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Guaranteed- and high-precision evaluation of the Lambert W function☆

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ABSTRACT

Solutions to a wide variety of transcendental equations can be expressed in terms of the Lambert W function. The W function, also occurring frequently in many branches of science, is a non-elementary but now standard mathematical function implemented in all major technical computing systems. In this work, we analyze an efficient logarithmic recursion with quadratic convergence rate to approximate its two real branches, W_0 and W_{-1}. We propose suitable starting values that ensure monotone convergence on the whole domain of definition of both branches. Then, we provide a priori, simple, explicit and uniform estimates on the convergence speed, which enable guaranteed, high-precision approximations of W_0 and W_{-1} at any point.

1. Introduction

The Lambert W function, first investigated in the 18th century, is defined implicitly by the transcendental equation

$$\mathrm{W}(x)e^{\mathrm{W}(x)} = x.$$

It has now become a standard mathematical function and it is included in all major technical computing systems. It appears in an increasingly growing number of applications—currently, there are several thousand papers that use the W function in one way or another [27]—due to the fact that solutions to a wide variety of polynomial-exponential-logarithmic equations can be expressed in terms of the W function.

The W function has found numerous applications in physics [19,52,61,63,64], including statistical mechanics [11], cosmology [55], relativity, or quantum mechanics, in statistics [53], in signal processing [13], electrical engineering [3], materials science, chemistry [6], molecular biology, pharmacology, earth sciences, ecological and evolutionary models [40], economics, epidemiology, or in sociology [66]; see also, e.g., Barry et al. [4, Table 1]. Some classical references on or general overview of the W function include [9,15,17,20,27,30,58,68]. Various further properties and representations of the Lambert W function are described in, e.g., [38,39,59], including its monotonicity properties [37], or continued fraction expansions [14]. The W

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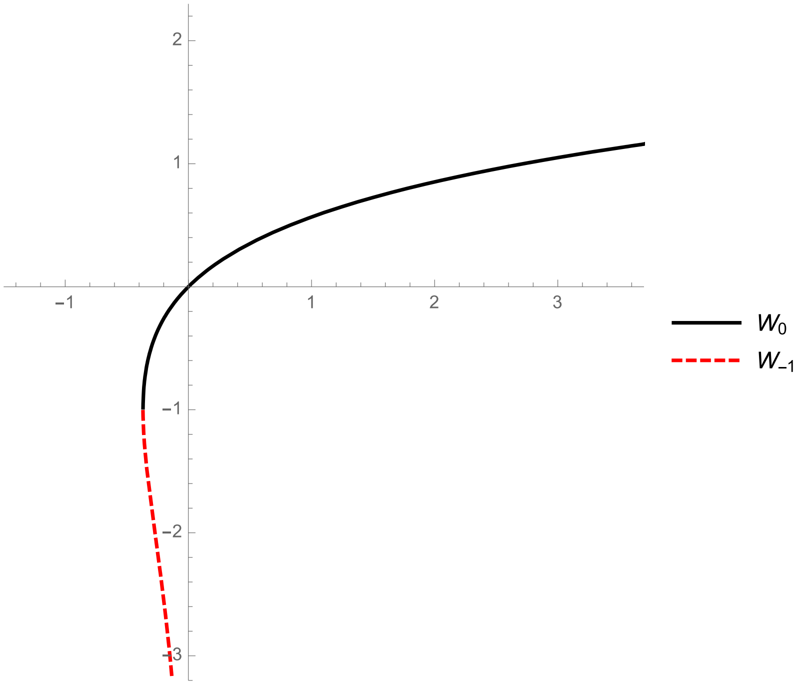


Fig. 1. The real branches of the W function.

function is related to several other mathematical functions or expressions—for example, to the function $x \mapsto x^x$ [29], the solutions to the equation $x^y = y^x$ [5,41], the infinite power tower [21,26], the Wright ω function [16], or the Euler T function [36]—and it finds many applications within mathematics itself [7,8]. The W function has been extended also for matrix arguments [22,51] to solve, for example, certain systems with time delay [12,32,43,57,69,70]. Some generalizations of the (scalar) W function, motivated by concrete applications, are described, e.g., in Maignan [44], Maignan and Scott [45], Mezö [46], Mezö and Baricz [49], Mezö et al. [50], Scott et al. [56], Teodorescu [60]. The first monograph (in English) on the W function has been published quite recently [47] (and see also his webpage [48] containing an expanding list of references).

