

Analysis of $MAP/G/1$ queue with inventory as the model of the node of wireless sensor network with energy harvesting

Аннотация

Queueing systems where certain inventory items are required to provide service to a customer become popular in the literature. Such systems are similar to the analysed in the literature models with paired customers, assembly-like queues, passenger-taxi models, etc. During the last few years they are considered in context of modelling operation of the nodes of a wireless sensor network with energy harvesting. Distinguishing feature of the model considered in this paper, besides the suggestion that arrival flow of customers is described by the Markovian Arrival Process, is the assumption about a general distribution of the service time while only exponential or phase-type distribution was previously assumed in the existing literature. We apply the well-known technique of $M/G/1$ type Markov chains and semi-regenerative processes to obtain the ergodicity criterion in a transparent form and stationary distribution of the system under study and sojourn time distribution. This creates an opportunity to formulate and solve various optimization problems.

Energy harvesting and queueing and inventory.

1 Introduction

Nowdays, one of the popular directions of research in queueing theory is analysis of queueing / inventory models in which service of customers is possible only conditional that some additional consumable resource (inventory) is available. This direction is close to the so called assembly-like systems or queues with paired customers or taxi / passenger queues where service of a customer requires presence of some items. The lack of available item causes temporal interruption of service process. In pure queueing / inventory models, see, e.g., [17], [18], classical assumptions about the inventory delivering are made like that the inventory is ordered when the number of available inventory units reaches some level s and after delivering the number of inventory units reaches the level S , $s < S$. Another possible scenario suggests that the units are delivered in some random moments defined by some arrival process. In our paper we interpret inventory units as the units of energy, presence of which is mandatory for service of customers. the energy required for customers service. This interpretation is suitable, e.g., for analytical modelling of operation of the nodes of wireless sensor networks. Such nodes usually have small batteries with limited power and storage space. Recent advances in energy harvesting

technology have resulted in the design of new types of sensor nodes which are able to extract energy from the surrounding environment (solar, wind, sound, vibration, thermal, and electromagnetic power). Because the service cannot be provided when there is no energy, it is desirable to apply some schemes to adjust operation of the node to the available amount of accumulated energy. As the relevant papers, we can mention [1, 2, 3, 6, 7, 8, 9, 10, 12, 13, 15, 16, 21, 22, 23, 24, 25, 26, 28, 29].

Majority of the papers assume that arrival flow of customers is defined by the stationary Poisson process. However, such a process is characterized by the constant instantaneous rate and independent inter-arrival times often is a bad descriptor of real world information flows in modern telecommunication networks. A most widely used and popular now point process that suits for modeling such flows is the Markovian arrival process (*MAP*), a special case of batch Markovian arrival process (BMAP), introduced first Neuts, M.F., as a Versatile Markovian Point Process (VMPP). For more information about the *MAP* see, e.g, [4], [19] and [27]. The models of queues with energy harvesting with *MAP* were considered only in [3, 7, 8, 9, 10, 16]. In this paper we also assume that arrival of customers is described by the *MAP*.

The main distinguishing feature of the considered model is that we assume an arbitrary distribution of customers service time. Only exponential or phase type distributions very previously considered in the literature. Phase type distribution can be used for approximation of an arbitrary distribution. E.g., the degenerate distribution (corresponding to the deterministic service time) may be approximated by Erlangian distribution with sufficiently many phases. But when the number of the phases is large, computational problems can arise. Therefore, it is desirable to analyse the model for arbitrary distribution of service time. The results for the degenerate distribution are easily obtained as the particular case.

It is worth to note that if the service of customers does not depend on the availability of energy units, the considered queueing model is reduced to the known *MAP/G/1* queue comprehensively analysed in [19]. The analysis presented in [19] (see also [11]) is far from being trivial, especially analysis of queue length at arbitrary time and sojourn time distribution. Additional account of the number of available energy units does not create principal difficulties, however, creates essential technical difficulties to be overcome.

The rest of the paper consists of the following. In Section 2, the mathematical model is described. In Section 3, the stationary distribution of the number of customers and energy units in the system at the service completion moments is analysed. This analysis includes derivation of the matrices of one-step transition probabilities of three-dimensional embedded Markov chain (subsection 3.1), derivation of ergodicity criterion for this Markov chain (subsection 3.2),

brief discussion of the ways for computation of the stationary distribution of embedded Markov chain (subsection 3.3) and computation of some performance measures of the system (subsection 3.4). Section 4 is devoted to computation of the stationary distribution of the number of customers and energy units in the system at arbitrary moments. In Section 5, Laplace-Stieltjes transforms of the stationary distribution of the virtual and actual sojourn times are obtained. Section 6 contains the results of numerical experiments and Section 7 concludes the paper.

2 Mathematical model

We consider the single server queueing system of $MAP/G/1$ type. Arrival flow is described by the Markovian Arrival Process (MAP), see [4],[19],[27]. This process assumes that the customers can arrive at the moments of jumps of the underlying Markov chain ν_t , $t \geq 0$, having a finite state space $\{0, 1, \dots, W\}$ and the generator $D(1) = D_0 + D_1$. The entries of the matrix D_1 of size $\bar{W} = W + 1$ define the intensities of transitions of the chain ν_t that are accompanied by customers arrival. The non-diagonal entries of the matrix D_0 define the intensities of transitions of the chain ν_t that are not accompanied by customer arrival.

The vector θ of the stationary distribution of the Markov chain ν_t is the unique solution to the system $\theta D(1) = \mathbf{0}$, $\theta \mathbf{e} = 1$. Here and throughout this paper \mathbf{e} is a column vector of appropriate size consisting of 1's, and $\mathbf{0}$ is a row vector of appropriate size consisting of zeroes. The average intensity of customers arrival λ is defined by the formula $\lambda = \theta D_1 \mathbf{e}$.

Service of arriving customer is possible only if so-called energy unit is available. Customers arriving when the server is busy or the server is idle but energy units are not available are stored in the buffer of an infinite capacity. Customers are picked up from the buffer according to First-In First-Out discipline. Service time of an arbitrary customer has distribution function $B(t)$ with Laplace-Stieltjes transform $\beta(s) = \int_0^\infty e^{-st} dB(t)$, $Re\ s > 0$, and finite initial moments $b_k = \int_0^\infty t^k dB(t)$, $k \geq 1$.

Energy units arrive according to the stationary Poisson process with the intensity γ and are stored in the stock of a finite capacity K . If the stock is full at the energy unit arrival, this unit is lost. One energy unit disappears from the stock at the instant of starting the service of a customer.

Our goal is to analyse the stationary behavior of the described queueing model. We are interested in analysis of the stationary behavior of two-dimensional

stochastic process $\tilde{\zeta}_t = \{i_t, k_t\}$, $t \geq 0$, where i_t is the number of customers and k_t is the number of energy units in the system at the moment t , $i_t \geq 0$, $k_t = \overline{0, K}$, where the notation $k_t = \overline{0, K}$ means that the variable k_t admits the integer values from the set $\{0, 1, \dots, K\}$. The process $\tilde{\zeta}_t$ is non-Markovian. Therefore, to analyse this process we will first consider the embedded Markov chain.

3 Stationary distribution of the number of customers and energy units in the system at the service completion moments

3.1 Embedded Markov Chain. Transition probabilities

Let t_n be the n th service completion moment, $n \geq 1$. Let i_n be the number of customers in the system at the moment $t_n + 0$, $i_n \geq 0$, and k_n be the number of energy units in the system at the moment $t_n - 0$, $k_n = \overline{0, K}$. Let us consider the three-dimensional process

$$\xi_n = \{i_n, k_n, \nu_n\}, \quad n \geq 1,$$

where ν_n is the state of the underlying process of the *MAP* at the moment t_n , $\nu_n = \overline{0, W}$.

It is easy to see that the process ξ_n is a discrete-time Markov chain. To prove this formally, we have to present expressions for one-step transition probabilities of this chain. Let us call the set of the states of the process ξ_n having the value (i, k) of the first two components as the level (i, k) . Each level consists of $W + 1$ states (i, k, ν) , $\nu = \overline{0, W}$.

Let $P\{(i, k) \rightarrow (j, k')\}$, be the matrix of transition probabilities from the level (i, k) to the level (j, k') , i.e., the matrix whose (ν, ν') th entry is the one-step transition probability

$$P\{(i, k, \nu) \rightarrow (j, k', \nu')\} = P\{i_{n+1} = j, k_{n+1} = k', \nu_{n+1} = \nu' | i_n = i, k_n = k, \nu_n = \nu\}.$$

To present the expressions for the probabilities $P\{(i, k) \rightarrow (j, k')\}$, we need the following notation:

- The probability of k energy units arrival during time t

$$\varphi_k(t) = \frac{(\gamma t)^k}{k!} e^{-\gamma t}, \quad k \geq 0.$$

- The probability of at least k energy units arrival of during time t

$$\hat{\varphi}_k(t) = \sum_{i=k}^{\infty} \varphi_i(t), \quad k \geq 0.$$

- The probability of k energy units arrival during a service time

$$\varphi_k = \int_0^\infty \varphi_k(t) dB(t), \quad k \geq 0.$$

- The probability of arrival of at least k energy units during service time

$$\hat{\varphi}_k = \sum_{i=k}^\infty \varphi_i, \quad k \geq 0.$$

- The matrix $P(i, t)$ entries of which define the probabilities of arrival of i customers and the corresponding transitions of the underlying process of the *MAP* during time t . The matrix generating function of these matrices is given by $\sum_{i=0}^\infty P(i, t) z^i = e^{(D_0 + D_1 z)t}$, $|z| < 1$.

