TU Delft

Networked and Distributed Control Systems

SC42100

Second Assignment

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Problem 1

(a) Applying primal decomposition and discussing the role of the master problem.

This section discusses the following convex optimisation problem with a complicating constraint:

$$\begin{aligned} \min_{\theta_1, \theta_2} \quad & f_1(\theta_1) + f_2(\theta_2) \\ \text{subject to} \quad & \theta_1 \in \Theta_1, \quad \theta_2 \in \Theta_2 \\ & h_1(\theta_1) + h_2(\theta_2) \leq 0 \end{aligned} \tag{1}$$

where Θ_1 and Θ_2 are convex sets and all the functions are convex. Primal decomposition is applied in order to solve the optimisation problem.

The two optimisation variables θ_1 and θ_2 are coupled only through the last constraint, $h_1(\theta_1) + h_2(\theta_2) \leq 0$. Note that if this last constraint did not hold, the problem would consist out of two uncoupled optimisation problems. The problem is thus rewritten into the following form with the introduction of a coupling variable r:

$$\begin{aligned} \min_{\theta_1, \theta_2} \quad & f_1(\theta_1) + f_2(\theta_2) \\ \text{subject to} \quad & \theta_1 \in \Theta_1, \quad \theta_2 \in \Theta_2 \\ & h_1(\theta_1) \leq r \\ & h_2(\theta_2) \leq -r \\ & r \in \mathbb{R} \end{aligned} \tag{2}$$

This optimisation problem is decomposed into the following two subproblems:

$$\begin{aligned} \min_{\theta_1} \quad & f_1(\theta_1) \\ \text{subject to} \quad & \theta_1 \in \Theta_1 \\ & h_1(\theta_1) \leq r \\ & r \in \mathbb{R} \end{aligned} \qquad\qquad \begin{aligned} \min_{\theta_2} \quad & f_2(\theta_2) \\ \text{subject to} \quad & \theta_2 \in \Theta_2 \\ & h_2(\theta_2) \leq -r \\ & r \in \mathbb{R} \end{aligned} \tag{3}$$

Then, a higher-level master problem is responsible for minimising the coupling variable:

$$\begin{aligned} \min_{r} \quad & \phi_1(r) + \phi_2(r) \\ \text{subject to} \quad & \phi_1(r) = \inf_{\theta_1} f_1(\theta_1) \; : \; h_1(\theta_1) \leq r \\ & \phi_2(r) = \inf_{\theta_2} f_2(\theta_2) \; : \; h_2(\theta_2) \leq -r \\ & r \in \mathbb{R} \end{aligned} \tag{4}$$

The role of the master problem is to find the value of r for which the cost function of Equation (2) is minimal.

(b) Show that $-\lambda^*$ is a subgradient of p at z.

As stated in the question, a subgradient can be found for the optimal value of each subproblem in Equation (3) from an optimal dual variable associated with the coupling constraint. The convex optimisation problem is given as:

$$
\begin{aligned}
\min_{\theta} \quad & f(\theta) \\
\text{subject to} \quad & \theta \in \Theta \\
& h(\theta) \leq z
\end{aligned}
\tag{5}
$$

After the introduction of multipliers, the Lagrangian becomes:

$$L(\theta, \lambda) = f(\theta) + \lambda^{\top} \big(h(\theta) - z\big) \tag{6}$$

The dual function $d(\lambda)$ is obtained as the maximum value of the Lagrangian for a given λ:

$$d(\lambda) = \inf_{\theta} \Big[f(\theta) + \lambda^{\top} h(\theta) - \lambda^{\top} z \Big] \tag{7}$$

Then, $\lambda^* = \arg\max_{\lambda} d(\lambda)$, $\lambda \geq 0$. The optimal value to the primal problem at z is:

$$
\begin{aligned}
p(z) &= d(\lambda^*) \\
&= \inf_{\theta} f(\theta) + (\lambda^*)^{\top} h(\theta) - (\lambda^*)^{\top} z
\end{aligned}
\tag{8}
$$

Then, for any other point $\tilde{z}$, this becomes:

$$
\begin{aligned}
p(\tilde{z}) &\geq \inf_{\theta} f(\theta) + (\lambda^*)^{\top} h(\theta) - (\lambda^*)^{\top} \tilde{z} \\
&= \inf_{\theta} f(\theta) + (\lambda^*)^{\top} \big(h(\theta) - z\big) - (\lambda^*)^{\top} (\tilde{z} - z) \\
&= \inf_{\theta} f(\theta) + (\lambda^*)^{\top} h(\theta) - (\lambda^*)^{\top} z - (\lambda^*)^{\top} \tilde{z} + (\lambda^*)^{\top} z \\
&= p(z) - (\lambda^*)^{\top} (\tilde{z} - z)
\end{aligned}
\tag{9}
$$

The subgradient g of p at z is defined as the following, with $g \in \partial p(z)$:

$$\partial p(z) = \{ g \in \mathbb{R} \mid p(\tilde{z}) \geq p(z) + g^{\top} (\tilde{z} - z), \ \forall \tilde{z} \in \mathbb{R} \} \tag{10}$$

