Approximation Methods

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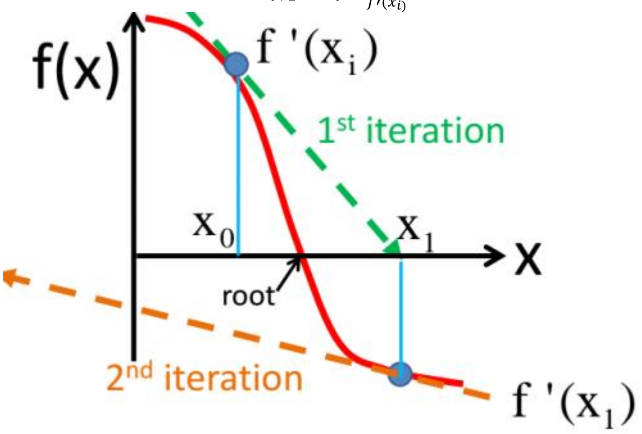
Linear Approximation

- Near the point x=a. The function looks a lot like its tangent line. We can use this to create a linear approximation to describe the behavior near a.
  - The equation is $L_a(x)=f'(x-a)a+f(a)$.
  - f'(a) describes the slope of the line. x-a describes the distance that the slope should travel from a and f(a) indicates our starting point

Finding Roots

- Bisection Method
  - 3 Step Method: Take a lower and upper bound. The lower bound ($f(x_{low})$) must be less than 0 and the upper bound ($f(x_{high})$) must be greater than 0.
  - The average of these two points is taken. If the $f(x_{avg})$ is greater than 0 assign it to be the upper bound, if it is less than 0 it is the lower bound. If it is 0, we have found a root.
  - Keep repeating the above process
- Newton-Raphson Method is a faster technique but there is a risk of divergence(increase in error and getting further away from root)
  - Find the tangent/linear approximation and x_i(initial guess). The intercept of the tangent line. This is the next iteration. Repeat this process until it is completed.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



- The answer can diverge if f'(x) does not exist or if f'(x) or f''(x) =0 or if the initial guess is "bad"
- We can use the bisection method to first get a good estimate THEN apply newton raphson.

Fixed Point Iteration

- This works for equations of the form f(x)=x (We can rearrange to create this equation). Since the result of f(x) should be close to x. We keep applying f(x), until we converge to an answer.
  - f(x)-> f(f(x))-> f(f(f(x)))...
  - This will work if |f'(x)|<1
- For this method there are lots of x's in the equation and sometimes choosing another x will let the it converge

Common Taylor Series

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$f(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$f(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^{2n} x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{6!} + \cdots$$

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

Approximating Integrals

Integrate e^{t^2} between 0 and 2 using a seventh order polynomial

$h(x)=e(x)= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \ -\infty < x < \infty$

Let f(x) be h(t²)

$$\int_0^2 1 + t^2 + \frac{t^4}{6} + \frac{t^6}{42} + R_n(t^2) = x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \int_0^x R(t^2)$$

Let's estimate the error : Since we used a 3rd order polynomial of our original

Polynomial Interpolations

- Consider 4 points $(0,y_0)(h,y_1)(2h,y_2)(3h,y_3)$
- We want to build a polynomial $a+bx+cx^2+dx^3$.
  - We know that $y_0=a$, $y_1=a+hb+h^2c+h^3d$. $y_2=a+2hb+4h^2c+8h^3d$, $y_4=a+3hb+9h^2c+27h^3d$
- We need to define ordered differences
  - First ordered differences are generally shown as $\Delta y_n = \frac{y_{n+1}-y_n}{h}$
  - The second order differences take the results of the first order differences and applies the same formula $\Delta^2 y_n = \frac{\Delta y_{n+1} - \Delta y_n}{h}$
- We can use these divided differences to help us solve the equation of the polynomial. In the nth order coefficient (d is the 3rd order polynomial).
  $\frac{\Delta^n y(x-x_0)(x-x_1)\ldots(x-x_n)}{n!\,h^n}$
  Note h, is the interval size from above.
- We can use this to find the coefficients to deteremine the equation of the polynomial.
  This is only true near the points, the function may diverge elsewhere.

Taylor Series

- We can use the above polynomial interpolation to begin building a taylor series. Since h is the difference of x and the ordered difference is the difference is y. We can rewrite this as a derivative as h approaches 0.
  Taylor Series: $\sum \frac{f^n(x)}{n!}(x-x_0)^n$
- Maclaurins Polynomial is a Taylor polynomial centered at 0 or $x_0=0$.
- When we say we take the nth order taylor polynomial, we expand our summation notation up to n terms.