- The matrix entries of which define the probabilities of arrival of i customers and k energy units and the corresponding transitions of the underlying process ν_t of the *MAP* during service time

$$\Phi(i, k) = \int_0^\infty P(i, t) \varphi_k(t) dB(t), \quad i \geq 0, \quad k \geq 0.$$

- The matrix whose entries define the probabilities of arrival of i customers and at least k energy units and transitions of the underlying process ν_t of the *MAP* during service time

$$\hat{\Phi}(i, k) = \int_0^\infty P(i, t) \hat{\varphi}_k(t) dB(t), \quad i \geq 0, \quad k \geq 0.$$

- The matrix whose entries define the probabilities of arrival of m customers and the corresponding transitions of the underlying process of the *MAP* during the interval between energy units arrival

$$N(m) = \int_0^\infty P(m, t) \gamma e^{-\gamma t} dt, \quad m \geq 0.$$

- The matrix whose entries define the probabilities of arrival of r energy units and transitions of the underlying process of the *MAP* during the interval between successive customers arrival

$$M(r) = \int_0^\infty e^{D_0 t} \varphi_r(t) D_1 dt = \int_0^\infty e^{D_0 t} \frac{(\gamma t)^r}{r!} e^{-\gamma t} D_1 dt = \gamma^r (-D_0 + \gamma I)^{-(r+1)} D_1.$$

- The matrix whose entries define the probabilities of arrival of at least r energy units and transitions of the underlying process of the *MAP* during the interval between successive customers arrival

$$\hat{M}(r) = \sum_{l=r}^{\infty} M(l) = \gamma^r (-D_0 + \gamma I)^{-r} (-D_0)^{-1} D_1, \quad r \geq 0.$$

Lemma Transition probability matrices $P \{(i, k) \rightarrow (j, k')\}, j \geq \max\{0, i-1\}$, are defined as follows:

$$P \{(0, 0) \rightarrow (j, k')\} = M(0) \sum_{n=0}^j N(n) \Phi(j-n, k') + N(0) \sum_{m=0}^{k'} M(m) \Phi(j, k'-m),$$

$$k' = \overline{0, K-2};$$

$$P \{(0, 0) \rightarrow (j, K-1)\} = N(0) \left[\sum_{m=0}^{K-1} M(m) \Phi(j, K-1-m) + \hat{M}(K) \Phi(j, 0) \right]$$

$$+ M(0) \sum_{n=0}^j N(n) \Phi(j-n, K-1);$$

$$P \{(0, 0) \rightarrow (j, K)\} = N(0) \left[\sum_{m=0}^{K-1} M(m) \hat{\Phi}(j, K-m) + \hat{M}(K) \hat{\Phi}(j, 1) \right]$$

$$+ M(0) \sum_{n=0}^j N(n) \hat{\Phi}(j-n, K);$$

$$P \{(0, k) \rightarrow (j, k')\} = \sum_{m=0}^{k'-k+1} M(m) \Phi(j, k'-k+1-m), \quad k = \overline{1, K}, k' = \overline{k-1, K-2};$$

$$P \{(0, k) \rightarrow (j, K-1)\} = \sum_{m=0}^{K-k} M(m) \Phi(j, K-k-m) + \hat{M}(K-k+1) \Phi(j, 0), \quad k = \overline{1, K};$$

$$P \{(0, k) \rightarrow (j, K)\} = \sum_{m=0}^{K-k} M(m) \hat{\Phi}(j, K-k+1-m) + \hat{M}(K-k+1) \hat{\Phi}(j, 1), \quad k = \overline{1, K};$$

$$P\{(i, 0) \rightarrow (j, k')\} = \sum_{m=0}^{j-i+1} N(m)\Phi(j-i+1-m, k'), \quad i \geq 1, \quad k' = \overline{0, K-1};$$

$$P\{(i, 0) \rightarrow (j, K)\} = \sum_{m=0}^{j-i+1} N(m)\hat{\Phi}(j-i+1-m, K), \quad i \geq 1;$$

$$P\{(i, k) \rightarrow (j, k')\} = \Phi(j-i+1, k'-k+1), \quad i \geq 1, \quad k' = \overline{k-1, K-1}, \quad k = \overline{1, K};$$

$$P\{(i, k) \rightarrow (j, K)\} = \hat{\Phi}(j-i+1, K-k+1), \quad i \geq 1, \quad k = \overline{1, K}.$$

Proof is easily implemented by means of analysis of possible transitions of the components of the Markov chain ξ_n between two successive service completion moments and taking into account the probabilistic meaning of the denotations explained above.

Corollary Let

$$\Psi_1(t) = \begin{pmatrix} O & O & \dots & O & O \\ \varphi_0(t) & \varphi_1(t) & \dots & \varphi_{K-1}(t) & \hat{\varphi}_K(t) \\ O & \varphi_0(t) & \dots & \varphi_{K-2}(t) & \hat{\varphi}_{K-1}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \dots & \varphi_0(t) & \hat{\varphi}_1(t) \end{pmatrix},$$

$$\Psi_0(t) = \begin{pmatrix} \varphi_0(t) & \varphi_1(t) & \dots & \varphi_{K-1}(t) & \hat{\varphi}_K(t) \\ O & O & \dots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \dots & O & O \end{pmatrix}.$$

Then, the block matrices $P_{i,j}$ composed by the blocks $P\{(i, k) \rightarrow (j, k')\}$, $k' = \overline{\max\{0, k-1\}, K}$, $k = \overline{0, K}$, for $i \geq 1$ are defined by

$$P_{i,j} = Y_{j-i+1}, \quad j \geq i-1,$$

where the matrices Y_l are defined by formula

$$Y_l = \int_0^\infty \Psi_1(t) \otimes P(l, t) dB(t) + \sum_{m=0}^l \int_0^\infty \Psi_0(t) \otimes N(m)P(l-m, t) dB(t), \quad l \geq 0.$$

The matrix generating function $Y(z) = \sum_{l=0}^{\infty} Y_l z^l$, $|z| < 1$, of matrices Y_l , $l \geq 0$, has the following form:

$$Y(z) = \int_0^{\infty} \Psi_1(t) \otimes e^{D(z)t} dB(t) + \gamma \int_0^{\infty} \Psi_0(t) \otimes (\gamma I - D(z))^{-1} e^{D(z)t} dB(t). \quad (1)$$

Here \otimes is the symbol of Kronecker product of matrices, see [14].

In turn, the block matrices $P_{0,j}$ composed by the blocks $P\{(0, k) \rightarrow (j, k')\}$, $k' = \max\{0, k-1\}, K$, $k = \overline{0, K}$, will be denoted by $P_{0,j} = V_j$, $j \geq 0$. Also denote $V(z) = \sum_{l=0}^{\infty} V_l z^l$, $|z| < 1$.

3.2 Ergodicity condition

It follows from Corollary 1 that the Markov chain ξ_n belongs to the class of $M/G/1$ type Markov chains, see [20]. It follows from [20] that the criterion of ergodicity for the Markov chain ξ_n is fulfillment of inequality

$$\mathbf{y} \sum_{k=1}^{\infty} k Y_k \mathbf{e} = \mathbf{y} Y'(1) \mathbf{e} < 1 \quad (2)$$

where the vector \mathbf{y} is the unique solution to the system

$$\mathbf{y} \sum_{k=0}^{\infty} Y_k = \mathbf{y} Y(1) = \mathbf{y}, \quad \mathbf{y} \mathbf{e} = 1. \quad (3)$$

By direct substitution into (3) where the matrix $Y(1)$ is obtained from (1), it is possible to verify that solution of system (3) has the form

$$\mathbf{y} = \mathbf{u} \otimes \boldsymbol{\theta} \quad (4)$$

where $\boldsymbol{\theta}$ is the vector of stationary distribution of the underlying process ν_t of the MAP and $\mathbf{u} = (u_0, u_1, \dots, u_K)$ is the unique solution to the system

$$\mathbf{u} \int_0^{\infty} (\Psi_0(t) + \Psi_1(t)) dB(t) = \mathbf{u}, \quad \mathbf{u} \mathbf{e} = 1. \quad (5)$$

It is easy to check that the vector \mathbf{u} defines the stationary distribution of the embedded Markov chain for $M/G/1/K$ type queueing model with the stationary arrival process with intensity γ and service time distribution function $B(t)$.