To show that $-\lambda^*$ is a subgradient of p at z, $-\lambda^*$ is substituted for g in Equation 10:

$$
\begin{aligned}
\partial p(z) &= \{ g \in \mathbb{R} \mid p(\tilde{z}) \geq p(z) + g^{\top} (\tilde{z} - z), \ \forall \tilde{z} \in \mathbb{R} \} \\
\partial p(z) &= \{ \lambda^* \in \mathbb{R} \mid p(\tilde{z}) \geq p(z) - (\lambda^*)^{\top} (\tilde{z} - z), \ \forall \tilde{z} \in \mathbb{R} \}
\end{aligned}
\tag{11}
$$

Therefore, $-\lambda^*$ is a subgradient of p at z. $\square$

Problem 2

$$\theta_{k+1}^i = \mathcal{P}_\Theta \left[\sum_{j=1}^{N} [W^\varphi]_{ij} \left(\theta_k^j - \alpha_k \, g_k^j \left(\theta_k^j \right) \right) \right], \quad i = 1, \dots, N \tag{12}$$

The communicated graph used for consensus is strongly connected, balanced, and the matrix W is doubly stochastic.

For $\varphi \to \infty$, there is an infinite number of consensus steps from one iteration to the next. Thus, for a strongly connected graph, consensus is reached in every iteration of the algorithm. This is because, as $\varphi \to \infty$, W^φ converges to the following form, where W corresponds to a strongly connected graph:

$$W^\infty = \begin{bmatrix} \frac{1}{N} & \cdots & \frac{1}{N} \\ \vdots & \ddots & \vdots \\ \frac{1}{N} & \cdots & \frac{1}{N} \end{bmatrix} \in \mathbb{R}^{N \times N} \tag{13}$$

Since consensus is reached in every iteration, $\theta_{k+1}^i = \bar{\theta}_{k+1}$ for every agent i.
For the dynamics of this consensus value $\bar{\theta}$, the projected incremental subgradient method simplifies to:

$$\begin{aligned} \bar{\theta}_{k+1} &= \mathcal{P}_\Theta \left[N \cdot \frac{1}{N} \cdot \left(\bar{\theta}_k - \alpha_k \, g_k(\bar{\theta}_k) \right) \right] \\ \bar{\theta}_{k+1} &= \mathcal{P}_\Theta \left[\bar{\theta}_k - \alpha_k \, g_k(\bar{\theta}_k) \right] \end{aligned} \tag{14}$$

Thus, the evolution of the consensus value is of the form of the standard subgradient method. $\square$

Problem 3

The objective of this exercise is to find a decomposition-based algorithm for the coordination of each aircraft towards a common target state at T_{final}. The optimisation problem can be seen as a Finite-Time Optimal Rendez Vouz (FTOR) problem, described by:

$$\begin{aligned}
\min_{u_i, x_f} \quad & \sum_i \sum_t x_i(t)^\top x_i(t) + u_i(t)^\top u_i(t) \\
\text{subject to} \quad & x_i(T_{\text{final}}) = x_f, \quad \forall i \\
& \sum_t u_i(t)^\top u_i(t) \leq u_{max}^2, \quad \forall i
\end{aligned} \tag{15}$$

The goal is to derive a solution to the problem based on dual decomposition using MATLAB. The rendezvous point is considered to be x_f and is not fixed a priori, but needs to be obtained through the optimisation problem.
The dynamics of the states of each aircraft are described by:

$$x_i(t+1) = A_i\, x_i(t) + B_i\, u_i(t), \quad i = 1,\, 2,\, 3,\, 4, \quad t = 0,\, \dots,\, T_{\text{final}} \tag{16}$$

With $x_i(t) = \begin{bmatrix} x_i^{\text{pos}}(t) & y_i^{\text{pos}}(t) & x_i^{\text{vel}}(t) & y_i^{\text{vel}}(t) \end{bmatrix}$.

The optimisation problem in Equation (15) can be separated into a lower and higher level of optimisation. For this, the Lagrangian of the problem is needed. In the Lagrangian, $\lambda_i \in \mathbb{R}^4$ is a Lagrange multiplier.

The Lagrangian of the problem is, for all four aircraft:

$$\begin{aligned}
L(x_i, u_i, \lambda_i) = & \sum_t \Big(x_1(t)^\top x_1(t)\Big) + \lambda_1^\top\, x_1(T_{\text{final}}) + \sum_t \Big(u_1(t)^\top u_1(t)\Big) - \lambda_1^\top\, x_f \\
& + \sum_t \Big(x_2(t)^\top x_2(t)\Big) + \lambda_2^\top\, x_2(T_{\text{final}}) + \sum_t \Big(u_2(t)^\top u_2(t)\Big) - \lambda_{2,1}^\top\, x_f \\
& + \sum_t \Big(x_3(t)^\top x_3(t)\Big) + \lambda_3^\top\, x_3(T_{\text{final}}) + \sum_t \Big(u_3(t)^\top u_3(t)\Big) - \lambda_{3,1}^\top\, x_f \\
& + \sum_t \Big(x_4(t)^\top x_4(t)\Big) + \lambda_4^\top\, x_4(T_{\text{final}}) + \sum_t \Big(u_4(t)^\top u_4(t)\Big) - \lambda_{4,1}^\top\, x_f
\end{aligned} \tag{17}$$