Remainder Theorem

- When we make a taylor approximation. We want to know how good is our approximation.
- First we need to recall some Triangle Inequalities: We can extend this to integration: $\int f(x) \leq \int |f(x)|$
- We can say that the remainder is equal to $\int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{n+1}(n)$, but $f^{n+1}(t)$ is challenging to find $f^{n+1}(t)$.
  - In order to be safe we can approximate the value k, which is the maximum over the range x_0 and x.
- We say that the remainder for the nth order polynomial is equal to $|k(x-x_0)^{n+1}/(n+1)!|$, where k is the maximum value of f^{n+1} over the range x_0 to x.
  - K can be greater than the maximum value as long as it is greater than any value in the range of integration.
  - When in doubt you can absolute value EVERYTHING! To make things easy.
- We can do the reverse as well as long as we can create a generalization for the derivative. Say we want to know how many terms to take for an error of 0.001. We create a generalized formula for the remainder and set that equal to less than or equal to 0.001.
- Then we solve the equation for n. Often times we will need to guess and check because of factorial makes it difficult to solve exactly.
- It makes sense for more terms in a Polynomial to make it better, but will we get an exact answer using an infinite series.
  - An infinite number of terms in a Taylor Polynomial is when its actually called a series.
  - We will get a max error of infinity, but that does not mean the error is always infinity. So HOW do we determine it? Convergence Tests in the next chapter.
- To determine taylor series, we can sub into our common taylor series most of the time or take the derivative or integral of one of the common taylor series.

Binomial Taylor Expansion

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \qquad (1)$$
$$= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots,$$

Approximating Integrals using Taylor Series 2

Approximate the integral between 0 and 1/2 with a precision of 0.01.
The first thing we need to do is find an expression for remainder .
$f(x)=1/(1+x^4)=1/(1-(-x^4)=1-x^4+x^8\ldots R_n(-x^4)$. We'll need to set up a dummy variable since our integral will be between 0 and x. So we rewrite this as.
$1-t^4+t^8\ldots R_n(-t^4)$.
We know that we are going to integrate this so we'll have an integral of the remainder.

$$\int^x R_n(-t^4) = \int^x \left| \frac{k}{\ldots}((-t^4)-0)^{n+1} \right|$$

$$\int_0^2 1+t^2+\frac{t^4}{6}+\frac{t^6}{42}+R_n(t^2)=x+\frac{x^3}{3}+\frac{x^5}{10}+\frac{x^7}{42}+\int_0^x R(t^2)$$

Let's estimate the error : Since we used a 3rd order polynomial of our original expansion we need to find the greatest value for the 4th order derivative of e^x, which is e^x. The range is between 0 and 2, but since it is t^2, the bound for this is between 0 and 4. The max value would be e^4.

Therefore the remainder is

$$\int_0^x \frac{e^4}{6}(t^2)^3=\frac{e^4}{6(7)}x^7|\begin{matrix}2\\0\end{matrix}=\frac{128e^4}{42}$$

This is the maximum error for this integration (which is actually quite big.)

If we sub in 2 to the original series , we get the integral is equal to approx. 382/35 with a maximum error of $128e^4/42$

We know that we are going to integrate this so we'll have an integral of the remainder.

$$\int_0^x R_n(-t^4)=\int_0^x |\frac{k}{n+1!}((-t^4)-0)^{n+1}|$$

Since k is greater than the derivative, we need to create a formula for the derivative. We are tempted to use the original equation, but since we used the taylor expansion for 1/1-x. We find a k value that is greater than the nth derivative of that!

The general derivative is equal to $n+1!/(1-u)^{n+1}$. We know that u is between 0 and $-t^4$, since t is our dummy variable it has the same bounds as x. Therefore u is between 0 and -1/16.

We can reason that 0 is the upper bound since we want the smallest possible denominator. Therefore the value of k can be n+1!

Subbing back into the equation

$$\int_0^x |\frac{n+1!}{n+1!}((-t^4))^{n+1}|=\frac{x^{4n+5}}{4n+5}\ Sub\ in\ the\ value\ of\ x$$

$$=\frac{1}{2}. The\ remainder\ is\ equal\ to\ \frac{(\frac{1}{2})^{4n+5}}{4n+5}$$

Now we have a general expression for the remainder. We guess till we have a value for n. The first one just happens to be enough so we only need one term for this maximum error!

Convergence Tests

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Series vs Sequences

- A series is an infinite series of numbers. The sequence is the list of the partial sums.
- We say that a series converges if the limit of the partial sums approaches a certain value.
- Convergence is unaffected by removing a few finite terms. We can subtract the first few terms from a series and it will still converge.
  - This is because the series as a whole is an infinite so removing a few finite terms will not affect it.