The system (5) can be rewritten in the form

$$u_k = u_0 \varphi_k + \sum_{l=1}^{k+1} u_l \varphi_{k+1-l}, k = \overline{0, K-1}, u_K = u_0 \hat{\varphi}_K + \sum_{l=1}^K u_l \varphi_{K+1-l},$$

and its solution is given by

$$u_k = u_0 \psi_k, k = \overline{1, K}, \quad (6)$$

and

$$u_0 = \left(\sum_{k=0}^K \psi_k \right)^{-1}, \quad (7)$$

where the numbers ψ_k can be recursively computed, e.g., from the following recursion

$$\psi_0 = 1, \psi_{k+1} = (\psi_k - \varphi_k - \sum_{i=1}^k \psi_i \varphi_{k+1-i}) \varphi_0^{-1}, k = \overline{0, K-1}. \quad (8)$$

By substitution of the vector \mathbf{y} in form (4) into inequality (2), we obtain inequality

$$\lambda b_1 + u_0 \frac{\lambda}{\gamma} < 1. \quad (9)$$

Therefore, we proved the following statement. **Theorem** The embedded Markov chain ξ_n for the queueing model under study is ergodic if and only if the inequality (9) is fulfilled where the probability u_0 is given by (7), (8).

Remark Condition (9) is easily tractable if it is rewritten in the form

$$\lambda(b_1 + u_0 \frac{1}{\gamma}) < 1. \quad (10)$$

Left hand side of (10) defines the mean number of customers arriving into the system between the successive service completion moments when the system is overloaded. Indeed, such a time includes the service time and the time until an energy unit arrival when the buffer of energy is empty. Mean value of this time is $b_1 + u_0 \frac{1}{\gamma}$ and the mean number of arriving customers during this time is $\lambda(b_1 + u_0 \frac{1}{\gamma})$. It is well known that the system is ergodic if this mean number is less than 1 (average number of arriving customers is less than the number of departing customers).

In what follows we assume that condition (9) is fulfilled.

3.3 Computation of the stationary distribution of embedded Markov chain

Let us denote

$$\pi(i, k, \nu) = \lim_{n \rightarrow \infty} P\{i_n = i, k_n = k, \nu_n = \nu\},$$

$$\boldsymbol{\pi}(i, k) = (\pi(i, k, 0), \dots, \pi(i, k, W)),$$

$$\boldsymbol{\pi}_i = (\boldsymbol{\pi}(i, 0), \boldsymbol{\pi}(i, 1), \dots, \boldsymbol{\pi}(i, K)), \quad i \geq 0.$$

It is well-known that the vectors $\boldsymbol{\pi}_i$, $i \geq 0$, satisfy the following system of linear algebraic equations:

$$\boldsymbol{\pi}_j = \boldsymbol{\pi}_0 V_j + \sum_{i=1}^{j+1} \boldsymbol{\pi}_i Y_{j-i+1}, \quad j \geq 0. \quad (11)$$

There are several possibilities to compute the vectors $\boldsymbol{\pi}_j$ $j \geq 0$. One of them is based on the use of the vector generating function $\boldsymbol{\pi}(z) = \sum_{j=0}^{\infty} \boldsymbol{\pi}_j z^j$, $|z| < 1$.

1. It is easy to derive from the system (11) the following functional equation for the vector generating function $\boldsymbol{\pi}(z)$:

$$\boldsymbol{\pi}(z)(Y(z) - zI) = \boldsymbol{\pi}_0(Y(z) - zV(z)). \quad (12)$$

Equation (12) can be solved using the reasonings of the analyticity of the generating function $\boldsymbol{\pi}(z)$ in the unit disc of the complex plane and existence, under fulfillment of ergodicity condition, of the required number of zeroes of the determinant of the matrix $Y(z) - zI$ in this unit disc. For more details see, e.g., [11].

Another possibility uses the sequential consideration of the series of censored Markov chains for the Markov chain ξ_n , for derivation of an alternative system of linear algebraic equations for the vectors $\boldsymbol{\pi}_j$ $j \geq 0$, and its solving based on extension of M. Neuts method with the use of the matrix G .

Algorithm.

1. Вычисляем матрицу G как минимальное неотрицательное решение матричного уравнения $G = Y(G)$.

2. Вычисляем матрицы $\bar{P}_{i,j}$ следующим образом:

$$\bar{P}_{0,j} = \sum_{n=j}^{\infty} V_n G^{n-j}, \quad j \geq 0, \quad \bar{P}_{i,j} = \sum_{n=j-i+1}^{\infty} Y_n G^{n-j+i-1}, \quad j \geq i, \quad i \geq 0.$$

3. Вычисляем матрицы $\Phi_j, j \geq 0$, используя рекуррентную формулу

$$\Phi_j = I_{(W+1)(K+1)}, \quad \Phi_j = (\bar{P}_{0,j} + \sum_{i=1}^{j-1} \Phi_i \bar{P}_{i,j})(I - \bar{P}_{j,j})^{-1}, \quad j \geq 1.$$

4. Вычисляем вектор π_0 как единственное решение системы

$$\pi_0(I - \bar{P}_{0,0}) = \mathbf{0}, \quad \pi_0(I + \sum_{j=1}^{\infty} \Phi_j)e = 1.$$

5. Вычисляем векторы π_j следующим образом: $\pi_j = \pi_0 \Phi_j, j \geq 1$.

3.4 Computation of some performance measures of the system

Having computed the vectors $\pi_j, j \geq 0$, we have an opportunity to compute the values of some performance measures of the system.

Average number L of customers in the system at service completion moments is given by

$$L = \sum_{i=0}^{\infty} i \pi_i e.$$

Average number N of energy units in the system just before the service completion moments is given by

$$N = \sum_{i=0}^{\infty} \sum_{k=1}^K k \pi(i, k) e.$$

The probability that customers are absent in the system at service completion moments is given by

$$P_0^{custom} = \pi_0 e.$$

The probability that energy is absent in the system just before the service completion moments is given by:

$$P_0^{energy} = \sum_{i=0}^{\infty} \pi(i, 0) e.$$

The probability that the server will terminate operation at service completion moment due to the absence of energy is given by:

$$P_{idle} = \sum_{i=1}^{\infty} \pi(i, 0) e.$$

The mean inter-departure time τ in the considered model is defined by

$$\begin{aligned} \tau = & b_1 + \boldsymbol{\pi}(0, 0)(-D_0 + \gamma I)^{-1}(D_1\gamma^{-1} + \gamma(-D_0)^{-1})\mathbf{e} + \gamma^{-1} \sum_{i=1}^{\infty} \boldsymbol{\pi}(i, 0)\mathbf{e} \\ & + \sum_{k=1}^K \boldsymbol{\pi}(0, k)(-D_0)^{-1}\mathbf{e}. \end{aligned}$$

Remark Due to the absence of customer loss in the considered system, the relation $\tau = \lambda^{-1}$ holds true. This relation is useful for control of accuracy of computation of the stationary probability vectors $\boldsymbol{\pi}_j$ $j \geq 0$.

In what follows, we consider the vectors of stationary probabilities of the embedded Markov chain $\boldsymbol{\pi}_j$ $j \geq 0$, be known and compute the vectors of stationary probabilities of the system states at an arbitrary moment as well as the stationary distribution of the virtual and actual sojourn time of a customer in the system.

4 Stationary distribution of the number of customers and energy units in the system at arbitrary moments

In this section, we consider the process $\zeta_t = \{i_t, k_t, \nu_t\}$, $t \geq 0$, whose components i_t and k_t are the number of customers and the number of energy units in the system, respectively, ν_t is the state of the *MAP* at an arbitrary time t .

Using the definition of semi-regenerative processes given in [5], it can be verified that the process ζ_t is a semi-regenerative one with the embedded Markov renewal process $\{\xi_n, t_n\}$, $n \geq 1$.

Let

$$p(j, k, \nu) = \lim_{t \rightarrow \infty} P\{i_t = i, k_t = k, \nu_t = \nu\}, i \geq 0, r = \overline{0, K}, \nu = \overline{0, W}, \quad (13)$$

be the stationary distribution of the process ξ_t , $t \geq 0$.

From [5], the limits in (13) exist if the process $\{\xi_n, t_n\}$, $n \geq 1$, is irreducible aperiodic recurrent and the value of the mean inter-departure time is finite. All these conditions hold if inequality (10) is satisfied.

To find the steady state probabilities (13), we use the results for semi-regenerative processes which allow to express the stationary distribution of such processes via the stationary distribution of their embedded Markov chain.

Let us denote

$$\mathbf{p}(i, k) = (p(i, k, 0), \dots, p(i, k, W)),$$

$$\mathbf{p}_i = (\mathbf{p}(i, 0), \mathbf{p}(i, 1), \dots, \mathbf{p}(i, K)), \quad i \geq 0.$$

To compute the vectors \mathbf{p}_i , $i \geq 0$, we express these vectors via the vectors of stationary distribution $\boldsymbol{\pi}_i$, $i \geq 0$, of the embedded Markov chain ξ_n , $n \geq 1$.

