1 Signals

- A quantity that can be varied to convey information
- Converted into electrical form using a transducer
- e.g. sine waves

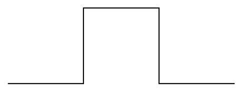


Figure 1: A square wave for some reason

1.1 Laplace transforms (LT)

- For modeling a linear system sing a transfer function
- LT of function $f(t)$ in time domain is

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \mathcal{L}\{f(t)\} \tag{1}$$

with Laplace variable $s = \sigma j\omega$ with dimension $time^{-1}$

Example 1.1.1.

$$f(t) = e^{\alpha t} \tag{2}$$

$$\begin{aligned} F(s) &= \int_0^\infty e^{\alpha t}e^{-st}dt = \mathcal{L}\{f(t)\} \\ &= \int_0^\infty e^{-(s-\alpha)t}dt \\ &= -\frac{1}{s-\alpha}\left[e^{-(s-\alpha)t}\right]_0^\infty \\ &= \frac{1}{s-\alpha} \end{aligned} \tag{3}$$

1.2 Inverse LT

- $\mathcal{L}^{-1}F(s) = f(t)$
- F(s) and f(t) are LT pairs
- Obtained using partial fraction method and table of LT pairs

Example 1.2.1. Determine the signal given

$$\begin{aligned} F(s) &= \frac{s+4}{s(s+2)} \\ &\quad \text{using partial fraction method} \\ F(s) &= \frac{2}{s} - \frac{1}{s+2} \\ &\quad \text{from databook table 1.1 2nd \& 4th rows} \\ f(t) &= 2u(t) - e^{-2t} \end{aligned} \tag{4}$$

for $t \geq 0$, where $u(t)$ is a unit step

Figure 2: Step response to something I think

1.3 Properties of LT

Property 1 if $x(t) \leftrightarrow X(s)$ and $y(t) \leftrightarrow Y(s)$ then $x(t) + -y(t) \leftrightarrow X(s) + -Y(s)$

Property 2 if $x(t) \leftrightarrow X(s)$ and K is constant, then $Kx(t) \leftrightarrow KX(s)$

Example 1.3.1. Determine LT of $v(t) = 3cos4t$ From table 1.1 7th row

$$\mathcal{L}\{cos\omega t\} = \frac{s}{s^2 + \omega^2} \tag{5}$$

i.e., $\omega = 4$, and using property 2 gives

$$V(s) = \frac{3s}{s^2 + 16} \tag{6}$$

Property 3 Derivatives

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^1(0)...f^{n-1}(0) \tag{7}$$

where $f^n(t)$ denotes the n^{th} derivative of f(t) Assume quiescent state, i.e. all system variables and their derivatives are 0 at $t = 0$,

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) \tag{8}$$

valid assumption in all practical systems (no power→off)

Example 1.3.2. Given

$$\tau \frac{dy(t)}{dt} + y(t) = kx(t)$$

where $x(t)$ and $y(t)$ are input and output of a system respectively.

$$\tau s Y(s) + Y(s) = kX(s) Y(s) = X(s)\left[\frac{k1}{1 + s^{\tau}}\right] \tag{9}$$

Property 4 Integration

$$\int_0^t f(t)dt \leftrightarrow \frac{F(s)}{s} \tag{10}$$

Property 5 Time-shift (delay)

$$\mathcal{L}\left\{f(t - T)\right\} = e^{-sT} F(s) \tag{11}$$

Property 6 If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{e^{at} f(t)\} = F(s - a)$

2 Laplace transfer function (TF)

- For a linear and stationary system

$$TF = \frac{\mathcal{L}\{output\}}{\mathcal{L}\{input\}} \tag{12}$$

ie $\mathcal{L}\{output\} = TFx\mathcal{L}\{input\}$ with all initial conditions assumed zero.

- TF describes the dynamics of the system

A linear system obeys the principle of superposition, i.e. if $x_1 \to y_1$ and $x_2 \to y_2$ then $x_1+x_2 \to y_1 + y_2$ where x_n and y_n are respectively the input and output of the system.

