ECON 833

Jason DeBacker

Dynamic Programming Lecture Notes #1

<u>The Cake Eating Problem:</u>

- discrete time, $t = 1, 2$
- $c_t \equiv$ consumption of cake in period t
- <u>Preferences:</u> $u(c_1) + \beta u(c_2)$
  - $u'(\cdot) > 0$
  - $u''(\cdot) < 0$ (i.e., strictly concave utility function)
  - $0 \leq \beta \leq 1$ discount factor
  - $u'(0) = \infty$
    * Inada condition (first derivative approaches infinity as c approaches zero), always keeps you away from boundary conditions/corner solutions
- <u>Endowment:</u>
  - $w_1 > 0$ given (start of period one)
  - No endowment in period 2 (it's important that agent knows this at outset)
- <u>Technology:</u>
  - <u>Storage technology:</u> $w_2 = w_1 - c_1$ (this is called the "transition equation")
    * Storage technology is: "how much of that stuff that I put in today is there tomorrow"
- <u>Markets:</u>
  - None here
- <u>Information:</u>
  - No uncertainty
- The problem:
  - $\max_{c_1, c_2, w_2, w_3} u(c_1) + \beta u(c_2)$
    * subject to:
    * $w_2 = w_1 - c_1$
    * $w_3 = w_2 - c_2$
    * $c_t \geq 0$, $t = 1, 2$
      · Inada condition takes care of this condition and ensures interior solution
    * $w_t \geq 0$, $t = 2, 3$
    * Note that there will be 6 Lagrange multipliers for the 6 constraints
    * However, with some substitutions, we can eliminate some constraints
    * As noted, the Inada condition takes care of two constraints ($c_1 \geq 0, c_2 \geq 0$)
    * Then one can combine the first two constraints into one: $w_3 + c_1 + c_2 = w_1$ and we'll use λ as the Lagrangian multiplier on this constraint. Note this also gets rid of w_2 as a choice variable
    * Which leaves only one more constraint, $w_3 \geq 0$, we'll use ϕ as the Lagrangian multiplier on this constraint (and this covers it since $w_2 \geq 0$ is implied by the two remaining constraints)
  - Lagrangian: $L = \max_{c_1, c_2, w_3} u(c_1) + \beta u(c_2) + \lambda(w_1 - c_1 - c2 - w_3) + \phi(w_3)$
  - <u>FOCs:</u>

        * w.r.t. c_1: $u'(c_1) = \lambda$
        * w.r.t. c_2: $\beta u'(c_2) = \lambda$
            · Note that the two conditions above imply the "Euler" equation : $u'(c_1) = \beta u'(c_2)$
            · $\implies$ We'll see these Euler equations all the time.
            · $\implies$ They relate two variables across time.
            · $\implies$ They are a condition of inter-temporal optimization.
            · This condition is a necessary, but not sufficient, condition for choices along an optimal path in a dynamic optimization problem
            · Interpretation: If (discounted) marginal utilities are not equal, then agent can improve utility by rearranging the amounts consumed in different periods
            · DRAW inter-temporal budget constraint and indifference curve (whose slope is the ratio of marginal utilities).
        * w.r.t. w_3: $\phi = \lambda$
            · If $\phi > 0$, then that means the non-negativity constraint on w_3 binds, thus $w_3 = 0$
            · We assumed that the marginal utility of consumption was positive (i.e., $u'(c) > 0$), thus $\lambda > 0$ and so $\phi > 0$
            · Thus we know that $w_3 = 0$ (i.e., we don't leave any cake left over for period in which we get no utility from consuming it)
    - Since agents only receive an endowment in period 1 and get no utility from period 3 consumption, we can rewrite this problem in a more simple way:
        * $c_1 + c_2 = w_1$
        * $w_1 - c_1 = s$, where s=savings
        * $c_2 = s$
        * now the maximization problem becomes: $\max_{s:w_1 \geq s \geq 0} u(w_1 - s) + \beta u(s)$
        * the FOC (now just w.r.t. s) becomes the Euler equation: $u'(w_1 - s) = \beta u'(s)$
        * We can write the optimization problem as a Bellman equation: $V_2 \equiv \max_s u(w_1 - s) + \beta u(s)$
            · $\implies u'(w_1 - s) = \beta u'(s) \implies$ how agent acts optimally is given by the Euler equation
            · $s(w_1) \implies c_1$ and c_2 as a function of w_1
            · This is the policy function or decision rule (demand function is a specific example of this)
            · describes how agents chose endogenous variables as a function of exogenous variables and parameters
            · $V_2(w_1) = u(w_1 - s(w_1)) + \beta u(s(w_1))$ (where V_2 is the value once I know how the agent will optimize (from policy function above))