As it was mentioned above, the moments t_n of service completions are considered as the moments of renewals and the process $\{\xi_n, t_n\}$, $n \geq 1$, is the embedded Markov renewal process for the semi-regenerative process ξ_t . Denote by \tilde{V}_j the matrix of transition probabilities of the process ξ_t from the moment of renewal when the process was in a states with the value 0 of the component i_n to an arbitrary moment preceding the next renewal when the process is in a state with the value j of the denumerable component i_n .

Let also $\tilde{Y}_{i,j}$ be the matrix of transition probabilities of the semi-regenerative process ξ_t from the moment of the renewal when the process was in the states with the value $i > 0$ of the component i_n to an arbitrary moment preceding the next renewal when the process is in the states with the value j of the component i_n .

Lemma The matrices \tilde{V}_j , $j \geq 0$, are calculated as follows.

$$\tilde{V}_0 = \begin{pmatrix} \tilde{V}_{0,0}^{(0)} & \tilde{V}_{0,1}^{(0)} & \tilde{V}_{0,2}^{(0)} & \cdots & \tilde{V}_{0,K-1}^{(0)} & \tilde{V}_{0,K}^{(0)} \\ O & \tilde{V}_{1,1}^{(0)} & \tilde{V}_{1,2}^{(0)} & \cdots & \tilde{V}_{1,K-1}^{(0)} & \tilde{V}_{1,K}^{(0)} \\ O & O & \tilde{V}_{2,2}^{(0)} & \cdots & \tilde{V}_{2,K-1}^{(0)} & \tilde{V}_{2,K}^{(0)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & \cdots & O & \tilde{V}_{K,K}^{(0)} \end{pmatrix}$$

where

$$\tilde{V}_{k,k'}^{(0)} = \gamma^{k'-k} (\gamma I - D_0)^{-(k'-k+1)}, \quad 0 \leq k \leq k' \leq K-1,$$

$$\tilde{V}_{k,K}^{(0)} = (-D_0)^{-1} - \sum_{l=0}^{K-k-1} \gamma^l (\gamma I - D_0)^{-(l+1)}.$$

For $j > 0$, the matrices \tilde{V}_j have the following block structure:

$$\tilde{V}_j = \begin{pmatrix} \tilde{V}_{0,0}^{(j)} & \tilde{V}_{0,1}^{(j)} & \tilde{V}_{0,2}^{(j)} & \cdots & \tilde{V}_{0,K-2}^{(j)} & \tilde{V}_{0,K-1}^{(j)} & \tilde{V}_{0,K}^{(j)} \\ \tilde{V}_{1,0}^{(j)} & \tilde{V}_{1,1}^{(j)} & \tilde{V}_{1,2}^{(j)} & \cdots & \tilde{V}_{1,K-2}^{(j)} & \tilde{V}_{1,K-1}^{(j)} & \tilde{V}_{1,K}^{(j)} \\ O & \tilde{V}_{2,1}^{(j)} & \tilde{V}_{2,2}^{(j)} & \cdots & \tilde{V}_{2,K-2}^{(j)} & \tilde{V}_{2,K-1}^{(j)} & \tilde{V}_{2,K}^{(j)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & \cdots & O & \tilde{V}_{K,K-1}^{(j)} & \tilde{V}_{K,K}^{(j)} \end{pmatrix}$$

where

$$\tilde{V}_{0,k'}^{(j)} = \delta_{0,k'} \gamma N(j) + N(0) \sum_{l=0}^{k'} M(l) \tilde{\Phi}(j-1, k'-l) + M(0) \sum_{l=0}^{j-1} N(l) \tilde{\Phi}(j-l-1, k'),$$

$$0 \leq k' \leq K-2,$$

$$\tilde{V}_{0,K-1}^{(j)} = \delta_{0,K-1} \gamma N(j) + M(0) \sum_{l=0}^{j-1} N(l) \tilde{\Phi}(j-l-1, K-1)$$

$$+ N(0) \left[\sum_{l=0}^{K-1} M(l) \tilde{\Phi}(j-1, K-1-l) + \hat{M}(K) \tilde{\Phi}(j-1, 0) \right], \quad 0 \leq k' \leq K-2,$$

$$\tilde{V}_{0,K}^{(j)} = N(0) \left[\sum_{l=0}^{K-1} M(l) \tilde{\Phi}(j-1, K-l) + \hat{M}(K) \bar{\Phi}(j-1, 1) \right]$$

$$+ M(0) \sum_{l=0}^{j-1} N(l) \bar{\Phi}(j-l-1, K),$$

$$\tilde{V}_{k,k'}^{(j)} = \sum_{l=0}^{k'-k+1} M(l) \tilde{\Phi}(j-1, k'-k-l+1), \quad k = \overline{1, K}, \quad k-1 \leq k' \leq K-2,$$

$$\tilde{V}_{k,K-1}^{(j)} = \sum_{l=0}^{K-k} M(l) \tilde{\Phi}(j-1, K-k-l) + \hat{M}(K-k+1) \tilde{\Phi}(j-1, 0), \quad k = \overline{1, K},$$

$$\tilde{V}_{k,K}^{(j)} = \sum_{l=0}^{K-k} M(l) \bar{\Phi}(j-1, K-k-l+1) + \hat{M}(K-k+1) \bar{\Phi}(j-1, 1), \quad k = \overline{1, K},$$

where

$$\tilde{\Phi}(i, k) = \int_0^\infty P(i, t) \varphi_k(t) (1 - B(t)) dt,$$

$$\bar{\Phi}(i, k) = \int_0^\infty P(i, t) \hat{\varphi}_k(t) (1 - B(t)) dt, \quad i \geq 0, \quad k \geq 0.$$

Now we will focus on finding the matrices $\tilde{Y}_{i,j}$. As it will be seen, the matrix $\tilde{Y}_{i,j}$ depends on the values i, j only via the difference $r = j - i$. Denote $\tilde{Y}_{i,i+r} = \tilde{Y}_r = (\tilde{Y}_{k,k'}^r)_{k,k'=0,K}$.

Then the following statement is true.

Lemma. The matrices \tilde{Y}_r have the following form:

$$\tilde{Y}_r = \begin{pmatrix} \tilde{Y}_{0,0}^{(r)} & \tilde{Y}_{0,1}^{(r)} & \tilde{Y}_{0,2}^{(r)} & \cdots & \tilde{Y}_{0,K-2}^{(r)} & \tilde{Y}_{0,K-1}^{(r)} & \tilde{Y}_{0,K}^{(r)} \\ \tilde{Y}_{1,0}^{(r)} & \tilde{Y}_{1,1}^{(r)} & \tilde{Y}_{1,2}^{(r)} & \cdots & \tilde{Y}_{1,K-2}^{(r)} & \tilde{Y}_{1,K-1}^{(r)} & \tilde{Y}_{1,K}^{(r)} \\ O & \tilde{Y}_{2,1}^{(r)} & \tilde{Y}_{2,2}^{(r)} & \cdots & \tilde{Y}_{2,K-2}^{(r)} & \tilde{Y}_{2,K-1}^{(r)} & \tilde{Y}_{2,K}^{(r)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & \cdots & O & \tilde{Y}_{K,K-1}^{(r)} & \tilde{Y}_{K,K}^{(r)} \end{pmatrix}, \quad r \geq 0,$$

where

$$\tilde{Y}_{0,k'}^{(r)} = \sum_{l=0}^r N(l) \tilde{\Phi}(r-l, k'), \quad k' = \overline{0, K-1},$$

$$\tilde{Y}_{0,K}^{(r)} = \sum_{l=0}^r N(l) \bar{\Phi}(r-l, K),$$

$$\tilde{Y}_{k,k'}^{(r)} = \tilde{\Phi}(r, k' - k + 1), \quad k = \overline{1, K}, \quad k' = \overline{k-1, K-1},$$

$$\tilde{Y}_{k,K}^{(r)} = \bar{\Phi}(r, K - k + 1), \quad k = \overline{1, K}.$$

Theorem The vectors \mathbf{p}_j , $j \geq 0$, are calculated via the vectors $\boldsymbol{\pi}_i$, $i \geq 0$, by the formula

$$\mathbf{p}_j = \lambda \left(\boldsymbol{\pi}_0 \tilde{V}_j + \sum_{r=0}^{j-1} \boldsymbol{\pi}_{j-r} \tilde{Y}_r \right), \quad j \geq 0.$$

Proof follows from [5].