Geometric Series Test

- Geometric Series are of the form $\sum ar^k$
  - Remember r, can be a fraction and it is fine, so negative exponents aren't an issue, in fact they're preferable!
- If |r| is greater than 1, the series diverges and goes to infinity
- If |r| is less than 1 it converges towards the value a/(1-r)
- If r=1 the test will not work and we must try another

Divergence Test

- The divergence test can only shows that a sequence diverges and it cannot be used to show a sequence converges. If $\lim_{k\to\infty} a_k \neq 0$ then the series diverges.

Integral Test

- A sequence for which the kth term can be represented by f(k) will converge if and only if the integral f(x) has a finite value.
  - The integral ranges between the first term and infinity.
- Note just because the integral equals a certain value, does not mean f(x) converges to that value.

Ex. $\frac{1}{kln(k)^2}$

$$Converges\ if\ \int_2^\infty \frac{1}{k\ln(k)^2}dk = \int_2^\infty \frac{1}{kx^2}k\,dx = \int_2^\infty \frac{1}{x^2} = -\frac{1}{x}\Big|_2^\infty = -\frac{1}{\ln 2}$$

Therefore the series converges

P-Series Test

- $1/x^p$ diverges if p is less than or equal to 1, it converges if it is greater than 1.

Comparison Test

- Suppose we take an unknown series a_k and we know a series b_k converges. If $a_k<b_k$ we know that a_k converges.
- The opposite is true if $a_k>b_k$ and b_k diverges, than a_k converges.
- We can ignore the first few terms of the series as long as an infinite number of terms are less than or greater than the converging series

Ex. Consider the series $\ln k/k^{1/2}$. We think it diverges because it is very similar to p-series but we need to prove it.
We need to show that $\ln k/k^{1/2} > 1/k^{1/2}$. We know this is true if k>3, but our series starts from 1.
This is okay because we are removing a finite terms from an infinite series

Limit Comparison Test

- If a_k is a positive series and $\lim_{k\to\infty}\frac{a_k}{b_k}$=L. Then a_k and b_k must either both converge or both diverge.

Ex. See if 2/k+√k

$$\lim_{k\to\infty}\frac{\frac{2}{k+\sqrt{k}}}{\frac{2}{k}} = \lim_{k\to\infty}\frac{k}{k+\sqrt{k}} = 1$$

Since 2/k diverges than 2/k+√k must also diverge.

Alternating Series Test

- Let a^k be positive and consider the series $\sum(-1)^k a_k$ .We say this series is alternating because the sign is alternating. We say this series converges if a_k approaches 0 as k approaches infinity and the series is decreasing
- We can determine if the series is decreasing by checking if the derivative is less than 0 in the interval we are converging in.

Ex. Consider the series $(-1)^k 1/k$
We know that 1/k goes to 0 as k approaches infinity. When we take the derivative it is equal to $-k^{-2}$. This is always less than 0 so it decreasing over the whole interval.
Therefore this series converges

- The alternating series has advantages to finding error. Since it alternates up from positive to negative. The nth degree polynomial for an alternating series has a maximum error equal to the n+1th term of the series.

Absolute and Conditional Convergence

- We say that a series is absolutely convergent if $|a_k|$ converges. If this is not the case but a_k converges. We call that conditional convergence.
- We cannot rearrange the terms of a conditionally convergent series. This messes up the series. It is okay to rearrange terms in an absolutely convergent series.

Consider for a moment that we break an alternating series into a positive series and a negative series and we have a target number a. If we can permute, we can add a certain number of positive terms, till our series is greater than a, then we add the first few terms from the negative series till we are below a. Then we continue adding terms from the positive series till we are above a. We repeat this process.
Since these are infinite series we could do this forever and say that the alternating series is equal to any number we choose.
Therefore permuting alternating series is not allowed!

Ratio Test

- Consider a series a_k suppose that $\lim_{x\to\infty}\left|\frac{a_{x+1}}{a_x}\right| = L$. If L > 1 then the series diverges. If L<1 then the series converges absolutely. If L=1, then this test does not work and we need to use another test.
- Remember to use L'hopital rule to help solve these limits!

Power Series & The Big O Notation

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Power of Series & Radius of Convergence

- A power series is a series of the form $\sum c_k\,(x-x_0)^k$. The Taylor series is a power series. Depending on the value of x, this series will converge.
- We can do a ratio test on this
  $\lim_{k\to\infty}\frac{c_{k+1}(x-x_0)^k}{c_k(x-x_0)^k} = |x-x_0|\lim_{k\to\infty}\frac{c_{k+1}}{c_k} = L$
- Since we want L to be less than 1. We need x-x_0 to be less than or equal to c_k/c_{k+1}. Therefore this value is known as the radius of convergence
  - But what happens when x-x_0 is equal to R exactly. We would need to check ourselves using convergence tests to see if the limits work
- If R=0, it only converges at x_0, If R is infinity, the series converges for all values, if R is a finite number, then it converges in the interval x-R,x+R. The ends once again can only be checked manually.
- If we take the derivative, integrate, multiply by a constant. The radius of convergence is preserved
- If we add or multiply two series together, we will get a radius of convergence that is the lesser of the twos series.
  - The boundaries convergences may change when we do the above operations so check them again!

