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Question 1		Mark Allocation

(a) Transfer function (TF) $= \dfrac{C(s)}{R(s)}$

where $C(s)$ – $\mathcal{L}$ {system output}
$R(s)$ – $\mathcal{L}$ {system Input}
$\mathcal{L}$ – Laplace transform

For system output to be an impulse response, $R(s) = 1$

$\therefore$ TF $= C(s)$

$= \mathcal{L}\, h(t)$

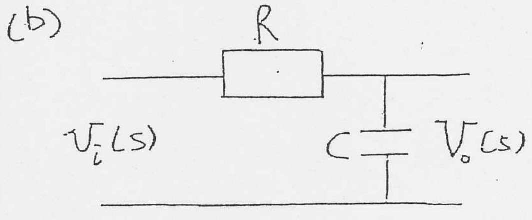
$= \dfrac{2}{(s+2)^2 + 2^2}$ (from Laplace Transform table)

$= \dfrac{2}{s^2+4s+8}$ 6

$\therefore$ The poles (roots of the denominator of TF) are at: $s = -2 \pm j2$ 2

Since the complex conjugate pair of poles are on left-half of s-plane, the system is stable. 2

(b)



$V_o(s) = \dfrac{I(s)}{sC}$

where $I(s)$ is Laplace transform of current in the circuit

$V_i(s) = R\,I(s) + \dfrac{I(s)}{sC}$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{I(s)}{sC}}{RI(s) + \frac{I(s)}{sC}}$$

$$= \frac{1}{sRC+1}$$

$$= \frac{1}{1+sT} \quad \text{where } T = RC.$$ 5

For a unit ramp input $V_i(s) = \frac{1}{s^2}$

i.e. $V_o(s) = \frac{1}{1+sT} \times \frac{1}{s^2}$

$$= \frac{T}{s+\frac{1}{T}} - \frac{T}{s} + \frac{1}{s^2}$$

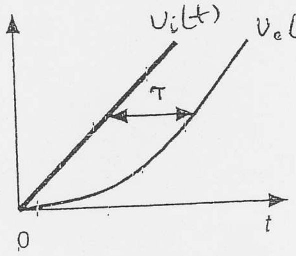
$$\therefore v_o(t) = Te^{-\frac{t}{T}} - T + t$$ (using Laplace Transform table)

$$= t - T(1 - e^{-\frac{t}{T}})$$ 5

The dynamic response of the servo to a unit ramp input is as show below, where the output $v_o(t)$ lags the input $v_i(t)$. As $t \to \infty$,

$v_o(t) \to v_i(t-T)$,

i.e., value of the output at time t is equal to value of the input at time $t-T$. In other words,



$v_o(t)$ always lags $v_i(t)$ by a time equal to T for large t. The smaller T is, the shorter is the time lag. 5

Total = 25 marks

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Question 2		Mark Allocation

Given poles at $z = \frac{1}{4} \pm j\frac{1}{4}$ and zeros at $z = -1 \pm j$,

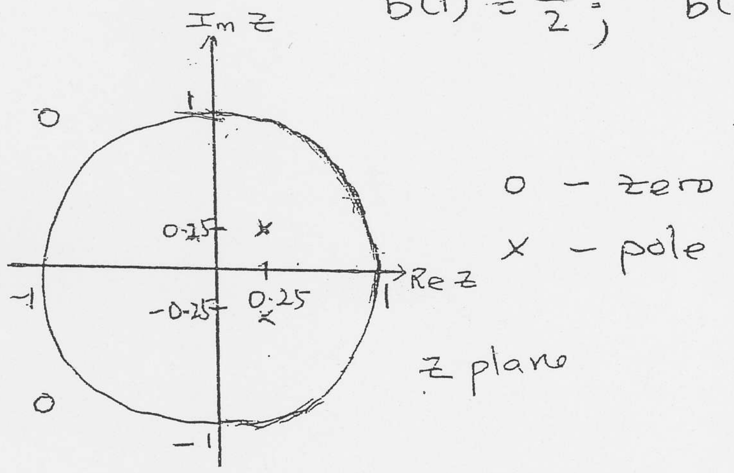
$$H(z) = \frac{(z+1-j)(z+1+j)}{(z-\frac{1}{4}-j\frac{1}{4})(z-\frac{1}{4}+j\frac{1}{4})}$$

