2012 Past Paper –Question 4

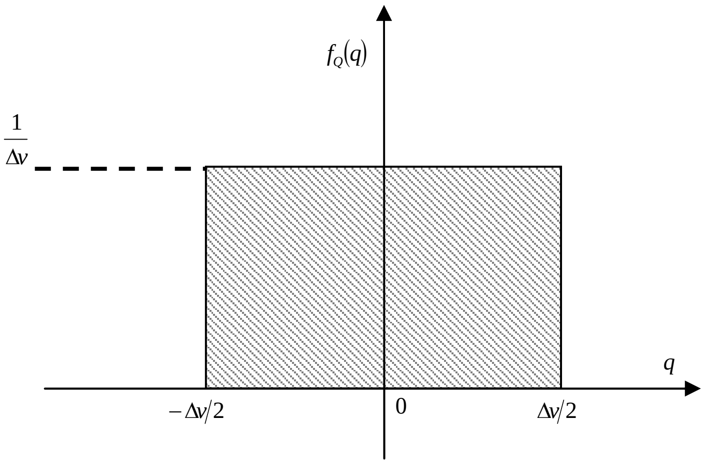
(a) Sampling is the process of taking slices in time from a signal that is time continuous to produce a set of discrete values that represent the signal.

Quantisation is the production of a set of discrete values that represent the amplitude of a continuous waveform by dividing it into a fixed number of subintervals each represented by a number in the subinterval.

Encoding for PCM is the act of representing the values obtained in quantisation as binary numbers, which are then transmitted.

(b) The transformation of an analogue input signal to its digital approximation requires mapping to a set of digital levels, taking the closest one to the signal as its value. This approximation process is known as quantisation. The error in the approximation of the signal by the levels is by default uniformly distributed between $-\Delta v/2$ and $\Delta v/2$ for a level spacing of Δv. Since the signal is not known a priori, the error is a random variable q, with PDF $f_Q(q)$ that is known as the quantisation noise.

For uniform sampling, the error is distributed as shown



$$f_Q(q) = \begin{cases} \dfrac{1}{\Delta v} & -\dfrac{\Delta v}{2} < \mathrm{q} < \dfrac{\Delta v}{2} \\ 0 & \text{otherwise} \end{cases}.$$

The mean is zero by inspection [Check: $E[q] = \frac{1}{\Delta v}\int_{-\frac{\Delta v}{2}}^{\frac{\Delta v}{2}} x\,dx = \frac{1}{\Delta}\frac{x^2}{2}\Big|_{-\frac{\Delta v}{2}}^{\frac{\Delta v}{2}} = 0$]:

$$\sigma_Q^2 = E\left[x^2\right] = \frac{1}{\Delta v}\int_{-\frac{\Delta v}{2}}^{\frac{\Delta v}{2}} x^2 dx = \frac{1}{\Delta v}\left.\frac{x^3}{3}\right|_{-\frac{\Delta v}{2}}^{\frac{\Delta v}{2}} = \frac{1}{\Delta v}\left\{\frac{1}{3}\frac{(\Delta v)^3}{8} + \frac{1}{3}\frac{(\Delta v)^3}{8}\right\} = \frac{(\Delta v)^2}{12}$$

(c) The sqer is the ratio of the mean squared signal level to the mean squared quantisation noise ratio.

$$\text{sqer} = \frac{\left\langle f^2(t)\right\rangle}{(\Delta v)^2/12}$$

For a linear quantiser with N levels and steps of Δv, these are related to the peak voltage V by:

$$\Delta v = \frac{2V}{N}$$

So

$$\text{sqer} = \frac{\left\langle f^2(t)\right\rangle}{(2V/M)^2/12} = \frac{3M^2\left\langle f^2(t)\right\rangle}{V^2} = \frac{3M^2}{\alpha} \text{ with } \alpha = \frac{V^2}{\left\langle f^2(t)\right\rangle}$$

(d) Expressing the sqer in dB:

$$\text{sqer}_{\text{dB}} = 10\log 3 + 20\log M - 10\log\alpha$$

Putting in the numbers given:

$$10\log 3 + 20\log 2^8 - 10\log\alpha = 40$$
$$10\log\alpha = 4.77 + 48.17 - 40$$
$$\alpha = 19.68$$

So

$$V^2 = \alpha \times \left\langle f^2(t)\right\rangle = 19.68 \times 5 \times 10^{-3}$$
$$V = 0.31 \text{ Volts}$$

2012 Past Paper –Question 5

(a) Given an input signal with frequency spectrum X(f) and an output signal spectrum Y(f), the ratio Y(f)/X(f) is the channel transfer function H(f). This may be written as:

$$\mathrm{H}(f) = \mathrm{A}(f)\exp[\phi(f)]$$

A(f) is the amplitude response and ϕ(f) is the phase response with group delay $\tau_{\mathrm{g}} = -\frac{d\phi(f)}{df}$

(b) In an ideal world we would like a signal to be transmitted through a particular transmission system without distortion. If x(t) is transmitted the output y(t) should be a delayed copy of this, i.e.

$$\mathrm{y}(t) = Kx(t - t_0)$$

where the t_0 is the transmission delay, and K is a constant (attenuation).