2.1 Resistors

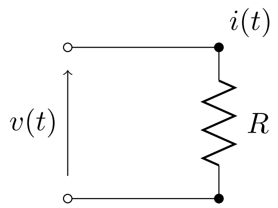


Figure 3: Simple resistor network

$$v(t) = Ri(t) \tag{13}$$

Taking LT and assume zero initial conditions

$$V(s) = RI(s) \tag{14}$$

2.2 Capacitors

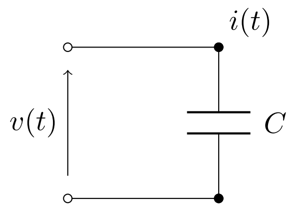


Figure 4: Simple capacitor network

$$v(t) = \frac{1}{C}\int i(t)dt \tag{15}$$

Taking LT and assume zero initial conditions

$$V(s) = \frac{I(s)}{sC} \tag{16}$$

2.3 Inductors

$$v(t) = L\frac{di(t)}{dt} \tag{17}$$

Take LT and assume zero initial conditions

$$V(s) = sLI(s) \tag{18}$$

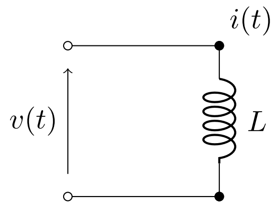


Figure 5: Simple inductor network

2.4 Kirchoff's Laws

1. The total current flowing towards a node is equal to the total current flowing from that node
2. In a closed circuit, the algebraic sum of the products of the current and the resistance of each part of the circuit is equal to the resultant e.m.f. in the circuit.

   Alternatively, in a given loop, the sum of voltage rises is equal to the sum of voltage drops.

The TF of a system can be found by finding the LT of each component and applying Kirchoff's laws.

Example 2.4.1. See Fig. 6

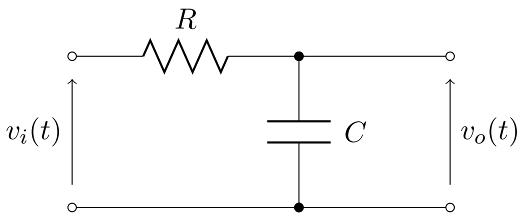


Figure 6: An integrating circuit

$$V_0(s) = \frac{I(s)}{sC} \tag{19}$$

From 2nd law,

$$V_i(s) = RI(s) + \frac{I(s)}{sC} \tag{20}$$

TF

$$\begin{aligned}\frac{V_o}{V_i} &= \frac{\frac{I(s)}{sC}}{RI(s) + \frac{I(s)}{sC}} \\ &= \frac{1}{sRC+1}\end{aligned} \tag{21}$$

Example 2.4.2. See Fig. 7

Applying 2nd law to 1st loop,

$$V_i(s) = R_1 I_1(s) + \frac{I_1(s)}{sC} - \frac{I_2(s)}{sC} \tag{22}$$

Applying 2nd law to 2nd loop,

$$\frac{I_2(s)}{sC} - \frac{I_1(s)}{sC} + sLI_2(s) + R_2 I_2(s) = 0 \tag{23}$$

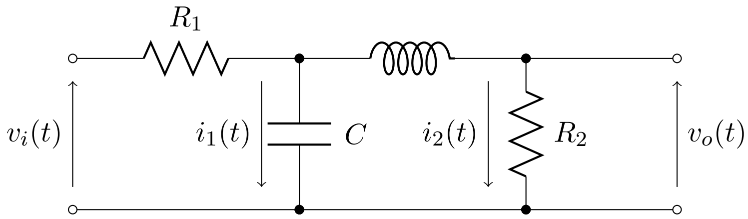


Figure 7: An example electrical network

Solving simultaneously,

$$V_i(s) = I_2(s)\left(s^2LCR_1 + s\left[CR_1R_2 + L\right] + \left[R_1 + R_2\right]\right) \tag{24}$$

note

$$V_o(S) = R_2 I_2(s) \tag{25}$$

$\therefore$ TF is

$$\frac{V_o}{V_i} = \frac{R_2}{s^2LCR_1 + s\left[CR_1R_2 + L\right] + \left[R_1 + R_2\right]} \tag{26}$$