$$= \frac{z^2+2z+2}{z^2-\frac{1}{2}z+\frac{1}{8}}$$

$$= \frac{1+2z^{-1}+2z^{-2}}{1-(\frac{1}{2}z^{-1}-\frac{1}{8}z^{-2})}$$ 7

cf. $H(z) = \dfrac{\sum_{m=0}^{2} a(m) z^{-m}}{1-\sum_{m=1}^{2} b(m) z^{-m}}$ where $a(m)$ and $b(m)$ are filter coefficients

$\Rightarrow$ $a(0) = 1$; $a(1) = 2$; $a(2) = 2$
$b(1) = \frac{1}{2}$; $b(2) = -\frac{1}{8}$ 5



3

pole-zero plot

$$|H(0)| = \frac{\sqrt{2^2+1^2} \times \sqrt{2^2+1^2}}{\sqrt{0.75^2+0.25^2} \times \sqrt{0.75^2+0.25^2}} = 8$$ 2

$$\left|H\left(\frac{\pi}{2}\right)\right| = \frac{1 \times \sqrt{2^2+1^2}}{\sqrt{0.25^2+0.75^2} \times \sqrt{0.25^2+1.25^2}} = 2.2188$$ 2

$$|H(\pi)| = \frac{1 \times 1}{\sqrt{1.25^2+0.25^2} \times \sqrt{1.25^2+0.25^2}} = 0.6154$$ 2

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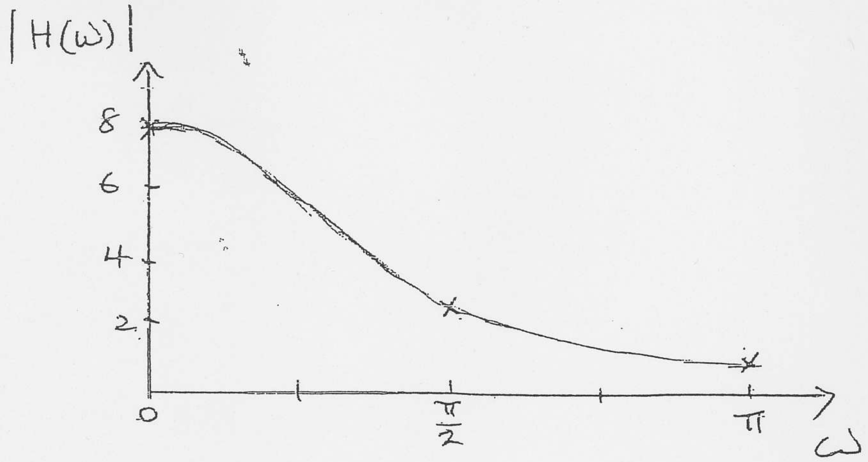
Question 2 (continued)		Mark Allocation



2

The filter is lowpass because it passes (amplifies) low frequencies and attenuates high frequencies. 2

Total = 25 marks

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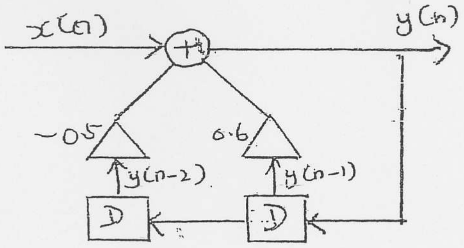
Question 3		Mark Allocation



Difference equation

$\{y(n)\} = \{x(n)\} + 0.6\{y(n-1)\} - 0.5\{y(n-2)\}$ — ① 5

For impulse response

$\{x(n)\} = \{1, 0, 0, 0, 0, 0, 0, 0, 0\}$ 2

Assuming quiescent condition:

$\{x(n)\}$	$\{y(n-2)\}$	$\{y(n-1)\}$	$\{y(n)\}$
1	0	0	1
0	0	1	0.6
0	1	0.6	−0.1
0	0.6	−0.1	−0.4
0	−0.1	−0.4	−0.2
0	−0.4	−0.2	0.1
0	−0.2	0.1	0.2
0	0.1	0.2	0.1
0	0.2	0.1	0.0