Which leads to the dual function:

$$\begin{aligned}
d(\lambda_i) = \inf_{u_1,\, u_2,\, u_3,\, u_4,\, x_f} & \sum_t \Big(u_1(t)^\top u_1(t)\Big) + \sum_t \Big(u_2(t)^\top u_2(t)\Big) + \sum_t \Big(u_3(t)^\top u_3(t)\Big) \\
& + \sum_t \Big(u_4(t)^\top u_4(t)\Big) + -(\lambda_1^\top + \lambda_2^\top + \lambda_3^\top + \lambda_4^\top)\, x_f \\
& + \lambda_1^\top\, x_1(T_{\text{final}}) + \lambda_2^\top\, x_2(T_{\text{final}}) + \lambda_3^\top\, x_3(T_{\text{final}}) + \lambda_4^\top\, x_4(T_{\text{final}}) \\
& + \sum_t \Big(x_1(t)^\top x_1(t)\Big) + \sum_t \Big(x_2(t)^\top x_2(t)\Big) + \sum_t \Big(x_3(t)^\top x_3(t)\Big) + \sum_t \Big(x_4(t)^\top x_4(t)\Big)
\end{aligned} \tag{18}$$

This function can be separated in the following way:

$$
\begin{aligned}
d(\lambda_i) = &\inf_{u_1, x_f} \left[\sum_t \Big(u_1(t)^\top u_1(t) \Big) + \sum_t \Big(x_1(t)^\top x_1(t) \Big) - \lambda_1^\top x_f + \lambda_1^\top x_1(T_{\text{final}}) \right] \\
&+ \inf_{u_2, x_f} \left[\sum_t \Big(u_2(t)^\top u_2(t) \Big) + \sum_t \Big(x_2(t)^\top x_2(t) \Big) - \lambda_2^\top x_f + \lambda_1^\top x_1(T_{\text{final}}) \right] \\
&+ \inf_{u_3, x_f} \left[\sum_t \Big(u_3(t)^\top u_3(t) \Big) + \sum_t \Big(x_3(t)^\top x_3(t) \Big) - \lambda_3^\top x_f + \lambda_1^\top x_1(T_{\text{final}}) \right] \\
&+ \inf_{u_4, x_f} \left[\sum_t \Big(u_4(t)^\top u_4(t) \Big) + \sum_t \Big(x_4(t)^\top x_4(t) \Big) - \lambda_4^\top x_f + \lambda_1^\top x_1(T_{\text{final}}) \right]
\end{aligned}
\tag{19}
$$

Each of the four lower-level subproblems is then described by:

$$
\begin{aligned}
\min_{u_i, x_f} \quad & \sum_t \Big(x_i(t)^\top x_i(t) + u_i(t)^\top u_i(t) \Big) + \lambda_i^\top \left(x_i(T_{\text{final}}) - x_f \right) \\
\text{subject to} \quad & x_i(t+1) = A_i \, x_i(t) + B_i \, u_i(t) \\
& \sum_t u_i(t)^\top u_i(t) \leq u_{\max}^2 \\
& u_i \in \mathbb{R}^2 \\
& x_i, \, x_f \in \mathbb{R}^4 \\
& i = 1, \, 2, \, 3, \, 4 \\
& t = 0, \, 1, \, 2, \, 3, \, 4
\end{aligned}
\tag{20}
$$

Since the dual problem acts as a lower bound to the solution of the primal problem as defined in Equation (15), it should be maximised. Maximisation ensures that the solution converges to the solution of the primal problem. Thus, the minimisation problem, as posed in Equation (20), should be maximised with λ_i as the decision variable. The algorithm for maximising the dual function is as follows [1]:

- Solve the four subproblems for the current λ_i
- Update λ_i by taking a gradient descent step

The gradient of the dual function with respect to λ_i is defined as:

$$
\frac{\partial d(\lambda_i)}{\partial \lambda_i} = \begin{bmatrix} x_i^{\text{pos}}(T_{\text{final}}) - x_f^1 \\ y_i^{\text{pos}}(T_{\text{final}}) - x_f^2 \\ x_i^{\text{vel}}(T_{\text{final}}) - x_f^3 \\ y_i^{\text{vel}}(T_{\text{final}}) - x_f^4 \end{bmatrix}
\tag{21}
$$

Of course, after optimising each subproblem, x_f will be equal to its respective $x_i(T_{\text{final}})$. However, the states of the aircraft must converge to each other. As a solution, the gradient can be calculated by comparing one aircraft's $x_i(T_{\text{final}})$ to another aircraft's $x_{j \neq i}(T_{\text{final}})$. In this way, a gradient descent step can be taken up to when the aircraft are at the same state. Thus, in the code, the following gradients are calculated:

$$\begin{aligned}
\frac{\partial d(\lambda_1)}{\partial \lambda_1} &= \begin{bmatrix} x_1^{\text{pos}}(T_{\text{final}}) - x_2^{\text{pos}}(T_{\text{final}}) \\ y_1^{\text{pos}}(T_{\text{final}}) - y_2^{\text{pos}}(T_{\text{final}}) \\ x_1^{\text{vel}}(T_{\text{final}}) - x_2^{\text{vel}}(T_{\text{final}}) \\ y_1^{\text{vel}}(T_{\text{final}}) - y_2^{\text{vel}}(T_{\text{final}}) \end{bmatrix} \\
\frac{\partial d(\lambda_2)}{\partial \lambda_2} &= \begin{bmatrix} x_2^{\text{pos}}(T_{\text{final}}) - x_3^{\text{pos}}(T_{\text{final}}) \\ y_2^{\text{pos}}(T_{\text{final}}) - y_3^{\text{pos}}(T_{\text{final}}) \\ x_2^{\text{vel}}(T_{\text{final}}) - x_3^{\text{vel}}(T_{\text{final}}) \\ y_2^{\text{vel}}(T_{\text{final}}) - y_3^{\text{vel}}(T_{\text{final}}) \end{bmatrix} \\
\frac{\partial d(\lambda_3)}{\partial \lambda_3} &= \begin{bmatrix} x_3^{\text{pos}}(T_{\text{final}}) - x_4^{\text{pos}}(T_{\text{final}}) \\ y_3^{\text{pos}}(T_{\text{final}}) - y_4^{\text{pos}}(T_{\text{final}}) \\ x_3^{\text{vel}}(T_{\text{final}}) - x_4^{\text{vel}}(T_{\text{final}}) \\ y_3^{\text{vel}}(T_{\text{final}}) - y_4^{\text{vel}}(T_{\text{final}}) \end{bmatrix} \\
\frac{\partial d(\lambda_4)}{\partial \lambda_4} &= \begin{bmatrix} x_4^{\text{pos}}(T_{\text{final}}) - x_1^{\text{pos}}(T_{\text{final}}) \\ y_4^{\text{pos}}(T_{\text{final}}) - y_1^{\text{pos}}(T_{\text{final}}) \\ x_4^{\text{vel}}(T_{\text{final}}) - x_1^{\text{vel}}(T_{\text{final}}) \\ y_4^{\text{vel}}(T_{\text{final}}) - y_1^{\text{vel}}(T_{\text{final}}) \end{bmatrix}
\end{aligned} \tag{22}$$

Unfortunately, the algorithm does not yield valid results after implementation in MATLAB. For aircraft 2 and 4, the optimisation algorithm runs into 'numerical problems' after time step 2. For the other two aircraft, this happens one timestep after that. For running the code, MATLAB R2020a and YALMIP were used. The optimisation solver has been set to GUROBI.
Due to time limitations, the error in the code has not been found. It is expected that the error lies in the updating of the Lagrange multipliers. In most literature, the procedure is done for two subproblems instead of four. Then, the gradient can be determined by comparing the two constraint variables. With four subproblems, there are six unique combinations of variables, which complicates the algorithm.

Problem 4

(a) Derive an expression for the next iteration.

The following positive definite quadratic function is given:

$$\begin{aligned} V(u_1, u_2) &= \frac{1}{2} \begin{bmatrix} u_1^\top & u_2^\top \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} c_1^\top & c_2^\top \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + d \\ &= \frac{1}{2} u_1^\top H_{11} u_1 + \frac{1}{2} u_2^\top H_{21} u_1 + \frac{1}{2} u_1^\top H_{12} u_2 + \frac{1}{2} u_2^\top H_{22} u_2 + c_1^\top u_1 + c_2^\top u_2 + d \end{aligned} \tag{23}$$

To optimise this function, the following algorithm is used:

1. Optimise $V(u_1, u_2)$ over u_1 with u_2 fixed;
2. Optimise $V(u_1, u_2)$ over u_2 with u_1 fixed;
3. Repeat.

Therefore, to compute the next iteration (u_1^{p+1}, u_2^{p+1}) of the initial point (u_1^p, u_2^p), $V(u_1, u_2)$ is optimised by setting the derivative equal to zero. As the given quadratic function $V(u)$ is positive definite, the matrix H is also positive definite. As it is contained in the definition for positive definite matrices that a positive definite matrix is symmetric, H is symmetric as well. Therefore, the following holds:

$$\begin{aligned} H_{12} &= H_{21}^\top \\ H_{21} &= H_{12}^\top \\ H_{11} &= H_{11}^\top \\ H_{22} &= H_{22}^\top \end{aligned} \tag{24}$$

First, the partial derivatives are computed with the use of the following rules for the derivation of matrices and vectors, where $A \in \mathbb{R}^{n \times n}$ is an $n \times n$ matrix and $a \in \mathbb{R}^{n \times 1}$ and $x \in \mathbb{R}^{n \times 1}$ both vectors of $n \times 1$ [2]:

$$\begin{aligned} \frac{d(x^\top A x)}{dx} &= (A + A^\top) x \\ \frac{d(x^\top A)}{dx} &= A \\ \frac{d(Ax)}{dx} &= A^\top \\ \frac{d(a^\top x)}{dx} &= a \end{aligned} \tag{25}$$