The (simplified) problem:

- $V_2(w_1) = \max_{w_1 \geq s \geq 0} u(w_1 - s) + \beta u(s)$
    - recall that V_2 is called the value function
    - recall that $w_3 = 0$ b/c eat all cake in two periods
- $\implies u'(w_1 - s) = \beta u'(s)$ (i.e., discounted marginal utilities are equal)
- $\implies s(w_1)$ policy function (or decision rule) $\implies$ so $\forall$ w we have solved the problem

Examples:

1. $\beta = 1$, any concave utility function

- $V_2(w_1) = u(w_1 - s) + u(s)$
- Euler equation says that $u'(c_1) = u'(c_2)$
- $\implies c_1 = c_2 = \frac{w}{2}$
- $V_2 = 2u\left(\frac{w}{2}\right)$
- Why?
  - Euler equation says that $u'(c_1) = u'(c_2)$
  - Because $u''(c) < 0$, agent is risk averse and would like to smooth consumption
  - This is an extreme example of consumption smoothing since $\beta = 1$, there is no discounting of future consumption
  - And since there is no uncertainty, there is no precautionary savings
  - Thus have $u'(c_1) = u'(c_2) \implies c_1 = c_2$
- Draw two axes with c_1 and c_2 and 45 degree line. Show preferences tangent right at 45 degree line, along budget constraint which goes from w_1 on one axis to the other.

2. $\beta < 1$, $u(c) = ln(c)$
   - $V_2(w_1) = ln(w_1 - s) + \beta ln(s)$
   - Euler equation $\implies u'(c_1) = \beta u'(c_2) < u'(c_2)$
     - $u''(\cdot) < 0 \implies c_1 > c_2$
     - Euler: $\frac{1}{c_1} = \frac{\beta}{c_2}$
     - or, $\frac{1}{w_1 - s} = \frac{\beta}{s} \implies s = \frac{\beta w_1}{1+\beta} = c_2$
     - $\implies c_1 = \frac{w_1}{1+\beta}$
     - $V_2(w_1) = ln\left(\frac{w_1}{1+\beta}\right) + \beta ln\left(\frac{\beta w_1}{1+\beta}\right)$
     - or $V_2(w_1) = ln\left(\frac{1}{1+\beta}\right) + \beta ln\left(\frac{\beta}{1+\beta}\right) + (1+\beta)ln(w_1)$ (get this by using properties of logs)
   - Draw two axes with c_1 and c_2 and 45 degree line. Show preferences tangent below 45 degree line, along budget constraint which goes from w_1 on one axis to the other.

<u>Extension #1</u>: endowment of cake in period 2 (y_2)

- Preferences: $u(c_1) + \beta u(c_2)$
- Endowment: w_1 when born, y_2 in period 2
- Constraints:
  - $w_2 = w_1 - c_1$
  - $c_2 = y_2 + w_2$
  - or $c_1 + c_2 = w_1 + y_2$ (combining first two constraints)
  - Note that we are imposing $w_3 = 0$, this is b/c $w_3 > 0$ is suboptimal and $w_3 < 0$ is not allowed
  - Note that $c_1 > w_1$ is ok; borrowing is allowed
  - Can rewrite first two constraints in terms of savings (where $s = w_2$):
    - $s = w_1 - c_1$
    - $c_2 = y_2 + s$
- Problem: $\max_s u(w_1 - s) + \beta u(y_2 + s)$

- FOC: $u'(w_1 - s) = \beta u'(y_2 + s)$
  - $s(w_1, y_2)$
  - $c_1(w_1, y_2) = c_1(w_1 + y_2)$
  - $c_2(w_1, y_2) = c_1(w_1 + y_2)$
    * Consumption just depends upon the sum of w_1 and y_2, not the timing (since can borrow/lend freely) - all that matters is the lifetime endowment
    * Savings does depend upon the timing of the endowments - that's the whole point of savings, to change the timing of consumption
- Draw two axes with c_1 and c_2 and 45 degree line. Show preferences tangent below 45 degree line, along budget constraint which goes from $w_1 + y_2$ on one axis to the other. Label this the lifetime endowment.