Impulse response

$\{y(n)\} = \{1, 0.6, -0.1, -0.4, -0.2, 0.1, 0.2, 0.1, 0.0\}$. 6

From ① :

$$Y(z) = X(z) + 0.6\,z^{-1}Y(z) - 0.5\,z^{-2}Y(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1-0.6z^{-1}+0.5z^{-2}}$$

$$= \frac{z^2}{z^2-0.6z+0.5}$$ 4

Zeros are roots of numerator of transfer function
∴ double zeros at $z = 0$ 2

poles are roots of denominator of transfer function
∴ poles are at $z = -0.3 \pm j0.64$ 2
Since the complex conjugate pair of poles are within the unit circle of the z-plane pole-zero plot, i.e. $|z| = 0.71 < 1$, the system is a stable system. 2

Since the impulse response tends to 0 as n tends to infinity, the system is stable. 2

Total = 25 marks

(a) $R_{xx}(\tau) = E(x(t) . x(t+\tau))$

At $\tau = 0$ $P(1,1) = 0.6$

$P(0,0) = 0.4$

$P(0,1) = P(1,0) = 0$

But only $x(t) = x(t+0) = 1$ will contribute to the ACF so $R_{xx}(0) = P(1,1) = 0.6$ — 4

After switching points $t = \lambda, 2\lambda, 3\lambda \text{ -- } n\lambda$

$P(1,1) = 0.6 \times 0.6 = 0.36$

$P(0,0) = 0.4 \times 0.4 = 0.16$

$P(1,0) = 0.6 \times 0.4 = 0.24$

$P(0,1) = 0.4 \times 0.6 = 0.24$

1.00 ✓

But again only $x(t) = x(t+\tau) = 1$ contributes to $R_{xx}(\tau)$, so $\tau \geq \lambda$, $R_{xx}(\tau) = E[x(t).x(t+\tau)]$

$= P(1,1) = 0.36$

For $\tau < \lambda$ the probability of an event in the interval $t \to t+\lambda = \frac{\tau}{\lambda}$

(ie passing a switching point) — 5

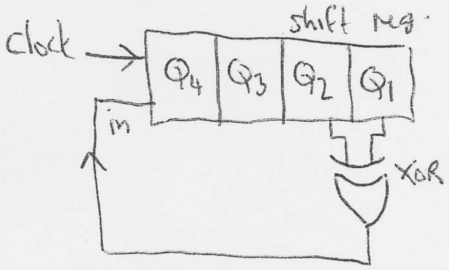
In which case $E[x(t) \cdot x(t+\tau)] = 0.36$, or the probability of no event is $\left(1 - \frac{\tau}{\lambda}\right)$, in which case

$$E[x(t) \cdot x(t+\tau)] = P(1,1)\Big|_{\tau=0} = 0.6$$

Thus $R_{xx}(\tau < \lambda) = \left(1 - \frac{\tau}{\lambda}\right)0.6 + \left(\frac{\tau}{\lambda}\right)0.36$

ACF is an even function so $R_{xx}(\tau < \lambda) = 0.6 - 0.24\left(\frac{|\tau|}{\lambda}\right)$ 5

