Differentially Private Marginals

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Abstract

We give a differentially private algorithm to output approximate k-way marginals of tabular data. Marginals are the counts of various tuples in the data, such as the number of user records x such that $(x_i, x_j) = (a, b)$ (which is a 2-way marginal). We are interested in computing approximate k-way marginals for $k = 1, 2, \ldots, T$. The algorithm is inspired by differentially private n-gram extraction [KGKY21].

1 An Algorithm for DP Marginals

In this section we describe our algorithm for DP Marginals. The pseudocode is presented in Algorithm 1. Given private tabular data, we will assume that the rows are indexed by users and the columns denote various attributes. A row is also called a record. A generic record is denoted by X, where the i^{th} column of the record is X_i. A *k-tuple* is defined by a set of k columns along with possible values for each of them, i.e., a k-tuple looks like $(X_{i_1} = a_1, X_{i_2} = a_2, \ldots, X_{i_k} = a_k)$. A record can have many empty columns, so we will denote the empty value with $\perp$. We denote by $\mathrm{wt}(X)$ to be the number of non-empty columns in X. The set of all non-empty[1] k-tuples contained in a record X is denoted by $M_k(X)$. Therefore $|M_k(X)| = \binom{\mathrm{wt}(X)}{k}$. For example if $X = (a, b, \perp, a)$, then $\mathrm{wt}(X) = 3$ and $M_2(X) = \{(X_1 = a, X_2 = b), (X_1 = a, X_4 = a), (X_2 = b, X_4 = a)\}$.

The algorithm iteratively extracts k-tuples for $k = 1, 2, \ldots, T$, i.e., the algorithm uses the already extracted $(k-1)$-tuples to extract k-tuples. Let S_k denote the extracted set of k-tuples.

1.1 Controlling spurious n-tuples using ρ_k:

In our privacy analysis (Section 1.2), we will show that the privacy of the DP Marginals algorithm depends only on ρ_1 and $\sigma_1, \sigma_2, \ldots, \sigma_T$. In particular, $\rho_2, \ldots, \rho_T$ do not affect privacy. Instead, they are used to control the number of *spurious* k-tuples that we extract, i.e., k-tuples which are not actually used by any user but output by the algorithm.

Proposition 1.1. For $k \geq 2$, the expected number of spurious k-tuples output by Algorithm 1 is at most $|V_k|(1 - \Phi(\rho_k/(\sigma_k \Delta_k^{1/2})))$ where Φ is the Gaussian CDF. And the algorithm will not output any spurious 1-tuples.

Proof. A spurious k-tuple will have zero weight in the histogram H_k that the algorithm builds. So after adding $N(0, \Delta_k \sigma_k^2)$ noise, the probability that it will cross the threshold ρ_k is exactly $1 - \Phi(\rho_k/(\sigma_k \Delta_k^{1/2}))$.

□

Larger we set ρ_k, smaller the number of spurious k-tuples. But setting large ρ_k will reduce the number of non-spurious k-tuples extracted by the algorithm. So $\rho_2, \ldots, \rho_T$ should be set delicately

[1]None of the k values should be empty.

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Algorithm 1: Algorithm for differentially private marginals

Input: A set of N users where each user u has a record X^u
T: maximum length of marginals to be extracted
Q: Percentile parameter to estimate maximum contributions (can set it to 99% for example)
η: fraction of spurious k-tuples in the output
$\sigma_1, \sigma_2, \ldots, \sigma_T$: Noise parameters
ε_Q: ε for DP Q^{th}-percentile
δ : final δ value in the (ε, δ)-DP guarantee

Output: $S_1, S_2, \ldots, S_T$ where S_k is a set of k-tuples extracted and $H_1, H_2, \ldots, H_T$ which are noisy marginals of $S_1, S_2, \ldots, S_T$