Example of Solving Limits using Taylor Series

$$\lim_{x\to0}\frac{x\cos(x)-\sin(x)}{x^3} = \lim_{x\to0}\frac{x(1-\frac{x^2}{2}+O(x^4)-(x-\frac{x^3}{6}+O(x^5)}{x^3}$$

$$= \lim_{x\to0}\frac{\frac{-x^3}{3}+O(x^5)}{x^3} = -\frac{1}{3}+\lim_{x\to0}O(x^2) = -\frac{1}{3}$$

The Big O Notation

- Consider 2 functions f and g. We say that f is of order g as $x\to x_0$ and f(x)=O(g(x)) as $x\to x_0$ if there is a positive constant c for which $|f(x)|<c|g(x)|$ for every x in an interval around x_0
- Ex. Consider the interval [-1 1]. $X^3 < X^2$ So we say that $x^3 = O(x^2)$ for x->0
- We can use this to represent Taylor's Inequality. Recall that k/n+1! is a constant and $(x-x_0)^{n+1}$ will be of order x^n. Therefore we can represent the remainder for an nth order taylor polynomial with $O(x^{n+1})$
- Operations with O()
  - In general $O(x^k)+O(x^l)$ where k<l can be represented by $O(x^k)$
  - $O(x^m)O(x^n)=O(x^{m+n})$
  - $O(x^m)^n)=O(x^{mn})$
  - $O(x^m)\,x^n=O(x^{m+n})$
  - $O(x^m)/O(x^n)$ IS UNDETERMINED
  - $O(x^m)/x^n=O(x^{m-n})$

Operations with Taylor Series

- Using this notation we can conveniently operate on taylor series and keep track of the error.
- Addition and multiplication work as expected, similar to polynomials. The O just keeps us keep track of the size of the remainder
  - As a side note if $O(x^m) + x^n$ and m < n. We can represent this as a whole as $O(x^m)$

Ex. Consider cos(x)/1+x^2

Taylor series for cos (x) we'll use $1-\frac{x^2}{2}+O(x^4)$

And 1/1+x^2 Taylor series is equal to 1-x^2+$O(x^4)$

When we multiply these two series we get

$$\frac{\cos(x)}{1+x^2} = 1-\frac{3x^2}{2}+\frac{x^4}{2}+O(x^4)-\frac{x^2}{2}O(x^4)+O(x^4)^2+O(x^4)-x^2O(x^4)$$

$$= 1-\frac{3x^2}{2}+\frac{x^4}{2}+O(x^4)-O(x^6)+O(x^8)+O(x^4)-O(x^6)$$

$$= 1-\frac{3x^2}{2}+O(x^4)$$

Note. Since our remainder is of order 4 there is no point to have $x^4/2$.

- Division is a little more complicated. We must do long division to determining the quotient.

Ex. Tan(x)=$\frac{\sin(x)}{\cos(x)} = \frac{x-\frac{x^3}{6}+\frac{x^5}{120}+O(x^7)}{1-\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)}$ AFTER SOME LONG DIVISION!

$$x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{O(x^7)}{\cos(x)} = x+\frac{1}{3}x^3+\frac{2}{15}+O(x^7)$$

We do not consider the cos(x) since it approaches 1 when we are near 0, which is wear this Maclurin Polynomial is valid.

- We can solve limits using this method by representing complex functions like e, sin, cos etc… with taylor series.

Multivariable Calculus

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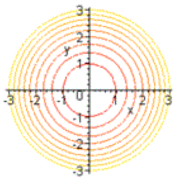
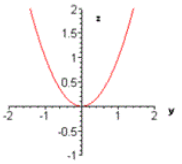
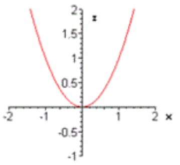
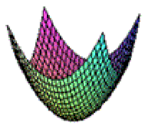
Multivariable Functions

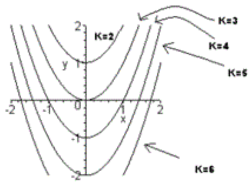
- Functions can have a domain that is greater than one dimension. We can describe these functions as z=f(x,y,)
- To sketch these functions we use the cross sections x. We graph 2D plots of (x,y) (y,z) and (x,z). Setting the third variable constant.
- This gives us cross sections of our graph and by plotting each of these shapes on a 3D plot we can sketch 3D graphs
- Rather than plotting 3D plots, we can plot level curves. These are a series of functions where z or the value of the function is constant across each curve.
- Two types of functions to know:
  - Paraboloid: $z=x^2+y^2$

3D Plot



Level Curve



  - Cone: $z=\sqrt{x^2+y^2}$
  - Hyperbola $x^2-y^2=c$, where c is a constant. This is no a 3D graph, but it is useful to remember for cross sections.