(b) Generator



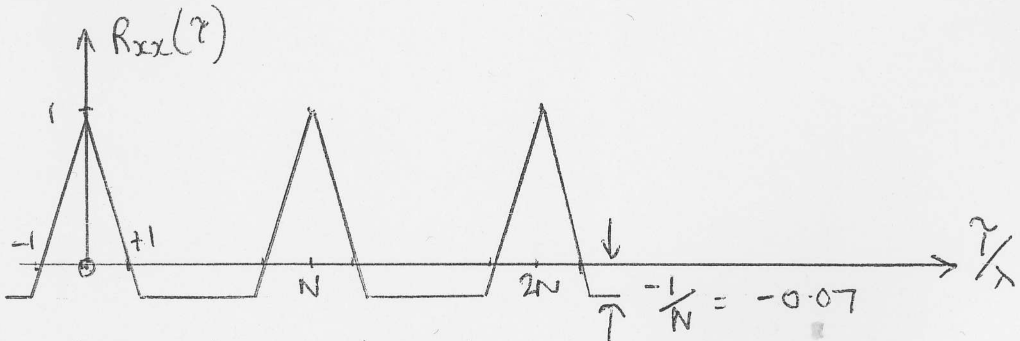
Sequence

Q_1	Q_2	Q_3	Q_4	Decimal
0	0	0	1	1
0	0	1	0	2
0	1	0	0	4
1	0	0	1	9

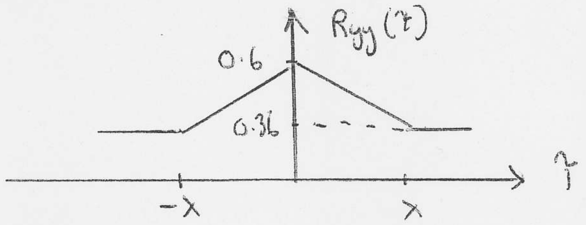
etc ...

Take one bit, say Q_1, as binary output. Start with any number except 0000 to get maximum length. 4

ACF of PRBS is



ACF of discrete interval BS is



- ACF of PRBS repeats periodically whereas for Discrete interval it does not.
- ACF of PRBS has max = 1, DIBS max. = 0.6
- ACF of PRBS has min = −0.07, DIBS min = +0.36
- ACF of PRBS crosses horizontal axis at $\pm\lambda$, DIBs, bottom of slope at $\pm\lambda$ — 4

<u>Examples of Use</u> – one from say :-

Monte Carlo

Injection into fluid flow to measure speed

Seismology

Determine noise transition path in vehicles

System identification

One paragraph sufficient — 3

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Question 5	Mark Allocation

(a) Using $G(e^{j\omega T}) = \sum_{k=-\infty}^{\infty} g[k] e^{-jk\omega T}$

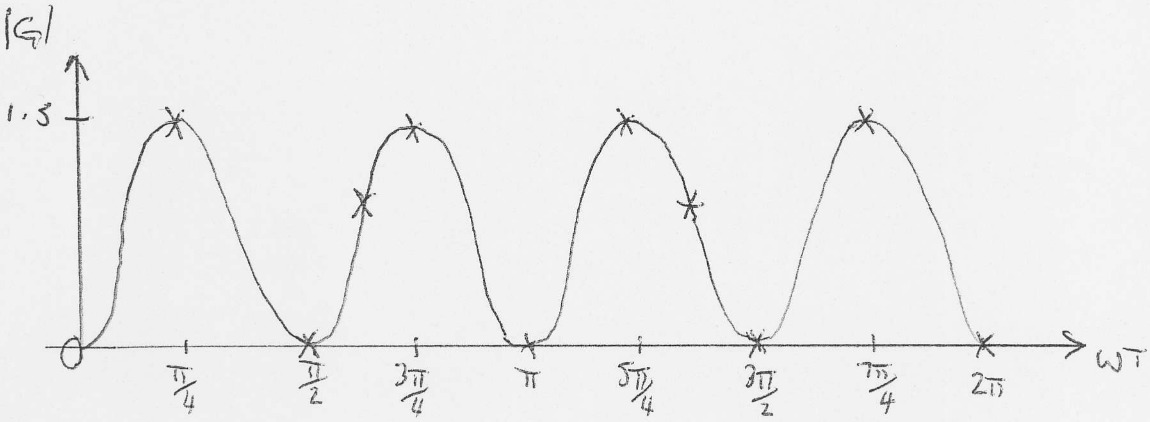
$$G(e^{j\omega T}) = -0.5 e^{+j\omega 3T} - 0.5 e^{+j\omega T} + 0.5 e^{-j\omega T} + 0.5 e^{-j\omega 3T}$$

$$= -0.5 \left\{ \left(e^{j\omega 3T} - e^{-j\omega 3T} \right) + \left(e^{j\omega T} - e^{-j\omega T} \right) \right\}$$

