1 Signals

- A quantity that can be varied to convey information
- Converted into electrical form using a transducer
- e.g. sine waves

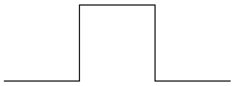

Figure 1: A square wave for some reason

1.1 Laplace transforms (LT)

- For modelling a linear sstem sing a transfer function
- LT of function $f(t)$ in time domain is

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \mathcal{L}\{f(t)\} \tag{1}$$

with laplace variable $s = \sigma j\omega$ with dimension $time^{-1}$

Example 1.1.1.

$$f(t) = e^{\alpha t} \tag{2}$$

$$\begin{aligned} F(s) &= \int_0^\infty e^{\alpha t}e^{-st}dt = \mathcal{L}\{f(t)\} \\ &= \int_0^\infty e^{-(s-\alpha)t}dt \\ &= -\frac{1}{s-\alpha}\left[e^{-(s-\alpha)t}\right]_0^\infty \\ &= \frac{1}{s-\alpha} \end{aligned} \tag{3}$$

1.2 Inverse LT

- $\mathcal{L}^{-1}F(s) = f(t)$
- F(s) and f(t) are LT pairs
- Obtained using partial fraction method and table of LT pairs

Example 1.2.1. Determine the signal given

$$\begin{aligned} F(s) &= \frac{s+4}{s(s+2)} \\ &\quad \text{using partial fraction method} \\ F(s) &= \frac{2}{s} - \frac{1}{s+2} \\ &\quad \text{from databook table 1.1 2nd \& 4th rows} \\ f(t) &= 2u(t) - e^{-2t} \end{aligned} \tag{4}$$

for $t \geq 0$, where $u(t)$ is a unit step

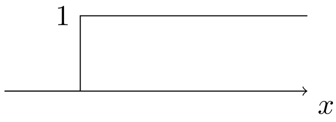

Figure 2:

1.3 Properties of LT

Property 1 if $x(t) \leftrightarrow X(s)$ and $y(t) \leftrightarrow Y(s)$ then $x(t) + -y(t) \leftrightarrow X(s) + -Y(s)$

Property 2 if $x(t) \leftrightarrow X(s)$ and K is constant, then $Kx(t) \leftrightarrow KX(s)$

Example 1.3.1. Determine LT of $v(t) = 3cos4t$ From table 1.1 7th row

$$\mathcal{L}\{cos\omega t\} = \frac{s}{s^2 + \omega^2} \tag{5}$$

i.e., $\omega = 4$, and using property 2 gives

$$V(s) = \frac{3s}{s^2 + 16} \tag{6}$$

Property 3 Derivatives

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^1(0) ... f^{n-1}(0) \tag{7}$$

where $f^n(t)$ denotes the n$^{\text{th}}$ derivative of f(t) Assume quiescent state, i.e. all system variables and their derivatives are 0 at $t = 0$,

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) \tag{8}$$

valid assumption in all practical systems (no power→off)

Example 1.3.2. Given

$$\tau \frac{dy(t)}{dt} + y(t) = kx(t)$$

where $x(t)$ and $y(t)$ are input and output of a system respectively.

$$\tau sY(s) + Y(s) = kX(s)Y(s) = X(s)\left[\frac{k1}{1 + s^\tau}\right] \tag{9}$$

Property 4 Integration

$$\int_0^t f(t)dt \leftrightarrow \frac{F(s)}{s} \tag{10}$$

Property 5 Time-shift (delay)

$$\mathcal{L}\{f(t - T)\} = e^{-sT} F(s) \tag{11}$$

Property 6 If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{e^{at} f(t)\} = F(s - a)$

2 Laplace transfer function (TF)

- For a linear and stationary system

$$TF = \frac{\mathcal{L}\{output\}}{\mathcal{L}\{input\}} \tag{12}$$

  ie $\mathcal{L}\{output\} = TFx\mathcal{L}\{input\}$

  with all intitial conditions assymed zero.

- TF describes the dynamics of the system

A linear system obeys the principle of superposition, i.e. if $x_1 \to y_1$ and $x_2 \to y_2$ then $x_1 + x_2 \to y_1 + y_2$ where x_n and y_n are respectively the input and output of the sytem.

2.1 Resistors

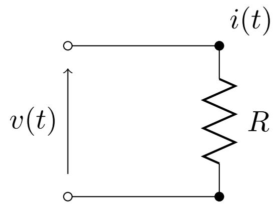

Figure 3: Simple resistor network

$$v(t) = Ri(t) \tag{13}$$

taking LT and assume zero initial conditions

$$V(s) = RI(s) \tag{14}$$

2.2 Capacitors

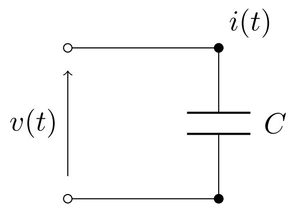

Figure 4: Simple capacitor network

$$v(t) = \frac{1}{C}\int i(t)dt \tag{15}$$

Taking LT and assume zero initial conditions

$$V(s) = \frac{I(s)}{sC} \tag{16}$$

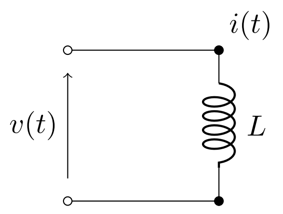

Figure 5: Simple inductor network

2.3 Inductors

$$v(t) = L\frac{di(t)}{dt} \tag{17}$$

Take LT and assume zero initial conditions

$$V(s) = sLI(s) \tag{18}$$

2.4 Kirchoff's Laws

1. The total current flowing towards a node is equal to the total current flowing from that node
2. In a closed circuit, the algebraic sum of the products of the current and the resistance of each part of the circuit is equal to the resultant e.m.f. in the circuit.