Using this we can obtain the derivatives at the initial point (u_1^p, u_2^p) and set them to zero:

$$
\begin{aligned}
\frac{\partial V(u_1,u_2)}{\partial u_1} &= \frac{1}{2}(H_{11}+H_{11}^\top)u_1^{p+1} + \frac{1}{2}H_{21}^\top u_2^p + \frac{1}{2}H_{12}u_2^p + c_1 \\
&= H_{11}u_1^{p+1} + \frac{1}{2}H_{21}^\top u_2^p + \frac{1}{2}H_{12}u_2^p + c_1 \\
&= H_{11}u_1^{p+1} + \frac{1}{2}H_{12}u_2^p + \frac{1}{2}H_{12}u_2^p + c_1 \\
&= H_{11}u_1^{p+1} + H_{12}u_2^p + c_1 = 0 \\
\frac{\partial V(u_1,u_2)}{\partial u_2} &= \frac{1}{2}(H_{22}+H_{22}^\top)u_2^{p+1} + \frac{1}{2}H_{21}u_1^p + \frac{1}{2}H_{12}^\top u_1^p + c_2 \\
&= H_{22}u_2^{p+1} + \frac{1}{2}H_{21}u_1^p + \frac{1}{2}H_{12}^\top u_1^p + c_2 \\
&= H_{22}u_2^{p+1} + \frac{1}{2}H_{21}u_1^p + \frac{1}{2}H_{21}u_1^p + c_2 \\
&= H_{22}u_2^{p+1} + H_{21}u_1^p + c_2 = 0
\end{aligned} \tag{26}
$$

By the formulation of the question, we assume that indeed the inverses of H_{11} and H_{22} exist (and therefore they both are square matrices), we can finally write:

$$
\begin{aligned}
u_1^{p+1} &= -H_{11}^{-1}\left(H_{12}u_2^p + c_1\right) \\
u_2^{p+1} &= -H_{22}^{-1}\left(H_{21}u_1^p + c_2\right) \qquad \square
\end{aligned} \tag{27}
$$

(b) Establish that the optimisation procedure converges.

The algorithm is summarised by:

$$
u^{p+1} = Au^p + b \tag{28}
$$

With iteration matrix A and constant b given by:

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & -H_{11}^{-1}H_{12} \\ -H_{22}^{-1}H_{21} & 0 \end{bmatrix} \\
b &= \begin{bmatrix} -H_{11}^{-1}c_1 \\ -H_{22}^{-1}c_2 \end{bmatrix}
\end{aligned} \tag{29}
$$

To establish that the optimisation algorithm converges, the iteration matrix A needs to be proven to be stable. The following can be proved in order to prove stability:

$$
\begin{gathered}
|\text{eig}\,(A) < 1 \\
-1 < \text{eig}(A) < 1 \\
\text{eig}(A) > -I \;\wedge\; \text{eig}(A) < I \\
\text{eig}(A) + I > 0 \;\wedge\; \text{eig}(A) - I < 0
\end{gathered} \tag{30}
$$

By the characteristics of the identity matrix, the following holds:

$$
\begin{aligned}
\text{eig}(A) + I = \text{eig}(A+I) \\
\text{eig}(A) - I = \text{eig}(A-I)
\end{aligned} \tag{31}
$$

Furthermore, it is known that for any negative definite matrix N, it holds that:

$$\mathrm{eig}(N) < 0 \tag{32}$$

And for any positive definite matrix P, it holds that:

$$\mathrm{eig}(P) > 0 \tag{33}$$

Therefore, by combining the Equations (31), (32) and (33), it remains to prove that in order for (30) to hold, it must hold that $(A - I)$ is negative definite and $(A + I)$ is positive definite. Because then, it holds that:

$$\begin{aligned} \mathrm{eig}(A - I) &< 0 \\ \mathrm{eig}(A) - I &< 0 \\ \mathrm{eig}(A) &< I \end{aligned} \tag{34}$$

And also:

$$\begin{aligned} \mathrm{eig}(A + I) &> 0 \\ \mathrm{eig}(A) + I &> 0 \\ \mathrm{eig}(A) &> I \end{aligned} \tag{35}$$

The matrices $(A - I)$ and $(A + I)$ are defined as:

$$\begin{aligned} (A - I) &= \begin{bmatrix} -I & -H_{11}^{-1}H_{12} \\ -H_{22}^{-1}H_{21} & -I \end{bmatrix} \\ (A + I) &= \begin{bmatrix} I & -H_{11}^{-1}H_{12} \\ -H_{22}^{-1}H_{21} & I \end{bmatrix} \end{aligned} \tag{36}$$

For showing that $(A + I)$ is positive definite, the Schur decomposition is obtained:

$$(A + I) = \begin{bmatrix} I & -H_{11}^{-1}H_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} I - H_{11}^{-1}H_{12}H_{22}^{-1}H_{21} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -H_{22}^{-1}H_{21} & I \end{bmatrix} \tag{37}$$

Furthermore, the Schur decomposition of H is given by:

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} I & H_{12}H_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} H_{11} - H_{12}H_{22}^{-1}H_{21} & 0 \\ 0 & H_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ H_{22}^{-1}H_{21} & 0 \end{bmatrix} \tag{38}$$

Furthermore, by the properties of the Schur complement, it holds that H is positive definite if and only if H_{22} is positive definite and the Schur complement of H_{22} is also positive definite. Thus, H_{22} is positive definite, and its Schur complement $H/H_{22} = H_{11} - H_{12}H_{22}^{-1}H_{21}$ is also positive definite. For $(A+I)$, it holds that I is positive definite, and it remains to check whether the Schur complement of I is also positive definite:

$$\begin{aligned} (A + I)/I &= I - H_{22}^{-1}H_{21}H_{11}^{-1}H_{12} \\ &= H_{11}^{-1}\underbrace{(H_{11} - H_{22}^{-1}H_{21}H_{12})}_{\text{Schur complement of H}} \end{aligned} \tag{39}$$

As H_{11} is positive definite, its inverse is also positive definite. Therefore, Eq. (39) is a multiplication of two positive definite matrices and therefore positive definite as well. Hence, it is proven that $(A + I)$ is positive definite.