Corollary The vector generating function $\mathbf{p}(z) = \sum_{j=0}^{\infty} \mathbf{p}_j z^j$ of the stationary distribution of the system states at an arbitrary moment is related with the vector generating function $\boldsymbol{\pi}(z)$ of the stationary distribution of the embedded Markov chain by

$$\mathbf{p}(z) = \lambda \left(\boldsymbol{\pi}(z) \tilde{Y}(z) + \boldsymbol{\pi}_0 (\tilde{V}(z) - \tilde{Y}(z)) \right)$$

where $\tilde{V}(z) = \sum_{j=0}^{\infty} \tilde{V}_j z^j$ and $\tilde{Y}(z) = \sum_{j=0}^{\infty} \tilde{Y}_j z^j$,

$$\tilde{Y}(z) = \int_0^{\infty} \Psi_1(t) \otimes e^{D(z)t} (1 - B(t)) dt + \gamma \int_0^{\infty} \Psi_0(t) \otimes (\gamma I - D(z))^{-1} e^{D(z)t} (1 - B(t)) dt.$$

Average number \tilde{L} of customers in the system at an arbitrary moment is given by

$$\tilde{L} = \sum_{i=0}^{\infty} i \mathbf{p}_i \mathbf{e}.$$

Average number \tilde{N} of energy units in the system at an arbitrary moment is given by

$$\tilde{N} = \sum_{i=0}^{\infty} \sum_{k=1}^K k \mathbf{p}(i, k) \mathbf{e}.$$

The probability that customers are absent in the system at an arbitrary moment is given by

$$\tilde{P}_0^{custom} = \mathbf{p}_0 \mathbf{e}.$$

The probability that energy is absent in the system at an arbitrary moment is given by

$$\tilde{P}_0^{energy} = \sum_{i=0}^{\infty} \mathbf{p}(i, 0) \mathbf{e}.$$

The probability that of an arbitrary energy unit loss is given by

$$P_{loss}^{energy} = \sum_{i=0}^{\infty} \mathbf{p}(i, K) \mathbf{e}.$$

5 Stationary distribution of the virtual and actual sojourn times

At the beginning, we analyze the stationary distribution of the virtual sojourn time, i.e., sojourn time of a customer if it will arrive at an arbitrary moment. The virtual sojourn time consists of the *residual service time* from an arbitrary time instant (associated with virtual customer arrival) to the next service completion epoch; the *generalized service times* of customers staying in the queue at an arbitrary time and the *generalized service time* of the virtual customer.

The generalized service time of an arbitrary customer is defined as just the service time of this customer if there is energy in the system at the moment when the customer enters to the server. If the energy at this moment is absent, the generalized service time includes additionally the time interval until an unit of energy arrival.

Let $\hat{\mathbf{B}}(x)$ be the matrix distribution function defined by

$$\hat{\mathbf{B}}(x) = (\hat{B}(x)_{k,k'})_{k,k'=0,\overline{K}}$$

where $\hat{B}(x)_{k,k'} = P\{t_{n+1} - t_n < x, k_{n+1} = k' \mid k_n = k\}$.

Lemma. The matrix Laplace-Stieltjes transform (*LST*) $\mathcal{B}(s) = \int_0^\infty e^{-sx} d\hat{\mathbf{B}}(x)$ of the matrix distribution function $\hat{\mathbf{B}}(x)$ is calculated as

$$\mathcal{B}(s) = \int_0^\infty e^{-st} (\sigma(s)\Psi_0(t) + \Psi_1(t)) dB(t) \quad (14)$$

where $\sigma(s) = \int_0^\infty e^{-st} \gamma e^{-\gamma t} dt = \frac{\gamma}{s+\gamma}$.

Proof. If there is energy in the system at the generalized service time beginning, then evidently the *LST* $(\mathcal{B}(s))_{k,k'}, k > 0$, is calculated by $\int_0^\infty e^{-st} \varphi_{k'-k+1}(t) dB(t)$, if $k' < K$ and by $\int_0^\infty e^{-st} \hat{\varphi}_K(t) dB(t)$, if $k' = K$.

If the energy at the generalized service time beginning in the system is absent, the corresponding *LST* $(\mathcal{B}(s))_{0,k'}$ is defined by $\sigma(s) \int_0^\infty e^{-st} \varphi_{k'}(t) dB(t)$, if $k' < K$ and by $\sigma(s) \int_0^\infty e^{-st} \hat{\varphi}_K(t) dB(t)$, if $k' = K$. The stated result (14) follows immediately. \square

Now we study the *residual service time*. To this end, we consider the process $\chi_t = \{j_t, k_t, \nu_t, \tilde{v}_t\}$, $t \geq 0$, whose components are defined as follows: j_t is the number of customers in the system at time t , k_t is the number of energy units in the system at the time immediately after the next service completion epoch, ν_t is the state of the *MAP*, \tilde{v}_t is the residual time from t to the next service completion epoch.

Using the definition of semi-regenerative processes given in [5], it can be verified that the process χ_t is a semi-regenerative one with the embedded Markov renewal process $\{\xi_n, t_n\}$, $n \geq 1$.

Let

$$\tilde{V}(j, k, \nu, x) = \lim_{t \rightarrow \infty} P\{i_t = j, k_t = k, \nu_t = \nu, \tilde{v}_t < x\}, \quad (15)$$

$$j \geq 0, r = \overline{0, K}, \nu = \overline{0, W}, x \geq 0,$$

be the stationary distribution of the process $\chi_t, t \geq 0$.

From [5], the limits in (15) exist if the process $\{\xi_n, t_n\}, n \geq 1$, is irreducible aperiodic recurrent and the value τ of the mean inter-departure time is finite. All these conditions hold if inequality (9) is satisfied.

Let $\tilde{\mathbf{V}}(j, x)$ be the row vector of the steady state probabilities $\tilde{V}(j, k, \nu, x)$ arranged according to the lexicographic order of the components (k, ν) , and let $\tilde{\mathbf{v}}(j, s)$ be the corresponding vector of the LST s, that is $\tilde{\mathbf{v}}(j, s) = \int_0^\infty e^{-sx} d\tilde{\mathbf{V}}(j, x), j \geq 0$.

To find the vectors $\tilde{\mathbf{v}}(j, s)$ we will express these vectors via the stationary distribution $\boldsymbol{\pi}_i, i \geq 0$, of the Markov chain $\xi_n, n \geq 1$.

To this end, we first calculate the conditional probabilities

$$p_{i,k,\nu}(j, k', \nu', x) = \lim_{t \rightarrow \infty} P\{j_t = j, k_t = k', \nu_t = \nu', \tilde{v}_t < x | j_{t_n} = i, k_{t_n} = k, \nu_{t_n} = \nu\}$$

where t_n is the previous service completion epoch before moment t .

Define the matrices $C_{k,k'}^{(j)}(x) = (p_{0,k,\nu,t}(j, k', \nu', x))_{\nu, \nu' = \overline{0, W}}$ and $\Omega_{k,k'}^{(r)}(x) = (p_{i,k,\nu,t}(i + r, k', \nu', x))_{\nu, \nu' = \overline{0, W}}$, the block matrices $C_j(x) = (C_{k,k'}^{(j)}(x))_{k,k' = \overline{0, K}}$ and $\Omega_r(x) = (\Omega_{k,k'}^{(r)}(x))_{k,k' = \overline{0, K}}$.

$$\text{Let } C_j(s) = \int_0^\infty e^{-sx} dC_j(x) \text{ and } \Omega_r(s) = \int_0^\infty e^{-sx} d\Omega_r(x).$$

Lemma.

The matrix $C_0(s)$ is calculated by

$$C_0(s) = \tilde{V}_0(\mathcal{B}(s) \otimes I_W).$$

The matrices $C_j(s), j \geq 1$, have the following block structure:

$$C_j(s) = \begin{pmatrix} C_{0,0}^{(j)}(s) & C_{0,1}^{(j)}(s) & C_{0,2}^{(j)}(s) & \dots & C_{0,K-1}^{(j)}(s) & C_{0,K}^{(j)}(s) & C_{0,K+1}^{(j)}(s) \\ O & C_{1,1}^{(j)}(s) & C_{1,2}^{(j)}(s) & \dots & C_{1,K-1}^{(j)}(s) & C_{1,K}^{(j)}(s) & C_{1,K+1}^{(j)}(s) \\ O & O & C_{2,2}^{(j)}(s) & \dots & C_{2,K-1}^{(j)}(s) & C_{2,K}^{(j)}(s) & C_{2,K+1}^{(j)}(s) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & \dots & O & C_{K,K}^{(j)}(s) & C_{K,K+1}^{(j)}(s) \end{pmatrix}, j \geq 1,$$

where

$$C_{0,k'}^{(j)}(s) = \gamma N(j)(\mathcal{B}(s))_{0,k'} + N(0) \sum_{l=0}^{k'} M(l) \Psi_{k'-l}(j-1, s)$$

$$+M(0) \sum_{l=0}^{j-1} N(l) \Psi_{k'}(j-l-1, s), \quad 0 \leq k' \leq K-2,$$

$$C_{0,K-1}^{(j)}(s) = \gamma N(j) (\mathcal{B}(s))_{0,K-1} + M(0) \sum_{l=0}^{j-1} N(l) \Psi_{K-1}(j-l-1, s)$$

$$+N(0) \left[\sum_{l=0}^{K-1} M(l) \Psi_{k-l-1}(j-1, s) + \hat{M}(K) \Psi_0(j-1, s) \right],$$

$$C_{0,K}^{(j)}(s) = N(0) \left[\sum_{l=0}^{K-1} M(l) \hat{\Psi}_{K-l}(j-1, s) + \hat{M}(K) \hat{\Psi}_1(j-1, s) \right]$$

$$+M(0) \sum_{l=0}^{j-1} N(l) \hat{\Psi}_K(j-l-1, s),$$

$$C_{k,k'}^{(j)}(s) = \sum_{l=0}^{k'-k+1} M(l) \Psi_{k'-k-l+1}(j-1, s), \quad k = \overline{1, K}, \quad k-1 \leq k' \leq K-2,$$

$$C_{k,K-1}^{(j)}(s) = \sum_{l=0}^{K-k} M(l) \Psi_{K-k-l}(j-1, s) + \hat{M}(K-k+1) \Psi_0(j-1, s), \quad k = \overline{1, K},$$

$$C_{k,K}^{(j)}(s) = \sum_{l=0}^{K-k} M(l) \hat{\Psi}_{K-k-l+1}(j-1, s) + \hat{M}(K-k+1) \hat{\Psi}_1(j-1, s), \quad k = \overline{1, K}.$$