A couple of applications of this extension:

1. Fiscal policy (Ricardian equivalence)
   - It doesn't matter if finance gov't spending with debt or current taxes
   - debt is future taxes
   - consumption will remain the same (lifetime endowments unchanged, just change in when taxes come), but savings changes, depending on if tax now and give back later or have no tax
2. Borrowing restrictions ($s \geq 0$)
   - With restrictions to borrowing, then consumption depends upon timing of income $\implies$ b/c can consumer more in c_1.
   - Draw two axes with c_1 and c_2 and 45 degree line. Show preferences tangent below 45 degree line, along budget constraint which goes from $w_1 + y_2$ on one axis to the other. Label this the lifetime endowment. But draw endowment of w_1 and y_2 to the left of the 45 degree line. Say that borrowing constraint means that can't get to where want to be.

<u>Extension #2</u>: Return on storage

- $w_1 > 0$, $y_2 = 0$
- $\rho \equiv$ return to storage (more like a real than nominal rate of return) $\implies$ it's a statement about storage technology
- $\implies$ transition equation $w_2 = \underbrace{(w_1 - c_1)}_{s} \rho$
- Problem: $\max_{w_1 \geq s \geq 0} u(w_1 - s) + \beta u(\rho s)$
  - FOC: $u' \underbrace{(w_1 - s)}_{c_1} = \beta \rho u' \underbrace{(\rho s)}_{c_2}$
  - $\implies s(\rho, w_1)$ is a policy function
  - $V_2(\rho, w_1)$ is the value function
  - Draw axes with c_1 and c_2, budget constraint going from ρw_1 on the c_2 axis to w_1 on the c_1 axis, and a 45 degree line
    * if $\beta\rho = 1$, then indifference curve hits at 45 degree line
    * $\beta\rho = 1 \implies c_1 = c_2$

- $\beta\rho < 1 \implies c_1 > c_2$
- $\beta\rho > 1 \implies c_1 < c_2$

- Comparative Statics: w.r.t. (ρ, w_1)
  - Euler: $u'(w_1 - s) - \beta\rho u'(\rho s) = 0 \implies G(s(\rho), \rho)$
  - IFT $\implies \frac{ds}{d\rho} = \frac{-G_2}{G_1} = \frac{\beta u'(\rho s) + \beta\rho s u''(\rho s)}{-u''(w_1 - s) - \beta\rho^2 u''(\rho s)}$
  - $\frac{ds}{d\rho} = \beta \frac{[\overbrace{u'(s\rho)}^{substitution} + \overbrace{s\rho u''(s\rho)}^{income}]}{z} = \beta \frac{u'(s\rho)[1 - R(s\rho)]}{z}$, where $R(x) = \frac{-xu''(x)}{u'(x)}$
    - $z = -u''(c_1) - \beta\rho^2 u''(s\rho)$
    - $R(x)$ is the coefficient of relative risk aversion (relative to wealth/consumption)
    - more curvature in $u(\cdot)$ increases the measure of relative risk aversion
  - if $R(s\rho) < 1$, then $\frac{ds}{d\rho} > 0$
  - if $R(s\rho) > 1$, then $\frac{ds}{d\rho} < 0$
    - depends if income or substitution effect dominates
    - subs $= u'(s\rho) \implies$ how utility changes with change in ρ
    - income $= u''(s\rho) * \underbrace{s\rho}_{\text{savings}\times\text{storage rate}} \implies$ how marginal utility changes for a change in ρ
  - Draw graph with s and ρ on axes and a backwards bending curve (i.e. s rises with ρ to some point, then declines as ρ gets sufficiently large
  - If substitution effect is bigger, then a higher ρ = save more $\implies$ c_1 is less
    - draw a graph with two budget constraints showing changes in ρ and have to indifference curves showing c_1 declining as ρ increases.
  - For comparative static w.r.t. change in w_1
    - Euler + IFT $\implies \frac{ds}{dw_1} = \frac{u''(w_1 - s)}{u''(w_1 - s) + \beta\rho^2 u''(\rho s)} = \frac{\leq 0}{< 0} > 0$
    - $\implies$ as endowment $\uparrow$, save more
  - Likewise, can do comparative statics w/ value function (an endogenous variable):
  - $V_2(w_1, \rho) = u(w_1 - s) + \beta u(\rho s) = u(w_1 - s(w_1, \rho)) + \beta u(\rho s(w_1, \rho))$
  - $\frac{dV_2}{d\rho} = -u'(c_1)\frac{ds}{d\rho} + \rho\beta u'(c_2)\frac{ds}{d\rho} + \beta s u'(c_2)$
  - $\implies \frac{dV_2}{d\rho} = \frac{ds}{d\rho}[\underbrace{-u'(c_1 + \rho\beta u'(c_2)}_{=0 \text{ by FOC}}] + \beta s u'(c_2)$
  - The above is an example of an envelope condition - change in ρ has no effect on s b/c V is at a maximum (and derivative at max=0)
  - only direct effect of change in ρ affects V_2
  - so just have: $\frac{dV_2}{d\rho} = \frac{\beta u(s(\cdot)\rho)}{d\rho}$
  - $= s(\cdot)\beta u'(s(\cdot)\rho) > 0$
  - Could do the same with $\frac{dV_2}{dw_1}$ and find value function increases as endowment increases...