Vector Functions

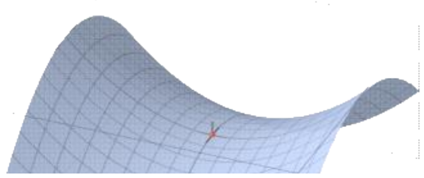
- Parametric Representation of a function is using a vector to represent each variable, usually with respect to a single variable. This is the parameter for the function
  Ex. Consider the unit circle: We can parameterize a variable t ranging between 0 and 2π. x(t)=cos(t) and y(t)=sin(t)

  We can consider this to be a position vector and t to be the unit time. Consider if we set x(t)=cos(2t) for t between 0 andπ. The function is the same. If we treat this as a position vector, we see the difference is that the "velocity" or first derivative is twice as fast.

- Vector Functions are a vector where the coordinates are based on functions.
  $$n(t)=\begin{pmatrix} x(t)=\cos(t) \\ y(t)=\sin(t) \end{pmatrix}$$
- If we have a vector function n'(t) is equal to the velocity and we get the direction as well. The second derivative is acceleration.
  Ex. Consider an ellipse which ranges from -2<x<2 and -3<y<3. We suspect that the parameterization is:
  $$n(t)=\begin{pmatrix} 2\cos(t) \\ 3\sin(t) \end{pmatrix}$$
  We can check this suspicion by subbing the parameterization into the function of the ellipse. If this is true then the function should hold.
  $$\frac{(2\cos(t))^2}{4}+\frac{(3\sin(t))^2}{9}=1.$$
  We can show that this is true and then these parameters are applicable.
- We can parameterize even a basic function consider y=2x. We can parameterize it as x(t)=t and y(t)=2t

Multivariable Optimization

- Unconstrained Optimization is when we are trying to find the maxes and minimums across the whole domain.
- Similar to one variable, we determine critical points and the critical points are defined as when $\nabla f(,x,y)=0$
  - These critical points may be maxes, minimums or saddle points.
  - Saddle points are neither maxes or mins, but they are cool to look at!



Limits

- Limits in multiple dimensions are significantly harder because we can approach from a variety of directions.
- Consider we can approach from the x-axis, the y-axis, a line of slope 1, a parabola etc. Even if in one direction the limit does not hold, the limit does not exist. We won't be expected to prove limits but only have to disprove it.

Ex. Evaluate $\lim_{(x,y)\to(0,0)}\frac{2x^2y}{x^4+y^2}$

Test out each possible direction till we find an anomaly

$$\lim_{(x,0)}\frac{2x^2y}{x^4+y^2}=\frac{0}{y^2}=0 \qquad \lim_{(0,y)}\frac{2x^2y}{x^4+y^2}=\frac{0}{x^4}=0$$
$$\lim_{(x,x)}\frac{2x^2y}{x^4+y^2}=\frac{2x^3}{x^4+x^2}=0 \qquad \lim_{(x,x^2)}\frac{2x^2x^2}{x^4+y^2}=\lim_{(x,0)}\frac{2x^2x^2}{x^4+y^2}=\frac{2x^4}{x^4+x^4}=1$$

Therefore the limit DNE.

Partial Derivatives

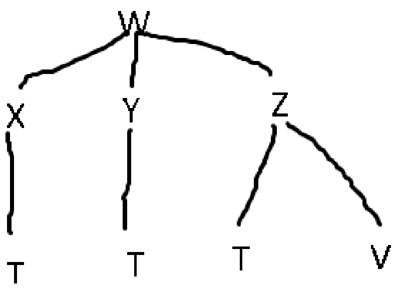
- Partial derivatives are for multi variable functions. We take a derivative with respect to a function and treat all variables as constants.
- Ex. Partial Derivative with respect to X of Y/X is equal to $-Y/X^2$
- Just like how we can take the second derivative of a function, we can also take the second partial derivative.
  - We can take the derivative with respect to x twice (xx) or y twice (yy), but it is also possible to take it to x then y or y then x.
    - We see that in general $f_{xy}=f_{yx}$ and the order is interchangeable.
    - This can be used to simplify derivatives. We hope that by taking the derivative with respect to a variable several times first, we bring the derivative to 0.