Using the time delay theorem of Fourier Transforms

$$\mathrm{Y}(f) = X(f)K\exp(-j2\pi f t_0)$$

So

$$\mathrm{H}(f) = K\exp(-j2\pi f t_0)$$

This has a constant amplitude response and group delay which must hold over the signal bandwidth.

(c)

(i) The input signal is an impulse and so the output spectrum P(f) will take the shape of the channel transfer function:

$$\mathrm{P}(f) = \begin{cases} 0 & f < -B/2 \\ 1 & -B/2 \le f \le B/2 \\ 0 & f > B/2 \end{cases}$$

(ii) The mean noise power N at the output of the channel is given in general by:

$$N = \frac{N_0}{2}\int_{-\infty}^{\infty}\left|H\left(f\right)\right|^2 df$$

So in this case:

$$N = \frac{N_0}{2}\int_{-B/2}^{B/2} df = \frac{N_0 B}{2}\text{Watts}$$

(iii) The matched filter is such that its product leaves the overall transfer function unchanged since it is the same rectangular function as the channel. For the SNR we need:

$$\Delta = \sqrt{\frac{2}{N_0}\int_{-\infty}^{\infty}\left|P(f)\right|^2 df}$$

$$\Delta = \sqrt{\frac{2}{N_0}\int_{-\infty}^{\infty}\left|P(f)\right|^2 df} = \sqrt{\frac{2}{N_0}\int_{-B/2}^{B/2} df} = \sqrt{\frac{2B}{N_0}}$$

(d) The mean noise power is

$$N = \frac{N_0}{2}\int_{-\infty}^{\infty}\left|H\left(f\right)\right|^4 df = 10^{-8}\int_{-2000}^{2000} df = 4\times 10^{-5}\ \text{W} = 40\,\mu\text{W}$$

2012 Past Paper –Question 6

(a) Compression aspires to reduce the amount of data that are transmitted or stored by seeking patterns within the data that can be represented by a reduced set of data points. In contrast, encryption is attempting to produce uniform data sets that have maximum entropy. Any attempts to compress encrypted data set will result in almost no compression so compression must be carried out prior to encryption.

(b) Text Assuming that a DMS model is used, the probabilities will be

$$\begin{aligned} p_0 = p_3 = p_4 = p_5 = p_6 &= 8/100 = 0.08 \\ p_1 &= 45/100 = 0.45 \\ p_2 &= 15/100 = 0.15 \end{aligned}$$

Entropy is thus:

$$\begin{aligned} H(Y) &= -5\times 0.08\lg(0.08) - 0.45\lg(0.45) - 0.15\lg(0.15) \\ &= -0.4\times\frac{\ln(0.08)}{\ln(2)} - 0.45\times\frac{\ln(0.45)}{\ln(2)} - 0.15\times\frac{\ln(0.15)}{\ln(2)} \\ &= -0.4\times\frac{-2.526}{0.693} - 0.45\times\frac{-0.799}{0.693} - 0.15\times\frac{-1.897}{0.693} \\ &= 1.458 + 0.519 + 0.411 \end{aligned}$$

= 2.388 bits/symbol.

(c) Using a DMS assumption for Z:

$$\begin{aligned} p_0 = p_4 = p_6 &= 15/100 = 0.15 \\ p_1 = p_2 = p_5 &= 14/100 = 0.14 \\ p_3 &= 13/100 = 0.13 \end{aligned}$$

$$\begin{aligned} H(Y) &= -3\times 0.15\lg(0.15) - 3\times 0.14\lg(0.14) - 0.13\lg(0.13) \\ &= -0.45\times\frac{-1.897}{0.693} - 0.42\times\frac{-1.966}{0.693} - 0.13\times\frac{-2.040}{0.693} \\ &= 1.232 + 1.192 + 0.383 \end{aligned}$$

= 2.807 bits/symbol.

(d) From the source coding theorem, the maximum compression ratio is:

$$\rho = H/\lg|\Gamma_{in}|$$

$$\lg|\Gamma_{\mathrm{in}}| = \lg(7) = \frac{\ln(7)}{\ln(2)} = \frac{1.946}{0.693} = 2.808 \text{ bits/symbol}$$

So we have $\rho_Y = H(Y) / \lg|\Gamma_{in}| = 2.388/\lg(7) = 0.85$

and $\rho_Z = H(Z) / \lg|\Gamma_{in}| = 2.807/\lg(7) \approx 1$

Encryption increases the entropy towards the limit of lg(7) for an equally distributed source, that is $\rho = 1$ and no compression possible.

The scheme employed has been very successful in raising the entropy and spreading the symbols out more evenly.