- Specific example:
  - if $u(c) = ln(c)$
    - FOC: $\frac{1}{c_1} = \frac{\beta\rho}{c_2} = \frac{\beta\rho}{\rho s} = \frac{\beta}{s}$
    - $\implies s = \frac{\beta w_1}{1+\beta}$
    - $\implies \frac{ds}{d\rho} = 0$
    - That is, if you have a log utility function, the income and substitution effects will cancel out

· subs $= u'(s\rho)$
· income $= s\rho u''(s\rho)$
· w/ log:
· subs $= \frac{1}{s\rho}$
· income $= \frac{-1}{s\rho}$
· $\implies$ subs+income $= 0$

* $\implies \frac{ds}{d\rho} = 0 \implies \varepsilon_{s,\rho} = 0 \implies$ elasticity of savings w.r.t. savings technology is 0
* finding a policy function from Euler equation and resource constraint:
  · Resource constraint: $c_1 = w_1 - s$
  · Euler: $\frac{1}{c_1} = \frac{\beta\rho}{c_2} = \frac{\beta\rho}{\rho s} = \frac{\beta}{s}$
  · $\implies \frac{1}{w_1 - s} = \frac{\beta}{s} \implies s = \left(\frac{\beta}{1+\beta}\right) w_1 = s(w_1)$ policy function
  · this implies $c_1 = \frac{w_1}{\beta+1}$, which is also a policy function

The T Period problem ($T < \infty$)

- $V_T(w_1) = \max_{(c_1,...,c_T)} \sum_{t=1}^{T} \beta^t u(c_t), \forall w_1$
  - s.t. $w_1 = \sum_{t=1}^{T} c_t$
  - $u' > 0, u'' < 0, u'(0) = \infty$
  - Note that V_T is the value function for the T period problem
  - Note that discounting period one has no effect on the Euler and will help us with the recursive solution to the $T+1$ period problem
  - This notation represents the sequence problem (the Bellman equation another way to represent the dynamic programming problem)
- The Lagrangian: $L = \max_{(c_1,...,c_T)} \sum_{t=1}^{T} \beta^t u(c_t) + \lambda \left(w_1 - \sum_{t=1}^{T} c_t\right)$
- FOCs: $\beta^t u'(c_t) = \lambda, t = 1, ..., T$ (there are T FOCs and T unknowns)
- The FOCs $\implies u'(c_t) = \beta u'(c_{t+1}), t = 1, ..., T-1$
  - $\implies$ can't be better off by switching around when consume
  - There are $T-1$ Euler equations
  - +1 necessary condition that says $w_1 = \sum_{t=1}^{T} c_t$)
- Policy Function: (generated by a given β and $u(\cdot)$)
  - θ is parameter describing $u(\cdot)$
  - β = discount factor
  - $(\theta, \beta) \implies$ policy function and value function
    * $c_t(w_1), t = 1, ..., T \implies$ optimal solution to the problem
- Value function
  - $V_T(w_1) = \sum_{t=1}^{T} \beta^t u(c_t(w_1)), \forall w_1$
    * put in the optimal c_t's and get the total value of the problem $\implies$ the max utility
    * e.g., $u(c) = \frac{c^{1-\gamma}}{1-\gamma} \implies R(c) = \gamma$
- Example: $u(c) = ln(c)$
  - $V_1(w_T) = ln(w_T), \forall w_T$ (draw time line from 0 to T and show how work backwards)