$$= -j \left(\sin 3\omega T + \sin \omega T \right)$$

<u>Magnitude</u> $|G(e^{j\omega T})| = |\sin 3\omega T + \sin \omega T|$ 7

ωT	0	$\pi/4$	$\pi/2$	$2\pi/3$	$3\pi/4$	π	$5\pi/4$	$4\pi/3$	$3\pi/2$	$7\pi/4$	2π
$\sin 3\omega T$	0	0.707	−1	0	0.707	0	−0.707	0	1	−0.707	0
$\sin \omega T$	0	0.707	1	0.866	0.707	0	−0.707	−0.866	−1	−0.707	0
$\lvert G(e^{j\omega T})\rvert$	0	1.414	0	0.866	1.414	0	+1.414	+0.866	0	1.414	0



6

(b) $y = [1.000 \;\; 0.760 \;\; 0.156 \;\; -0.522 \;\; -0.951 \;\; -0.924 \;\; -0.454 \;\; +0.233]$

$N = 8$

$$Y[m] = \sum_{n=0}^{7} y[n] \exp\left(-j\frac{mn\pi}{4}\right)$$

$$Y[0] = \sum_{n=0}^{7} y[n] = \underline{\underline{-0.702}} \quad \text{note } m=0$$ 2

$$Y[1] = y[0] + y[1]e^{-j\pi/4} + y[2]e^{-j\frac{\pi}{2}} + y[3]e^{-j\frac{3\pi}{4}} + y[4]e^{-j\pi} + y[5]e^{-j\frac{5\pi}{4}} + y[6]e^{-j\frac{3\pi}{2}} + y[7]e^{-j7\pi/4}$$

$y[0] = 1.000$

$y[1]e^{-j\pi/4} \rightarrow 0.760\left(\cos\frac{\pi}{4} - j\sin\frac{\pi}{4}\right) = 0.760(0.707 - j0.707) = 0.537 - j0.537$

$y[2]e^{-j\frac{\pi}{2}} \rightarrow 0.156\left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right) = 0.156(0 - j) = -j0.156$

$y[3]e^{-j\frac{3\pi}{4}} \rightarrow -0.522\left(\cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}\right) = 0.369 + 0.369j$

$y[4]e^{-j\pi} \rightarrow -0.951\left(\cos\pi - j\sin\pi\right) = 0.951 + 0j$

$y[5]e^{-j\frac{5\pi}{4}} \rightarrow -0.924\left(\cos\frac{5\pi}{4} - j\sin\frac{5\pi}{4}\right) = 0.653 - 0.653j$

$y[6]e^{-\frac{3\pi}{2}} \rightarrow -0.454\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right) = 0 - 0.454j$

$y[7]e^{-j\frac{7\pi}{4}} \rightarrow 0.233\left(\cos\frac{7\pi}{4} - j\sin\frac{7\pi}{4}\right) = 0.165 + 0.165j$

Sum to get $Y[1] = \underline{\underline{3.675 - 1.268j}}$ 3

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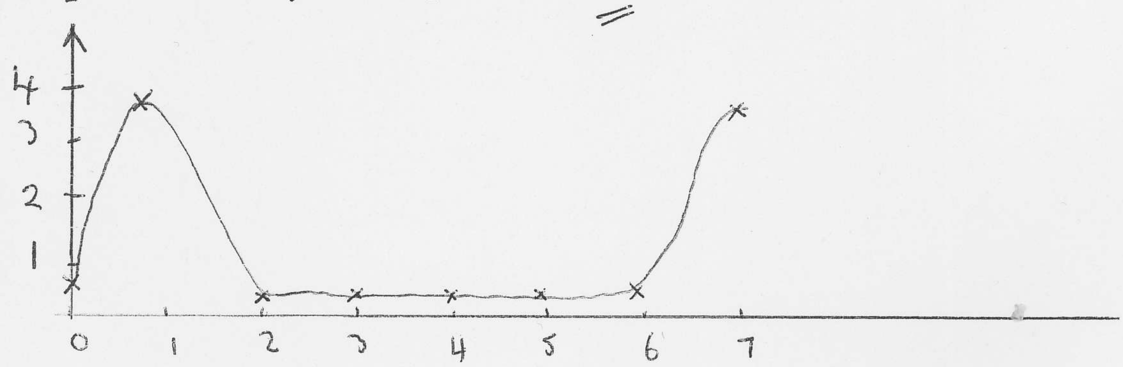
Question 5 cont.		Mark Allocation