2.5 Test signals and dynamic response

2.5.1 Unit step input

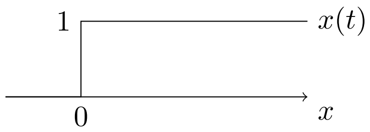


Figure 8: Step response

$$X(s) = \frac{1}{s} \tag{27}$$

Example 2.5.1. Given $H(s) = \frac{1}{1+\tau s}$ (a servo)

$$\begin{aligned} Y(s) &= \frac{1}{1+\tau s} \times \frac{1}{s} \\ &= \frac{\frac{1}{\tau}}{s\left(s + \frac{1}{\tau}\right)} \end{aligned} \tag{28}$$

Figure 9: Dynamic response of a servo when subject to a unit step

2.5.2 Unit ramp input

Where $\tan(\theta) = 1$ ie

$$\begin{aligned} x(t) &= t \\ X(s) &= \frac{1}{s^2} \end{aligned} \tag{29}$$

Figure 10: Dynamic response of a servo when subject to a unit step

Example 2.5.2. Given $H(s) = \frac{1}{1+\tau s}$

$$\begin{aligned} Y(s) &= \frac{1}{1+\tau s} \times \frac{1}{s^2} \\ &= \frac{\tau}{s+\frac{1}{\tau} - \frac{\tau}{s} + \frac{1}{s^2}} \end{aligned} \tag{30}$$

Thus,

$$\begin{aligned} y(t) &= \tau e^{-\frac{t}{\tau}} - \tau + t \\ &= t - \tau\left(1 - e^{-\frac{t}{\tau}}\right) \end{aligned} \tag{31}$$

3 Laplace Poles and Zeros

TF

$$\begin{aligned} G(s) &= \frac{b_m s_m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s_n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \\ G(s) &= k\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_m)} \\ &= k\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_n)} \end{aligned} \tag{32}$$

- where $b_0, b_1 \cdots b_m$ and $a_0, a_1 \cdots a_n$ are real
- $m < n$ for a practical system
- z_1 to z_m are roots of numerator (zeros)
- p_1 to p_n are roots of denominator (poles)
- k is a gain factor

Example 3.0.3.

$$G(s) = \frac{(s+1)}{s} \tag{33}$$

$\implies k = 1$ pole at $s = 0$, zero at $s = -1$

3.1 Poles and Responses

TF

$$G(s) = \frac{N(s)}{D(s)} \tag{34}$$

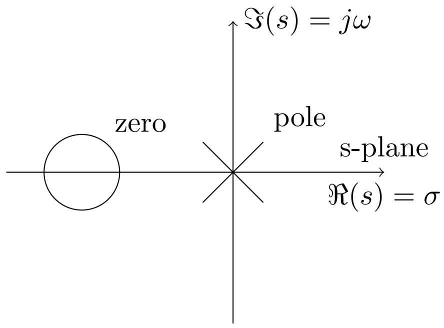


Figure 11: Pole zero plot of a transfer function, with $k = 1$

3.1.1 Case of real poles

Suppose

$$G(s) = \frac{kN(s)}{(s+a)(s+b)(s+c)} \tag{35}$$

Where $a, b, c\cdot$ are real and k is a constant.

The output when a step input (ie, $\frac{1}{s}$) is applied (i.e. the step response) is

$$\begin{aligned} \frac{kN(s)}{s(s+a)(s+b)(s+c)} &= \frac{k_0}{s} + \frac{k_1}{s+a} + \frac{k_2}{s+b} + \frac{k_3}{s+c} + \cdot \\ &= \frac{k_0}{s} + \text{remainder terms} \end{aligned} \tag{36}$$

where k_n are constants.

Note:

- Poles determine the form of response
- Zeros determine the relative components (due to input and poles) to the overall response through k_n
- $\frac{k_0}{s}$ is due to input and remainder term is due to poles

Inverse LT of remainder terms gives:

$$k_1 e^{-at} + k_2 e^{-bt} + k_3 e^{-ct} + \cdot \tag{37}$$

Where $k_1, k_2, k_3\cdot$ are affected by $N(s)$ and exponential terms depend on inputs. This is an exponential response.

3.1.2 Case of complex poles

Suppose $D(s)$ also has complex poles at $s = -\alpha \pm j\beta$. Each pair of xomplexce poles gives rise to

$$\frac{As + B}{(s+\alpha)^2 + \beta^2} \tag{38}$$

in the step response, where A and B are constants.