To prove that $(A-I)$ is negative definite, it is possible to prove that $-(A-I)$ is positive definite.

$$-(A-I) = \begin{bmatrix} I & H_{11}^{-1}H_{12} \\ H_{22}^{-1}H_{21} & I \end{bmatrix} \tag{40}$$

Again, I is positive definite. It remains to prove that the Schur complement of I is also positive definite:

$$-(A-I)/I = I - H_{11}^{-1}H_{12}H_{22}^{-1}H_{21} \tag{41}$$

As it turns out, the Schur complement $-(A-I)/I$ is the same as the Schur complement of $(A+I)/I$ in Eq. (39). Thus, it can be concluded that the Schur complement $-(A-I)/I$ is also positive definite, and therefore $-(A-I)$ is positive definite. Hence $(A-I)$ is negative definite, and with this, it is proven that:

$$\begin{aligned} &\mathrm{eig}(A+I) > 0 \wedge \mathrm{eig}(A-I) < 0 \\ &\Rightarrow \mathrm{eig}(A) > -I \wedge \mathrm{eig}(A) < I \\ &\Rightarrow -I < \mathrm{eig}(A) < I \\ &\Rightarrow \left|\mathrm{eig}(A)\right| < I \qquad \square \end{aligned} \tag{42}$$

(c) Show the solution of the iteration convergence

Using the fact that the algorithm converges, it remains to show that the algorithm converges to:

$$u^* = -H^{-1}c \tag{43}$$

Therefore, on the solution of the convergence, it holds that:

$$u^* = Au^* + b \tag{44}$$

The following is obtained, using the Schur complement:

$$
\begin{aligned}
(I-A)^{-1}b &= \left(\begin{bmatrix} I & H_{11}^{-1}H_{12} \\ H_{22}^{-1}H_{21} & I \end{bmatrix}\right)^{-1} \\
&= \begin{bmatrix} I & -H_{11}^{-1}H_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (I-H_{22}^{-1}H_{21}H_{11}^{-1}H_{12})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -H_{22}^{-1}H_{21} & I \end{bmatrix} \begin{bmatrix} -H_{11}^{-1}c_1 \\ -H_{22}^{-1}c_2 \end{bmatrix} \\
&= \begin{bmatrix} I & -H_{11}^{-1}H_{12}(I-H_{22}^{-1}H_{21}H_{11}^{-1}H_{12})^{-1} \\ 0 & (I-H_{22}^{-1}H_{21}H_{11}^{-1}H_{12})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -H_{22}^{-1}H_{21} & I \end{bmatrix} \begin{bmatrix} -H_{11}^{-1}c_1 \\ -H_{22}^{-1}c_2 \end{bmatrix} \\
&= \begin{bmatrix} I + H_{11}^{-1}H_{12}(I-H_{22}^{-1}H_{21}H_{11}^{-1}H_{12})^{-1}H_{22}^{-1}H_{21} & -H_{11}^{-1}H_{12}(I-H_{22}^{-1}H_{21}H_{11}^{-1}H_{12})^{-1} \\ -(I-H_{22}^{-1}H_{21}H_{11}^{-1}H_{12})^{-1}H_{22}^{-1}H_{21} & (I-H_{22}^{-1}H_{21}H_{11}^{-1}H_{12})^{-1} \end{bmatrix} \\
&\qquad \cdot \begin{bmatrix} -H_{11}^{-1}c_1 \\ -H_{22}^{-1}c_2 \end{bmatrix} \\
&= \begin{bmatrix} I + H_{11}^{-1}H_{12}(H_{22}-H_{21}H_{11}^{-1}H_{12})^{-1}H_{21} & -H_{11}^{-1}H_{12}(I-H_{22}^{-1}H_{21}H_{11}^{-1}H_{12})^{-1} \\ -(H_{22}-H_{21}H_{11}^{-1}H_{12})^{-1}H_{21} & (I-H_{22}^{-1}H_{21}H_{11}^{-1}H_{12})^{-1} \end{bmatrix} \\
&\qquad \cdot \begin{bmatrix} -H_{11}^{-1}c_1 \\ -H_{22}^{-1}c_2 \end{bmatrix} \\
&= \begin{bmatrix} H_{11}^{-1} + H_{11}^{-1}H_{12}(H_{22}-H_{21}H_{11}^{-1}H_{12})^{-1}H_{21}H_{11}^{-1} & -H_{11}^{-1}H_{12}(H_{22}-H_{21}H_{11}^{-1}H_{12})^{-1} \\ -(H_{22}-H_{21}H_{11}^{-1}H_{12})^{-1}H_{21}H_{11}^{-1} & (H_{22}-H_{21}H_{11}^{-1}H_{12})^{-1} \end{bmatrix} \\
&\qquad \cdot \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}
\end{aligned} \tag{45}
$$