$|Y[0]| = 0.702$, $|Y[1]| = |3.675 - 1.267j| = 3.89$,

$|Y[2]| = 0.37$, $|Y[3]| = 0.23$, $|Y[4]| = 0.20$, $|Y[5]| = 0.23$,

$|Y[6]| = 0.37$, $|Y[7]| = 3.89$. 1



2

This is not a line spectrum because we have broadening of the peaks. Values away from $|Y[1]|$ and $|Y[7]|$ not zero. Spectral leakage because start and end samples not identical. 2 Could use a window function to reduce end samples to zero. 2

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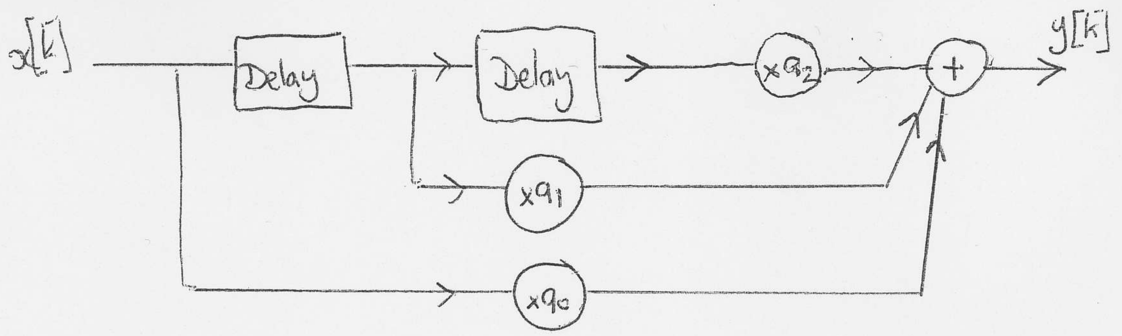
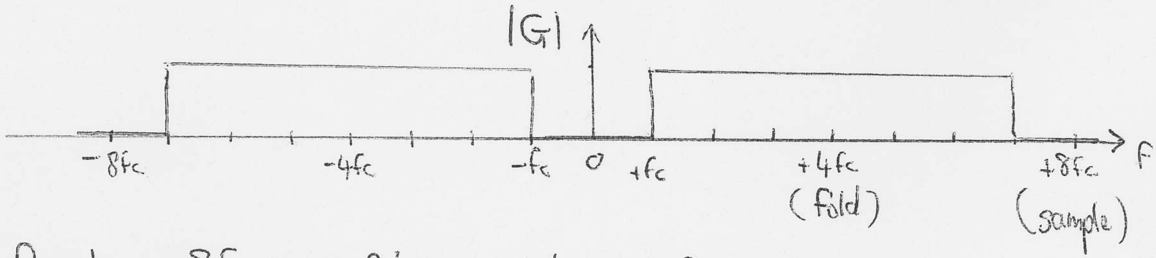
Question 6 | Mark Allocation



$$y[k] = x[k]a_0 + x[k-1]a_1 + x[k-2]a_2$$ 3

Periodically extended filter response



Period $= 8f_c = 2L$, $L = 4f_c$

Fourier series

$$a_n = \frac{1}{4f_c}\int_{-4f_c}^{4f_c} G(f)\cos\frac{n\pi f}{4f_c}\,df = \frac{2}{4f_c}\int_{f_c}^{4f_c}\cos\frac{n\pi f}{4f_c}\,df$$

$$= \frac{1}{2f_c}\left[\frac{4f_c}{n\pi}\sin\frac{n\pi f}{4f_c}\right]_{f_c}^{4f_c}$$