The matrices $\Omega_r(s)$, $r \geq 0$, have the following form:

$$\Omega_r(s) = \begin{pmatrix} \Omega_{0,0}^{(r)}(s) & \Omega_{0,1}^{(r)}(s) & \Omega_{0,2}^{(r)}(s) & \dots & \Omega_{0,K-2}^{(r)}(s) & \Omega_{0,K-1}^{(r)}(s) & \Omega_{0,K}^{(r)}(s) \\ \Omega_{1,0}^{(r)}(s) & \Omega_{1,1}^{(r)}(s) & \Omega_{1,2}^{(r)}(s) & \dots & \Omega_{1,K-2}^{(r)}(s) & \Omega_{1,K-1}^{(r)}(s) & \Omega_{1,K}^{(r)}(s) \\ O & \Omega_{2,1}^{(r)}(s) & \Omega_{2,2}^{(r)}(s) & \dots & \Omega_{2,K-2}^{(r)}(s) & \Omega_{2,K-1}^{(r)}(s) & \Omega_{2,K}^{(r)}(s) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & \dots & O & \Omega_{K,K-1}^{(r)}(s) & \Omega_{K,K}^{(r)}(s) \end{pmatrix},$$

$$\Omega_{0,k'}^{(r)}(s) = \sum_{l=0}^r N(l) \Psi_{k'}(r-l, s), \quad k' = \overline{0, K-1},$$

$$\Omega_{0,K}^{(r)}(s) = \sum_{l=0}^r N(l) \hat{\Psi}_K(r-l, s),$$

$$\Omega_{k,k'}^{(r)}(s) = \Psi_{k'-k+1}(r, s), \quad k' = \overline{k-1, K-1},$$

$$\Omega_{k,K}^{(r)}(s) = \hat{\Psi}_{K-k+1}(r, s), \quad k = \overline{1, K},$$

where

$$\Psi_k(i, s) = \int_0^\infty \varphi_k(t) \int_0^t P(i, x) e^{-s(t-x)} dx dB(t),$$

$$\hat{\Psi}_k(i, s)(s) = \int_0^\infty \hat{\varphi}_k(t) \int_0^t P(i, x) e^{-s(t-x)} dx dB(t) \quad i \geq 0, \quad k \geq 0.$$

Proof of the Lemma 5 is presented in Appendix A.

Using Lemma 5, we express the vectors $\tilde{\mathbf{v}}(j, s)$ via the stationary distribution $\boldsymbol{\pi}_i$, $i \geq 0$, of the Markov chain ξ_n , $n \geq 1$.

Lemma The vectors $\tilde{\mathbf{v}}(j, s)$ are related to the vectors $\boldsymbol{\pi}_i$, $i \geq 0$, of the stationary distribution of the Markov chain ξ_n , $n \geq 1$, as follows:

$$\tilde{\mathbf{v}}(j, s) = \tau^{-1} \left[\boldsymbol{\pi}_0 C_j(s) + \sum_{r=0}^{j-1} \boldsymbol{\pi}_{j-r} \Omega_r(s) \right], \quad j \geq 0,$$

where the mean inter-departure time τ is defined above.

Let $p_j(k, \nu)$ be the steady state probability that at an arbitrary time j customers stay in the system and at the time immediately after the next service completion epoch there are k energy units in the system. Denote by $\mathbf{p}_j(k)$ the vector of probabilities $p_j(k, \nu)$ listed in lexicographic order of component ν , $\nu = \overline{0, W}$.

Corollary The vectors $\mathbf{p}_j(k)$, $j \geq 0$, $k = \overline{0, K}$, are calculated as

$$\mathbf{p}_j(k) = \tilde{\mathbf{v}}(j, 0)(I_{K+1} \otimes \mathbf{e}_{\bar{W}}), \quad j \geq 0, \quad k = \overline{0, K}.$$

Now we are ready to derive the equation for the vector $LST \mathbf{v}(s)$ of the distribution of the virtual sojourn time in the system. Let $v(k, \nu, x)$ be the probability that, at an arbitrary time, the number of energy units just after the end of the virtual sojourn time is k , MAP is in the state ν and the virtual sojourn time in the system is less than x . Then $\mathbf{v}(s)$ is defined as a

vector of Laplace-Stieltjes transforms $v(k, \nu, s) = \int_0^\infty e^{-sx} dv(k, \nu, x)$ written in lexicographic order.

As mentioned above, the virtual sojourn time in the system consists of the residual time from an arbitrary time t to the next service completion epoch, the generalized service times of customers that await for a service at time t , and the generalized service time of the virtual customer. Taking into account the structure of the virtual sojourn time and using the law of total probability we express the vector LST $\mathbf{v}(s)$ in the following theorem

Theorem The vector LST of virtual sojourn time in the system is calculated by

$$\mathbf{v}(s) = \tilde{\mathbf{v}}(0, s) + \sum_{j=1}^{\infty} \tilde{\mathbf{v}}(j, s)(\mathcal{B}^j(s) \otimes I_{\bar{W}}).$$

Having calculated the LST of the virtual sojourn time, we are able to calculate the LST of the distribution of the actual sojourn time in the system.

Theorem The LST of the actual sojourn time distribution in the system is calculated as follows:

$$v^{(a)}(s) = \lambda^{-1} \mathbf{v}(s)(I_{K+1} \otimes D_1) \mathbf{e}.$$

Corollary The mean sojourn time in the system is calculated by

$$\bar{v}^{(a)} = -\lambda^{-1} \frac{d\mathbf{v}(s)}{ds} \Big|_{s=0} (I_{K+1} \otimes D_1) \mathbf{e}.$$

6 Numerical examples

The main aim of the numerical examples is to demonstrate the dependence of some performance measures of the considered system on the capacity K of the stock for energy accumulation and intensity λ of arrival process. The second aim is to show importance of account of correlation in arrival process.

We assume that the MAP is defined by the matrices

$$D_0 = \begin{pmatrix} -0.405780 & 0 \\ 0 & -0.013173 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 0.403080 & 0.002700 \\ 0.007338 & 0.005835 \end{pmatrix}.$$

The intensity of customers arrival is $\lambda = 0.3$. Squared coefficient of variation of inter-arrival times is $c_{var} = 14.33$ and the coefficient of their correlation is $c_{cor} = 0.1724$.

$$\gamma = 0.4$$

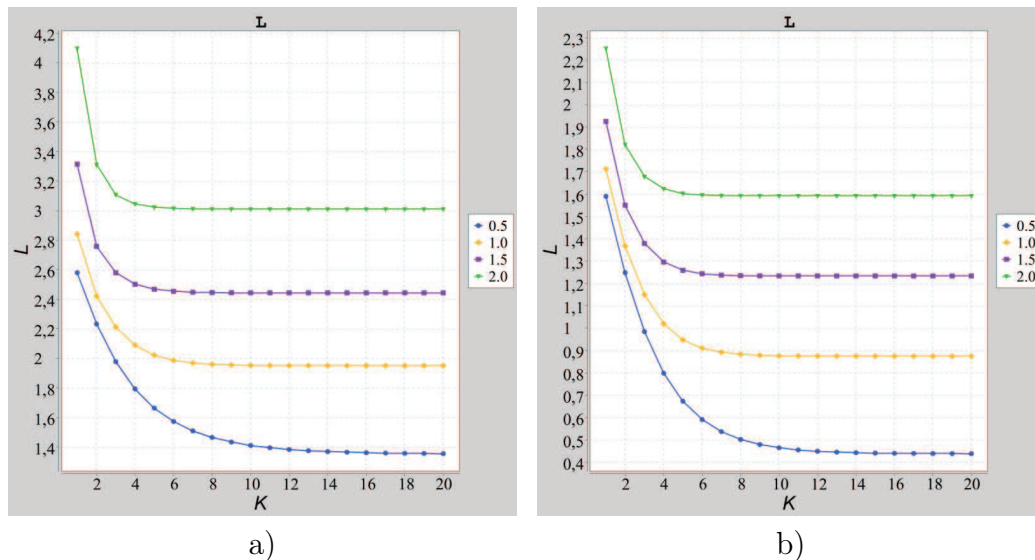


Рис. 1: Dependence of the average number of customers in the system at service completion moments on the capacity K of the stock in cases of the MAP (a) and stationary Poisson arrival process (b)