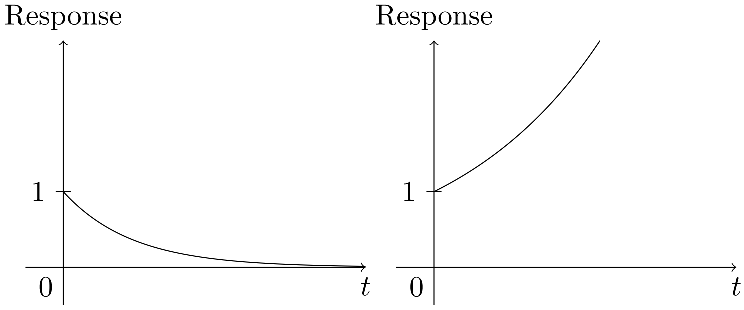
Let $B = A\alpha + C\beta$ and expand

$$= \frac{A(s+\alpha)}{(s+\alpha)^2 + \beta^2} + \frac{C\beta}{(s+\alpha)^2 + \beta^2} \tag{39}$$

Inverse LT gives

$$Ae^{-\alpha t}\cos\beta t + Ce^{-\beta t}\sin\beta t = Re^{-\alpha t}\sin(\beta t + \phi) \tag{40}$$

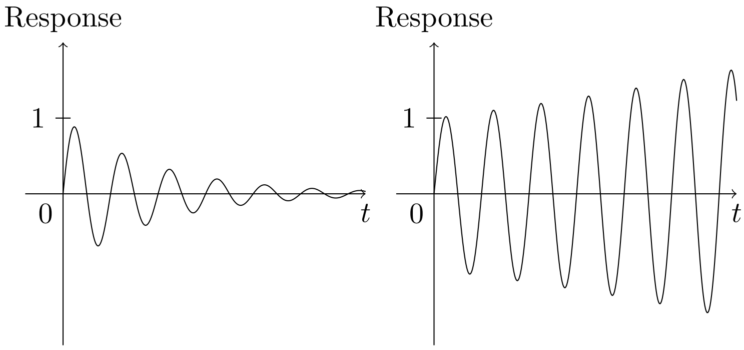
which gives rise to an oscillatory response.



(a) For $s = -a$ (b) For $s = a$

Figure 12: Time waveforms corresponding to a pole at $s = \pm a$ where a is positive

From equation 37, a pole at $s = -a$ (or $s = a$) gives rise to e^{-at} (or e^{at}) in the system response which is non oscillatory and decaying (or growing). The larger the value of a, the faster the rate of decay (or growth).



(a) Negative real part (b) Positive real part

Figure 13: Time waveforms corresponding to a complex pole with a real part

From equation 40, a pair of complex poles at $s = -\alpha \pm j\beta$ (or $s = \alpha \pm j\beta$) gives rise to $e^{-\alpha t} \sin(\beta t + \phi)$ (or $e^{\alpha t} \sin(\beta t + \phi)$) that is sinusoidal and decays (or grows) exponentially. The larger the value of α the faster the rate of decay (or growth). β determines the frequency of the oscillation.

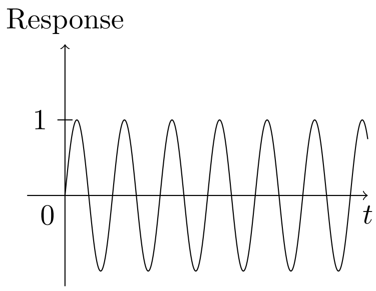


Figure 14: Time waveform corresponding to a complex pole on the imaginary axis of the Laplace s-plane

From equation 40, if a pair of complex poles are on the imaginary axis (i.e. $\alpha = 0$) then response is sinusoidal where β determines the frequency of oscillation.

4

5 Z Transform (ZT)

For a causal sequence $\{x(n)\}$, ie $x(n) = 0$ for $n < 0$,

$$\{x(n)\} = \{x(0), x(1), x(2) \cdot x(n) \cdot\} \tag{41}$$

its ZT is

$$X(z) = \sum_{n=0}^{\infty} x(n) z^- n \tag{42}$$

Where z is a complex variable, and z^{-n} denotes the position in time of $x(n)$.