- $V_2(w_{T-1}) = \max ln(w_{T-1} - w_T) + \beta ln(\underbrace{w_T}_{V_1(w_T)})$
- FOC: $\frac{1}{w_{T-1}-w_T} = \frac{\beta}{w_T} \implies w_T = \left(\frac{\beta}{1+\beta}\right) w_{T-1}$
- so you can find c_{T-1}: $c_{T-1} = w_{T-1} - w_T = \frac{w_{T-1}}{1+\beta}$
- plugging this into the Bellman equation, we can solve for V_2:
- $V_2(w_{T-1}) = ln\left(\frac{w_{T-1}}{1+\beta}\right) + \beta ln\left(\frac{\beta w_{T-1}}{1+\beta}\right)$
- simplifying: $= \underbrace{ln\left(\frac{1}{1+\beta}\right) + \beta ln\left(\frac{\beta}{1+\beta}\right)}_{A_2} + \underbrace{(1+\beta)}_{B_2} ln(w_{T-1})$
- $= A_2 + B_2 ln(w_{T+1})$
- One can keep going, backwards, to find solutions:
- $V_3(w_{T-2}) = \max_{w_{T-1}} ln(w_{T-2} - w_{T-1}) + \beta V_2(w_{T-1})$
- $= \max_{w_{T-1}} ln(w_{T-2} - w_{T-1}) + \beta(A_2 + B_2 ln(w_{T-1}))$
- To Solve:
  * Find FOC's: $\frac{1}{w_{T-2}-w_{T-1}} = \beta\frac{B_2}{w_{T-1}} \implies w_{T-2} = \frac{w_{T-1}(1+\beta B_2)}{\beta B_2} \implies w_{T-1} = \frac{w_{T-2}\beta B_2}{(1+\beta B_2)}$
  * Find w_{T-1} as a function of w_{T-2}
  * Solve for $V_3(w_{T-2}) = A_3 + B_3 ln(w_{T-2})$
- With this approach to solve the problem, all that matters is what you start with $\implies$ past consumption not important per se $\implies$ only good in that it tells you what you have today
  * State of the system is defined by how much cake you start with

The T+1 Period Problem

- The problem: $V_{T+1}(w_0) = \max_{\{c_0,\ldots,c_T\}} \sum_{t=0}^{T} \beta^t u(c_t)$
  - s.t. $\sum_{t=0}^{T} c_t = w_0$
  - Note: If we didn't discount period 1 in the value function above, we would go from $t = 1$ to $T+1$ here, adding a term at the end rather than the beginning
- Suppose you have solved the T-period problem and now you must solve the T+1 period problem, 2 options:
  1. Solve $\max_{c_1,\ldots,c_T} \sum_{t=0}^{T} \beta^t u(c_t)$
     - s.t. $w_0 = \sum_{t=0}^{T} c_t$
  2. solve $\max_{c_0} u(c_0) + V_T(w_1) \implies V_T$ is the value function for the T period problem
     - $w_1 = w_0 - c_0 \implies$ the 2-period problem
     - $\implies$ the "principle of optimality"
- Taking the latter approach, we have the Bellman equation:

$$V_{T+1}(w_0) = \max_{c_0} u(c_0) + V_T(w_0 - c_0) \tag{1}$$

  - Note that there is no β in front of V_T because in our specification of V_T above we discounted the first period

