4. (a) The probabilities must sum to 1 so $\alpha = 1 - 0.2 - 0.3 - 0.4 = 0.1$.

(b) The expected value of *n(t)*

$$E\{n(t)\} = \sum_{i=1}^{4} n_i p_i = 0.1 \times 0.2 + 0.15 \times 0.3 + 0.2 \times 0.4 + 0.25 \times 0.1$$

$$= 0.02 + 0.045 + 0.08 + 0.025 = 0.17V$$

The expected value of $n^2(t)$

$$E\{n^2(t)\} = \sum_{i=1}^{4} n_i^2 p_i = 0.01 \times 0.2 + 0.0225 \times 0.3 + 0.04 \times 0.4 + 0.0625 \times 0.1$$

$$= 0.001 + 0.00675 + 0.016 + 0.00625 = 0.03V^2$$

(c) The power into a load R is

$$N = \frac{E\{n^2(t)\}}{R} = \frac{0.03}{50} = 0.0006W = 0.6 \text{ mW}$$

(d) Here the integral of the probability density function must be 1, so the distribution is a rectangle of height 1/6 between voltages -3V to $+3\text{V}$

(e) Here we need the expected value of $s^2(t)$

$$E\{s^2(t)\} = \frac{1}{6}\int_{-3}^{3} v^2(t)dt = \frac{1}{6}\left[\frac{v^3}{3}\right]_{-3}^{3} = \frac{54}{18} = 3$$

$$S = \frac{E\{s^2(t)\}}{R} = \frac{3}{50} = 0.06W = 60 \text{ mW}$$

(f) Assuming that s(t) > 0 is one and s(t) < 0 is zero, an error occurs when

$\{-0.1 \le s(t) \le 0\}$ and $\{n(t) = 0.1$ or $n(t) = 0.15$ or $n(t) = 0.2$ or $n(t) = 0.25\}$

$\{-0.15 \le s(t) \le -0.1\}$ and $\{n(t) = 0.15$ or $n(t) = 0.2$ or $n(t) = 0.25\}$

$\{-0.2 \le s(t) \le -0.15\}$ and $\{n(t) = 0.2$ or $n(t) = 0.25\}$

$\{-0.25 \le s(t) \le -0.2\}$ and $\{n(t) = 0.25\}$

The probability of s(t) being in an interval of size 0.05 is 0.05/6, so the total is

$$P_e = \frac{0.05}{6} \times \left\{ \Pr\left(n(t) \geq 0.1\right) + \Pr\left(n(t) \geq 0.2\right) + \left(n(t) \geq 0.3\right) + \left(n(t) \geq 0.4\right) \right\}$$

We just sum the relevant probabilities:

$$P_e = \frac{0.05}{6} \times \left\{ 1 + 0.9 + 0.5 + 0.1 \right\} = 0.021$$

5. (a) In algebraic coding, redundant parity bits are added to a message to aid the process of error correction. The presence of such redundant bits increases the number of possible bit sequences that may be transmitted thus allowing there to be an increased distance (in terms of differences in bit positions) between the messages used. At the receiver, the message that is most similar to that received is used and this allows several errors to occur before the received message becomes closer to another message other than that which was sent.

(b) Rather than sending 0 or 1 to represent the data, we use 000 and 111. This means that a message will only be erroneously decoded if two or three bits are corrupted. If only one error occurs, the received message is still closest to the correct one. For example, 010 will be interpreted as 000 assuming that the middle bit is in error. The addition of two extra bits for this limited amount of error correction is not efficient since it requires three times the data rate compared with the uncoded data.

(c) Given a binary message, even (odd) parity indicates that the number of 1s in the message is even (odd). The utilisation of parity checks on combinations of the bits from a block of data enables more powerful codes to be realised since more use is being made of the message bits than just their repetition. Linear algebraic codes are formed from the product of a generator matrix with the message. At the receiver, the product of the received message and a parity check matrix is formed. The result of the product produces the information required to find and correct errors.

(d) (i) The number of code words is be $2^4 = 16$ and the rate is 4/7

(ii) The parity check matrix is formed thus

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{H^T} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The syndrome is

$$\mathbf{S} = \mathbf{H}.\mathbf{R} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(iii) The syndrome should be zero for a correct message to conform to the parity process.