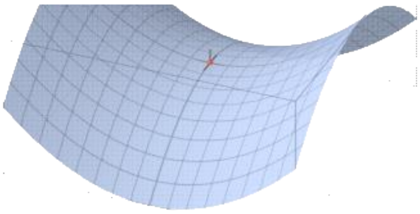
Taylor Series in Multiple Variables

- Taylor series can be expanded to multiple variables. Consider a situation where y is fixed centered around (a,b).
  $$f(x,y)=f(a,y)+\frac{\partial f}{\partial x}(a,y)(x-a)+\frac{\partial^2 f}{\partial x^2}(a,y)(x-a)^2\ldots$$
- We no can unfix y, and expand each term into its own Taylor series
  $$f(x,y)=f(a,b)+\frac{\partial f}{\partial y}(a,b)(y-b)+\frac{1}{2}\frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2+\cdots$$
  $$\frac{\partial f}{\partial x}(a,y)(x-a)$$
  $$=\frac{\partial f}{\partial x}(a,y)(x-a)+\frac{\partial^2 f}{\partial y\partial x}(a,b)(y-b)(x-a)$$
  $$+\frac{1}{2}\frac{\partial^3 f}{\partial y^2\partial x}(a,b)(y-b)^2(x-a)+\cdots$$
- We can rearrange all the Taylor series into an easier form.
  $$f(x,y)$$
  $$=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)+\frac{1}{2}(f_{xx}(a,b)(x-a)^2$$
  $$+f_{xy}(a,b)(x-a)(y-b)+f_{yy}(a,b)(y-b)^2$$
  $$+\cdots\frac{1}{n!}\sum_{i=1}^{n} nCi\frac{\partial^n f}{\partial x^i\partial y^{n-i}}(a,b)(x-a)^i(y-b)^{n-i}$$
- Alternatively we can create a dummy variable that reduces x and y to a single variable. Let u=x+y for the function sin(x+y).
- We can create a taylor series for the variable u, then sub (x+y) back into the series.
- Tangent Plane is similar to the linear approximation and can be seen in the taylor series. We get a simple planar approximation for f(x,y) centered at (a,b) with:
  $f_x(a,b)(x-a)+f_y(a,b)(y-b)$
- The error for these get smaller as we approach (a,b)

Chain Rule for Multiple Variables

- Consider a function w=f(x,y,z), where x(t), y(t) and z(t,v)
- The first thing we should draw is a tree diagram, where the top function is the function we are taking the derivative of. We then draw branches to all the variables that change z. (In this case (x, y and z). Then we branch x,y and z, into the variables that affect them. So we branch x and y down to t and we branch z into t and v.





- For maximums in all directions the function is sloping downwards. For minimums in all directions the function is sloping upwards.

- We can determine if critical points are maximum, minimums or saddle points by applying the second derivative test.
  $D=F_{xx}(a,b)F_{yy}(a,b)-(f_{xy}(a,b))^2$
  - If D<0 we have a saddle point
  - If D<0 and $f_{xx}(a,b)$ or $f_{yy}(a,b)$ < 0 then it is a max.
  - If D<0 and$f_{xx}(a,b)$ or $f_{yy}(a,b)$ >0 then it is a minimum.
- Constrained Optimization
  - If we need to limit the domain, we also need to test the boundaries. In one variable it was easy because these we the end points. In multiple variables we have functions instead.
- We know that if a maximum or minimum sits on a boundary it must be tangent to the level curves, because otherwise moving further or back along this curve we will get a higher or lower value.
- We describe our bounding function as g(x). The critical points of the curve satisfy the equation $\nabla f(x,y) = \lambda \nabla g(x,y)$
  - Where λ is the LaGrange, which is a constant value. Since these are vector equations, we can equate the x components of the gradient, the y components. But we need 3 equations to solve the system. Our third equation is g(x,y), since the points must lay on the line.
  - Results that satisfy this are critical points
    - We must plug these critical points and see which one is the greatest to determine maxes and mins.

General Method

1. If there is a boundary find the critical points on the boundary, if the boundary abruptly ends. Then the locations of those ends are also end points.
   a. If you require multiple curves to draw the boundary that is okay.
2. Perform an unconstrained optimization to determine if there is an absolute maximum or minimum within the region allow.
3. Test all of the critical points to determine the maximum or minimum.
   a. Plug them into the f(x,y) and see what gives the highest value

The Jacobian

- This the adjustment factor when we substitute our variables for integrals. If we replace x and y with u and v. The jacobian is equal to the absolute value of the determinant of the following matrix.

$$\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}$$

- A useful property is that the Jacobian is also equal to the inverse of the absolute value of.

$$\begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{matrix}$$

- We can always change variables of integration. It is challenging to find how we should find these changes.
  - A general tip is to have all the variables equaling a constant and look for 2 functions that are the same but equal 2 different constants.