And since the inverse of H is given by the Schur complement as:

$$
\begin{aligned}
H &= \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} I & 0 \\ -H_{22}^{-1}H_{21} & I \end{bmatrix} \begin{bmatrix} (H_{11}-H_{12}H_{22}^{-1}H_{21})^{-1} & 0 \\ 0 & H_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & -H_{12}H_{22}^{-1} \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} H_{11}^{-1} + H_{11}^{-1}H_{12}(H_{22}-H_{21}H_{11}^{-1}H_{12})^{-1}H_{21}H_{11}^{-1} & -H_{11}^{-1}H_{12}(H_{22}-H_{21}H_{11}^{-1}H_{12})^{-1} \\ -(H_{22}-H_{21}H_{11}^{-1}H_{12})^{-1}H_{21}H_{11}^{-1} & (H_{22}-H_{21}H_{11}^{-1}H_{12})^{-1} \end{bmatrix}
\end{aligned} \tag{46}
$$

And it can be observed that the inverse of H is the same as the left matrix in the final line in Eq. (46).
Thus indeed we have:

$$
u^* = (I-A)^{-1}b = -H^{-1}c \qquad \square \tag{47}
$$

Problem 5

(a) Prove that the cost function is monotonically decreasing.

When introducing a coordinate change $u^p = u^s + u^* = u^s - H^{-1}c$, the cost function $V(u^s)$ can be written as:

$$\begin{aligned} V(u^s) &= \frac{1}{2}\,(u^s - H^{-1}c)^\top H\,(u^s - H^{-1}c) + c^\top (u^s - H^{-1}c) + d \\ &= \frac{1}{2}\,(u^s)^\top H u^s - \frac{1}{2}\,c^\top H^{-\top}\,H\,u^s - \frac{1}{2}(u^s)^\top c + \frac{1}{2}\,c^\top H^{-\top}\,c + c^\top u^s - c^\top H^{-1}\,c + d \end{aligned} \tag{48}$$

Given that H is positive definite and thus symmetric, the cost function simplifies to:

$$\begin{aligned} V(u^s) &= \frac{1}{2}\,(u^s)^\top H u^s - \frac{1}{2}\,c^\top u^s - \frac{1}{2}(u^s)^\top c + \frac{1}{2}\,c^\top H^{-1}\,c + c^\top u^s - c^\top H^{-1}\,c + d \\ &= \frac{1}{2}\,(u^s)^\top H\,u^s - \frac{1}{2}\,c^\top H^{-1}\,c + d \end{aligned} \tag{49}$$

To investigate whether the cost function is monotonically decreasing, it is useful to look at the cost function in the next timestep. If $u^{p+1} = u^{s+1} + u^*$, then:

$$\begin{aligned} u^{s+1} + u^* &= A\,u^p + b \\ u^{s+1} &= A\,u^p + b - u^* \\ &= A\,(u^s + u^*) + \begin{bmatrix} -H_{11}^{-1} c_1 \\ -H_{22}^{-1} c_2 \end{bmatrix} - u^* \\ &= A\,u^s + (A - I)\,u^* + \begin{bmatrix} -H_{11}^{-1} c_1 \\ -H_{22}^{-1} c_2 \end{bmatrix} \end{aligned} \tag{50}$$

c_1 and c_2 can be substituted as:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -H\,u^* = -\begin{bmatrix} H_{11}\,u_1^* + H_{12}\,u_2^* \\ H_{21}\,u_1^* + H_{22}\,u_2^* \end{bmatrix} \tag{51}$$

Then, u^{s+1} is:

$$\begin{aligned} u^{s+1} &= A\,u^s + (A - I)\,u^* + \begin{bmatrix} H_{11}^{-1}(H_{11}\,u_1^* + H_{12}\,u_2^*) \\ H_{22}^{-1}(H_{21}\,u_1^* + H_{22}\,u_2^*) \end{bmatrix} \\ &= A\,u^s + (A - I)\,u^* + \begin{bmatrix} u_1^* + H_{11}^{-1} H_{12}\,u_2^* \\ H_{22}^{-1} H_{21}\,u_1^* + u_2^* \end{bmatrix} \\ &= A\,u^s + (A - I)\,u^* + (A + I)\,u^* \\ &= A\,u^s \end{aligned} \tag{52}$$

The cost function at the next timestep, $V(u^{s+1})$, is then equal to:

$$\begin{aligned} V(u^{s+1}) &= \frac{1}{2}\,(A\,u^s)^\top H\,A\,u^s - \frac{1}{2}\,c^\top H^{-1}\,c + d \\ &= \frac{1}{2}\,(u^s)^\top A^\top H\,A\,u^s - \frac{1}{2} c^\top H^{-1}\,c + d \end{aligned} \tag{53}$$

The difference between the cost function at step $s+1$ and at step s is given by:

$$V(u^{s+1}) - V(u^s) = \frac{1}{2}\,(u^s)^\top A^\top H\,A\,u^s - \frac{1}{2}\,(u^s)^\top H\,u^s \tag{54}$$

Since H is positive definite, $-\frac{1}{2}\,(u^s)^\top H\,u^s$ is negative for $u^s \neq 0$, and thus for $u^p \neq -H^{-1}\,c$. The first term, $\frac{1}{2}\,(u^s)^\top A^\top H\,A\,u^s$, is always smaller than the second one because the eigenvalues of A are strictly smaller than 1 in absolute value. Therefore, $V(u^{p+1}) - V(u^p)$ is always negative for $u^p \neq -H^{-1}\,c$ and the cost function $V(u^p)$ is monotonically decreasing. $\square$