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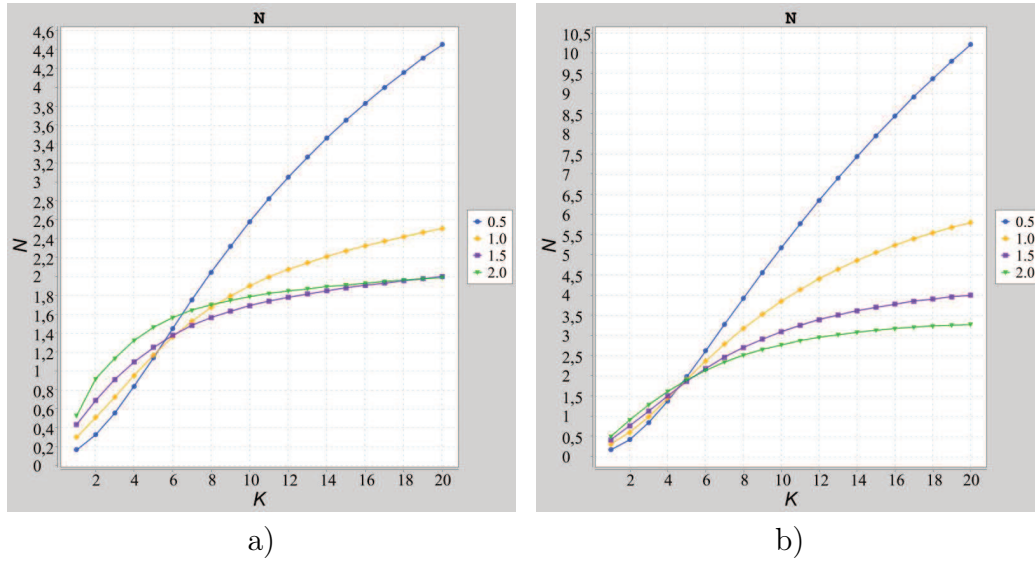


Рис. 2: Dependence of the average number of energy units at service completion moments on the capacity K of the stock in cases of the *MAP* (a) and stationary Poisson arrival process (b)

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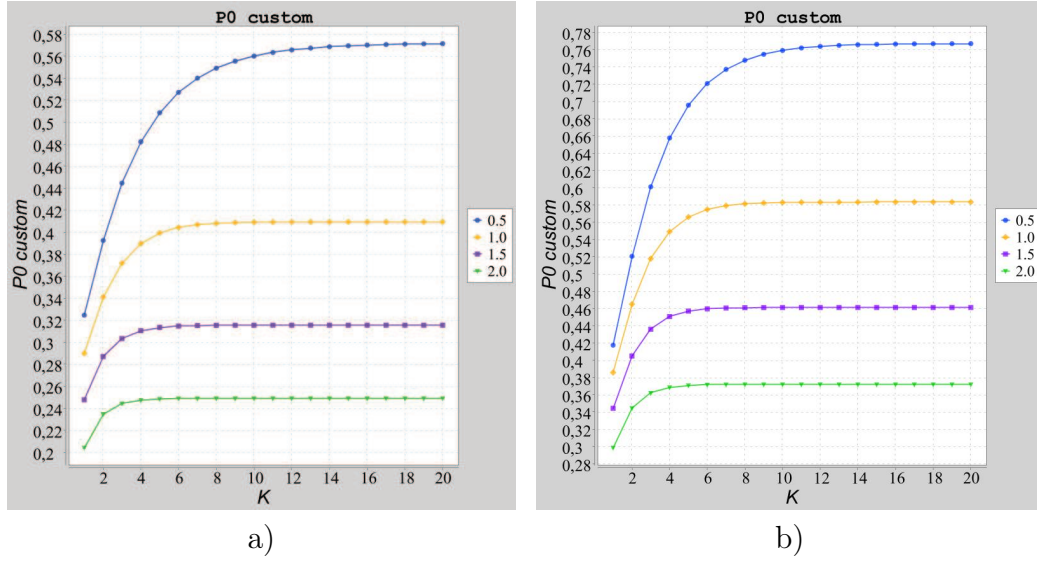


Рис. 3: Dependence of the probability that there is no customer in the system at service completion moments on the capacity K of the stock in cases of the *MAP* (a) and stationary Poisson arrival process (b)

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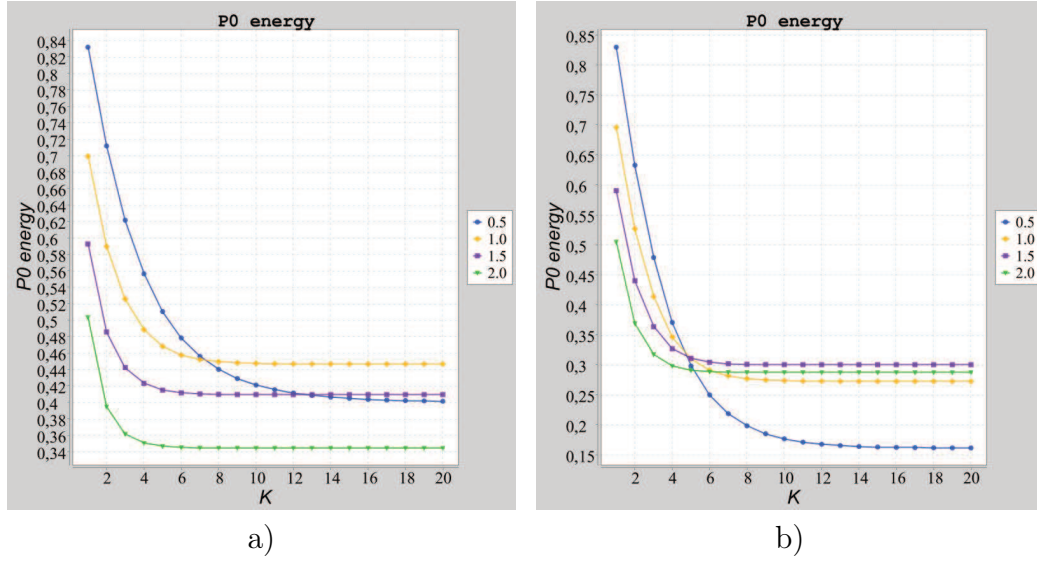


Рис. 4: Dependence of the probability that there is no energy in the system at service completion moments on the capacity K of the stock in cases of the *MAP* (a) and stationary Poisson arrival process (b)

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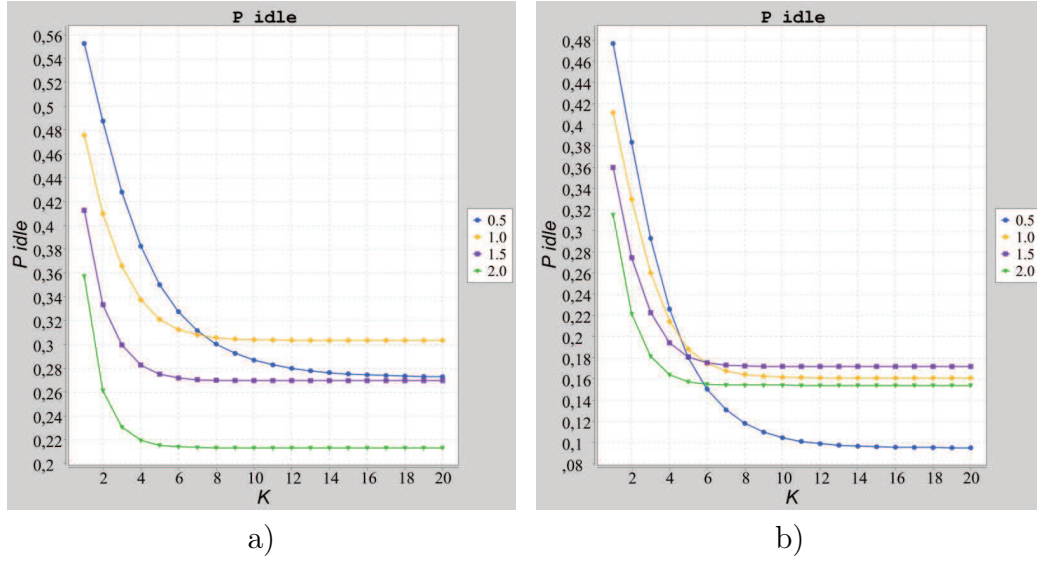


Рис. 5: Dependence of the probability that there the server cannot start service due to the lack of energy in the system at service completion moments on the capacity K of the stock in cases of the *MAP* (a) and stationary Poisson arrival process (b)

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Appendix A. Proof of Lemma 5.

