Since the construction of new freeways is not always a viable option or is too costly, other solutions have to be found. In many cases, the use of dynamic traffic control measures such as ramp metering, variable speed limits, reversible lanes, and route guidance may be a cost-efficient and effective solution.

In general, dynamic traffic control uses measurements of the traffic conditions over time and computes dynamic control signals to influence the behavior of the drivers and to generate a response in such a way that the performance of the network is improved, by reducing delays, emissions, fuel consumption etc.


Figure 1: Ramp Metering (left) and Variable Speed Limit (right) on the A13 in Delft

The most used control input in freeway traffic control is ramp metering, a device (usually a basic traffic light) located at an on-ramp that regulates the flow of traffic entering the freeway according to the current traffic conditions. Ramp metering systems have proved to be successful in decreasing traffic congestion and improving driver safety.

Another promising control measure in freeway traffic control is the use of Variable Speed

Limits (VSLs), overhead signs showing a varying speed limit that changes according to the current road conditions.

The goal of this assignment is to compute the values of the speed limits and the ramp metering rates (i.e. the percentage of ramp flow that is allowed to enter) that minimize the congestion that appears on a freeway stretch.

Let us assume that we want to model and control the behavior of the freeway stretch shown in Figure 2. The network considered has 4 segments, each of which has a length of 1 kilometer and 3 lanes. A ramp metering installation is located at the on-ramp on segment 4 and Variable Speed Limits are located at the start of segments 2 and 3.

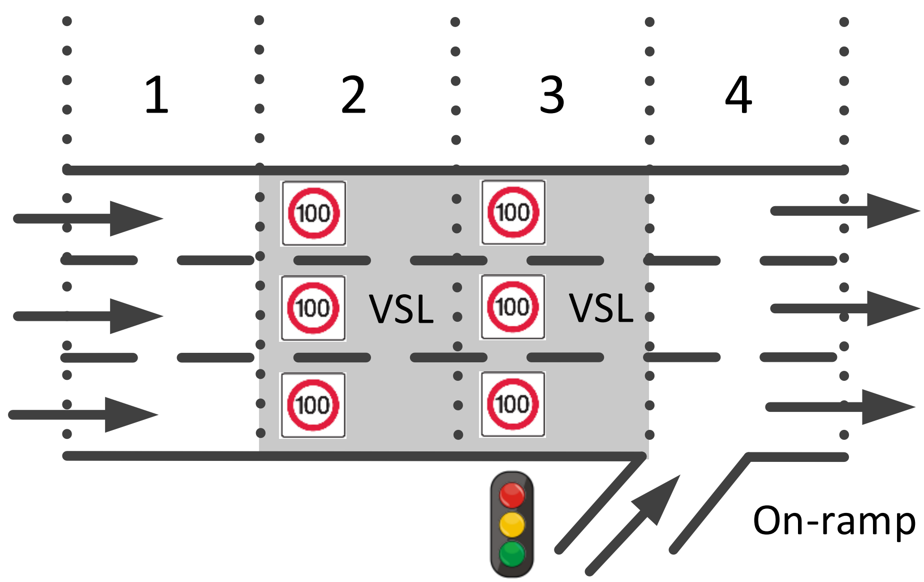


Figure 2: Schematic representation of the considered freeway stretch

The macroscopic model METANET will be used to model the behavior of the freeway. METANET represents the traffic network as a graph where the segments are indexed by an index $i$ and have a length of $L$ meters with $\lambda$ lanes.

The freeway is dynamically characterized by the traffic density of each segment ($\rho_i(k)$), the mean speed of each segment ($v_i(k)$) and the number of vehicles waiting on the on-ramp ($w_{\mathrm{r}}(k)$). The index $k$ is the time step corresponding to instant $t = kT$, where $T$ is the simulation time step (in this case, $T = 10$ s). The traffic density is defined as the number of vehicles per unit length of the roadway and is expressed in the units vehicles/(km lane). The mean speed and the queue are expressed in km/h and number of vehicles, respectively.

Two main equations describe the system dynamics of METANET model. The first one expresses the conservation of vehicles:

$$\rho_i(k+1) = \rho_i(k) + \frac{T}{\lambda L}(q_{i-1}(k) - q_i(k) + q_{\mathrm{r},i}(k)) \tag{1}$$

where $q_{\mathrm{r},i}(k)$ is the traffic flow that enters the freeway link $i$ from the connected on-ramp (if any). The traffic flow in each link $q_i(k)$ can be computed for each time step using $q_i(k) = \lambda\rho_i(k)v_i(k)$.