   Alternatively, in a given loop, the sum of voltage rises is equal to the sum of voltage drops.

The TF of a system can be found by finding the LT of each componenent and applying Kirchoff's laws.

Example 2.4.1. See Fig. 6

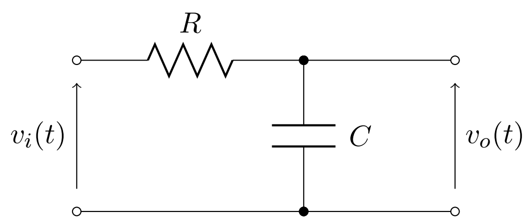

Figure 6: An integrating circuit

$$V_0(s) = \frac{I(s)}{sC} \tag{19}$$

From 2nd law,

$$V_i(s) = RI(s) + \frac{I(s)}{sC} \tag{20}$$

TF

$$\begin{aligned}\frac{V_o}{V_i} &= \frac{\frac{I(s)}{sC}}{RI(s) + \frac{I(s)}{sC}} \\ &= \frac{1}{sRC + 1}\end{aligned} \tag{21}$$

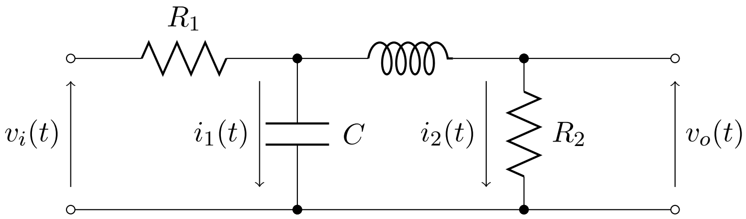

Figure 7: An example electrical network

Example 2.4.2. See Fig. 7

Applying 2nd law to 1st loop,

$$V_i(s) = R_1 I_1(s) + \frac{I_1(s)}{sC} - \frac{I_2(s)}{sC} \tag{22}$$

Applying 2nd law to 2nd loop,

$$\frac{I_2(s)}{sC} - \frac{I_1(s)}{sC} + sLI_2(s) + R_2 I_2(s) = 0 \tag{23}$$

Solving simultaneously,

$$V_i(s) = I_2(s)\left(s^2 LCR_1 + s\left[CR_1R_2 + L\right] + \left[R_1 + R_2\right]\right) \tag{24}$$

note

$$V_o(S) = R_2 I_2(s) \tag{25}$$

$\therefore$ TF is

$$\frac{V_o}{V_i} = \frac{R_2}{s^2 LCR_1 + s\left[CR_1R_2 + L\right] + \left[R_1 + R_2\right]} \tag{26}$$

3 Test signals and dynamic response

3.1 Unit step input

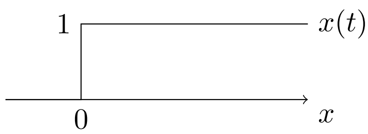

Figure 8: Step response

$$X(s) = \frac{1}{s} \tag{27}$$

Example 3.1.1. Given $H(s) = \frac{1}{1+\tau s}$ (a servo)

$$\begin{aligned} Y(s) &= \frac{1}{1+\tau s} \times \frac{1}{s} \\ &= \frac{\frac{1}{\tau}}{s\left(s + \frac{1}{\tau}\right)} \end{aligned} \tag{28}$$

Figure 9: Dynamic response of a servo when subject to a unit step

Figure 10: Dynamic response of a servo when subject to a unit step

3.2 Unit ramp input

Where $\tan(\theta) = 1$ ie

$$\begin{aligned} x(t) &= t \\ X(s) &= \frac{1}{s^2} \end{aligned} \tag{29}$$

Example 3.2.1. Given $H(s) = \frac{1}{1+\tau s}$

$$\begin{aligned} Y(s) &= \frac{1}{1+\tau s} \times \frac{1}{s^2} \\ &= \frac{\tau}{s + \frac{1}{\tau} - \frac{\tau}{s} + \frac{1}{s^2}} \end{aligned} \tag{30}$$

Thus,

$$\begin{aligned} y(t) &= \tau e^{-\frac{t}{\tau}} - \tau + t \\ &= t - \tau\left(1 - e^{-\frac{t}{\tau}}\right) \end{aligned} \tag{31}$$

4 Laplace Poles and Zeros

TF

$$\begin{aligned} G(s) &= \frac{b_m s_m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s_n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \\ G(s) &= k\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_m)} \\ &= k\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_n)} \end{aligned} \tag{32}$$

- where $b_0, b_1 \cdots b_m$ and $a_0, a_1 \cdots a_n$ are real
- $m < n$ for a practical systen
- z_1 to z_m are roots of numerator (zeros)
- p_1 to p_n are roots of denominator (poles)
- k is a gain factor

Example 4.0.2.

$$G(s) = \frac{(s+1)}{s} \tag{33}$$

$\implies k = 1$ pole at $s = 0$, zero at $s = -1$

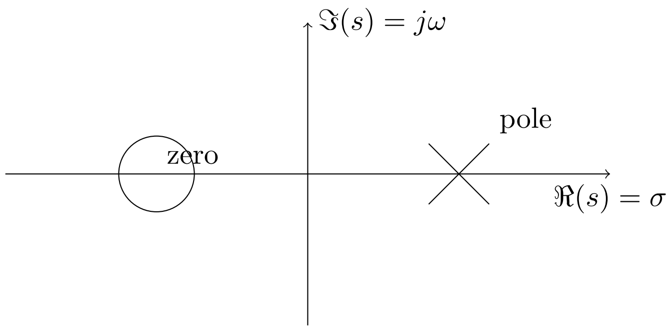

Figure 11: Pole zero plot of a transfer function, witk $k = 1$