- Noting that $w_1 = w_0 - c_0$, we can make w_1 the control variable (not c_0):

$$V_{T+1}(w_0) = \max_{w_1} u(\underbrace{w_0 - w_1}_{\substack{\text{Difference between} \\ \text{what I have now } (w_0) \\ \text{and what I leave} \\ \text{for tomorrow } (w_1)}}) + V_T(\underbrace{w_1}_{\substack{\text{we have already} \\ \text{found the solution to} \\ \text{this } \forall w_1}}) \tag{2}$$

  - This means that to solve the $T+1$ period problem, after solving the T period problem, we only have to find one thing - the optimal c_0 (i.e., $w_0 - w_1$)
  - Note: This is an application of the principal of optimality. Once we have a solution to V_T, we only need to maximize for the one additional period because we know that solution for V_T will give us the optimal choices for the next T periods.

- FOCs for T+1 problem:
  - $\beta^t u'(c_t) = \lambda, \ \forall \ t = 0, ..., T \implies T+1$ FOCs
  - $V_T'(w_0 - c_0) = \lambda$
  - $\implies u'(c_t) = \beta u'(c_{t+1}), \ \forall \ t = 0, ..., T-1$
  - $\implies u'(c_0) = \underbrace{V_T'(w_0 - c_0)}_{\substack{\text{the value of the } T \\ \text{period optimization problem}}} = \lambda = \beta u'(c_1) = \beta^2 u'(c_2) = ...$
  - Recall the 2-period problem:
    * $\frac{dV_2}{dw_1} = u'(c_1) = \beta u'(c_2) \implies$ we found this from applying the envelope theorem
    * This means that the change in the value of the value function is equal to the direct effect of the change in w_1 on the marginal utility in the first period (because we are at an optimal choice for our policy function - so the indirect effect is zero)
  - Principle of Optimality:
    * When $T+1$ periods, I only need to change one thing - just choose the right thing for period 0, then you know that the next T periods will be optimized, because $V_T(w_1)$ is the solution to that problem.

Infinite Horizon Problem; $T = \infty$

- The problem: $V(w) = \max_c u(c) + \beta V(w'), \ \forall w \in [0, \bar{w}]$
  - where $w' = w - c$ (primes will denote the next period - since we are dealing with an infinite horizon, we'll drop the time subscripts)

- Terminology:
  - **state variable(s)**: represent the state of the system, it's what you need to know to make a decision $\implies$ here we just need to know the about of cake (w)
  - **control variables**: what is chosen, here it is c, it is under the max argument
  - **stationarity**: no t's; there is no end, so there is no need to keep track of time, certain relationships don't change over time (e.g., preferences)
  - **transition equation**: equation describing the evolution of the state variable, tells the value of the state variable tomorrow as a function of the state variable today and the control variable; e.g., $w' = w - c$
  - **policy function**: specifies the control variables as a function of the state variables; it's time invariant (e.g., in $T = 2$, $w_1 - c_1 = w_2 \implies \underbrace{c_1}_{control} = \underbrace{w_1 - w_2}_{state}$)

- **value function**: $V(w)$, not indexed by time in infinite horizon problem
- **functional equation**: unknown $V(w)$, e.g., $V(w) = \max_c u(c) + \beta V(w'), \forall w \in [0, \bar{w}]$
  * Note: A functional equation relates a function to other function in an implicit way - by writing it as a function of itself at another point in time, say. These function can't be reduced to algebraic equations
  * Note: The functional equation for the value function is called a Bellman equation (it's Bellman's Principle of Optimality that is used to solve these problems recursively)
  * Note: Richard Bellman was an American mathematician in the 20th century who invented dynamic programming