$$= \frac{2}{n\pi}\Big(\underbrace{\sin n\pi}_{0} - \sin\frac{n\pi}{4}\Big) = -\frac{2}{n\pi}\sin\frac{n\pi}{4}$$

and $g_0 = a_0 =$ mean of $|G| = 6/8 = 0.75$; //

$$g_{\pm n} = \frac{a_n}{2} = \frac{-1}{n\pi}\sin\frac{n\pi}{4}$$ // 7

Truncating at $n = \pm 4$

n	g_n	Blackman weight	Windowed $g_n \cdot w_n$
0	0.75	$0.42 + 0.5 + 0.08 = 1.000$	0.750
± 1	-0.225	$0.42 + 0.5\cos\frac{2\pi}{9} + 0.08\cos\frac{4\pi}{9} = 0.817$	-0.184
± 2	-0.159	$0.42 + 0.5\cos\frac{4\pi}{9} + 0.08\cos\frac{8\pi}{9} = 0.432$	-0.069
± 3	-0.075	$0.42 + 0.5\cos\frac{6\pi}{9} + 0.08\cos\frac{12\pi}{9} = 0.130$	-0.010
± 4	0	– –	0.000

4

So $G(z) = -0.010z^3 - 0.069z^2 - 0.184z + 0.75 - 0.184z^{-1} - 0.069z^{-2} - 0.010z^{-3}$

$$G(e^{j\omega T}) = -2\left(0.01\cos 3\omega T + 0.069\cos 2\omega T + 0.184\cos \omega T\right) + 0.75$$ 1

At dc $(\omega = 0)$, $G = 0.75 - 2(0.01 + 0.069 + 0.184) = \underline{\underline{0.224}}$ 1

At folding frequency $\left(\omega = \frac{\pi}{T}\right)$ $G = 0.75 - 2(-0.01 + 0.069 - 0.184) = \underline{\underline{1.000}}$ 1

Causal Filter

$G(z) = -0.01z^0 - 0.069z^{-1} - 0.184z^{-2} + 0.75z^{-3} - 0.184z^{-4} - 0.069z^{-5} - 0.01z^{-6}$

$y[k] = -0.01x[k] - 0.069x[k-1] - 0.184x[k-2] + 0.75x[k-3] - 0.184x[k-4] - 0.069x[k-5] - 0.01[k-6]$ 2

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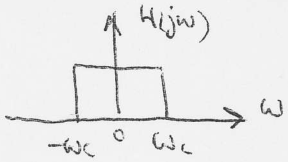
Question 6 cont.		Mark Allocation

Ideal low-pass



$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} (\cos\omega t + j\sin\omega t)\, d\omega$$

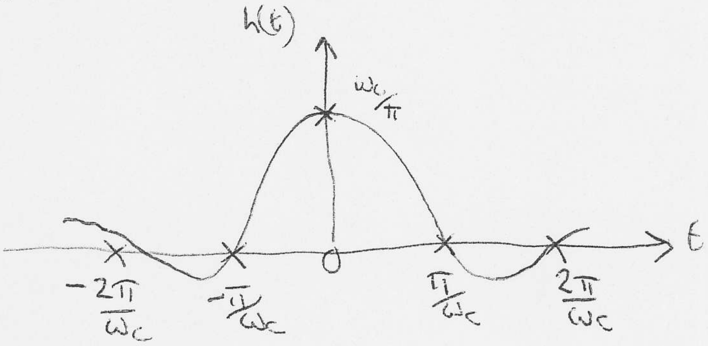
$$= \frac{1}{2\pi t}\Big[\sin\omega t - j\cos\omega t\Big]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi t}\cdot 2\sin\omega_c t$$

$$= \frac{\sin\omega_c t}{\pi t} \quad \text{and} \quad h(0) = \lim_{t\to 0}\frac{\sin\omega_c t}{\pi t} = \frac{\omega_c}{\pi}$$

$h(t)$ crosses time axis when $\sin\omega_c t = 0$

ie $\omega_c t = n\pi$ where n – integer $|n| > 0$

$t = \dfrac{n\pi}{\omega_c}$



Causality requires $h(t)$ to be zero for $t < 0$ so the filter is not realisable — 6

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