The second equation expresses the mean speed as a sum of the previous mean speed, a relaxation term, a convection term, and an anticipation term:

$$v_i(k+1) = v_i(k) + \frac{T}{\tau}(V_i(k) - v_i(k)) + \frac{T}{L}v_i(k)(v_{i-1}(k) - v_i(k)) - \frac{\mu T}{\tau L}\frac{\rho_{i+1}(k) - \rho_i(k)}{\rho_i(k) + K} \tag{2}$$

where $\tau$, $\mu$, and $K$ are model parameters that are assumed to be constant for all segments, and $V_i(k)$ is the desired speed for the drivers on segment $i$ that is modeled by the following equation:

$$V_i(k) = \min\left((1+\alpha)V_{\mathrm{SL},i}(k), v_{\mathrm{f}}e^{-\frac{1}{a}\left(\frac{\rho_i(k)}{\rho_{\mathrm{c}}}\right)^a}\right) \tag{3}$$

where $a$ is a model parameter, $v_{\mathrm{f}}$ is the free flow speed that the cars reach in steady state, $\rho_{\mathrm{c}}$ is the critical density (i.e. the density corresponding to the maximum flow), $V_{\mathrm{SL},i}(k), v_{\mathrm{f}}$ is the speed limit value applied on segment $i$, and $\alpha$ reflects the compliance of the drivers to the speed limit shown on the panels.

If the VSL is not activated, the default value for the speed limits is 120 km/h. If the VSL is activated, the new speed limit affects segments 2 and 3 ($V_{\mathrm{SL},2}(k) = V_{\mathrm{SL},3}(k) = V_{\mathrm{SL}}(k)$); the minimum and maximum values that can be applied for the VSL are 60 and 120 km/h, respectively. In real applications the implemented value for the VSL is limited to a discrete set (for example, $\{60, 80, 100, 120\}$ km/h). However, in order to facilitate the assignment, it will be assumed that $V_{\mathrm{SL}}(k)$ is a continuous variable.

In order to complete the model, the following equation defines the flow that enters from the controlled on-ramp on segment 4:

$$q_{\mathrm{r}}(k) = \min\left(r(k)C_{\mathrm{r}}, D_{\mathrm{r}}(k) + \frac{w_{\mathrm{r}}(k)}{T}, C_{\mathrm{r}}\frac{\rho_{\mathrm{m}} - \rho_4(k)}{\rho_{\mathrm{m}} - \rho_{\mathrm{c}}}\right) \tag{4}$$

where $r(k) \in [0,1]$ is the ramp metering rate applied at time step $k$, $\rho_{\mathrm{m}}$ and $C_{\mathrm{r}}$ are model parameters, $D_{\mathrm{r}}(k)$ is the demand of the on-ramp, and $w_{\mathrm{r}}(k)$ is the queue length the on-ramp of segment 4, the dynamics of which are defined by:

$$w_{\mathrm{r}}(k+1) = w_{\mathrm{r}}(k) + T(D_{\mathrm{r}}(k) - q_{\mathrm{r}}(k)) \tag{5}$$

It has to be noted that the ramp metering rate $r(k)$ is defined between 0 and 1 since it corresponds to the percentage of ramp flow that is allowed to enter and that the initial ramp queue is equal to 0 ($w_{\mathrm{r}}(0) = 0$).

Finally, some boundary conditions are defined:

- The downstream density of the final segment is considered to be equal to the density of the last segment: $\rho_5(k) = \rho_4(k)$.
- The speed upstream the first segment is considered to be equal to the speed of the first segment: $v_0(k) = v_1(k)$.
- The flow entering the first segment ($q_0(k)$) and the on-ramp demand ($D_{\mathrm{r}}(k)$) are inputs of the system.