// Learn 1-tuples and 1-way marginals
for $u \in [N]$ **do**
  $W_u \leftarrow M_1(X^u)$; // non-empty 1-tuples with user u
$\Delta_1 \leftarrow$ Compute Q^{th}-percentile of $\{|W_1|, |W_2|, \ldots, |W_N|\}$ using ε_Q-DP mechanism;
// Limit user contributions
if $|W_u| > \Delta_1$ **then**
  $W_u \leftarrow$ Randomly choose Δ_1 items from W_u;
$V_1 \leftarrow W_1 \cup W_2 \cup \cdots \cup W_N$
$\rho_1 \leftarrow 1 + \sigma_1 {\Delta_1}^{1/2} \Phi^{-1}\left(\left(1 - \frac{\delta}{2}\right)^{1/\Delta_1}\right)$;
$S_1, H_1 \leftarrow \mathsf{NoisyThresholding}\left((W_u : u \in [N]), \Omega = V_1, \Delta = \Delta_1, \rho = \rho_1, \sigma = \sigma_1 {\Delta_1}^{1/2}\right)$;
// Iteratively learn k-tuples and k-way marginals
for $k = 2$ ***to*** T **do**
  // Calculate valid k-tuples
  $V_k \leftarrow$ All possible k-tuples whose $(k-1)$-subtuples belong to S_{k-1};
  // Prune away invalid k-tuples
  for $u \in [N]$ **do**
    $W_u \leftarrow M_k(X^u) \cap V_k$; // non-empty k-tuples with user u which are also valid
  $\Delta_k \leftarrow$ Compute Q^{th}-percentile of $\{|W_1|, |W_2|, \ldots, |W_N|\}$ using ε_Q-DP mechanism;
  $\rho_k \leftarrow \sigma_k \Delta_k^{1/2} \Phi^{-1}\left(1 - \eta \min\left\{1, \frac{|S_{k-1}|}{|V_k|}\right\}\right)$;
  $S_k, H_k \leftarrow \mathsf{NoisyThresholding}\left((W_u : u \in [N]), \Omega = V_k, \Delta = \Delta_k, \rho = \rho_k, \sigma = \sigma_k \Delta_k^{1/2}\right)$;
Output $S_1, S_2, \ldots, S_T$ and $H_1, H_2, \ldots, H_T$;

to balance this tension. One convenient choice of ρ_k for $k \geq 2$ is to set,

$$\rho_k = \Delta_k^{1/2} \sigma_k \Phi^{-1}\left(1 - \eta \min\left\{1, \frac{|S_{k-1}|}{|V_k|}\right\}\right)$$

for some $\eta \in (0, 1)$. This implies that the expected number of spurious k-tuples output is at most $\eta \min\{|S_{k-1}|, |V_k|\}$ by Proposition 1.1. And the total number of spurious k-tuples output is at most $\eta(|S_1| + |S_2| + \cdots + |S_{T-1}|)$. Therefore spurious k-tuples output by the algorithm are at most an η-fraction of all the k-tuples output.

Remark 1.1. One can also completely eliminate spurious all spurious k-tuples if we set

$$\rho_k = 1 + \sigma_k \Delta_k^{1/2} \Phi^{-1}\left(\left(1 - \frac{\delta}{2T}\right)^{1/\Delta_k}\right)$$

for $k = 1, 2, \ldots, T$, and while extracting k-tuples, we set

$$\Omega = W_1 \cup W_2 \cup \cdots \cup W_N$$

where we prune each W_i to have size at most Δ_k. But typically, this will increase the thresholds beyond what we set in Algorithm 1 which will reduce the number of k-tuples we learn.

1.2 Privacy Analysis

We are now ready to prove the privacy of our DP Marginals algorithm.