- We need stationarity to solve an infinite horizon problem
  - In our notation, we'll lose all time subscripts, use primes (e.g., w') to indicate future variables
- In the finite horizon problem, we did't care how much we have left in period $T + 1$
  - In fact, that's how we solved the finite T problem (used this condition to start our backwards induction)
  - We do worry about the amount in "$T + 1$" in the infinite horizon problem
    * Will solve problem by using a fixed point theorem
- The Problem:
  - $V(w) = \max_c u(c) + \beta V(w'), \forall w \in [0, \bar{w}]$
  - or $V(w) = \max_{0 \leq w' \leq w} u(w - w') + \beta V(w'), \forall w$
  - control: w' (control the future state variable since it's determined with choice of c)
  - state: w
  - Here, we've substituted in the transition equation ($c = w - w'$)
    * The envelope theorem makes it easier to work with the latter than the former (this means that we don't have to look at $\frac{\partial c}{\partial w}$)
  - We are solving this problem for two **functions**:
    1. The value function, $V(w)$, $\forall w$
    2. The policy function, $w' = p(w)$ (or, equivalently, $c = \phi(w)$), $\forall w$
  - We need a few conditions to be satisfied in order to be able to solve a model like this:
    1. The problem is a convergent series, $0 < \beta < 1$
    2. The problem is stationary (time is irrelevant)
    3. The policy function is continuous, real-valued, and bounded
- FOCs in the ∞-horizon problem:
  - $u'(w - w') = \beta V'(w')$
  - where do we get $V'(w')$ (recall, we are solving for this)?
    * FOC says: $\frac{dV}{dw'} = -u'(w - w') + \beta V'(w') = 0$
    * $\frac{dV}{dw} = V'(w) = \underbrace{u'(w - w')}_{=\beta V'(w')} + \underbrace{[-u'(w - w') + \beta V'(w')]}_{=0,\text{by envelope condition}} \frac{dw'}{dw} = u'(w - w')$
    * what this means is that since $V(w')$ is the max of all future consumption $\forall w'$, the effect of getting more cake now on utility is only the direct effect on today's consumption (derivative of future years flat around max, so no change in marginal lifetime utility from that).
    * $V'(w') = u'(w' - w'')$... etc, $\implies$ the marginal value of extra cake today (tomorrow) is just the marginal value of consumption today (tomorrow)

- Thus the Euler can be written as: $u'(w-w') = \beta u'(w'-w'')$ - which looks just like what we've seen all along - that discounted, marginal utilities are equalized along the optimal consumption path
- What can we say about the solution to this problem:
    * $u'(c) = \beta u'(c') < u'(c) \implies c > c'$
    * $u'(\phi(w)) = \beta u'(\phi(w-\phi(w)))\forall w$
    * since $c' = \phi(w'), w' = w - c = w - \phi(w)$