Proof. Using the notations introduced in Lemma 1, we have:

$$\begin{aligned}
C_{k,k'}^0 &= \int_0^\infty e^{D_0 t} \phi_{k'-k}(t) dt, \quad 0 \leq k \leq k' \leq K-1, \\
C_{k,K}^0 &= \int_0^\infty e^{D_0 t} \hat{\phi}_{K-k}(t) dt, \quad 0 \leq k \leq K. \\
C_{0,k'}^{(j)}(s) &= \int_0^\infty P(j,t) \phi_0(t) dt (\mathcal{B}(s))_{0,k'} + \\
&+ \int_0^\infty \int_0^t e^{D_0 x} \gamma e^{-\gamma x} dx e^{D_0(t-x)} \sum_{l=0}^{k'} \phi_l(t-x) D_1 dt \Psi_{k'-l}(j-1, s) + \\
&+ \int_0^\infty \int_0^t e^{D_0 x} \phi_0(x) D_1 dx \sum_{l=0}^{j-1} P(l, t-x) \gamma e^{-\gamma(t-x)} dt \Psi_{k'}(j-l-1, s), \\
&\quad j \geq 1, 0 \leq k' \leq K-2, \\
C_{0,K-1}^{(j)}(s) &= \int_0^\infty P(j,t) \phi_0(t) dt (\mathcal{B}(s))_{0,K-1} + \\
&+ \int_0^\infty \int_0^t e^{D_0 x} \gamma e^{-\gamma x} dx e^{D_0(t-x)} \left[\sum_{l=0}^{K-1} \phi_l(t-x) D_1 dt \Psi_{K-l-1}(j-1, s) dt + \right. \\
&\quad \left. + \sum_{l=K}^\infty \phi_l(t-x) D_1 dt \Psi_0(j-1, s) \right] + \\
&+ \int_0^\infty \int_0^t e^{D_0 x} \phi_0(x) D_1 dx \sum_{l=0}^{j-1} P(l, t-x) \gamma e^{-\gamma(t-x)} dt \Psi_{K-1}(j-l-1, s), \quad j \geq 1, \\
C_{0,K}^{(j)}(s) &= \int_0^\infty \int_0^t e^{D_0 x} \gamma e^{-\gamma x} dx e^{D_0(t-x)} \left[\sum_{l=0}^{K-1} \phi_l(t-x) D_1 dt \bar{\Phi}_{K-l}(j-1, s) \right] dt +
\end{aligned}$$

$$\begin{aligned}
& + \sum_{l=K}^{\infty} \phi_l(t-x) D_1 dt \sum_{l=1}^{\infty} \bar{\Phi}_1(j-1, s)] dt] + \\
& + \int_0^{\infty} \int_0^t e^{D_0 x} \phi_0(x) D_1 dx \sum_{l=0}^{j-1} P(l, t-x) \gamma e^{-\gamma(t-x)} dt \bar{\Phi}_K(j-l-1, s)] dt, j \geq 1;
\end{aligned}$$

$$C_{k,k'}^{(j)}(s) = \int_0^{\infty} e^{D_0 t} \sum_{l=0}^{k'-k+1} \phi_l(t) D_1 dt \Psi_{k'-k-l+1}(j-1, s), \quad k \geq 1, k-1 \leq k' \leq K-2, j \geq 1.$$

$$C_{k,K-1}^{(j)}(s) = \int_0^{\infty} e^{D_0 t} \left[\sum_{l=0}^{K-k} \phi_l(t) D_1 dt \Psi_{K-k-l}(j-1, s) + \sum_{l=K-k+1}^{\infty} \phi_l(t) D_1 dt \Psi_0(j-1, s) \right],$$

$$k \geq 1, j \geq 1.$$

$$C_{k,K}^{(j)}(s) = \sum_{l=0}^{K-k} \int_0^{\infty} e^{D_0 t} \phi_l(t) D_1 dt \sum_{n=K-k-l+1}^{\infty} \bar{\Phi}_{K-k-l+1}(j-1, s) +$$

$$+ \sum_{l=K-k+1}^{\infty} \int_0^{\infty} e^{D_0 t} \phi_l(t) D_1 dt \sum_{n=1}^{\infty} \bar{\Phi}_1(j-1, s), \quad j > 0, k = \overline{1, K},$$

$$\Omega_{0,k'}^{(r)}(s) = \int_0^{\infty} \sum_{l=0}^{(r)} P(l, t) \gamma e^{-\gamma t} dt \tilde{\Phi}_{k'}(r, s), \quad k' = \overline{0, K-1}, r \geq 0,$$

$$\Omega_{0,K}^{(r)}(s) = \int_0^{\infty} \sum_{l=0}^{(r)} P(l, t) \gamma e^{-\gamma t} dt \bar{\Phi}_K(r, s), \quad k' = \overline{K-2, K-1}, r \geq 0,$$

$$\Omega_{k,k'}^{(r)}(s) = \tilde{\Phi}_{k'-k+1}(r, s), \quad k = \overline{1, K}, k' = \overline{k-1, K-1}, r \geq 0,$$

$$\Omega_{k,K}^{(r)}(s) = \bar{\Phi}_{K-k+1}(r, s), \quad k = \overline{k-1, K}, r \geq 0.$$

Appendix B. Computation of the matrices $\tilde{\Phi}(n, k)$, $n \geq 0, k = \overline{0, K}$

These matrices are easily computed, e.g., when service time has phase type distribution. Let us illustrate computation of the matrices $\tilde{\Phi}(n, k)$ in

the case when service time does not have phase type distribution. Namely, we consider the constant service time. In this case, the distribution function $B(t)$ has the form

$$B(t) = \begin{cases} 0, & t \leq b_1, \\ 1, & t > b_1. \end{cases}$$

and, therefore, we have

$$\tilde{\Phi}(n, k) = \int_0^\infty P(n, t) \varphi_k(t) (1 - B(t)) dt = \int_0^{b_1} P(n, t) \varphi_k(t) dt. \quad (16)$$

To calculate the integral in (16), we use the well-known uniformization procedure.

Denote $h = \max_{i=0, \bar{W}} (-D_0)_{ii}$.

Then the matrix $P(n, t)$, $n \geq 1$, can be represented in the following form:

$$P(n, t) = \sum_{j=0}^{\infty} e^{-ht} \frac{(ht)^j}{j!} K_n^{(j)}, \quad n \geq 0, \quad (17)$$

where the matrices $K_n^{(j)}$, $n \geq 1, j \geq 0$, are calculated by recursion

$$\begin{aligned} K_0^{(0)} &= I, \quad K_n^{(0)} = O, \quad n \geq 1, \\ K_0^{(j+1)} &= K_0^{(j)} (I + h^{-1} D_0), \\ K_n^{(j+1)} &= h^{-1} K_{n-1}^{(j)} D_1 + K_n^{(j)} (I + h^{-1} D_0), \quad n \geq 1, \quad j \geq 0. \end{aligned}$$

Substituting expression (17) for $P(n, t)$ and expression for $\varphi_k(t)$ in (16) and simplifying, we obtain the following relations:

$$\begin{aligned} \tilde{\Phi}(n, k) &= \sum_{j=0}^{\infty} K_n^{(j)} \frac{h^j \gamma^k}{j! k!} \int_0^{b_1} e^{-(h+\gamma)t} t^{j+k} dt = \\ &= \sum_{j=0}^{\infty} K_n^{(j)} \frac{h^j \gamma^k}{j! k!} \left[\frac{(j+k)!}{(h+\gamma)^{j+k+1}} - e^{-b_1(h+\gamma)} \sum_{l=0}^{j+k} \frac{(j+k)!}{l!} \frac{b_1^l}{(h+\gamma)^{j+k-l+1}} \right] = \\ &= \sum_{j=0}^{\infty} K_n^{(j)} \frac{h^j \gamma^k}{j! k!} \frac{(j+k)!}{(h+\gamma)^{j+k+1}} \left[1 - e^{-b_1(h+\gamma)} \sum_{l=0}^{j+k} \frac{[b_1(h+\gamma)]^l}{l!} \right] = \\ &= \frac{1}{(h+\gamma)k!} \left(\frac{\gamma}{h+\gamma} \right)^k \sum_{j=0}^{\infty} K_n^{(j)} \frac{(j+k)!}{j!} \left(\frac{h}{h+\gamma} \right)^j \left[1 - e^{-b_1(h+\gamma)} \sum_{l=0}^{j+k} \frac{[b_1(h+\gamma)]^l}{l!} \right]. \end{aligned}$$

Thus, the matrix $\tilde{\Phi}(n, k)$ is equal to

$$= \frac{1}{(h + \gamma)k!} \left(\frac{\gamma}{h + \gamma} \right)^k \sum_{j=0}^{\infty} K_n^{(j)} \frac{(j + k)!}{j!} \left(\frac{h}{h + \gamma} \right)^j \left[1 - e^{-b_1(h + \gamma)} \sum_{l=0}^{j+k} \frac{[b_1(h + \gamma)]^l}{l!} \right].$$