Ex. Evaluate the integral for $\cos(\frac{y-x}{y+x})$

For the domain:

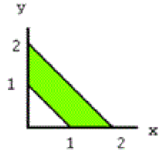
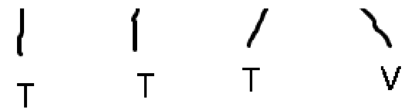


Figure 6:

- Notice how there is a y-x and y+x in the integral. These would make convenient variables. Therefore let u=x+y and v=y-x.
  - Notice how the region is bound by x=0, y=0, y=1-x and y=2-x
    - y=2-x -> x+y=2 -> u=2; 1-x -> x+y=1 -> u=1.
    - Now consider the bound for x. When x=0 v=y and u=y. Therefore v=u
    - Now consider the bound along the y axis. When y=0, We get that u=x and v=-x. Therefore u=-v.
    - Now we have developed that v sits between -u and u and u sits between 1 and 2. Now we need a Jacobian



- If we want to take the derivative of W with respect to T. Then we simply follow every branch and take a derivative. We multiply the derivative of W with respect to X and X with respect to T. That is one branch. Then we add that to the branch of Y (which would equal the derivative of Y with respect to T multiplied by the derivative of W with respect to Y). Finally add in the Z branch.
  - OR MATHEMATICALLY: $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$

Gradient Vector and Directional Derivatives

- The gradient vector for f(x,y) at the point a,b is defined as
  $\nabla f(a,b) = \begin{pmatrix} \frac{\partial f}{\partial x}(a,b) \\ \frac{\partial f}{\partial y}(a,b) \end{pmatrix}$
- Note the gradient vector produce a vector and not a scalar quantity.
- Directional derivatives are defined as the slope of the plane along the direction defined.
  - NOTE: If the direction is from the point (a,b) towards the origin. The direction is not (a,b), but (-a,-b)
- The directional derivative is equal to the dot product of the direction (c,d) and the gradient vector at the given point. Note (c,d) needs to be a unit vector.
  - If it isn't divide (c,d) by its magnitude ($\sqrt{c^2+d^2}$)

$D_{(c,d)}f(a,b) = \nabla f(a,b) \cdot (c,d)$

Double Integrals

- Integration for multi variable functions is used to find the volume under a plane. We'll begin with the simplistic case where we are dealing with a rectangular domain bound by a<x<b and c<y<d.
- We divide our rectangular domain into small squares of area Δx by Δy. Similar to one variable we take an estimated point f(x*,y*), which is a point inside the mini-region
  - The volume can be approximated as

$$\sum_x \sum_y f(x^*, y^*)\Delta x \Delta y$$

- As Δx and Δy approach 0 we get the integral:
  $\int_a^b \int_c^d f(x,y)dy\,dx$
- We can switch the order of integration, if it makes it easier dx dy, but we must integrate across a to b first if we switch the order.
- Now let's consider a domain that is bound by functions rather than constants.
- Plot the domain on a plot of x and y. Choose a variable to integrate first, find the lower constant and upper constant we will be integrating across.
  - Say we integrate across x and x is bound by a and b.
- Then we draw an arrow in the y direction form the x-axis towards the region of integration. The first curve the arrow hits is the lower bound of integration. We will call this curve $g_1(x)$. The curve is a function of x. We then continue drawing the arrow till we hit the curve where we leave the region. This is the upper bound and we will call this curve $g_2(x)$.
  The integral to solve is :
  $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx$
- We can switch the order of integration, but our upper and lower bounds will change. We will need to determine the constants by which y is bound.
  - We will then need to determine the curves that x is bound. These curves will be functions of y. Then we can reverse the order of integration

Ex Evaluate the integral

$\int_0^2 \int_{x^2}^4 x^3\sin(y^2)\,dy\,dx$

This integral looks really hard so let's reverse the order. Consider the graph:

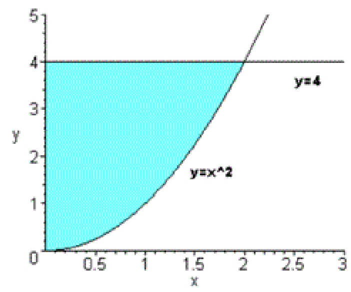


Figure 7:

We see that y varies between 0 and 4 as constants go. Now if we draw an arrow parallel to the x-axis. We see that the first curve we hit is the line x=0. Therefore this is the lower bound. As we exit the region , we hit the boundary x=√y.

Therefore we can rewrite the integral

- Now consider the bound along the y axis. When y=0, We get that u=x and v=-x. Therefore u=-v.
- Now we have developed that v sits between -u and u and u sits between 1 and 2. Now we need a Jacobian

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2. Therefore\ J = \frac{1}{2}$$

$$\int_1^2 \int_{-u}^{u} \cos(\frac{v}{u}) \frac{1}{2} dv\, du = \int_1^2 \sin\left(\frac{v}{u}\right) \frac{1}{2u} \Big|_{-u}^{u} = \int_1^2 \frac{u}{2} (\sin(1) - \sin(-1))\, du$$

$$= \frac{u^2}{4} 2\sin(1) \Big|_1^2 = 2\sin(1) - \frac{1}{2\sin(1)} = \frac{3}{2}\sin(1)$$