(b) Show the size of the decrease

As shown in question (a), the decrease is given by:

$$\begin{aligned} V(u^{s+1}) - V(u^s) &= \frac{1}{2}\,(u^s)^\top A^\top H\,A\,u^s - \frac{1}{2}\,(u^s)^\top H\,u^s \\ &= \frac{1}{2}\,(u^s)^\top \left(A^\top H\,A - H\right)\,u^s \\ &= \frac{1}{2}\,(u^p - u^*)^\top \left(A^\top H\,A - H\right)\,(u^p - u^*) \\ &= -\frac{1}{2}\,(u^p - u^*)^\top \left(H - A^\top H\,A\right)\,(u^p - u^*) \end{aligned} \tag{55}$$

For proving that $H - A^\top H\,A$ is equal to P, it is useful to expand P:

$$\begin{aligned} P &= H\,D^{-1}\,(D-N)\,D^{-1}\,H \\ &= \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \left(\begin{bmatrix} H_{11}^{-1} & 0 \\ 0 & H_{22}^{-1} \end{bmatrix} \begin{bmatrix} H_{11} & -H_{12} \\ -H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} H_{11}^{-1} & 0 \\ 0 & H_{22}^{-1} \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \right) \\ &= \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \left(\begin{bmatrix} H_{11}^{-1} & 0 \\ 0 & H_{22}^{-1} \end{bmatrix} \begin{bmatrix} H_{11} & -H_{12} \\ -H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} I & H_{11}^{-1}\,H_{12} \\ H_{22}^{-1}\,H_{21} & I \end{bmatrix} \right) \\ &= \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \left(\begin{bmatrix} I & -H_{11}^{-1}\,H_{12} \\ -H_{22}^{-1}\,H_{21} & I \end{bmatrix} \begin{bmatrix} I & H_{11}^{-1}\,H_{12} \\ H_{22}^{-1}\,H_{21} & I \end{bmatrix} \right) \\ &= \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \left(\begin{bmatrix} I - H_{11}^{-1}\,H_{12}H_{22}^{-1}\,H_{21} & 0 \\ 0 & I - H_{22}^{-1}\,H_{21}\,H_{11}^{-1}\,H_{12} \end{bmatrix} \right) \\ &= \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \left(I - \underbrace{\begin{bmatrix} H_{11}^{-1}\,H_{12}H_{22}^{-1}\,H_{21} & 0 \\ 0 & H_{22}^{-1}\,H_{21}\,H_{11}^{-1}\,H_{12} \end{bmatrix}}_{X} \right) \\ &= H - H\,X \end{aligned} \tag{56}$$

Now, it is necessary to prove that $H\,X = A^\top H\,A$ to show the equivalence between P and $H -$

$A^\top H\, A$:

$$
\begin{aligned}
A^\top H\, A &= \begin{bmatrix} 0 & -H_{11}^{-1} H_{12} \\ -H_{22}^{-1} H_{21} & 0 \end{bmatrix}^\top \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} 0 & -H_{11}^{-1} H_{12} \\ -H_{22}^{-1} H_{21} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -H_{21}^\top H_{22}^{-1} \\ -H_{12}^\top H_{11}^{-1} & 0 \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} 0 & -H_{11}^{-1} H_{12} \\ -H_{22}^{-1} H_{21} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -H_{12}\, H_{22}^{-1} \\ -H_{21}\, H_{11}^{-1} & 0 \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} 0 & -H_{11}^{-1} H_{12} \\ -H_{22}^{-1} H_{21} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -H_{12}\, H_{22}^{-1} \\ -H_{21}\, H_{11}^{-1} & 0 \end{bmatrix} \begin{bmatrix} -H_{12}\, H_{22}^{-1}\, H_{21} & -H_{12} \\ -H_{21} & -H_{21}\, H_{11}^{-1}\, H_{12} \end{bmatrix} \\
&= \begin{bmatrix} H_{12}\, H_{22}^{-1}\, H_{21} & H_{12}\, H_{22}^{-1}\, H_{21}\, H_{11}^{-1}\, H_{12} \\ H_{21}\, H_{11}^{-1}\, H_{12}\, H_{22}^{-1}\, H_{21} & H_{21}\, H_{11}^{-1}\, H_{12} \end{bmatrix} \\
&= \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} H_{11}^{-1}\, H_{12} H_{22}^{-1}\, H_{21} & 0 \\ 0 & H_{22}^{-1}\, H_{21}\, H_{11}^{-1}\, H_{12} \end{bmatrix} \\
&= H\, X
\end{aligned} \tag{57}
$$

Thus, $H - A^\top H\, A = P$. Therefore, the size of the decrease of the cost function, derived from Eq. (55) is given by:

$$
V(u^{s+1}) - V(u^s) = -\frac{1}{2}\,(u^p - u^*)^\top P\,(u^p - u^*) \tag{58}
$$

From Eq. 54, the decrease is nonzero and positive in size only when $u^s \neq 0$. Since u^s was defined to be equal to $u^p + H^{-1}\, c$ in question (a), the decrease is nonzero for $u^p \neq -H^{-1}\, C$. Due to the size of the decrease being always nonnegative, the optimum must be at $u^p = -H^{-1}\, c = u^*$. $\square$

References

[1] Y. Li. A tutorial on dual decomposition.

[2] M. S. P. K. B. Petersen. The matrix cookbook.