Algorithm 2: NoisyThresholding $((W_u : u \in [N]), \Omega, \Delta, \rho, \sigma)$

Input: A set of N users where each user u has some subset $W_u \subset \Omega$
Δ: maximum contribution parameter
ρ: Threshold parameter
σ: Noise parameter.
Output: S: Set of items extracted and H: a histogram of noisy counts of items in S

```
// Initialize histogram to 0 for all items in the universe Ω
```
for ω *in* Ω **do**
  $H[\omega] \leftarrow 0$;

```
// Build histogram by limiting user contributions to Δ
```
for $u = 1$ ***to*** N **do**
  `// Limit user contributions`
  if $|W_u| > \Delta$ **then**
    $W_u \leftarrow$ Randomly choose Δ items from W_u;
  for ω *in* W_u **do**
    $H[\omega] \leftarrow H[\omega] + 1$;

```
// Add noise to H
```
for $\omega \in \Omega$ **do**
  $H[\omega] \leftarrow H[\omega] + \sigma N(0,1)$

```
// Output items which cross the threshold ρ
```
$S = \{\}$ (empty set);
for $\omega \in \Omega$ **do**
  if $H[\omega] > \rho$ **then**
    $S \leftarrow S \cup \{\omega\}$;
$H \leftarrow H|_S$; `// Remove items not in` S `from` H
Output S, H

Theorem 1.1. Let $\varepsilon > 0, 0 < \delta < 1$. Then Algorithm 1 satisfies (ε, δ)-DP if

$$\frac{T\varepsilon_Q^2}{2} + \frac{1}{2}\sum_{i=1}^{T}\frac{1}{\sigma_i^2} \leq \left(\sqrt{\varepsilon + \log(2/\delta)} - \sqrt{\log(2/\delta)}\right)^2.$$

Proof. The privacy of learning 1-tuples in Algorithm 1 is given by the composition of a Gaussian mechanism with ℓ_2-sensitivity 1 and noise σ_1 composed with $(0, \delta/2)$-algorithm as shown in [GGK+20] if we set ρ_1 as shown. The construction of k-tuples is a Gaussian mechanism with ℓ_2-sensitivity 1 and noise σ_k. We now combine these privacy guarantees using zCDP framework from [BS16] to get $(\varepsilon, \delta/2)$-DP. Each quantile algoritm is $(\varepsilon_Q, 0)$-DP, and so satisfies $(\varepsilon_Q^2/2, 0)$-zCDP. The Gaussian mechanism to compute k-tuples satisfies $\frac{1}{2\sigma_k^2}$-zCDP. By additivity of zCDP under composition [BS16], we get that the final algorithm is ρ-zCDP with $\rho = \frac{T\varepsilon_Q^2}{2} + \frac{1}{2}\sum_{i=1}^{T}\frac{1}{\sigma_i^2}$. Finally a ρ-zCDP algorithm is $(\varepsilon, \delta/2)$-DP whenever $\sqrt{\rho} \leq \sqrt{\varepsilon + \log(2/\delta)} - \sqrt{\log(2/\delta)}$ by Prop 1.3 from [BS16]. □

Remark 1.2. Let $\rho = \left(\sqrt{\varepsilon + \log(2/\delta)} - \sqrt{\log(2/\delta)}\right)^2$. To begin with, we suggest using only 10% of the privacy budget for finding the Q^{th} percentiles, i.e., we set ε_Q and $\sigma_1 = \cdots = \sigma_T$ such that $T\varepsilon_Q^2/2 = \rho/10$ and $\frac{1}{2}\sum_{i=1}^{T} 1/\sigma_i^2 = 9\rho/10$.

Remark 1.3. One can improve the privacy bound using numerical composition from [GLW21]. We think this will further reduce ε by an additive 1 or 2 for a fixed δ.

Remark 1.4. Note that the choice of $\rho_2, \ldots, \rho_k$ doesn't affect the privacy parameters. It only affects the fraction of spurious k-tuples in the output. We are using η parameter to control the fraction of spurious k-tuples, and $\rho_2, \ldots, \rho_k$ are set automatically based on η.

References

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