The Total Time Spent (TTS) by the drivers during each interval $[kT, (k+1)T]$ is taken as the output of the system:

$$y(k) = Tw_{\mathrm{r}}(k) + TL\lambda \sum_{i=1,2,3,4} \rho_i(k) \tag{6}$$

For the freeway considered, the length of each segment ($L$) is one kilometer, the simulation time step ($T$) is 10 s and the METANET model parameters have the values shown in the following table:

Table 1: METANET parameters

| $\tau$ | $\mu$ | $C_{\mathrm{r}}$ | $\rho_{\mathrm{m}}$ | $\alpha$ | $K$ | $a$ | $v_{\mathrm{f}}$ | $\rho_{\mathrm{c}}$ |
|---|---|---|---|---|---|---|---|---|
| 10 s | 80 $\frac{\mathrm{km}^2}{\mathrm{h}}$ | 2000 $\frac{\mathrm{veh}}{\mathrm{h}}$ | 120 $\frac{\mathrm{veh}}{\mathrm{km\ lane}}$ | 0.1 | 10 $\frac{\mathrm{veh}}{\mathrm{km\ lane}}$ | 2 | 110 $\frac{\mathrm{km}}{\mathrm{h}}$ | 28 $\frac{\mathrm{veh}}{\mathrm{km\ lane}}$ |

Moreover, assume that, at $k = 0$, the densities of all the segments are equal to 20 veh/km lane and the speeds are equal to 90 km/h, that the ramp demand is constant and equal to $D_r(k) = 1500$ veh/h and that the flow entering the mainline is defined as follows:

$$q_0(k) = \begin{cases} 7000 + 100E_1 \text{ veh/h} & \text{if } k < 12 \\ 2000 + 100E_2 \text{ veh/h} & \text{if } k \geq 12 \end{cases}$$

Tasks:

1. Formulate the discrete-time state-space model (i.e. determine $f$ and $g$) that predicts the density and the speed of each segment, and the number of vehicles in the on-ramp queue for the next simulation time step $k+1$ as follows:

$$\begin{cases} x(k+1) = f(x(k), u(k)) \\ y(k) = g(x(k), u(k)) \end{cases} \tag{7}$$

   where $f$ and $g$ are vector-valued, nonlinear functions.

   Note: The state $x(k)$ includes 9 variables: 4 densities (one for each segment), 4 speeds (one for each segment), and the on-ramp queue $w_{\mathrm{r}}(k)$. They input $u(k)$ includes two variables: The VSL ($V_{\mathrm{SL}}(k)$) and the ramp metering rate ($r(k)$). The output $y(k)$ includes one variable: the TTS for time step $k$. All the parameters have to be expressed with the same units: h, km, veh,... .

2. Assuming for task 2 that the ramp metering installation is not used ($r(k) = 1\ \forall\ k$), formulate the optimization problem in order to find the values of the VSLs that minimize the Total Time Spent (TTS) by the drivers in the network for the following 10 minutes ($[kT, (k+60)T]$). Select an appropriate optimization algorithm (explaining your choice) and write the corresponding MATLAB code.

3. Assuming again for task 3 that the ramp metering installation is not used, run the chosen optimization algorithm using two different starting points: $V_{\mathrm{SL}}(k) = 120$ km/h $\forall k$ and $V_{\mathrm{SL}}(k) = 60$ km/h $\forall k$. Is there a substantial difference between the obtained solutions? Why? How can the solution be improved (if possible)? If possible, improve the solution as much as you can. Can you prove that the solution obtained is the global optimum?

4. Assuming for task 4 that both control measures (i.e. the VSLs and ramp metering) are used, formulate the optimization problem and find the values of the speed limits and the ramp metering rates that minimize the Total Time Spent (TTS) by the drivers in the network for the following 10 minutes ($[kT, (k+60)T]$) while limiting the maximum number of vehicles waiting on the queue to $20 - E_3$ vehicles.

5. Plot the queues, the speed limits and the ramp metering rates obtained for tasks 3 and 4 and compare them with the ones obtained for the no-control case ($r(k) = 1$ and $V_{\mathrm{SL}}(k) = 120$ km/h $\forall k$). Moreover, find the TTS values over the entire simulation period for the different cases (including the no-control case) and compare them. Explain the results obtained.

6. Optional (up to 2 additional points): Formulate and solve the problem of task 2 considering that the value of the VSL is limited for the following discrete set: $\{60, 80, 100, 120\}$.

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The written report on the practical exercise should be submitted to Brightspace before Tuesday, October 30, 2018 at 17.00 p.m. as one pdf file. The MATLAB codes used should also be uploaded to Brightspace as one .zip file containing all the needed .m files; please make sure that the code is error free. Please note that you will lose 0.5 point from your grade on the report for each (started) day of delay in case you exceed the deadline.