Triple Integrals

- Triple Integrals are very similar to double integrals except we have a third integration to perform. The hardest part is to determine the boundries of integration, if they are not all constants bounds.
- The key to this is to draw the cross sections. Say we want x to be our variable that is bound between 2 constants. We then choose the next variable to integrate y.
- We look at the cross section of x and y and draw an arrow along the y-axis. The lower bound it hits will be the lower bound for y and the upper curve it hits will be the upper bound. These will be functions of x.
  - This will hit points on the x-axis that are either related or unrelated to the constant z.
- Finally we draw an arrow parallel to z-axis on the y-z cross section. The first curve we cross is the lower bound for z and the curve while exiting the region is the upper bound for z. Z will most likely change with respect to both x and y.
  - If there is a constant dummy variable, such as k, in the cross section we must replace it with x again.

Ex. Consider the region bound by x>0, y>0, z>0, x=y, $x^2+y^2=1$. Find the boundries of integration

First thing we need to do is decide on a variable to set between 2 constants. If we set x between two constants its bounds will be 0 and 1.

We want to integrate across y next so we look at the x-y cross section. We see that y crosses first at y=0 and exits the region at √(1-x^2)

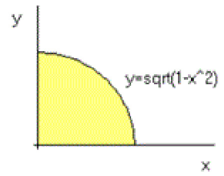
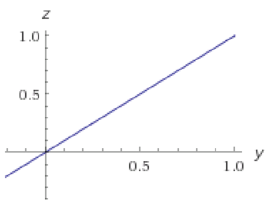


Figure 5:

We finally take a look at the y-z cross section. We see that it is has no constant variable that is related to x, so it is purely a function of y. We that it passes through z=0 first and exits along the curve z=y. Therefore the bounds of integration are 0 and y.



Now we can evaluate the integral:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y f(x,y,z) dz\, dy\, dx$$

We see that y varies between 0 and 4 as constants go. Now if we draw an arrow parallel to the x-axis. We see that the first curve we hit is the line x=0. Therefore this is the lower bound. As we exit the region , we hit the boundary x=√y.

Therefore we can rewrite the integral

$$\int_0^4 \int_0^{\sqrt{x}} x^3 \sin(y^3)\, dx\ dy = \int_0^4 \frac{y^2}{4} \sin(y^3) = -1/12(\cos(4^3) - \cos(0)$$

$$= \frac{1}{12} - \frac{1}{12}\cos(64)$$

Polar Coordinates for Integration

- Polar coordinates are useful if the region or parts of the region of integration are circular.
- Recall that we x=rcosθ and y=rsinθ. Therefore we can sub these values into the function and we can integrate across r and θ instead.
  - We need to decide on bounds for r and θ. For example if we are integrating across a quarter circle. Θ should vary between 0 and π/2.
  - The second thing we need to do is have a change factor. Recall when we normally perform substitution we have a factor that we had to adjust our integral by. For polar coordinates we multiply it by r.

$$\iint_R f(x,y) dx\, dy = \iint_R f(rsin(\theta), rcos(\theta)) r\, dr\, d\theta.$$

Cylindrical and Spherical Coordinates

- Cylindrical coordinates are very similar to polar coordinates replacing x and y with rcosθ and ysinθ, while keep z to be z. This is useful for regions of integration that are cylindrical. The jacobian for this is still r.
- Spherical coordinates have a radius ρ, an angle on the x-y plane,θ, and an angle from the z-axis,ϕ. The jacobian for this is ρ^2sinϕ. This is useful for regions of integration which are spherical.
  - ρ=√($x^2+y^2+z^2$)
  - θ=arctan(y/x)
  - ϕ=arccos(z/ρ)
  - x=ρ sinϕ cosθ
  - y=ρ sinθ sinϕ
  - z=ρ cosϕ
- These are useful for triple integrals, the method of substitution is a mix between the methods use for polar coordinates and triple integrals.

Final Notes

$$\iiint_V 1xdydz = Volume \qquad \iint_A 1\, dx\, dy = Area$$

$$\frac{1}{Area} \iint_A f(x,y) dx\, dy = Average\ Value\ of\ f(x,y)$$

$$\frac{1}{Volume} \iiint_V f(x,y,z)\, dx\, dy\, dz = Average\ Value\ of\ f(x,y,z)$$

Jacobian for Triple Integral: Replacing x,y,z with a,b,c

$$\begin{matrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} & \frac{\partial a}{\partial z} \\ \frac{\partial b}{\partial y} & \frac{\partial b}{\partial y} & \frac{\partial c}{\partial y} \\ \frac{\partial c}{\partial z} & \frac{\partial b}{\partial z} & \frac{\partial c}{\partial z} \end{matrix}$$