Example 5.0.1.

$$\begin{aligned} \{x(n)\} &= \{2, 4, 1, 0, 6 \cdots\} \\ X(z) &= 2z^0 + 4z^{-1} + z^{z-2} + 0z^{-3} + 6z^{-4} + \cdots \end{aligned} \tag{43}$$

Inverse ZT

$$\begin{aligned} X(z) = x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + \cdots + x(n)z^{-n} + \cdots \\ \{x(n)\} &= \{x(0), x(1), x(2), \cdots, x(n), \cdots\} \end{aligned} \tag{44}$$

Example 5.0.2.

$$X(z) = \frac{z^{-2} - z^{-7}}{1 - z{-1}} \tag{45}$$

$$\begin{aligned} X(z) &= (z^{-2} - z^{-7})(1 + z^{-1} + z^{-2} + \cdots + z^{-n} + \cdots \\ &= z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} \\ \{x(n)\} &= \{0, 0, 1, 1, 1, 1, 1\} \end{aligned} \tag{46}$$

5.1 Properties of ZT

1. Linearity $x_1(n) \leftrightarrow X_1(z)$ and $x_2(n) \leftrightarrow X_2(z)$ then $x_1(n) + x_2(n) \leftrightarrow X_1(z) + X_2(z)$
2. Time shifting

   $$\begin{aligned} y(n) &= x(n - m) \\ Y(z) &= z^{-m} X(z) \end{aligned} \tag{47}$$

   ie, multiplying by z^{-m} results in a delay of m sampling intervals.

   Example 5.1.1.

   $$\begin{aligned} \{x(n)\} &= \{5, 4, 3, 2, 1\} \\ \{y(n)\} &= \{-, 5, 4, 3, 2, 1\} \\ Y(z) &= 0z^0 + 5z^{-1} + 4z^{-2} + 3z^{-3} + 2z^4 + z^{-5} \\ &= z^{-1}(5 + 4z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}) \\ &= z^{-1}(X(z)) \end{aligned} \tag{48}$$

   $y(n)$ is $x(n)$ delayed by one.

3. Scaling

   $$k\{x(n)\} = \leftrightarrow kX(z) \tag{49}$$

5.2 ZT transfer function (ZT TF)

For a linear system ZT TF

$$H(z) = \frac{C(z)}{R(z)} \tag{50}$$

Where $\{c(n)\}$ is an output sequence and $\{r(n)\}$ is an input sequence

5.3 Pulse transfer function

For $H(z) = \frac{C(z)}{R(z)}$ if $\{r(n)\} = \{1, 0, 0, \cdots\}$ (a unit impulse, ie, $R(z) = 1$), then $C(z) = H(z)$ (pulse TF, which characterises the system).

$\{c(n)\}$ - Impulse response (IR)

$$\{c(n)\} = \{h(n)\} \leftrightarrow H(z) \tag{51}$$

ie pulse TF is ZT of IR.

5.4 Linear shift Invariant (LSI) operations on sequences

1. Multiplier (see Figure 15)

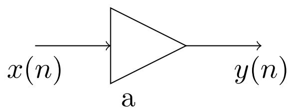


Figure 15: A multiplier

$$\{y(n)\} = \{ax(0), ax(1), \cdots, ax(n), \cdots\} \tag{52}$$

2. Adder (see Figure 16)

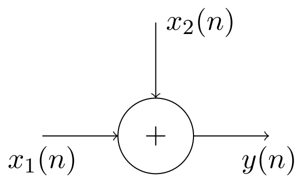


Figure 16: An adder

$$\{y(n)\} = \{x_1(0) + x_2(0), x_1(1) + x_2(1), \cdots\} \tag{53}$$

3. Delay unit (see fig 36)

$$\{y(n)\} = D\{x(n)\} = \{x(n-1)\} \tag{54}$$

Example 5.4.1. A moving averager (see fig 37)

$$\begin{aligned} y(n) &= \frac{z(n) + x(n-1)}{2} \\ \{y(n)\} &= \frac{\{z(n)\} + \{x(n-1)\}}{2} \\ Y(z) &= \frac{X(z) + Z^{-1}X(Z)}{2} \\ H(z) &= \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{2} = \frac{z+1}{2} \end{aligned} \tag{55}$$