- Example: $u(c) = ln(c)$, Solve by conjecturing a value function
    - recall: $V_T(w) = A_T + B_T ln(w)$
    - now, for the infinite horizon problem, guess that $V(w) = A + Bln(w)$
        * Here we are using the "guess and verify method" to solve this problem
        * Note that we are making a good guess here
        * While it can be helpful, won't need intuition like this to solve on computer- any initial guess will do. We will discuss more later.
    - Test our guess: $A + Bln(w) = \max_{w'} ln(\underbrace{w-w'}_{\text{policy function}}) + \beta(A + Bln(w')), \forall w$
    - FOC: $\frac{1}{w-w'} = \frac{\beta B}{w'} \implies w' = \frac{\beta B}{1+\beta B}w \implies$ this is the policy function
        * Which means $V(w) = ln\left(\frac{w}{1+\beta B}\right) + \beta(A + Bln\left(\frac{\beta B}{1+\beta B}w)\right), \forall w$
        * $= ln\left(\frac{1}{1+\beta B}\right) + ln(w) + \beta A + \beta Bln\left(\frac{\beta B}{1+\beta B})\right) + \beta Bln(w), \forall w$
        * $= \underbrace{ln\left(\dfrac{1}{1+\beta B}\right) + \beta A + \beta Bln\left(\dfrac{\beta B}{1+\beta B}\right)}_{A \implies \text{Find } A} + (\underbrace{1+\beta B}_{B \implies \text{find } B})ln(w), \forall w$
        * $B = 1 + \beta B \implies B = \frac{1}{1-\beta}$
        * The above then means that we can solve the policy function in terms of parameters: $w' = \frac{\beta B}{1+\beta B}w = \frac{1}{\frac{1}{\beta B}+1}w = \frac{1}{\frac{1}{\beta\frac{1}{1-\beta}}+1}w = \frac{1}{\frac{1-\beta}{\beta}+1}w = \frac{1}{\frac{1-\beta+\beta}{\beta}} = \frac{1}{\frac{1}{\beta}}w = \beta w = \beta w$ (same as T period problem)
        * Now that we have B in terms of β, we can plug in above and solve for A in terms of β... do this...
        * $A = ln\left(\frac{1}{1+\beta B}\right) + \beta A + \beta Bln\left(\frac{\beta B}{1+\beta B}\right)$
        * $A = ln\left(\frac{1}{1+\beta\frac{1}{1-\beta}}\right) + \beta A + \beta\frac{1}{1-\beta}ln\left(\beta\right)$
        * $A = ln\left(\frac{1}{\frac{1-\beta+\beta}{1-\beta}}\right) + \beta A + \beta\frac{1}{1-\beta}ln\left(\beta\right)$
        * $A = ln\left(\frac{1}{\frac{1}{1-\beta}}\right) + \beta A + \beta\frac{1}{1-\beta}ln\left(\beta\right)$
        * $A = ln\left(1-\beta\right) + \beta A + \beta\frac{1}{1-\beta}ln\left(\beta\right)$
        * $A - \beta A = ln\left(1-\beta\right) + \beta\frac{1}{1-\beta}ln\left(\beta\right)$
        * $A(1-\beta) = ln\left(1-\beta\right) + \beta\frac{1}{1-\beta}ln\left(\beta\right)$
        * $A = \frac{ln(1-\beta)+\beta\frac{1}{1-\beta}ln(\beta)}{1-\beta}$
    - Thus we have the solution to the value function: $V(w) = \frac{ln(1-\beta)+\beta\frac{1}{1-\beta}ln(\beta)}{1-\beta} + \frac{ln(w)}{1-\beta}$ – this guess works!
    - Should be able to prove the correct case (as above) works and that wrong cases of $V(w)$ don't
    - e.g., try $V(w) = Bln(w), V(w) = A + Bw \implies$ do these work?

- Example: $u(c) = ln(c)$, Solve by conjecturing a policy function
  - Guess $w' = \underbrace{\gamma}_{constant} * w$, so $c = (1-\gamma)w$
  - in ln case $c = (1-\beta)w, w' = \beta w$
    - This comes from fact that $c = w - w' = w - \beta w = (1-\beta)w$
  - $c' = (1-\beta)w' = (1-\beta)(w-c) = (1-\beta)(w-(1-\beta)w) = (1-\beta)(1-\beta)w$
  - Show that this guess works (we know it does, but to show how to do with with guess at policy function):
    - FOC implies: $\frac{1}{c} = \frac{\beta}{c'}$
    - Thus: $\frac{1}{(1-\beta)w} = \frac{\beta}{(1-\beta)w'} = \frac{\beta}{(1-\beta)\beta w}$
    - Thus we find: $\frac{1}{(1-\beta)w} = \frac{\beta}{(1-\beta)\beta w} = \frac{1}{(1-\beta)w}$, which is true - the guess is correct
  - Should be able to prove the correct case (as above) works and that wrong cases of $w' = \phi(w)$ don't
  - e.g., try $w' = (1-\beta)w$, or $w' = \beta ln(w) \implies$ do these work?
  - Draw a graph with time on the x-axis and c on the vertical. Draw a downward sloping curve.
  - You can think of the above sequence as coming from the policy functions ($c = \beta w$) for $ln(c) = u(c)$ or the stationary policy function $c = \phi(w)$ or in terms of the Euler equation: $u'(c) = \beta u'(c') < u'(c) \implies c > c'$
- Finding a solution in general: $[V(w), \phi(w)]$
  - may have an example that you can work out analytically (as above)
  - more likely - need to compute the solution
    - Will use a contraction mapping theorem and iteration
      - Proves that a sequence of functions will converge
      - $\{V_i(w)\} \implies V(w)$
    - Bellman operator: $V_{i+1}(w) = \max_{w'} u(w-w') + \beta V_i(w') \forall w$
      - $V_i(w)$ is arbitrary
      - solution is a function $(V(w))$ and a policy function $(\phi(w))$
      - The computer will compute these functions over a discrete grid that represents the state of the system (e.g., w)