The decoding process is represented as

$$\mathbf{S} = \mathbf{H}.\mathbf{R} = \mathbf{H}.(\mathbf{T} \oplus \mathbf{E})$$
$$= \mathbf{H}.\mathbf{T} \oplus \mathbf{H}.\mathbf{E}$$

Since the produce of the correct message and H is zero

$$\mathbf{S} = \mathbf{H}.\mathbf{E}$$

If a single error occurs the product **H.E**

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \\ E_7 \end{bmatrix}$$

Will copy one column of H into the syndrome because only one of the En values will be one. In this case, the syndrome is identical to the second column in R so bit two must be incorrect and the message should be

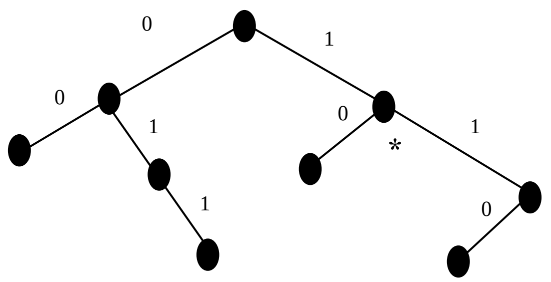
$$\mathbf{R}^{\mathbf{T}} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

6.

(a) A code is said to be a ***prefix code***, if none of the codewords is a prefix of any of the others.

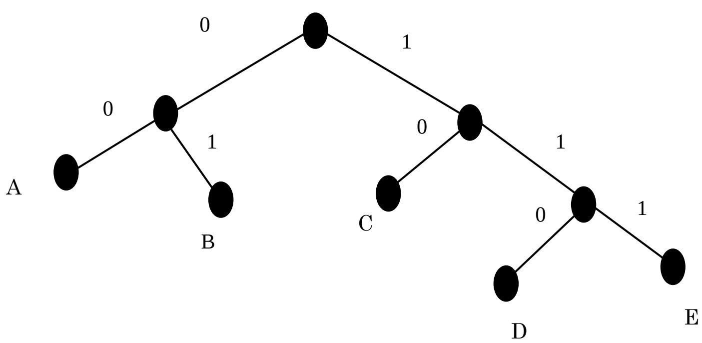
A code is *uniquely decodable* (UD) if any output string is the image of at most one input message.

(b) The coding $\{1,10,110,011,00\}$ has the tree



Since * falls within the tree and is a codeword, the code is not a prefix code (1 is a prefix of 10 and 110).

(c)
The tree for this source and coding (all symbols are leaves, therefore prefix)



Having equiprobable symbols makes the length:

$$\bar{L} = \frac{2+2+2+3+3}{5} = 2.4 \text{ bits/symbol}$$

(d) (i) The key to this is remembering that the probabilities must sum to 1.

Given $P(A) = 2P(B); P(B) = 2P(C); P(C) = 2P(D) = 2P(E)$, it follows that $P(B) = 4P(D) = 4P(E)$ and $P(A) = 8P(D) = 8P(E)$.

Thus $P(E)(8+4+2+1+1)=1 \Rightarrow P(E)=1/16$

Hence $P(D)=1/16; P(C)=1/8; P(B)=1/4; P(A)=1/2$

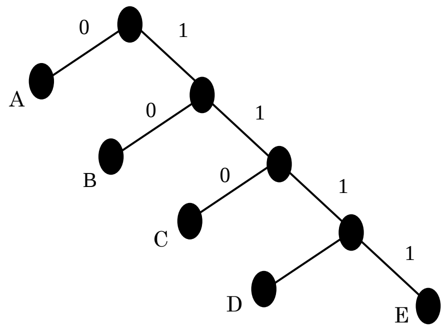
The entropy is:

$$H(X)=-\frac{1}{2}\log_2\left(\frac{1}{2}\right)-\frac{1}{4}\log_2\left(\frac{1}{4}\right)-\frac{1}{8}\log_2\left(\frac{1}{8}\right)-\frac{1}{16}\log_2\left(\frac{1}{16}\right)-\frac{1}{16}\log_2\left(\frac{1}{16}\right)$$
$$=\frac{1}{2}\times 1+\frac{1}{4}\times 2+\frac{1}{8}\times 3\times +2\times\frac{1}{16}\times 4$$

$=1{}^{7}\!/_{8}$ bits/symbol.

(ii) Code is as below with corresponding tree

$$\begin{array}{lcc} A & \mapsto & 0 \\ B & \mapsto & 10 \\ C & \mapsto & 110 \\ D & \mapsto & 1110 \\ E & \mapsto & 1111 \end{array}$$



(iii) The length is

$$\bar{L}=\frac{1}{2}\times 1+\frac{1}{4}\times 2+\frac{1}{8}\times 3\times +2\times\frac{1}{16}\times 4 = \mathrm{H(X)} = 1{}^{7}\!/_{8} \text{ bits/symbol.}$$

The length is the minimum possible by Shannon's Source Coding Theorem, as it is equal to the entropy. This occurs only because the probabilities here are powers of 2.