The W function has two real, and infinitely many complex branches [34,35]. The real branches are usually denoted by

$$\mathrm{W}_0 : [-1/e, \infty) \to [-1, \infty)$$

and

$$\mathrm{W}_{-1} : [-1/e, 0) \to (-\infty, -1],$$

see Fig. 1. Both of these are strictly monotone, and some simple special values include $\mathrm{W}_0(0) = 0$, $\mathrm{W}_0(e) = 1$, $\mathrm{W}_0(-1/e) = -1$, or $\mathrm{W}_{-1}(-1/e) = -1$.

The W function is not an elementary function [10], so it is natural to ask how one can approximate it efficiently with simpler functions. In the literature, one can find many different representations and approximations for the real branches of the W function on various intervals, see, e.g., Alzahrani and Salem [2], Corless et al. [18], Gautschi [28], Hoofar and Hassani [31], Iacono and Boyd [33], Johansson [35], Wolfram research [67]. These include

(i) series expansions
- Taylor expansions, e.g., about the origin

$$\sum_{k=1}^{\infty} \frac{(-k)^{k-1}}{k!} x^k = x - x^2 + \frac{3x^3}{2} - \frac{8x^4}{3} + \frac{125x^5}{24} + \mathcal{O}\left(x^6\right); \tag{1}$$

- Puiseux expansions, e.g., about the branch point $x = -1/e$;
- asymptotic expansions about $+\infty$, such as

$$\ln(x) - \ln(\ln(x)) + \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} c_{k,m} \frac{(\ln(\ln(x)))^m}{(\ln(x))^{m+k}} \tag{2}$$

where the coefficients $c_{k,m}$ are defined in terms of the Stirling cycle numbers;

(ii) recursive approximations

- the recursion

$$\lambda_{n+1}(x) := \ln(x) - \ln(\lambda_n(x)); \tag{3}$$

- the Newton-type iteration

$$\nu_{n+1}(x) := \nu_n(x) - \frac{\nu_n(x) - xe^{-\nu_n(x)}}{1+\nu_n(x)}; \tag{4}$$

- the iteration

$$\beta_{n+1}(x) := \frac{\beta_n(x)}{1+\beta_n(x)}\left(1+\ln\left(\frac{x}{\beta_n(x)}\right)\right); \tag{5}$$

- the Halley-type iteration

$$h_{n+1}(x) := h_n(x) - \frac{h_n(x)e^{h_n(x)} - x}{e^{h_n(x)}(h_n(x)+1) - \frac{(h_n(x)+2)\left(h_n(x)e^{h_n(x)}-x\right)}{2(h_n(x)+1)}}; \tag{6}$$

- the Fritsch–Shafer–Crowley (FSC) scheme;

(iii) analytic bounds on different intervals

- the bounds

$$\ln(x) - \ln(\ln(x)) + \frac{\ln(\ln(x))}{2\ln(x)} < \mathrm{W}_0(x) < \ln(x) - \ln(\ln(x)) + \frac{e\ln(\ln(x))}{(e-1)\ln(x)}, \tag{7}$$

valid for $x \in (e, +\infty)$;
- or, for example, the bounds

$$\frac{e\ln(-x)}{e-1} \le \mathrm{W}_{-1}(x) \le \ln(-x) - \ln(-\ln(-x)) \tag{8}$$

valid for $x \in [-1/e, 0)$.

As for the other (complex) branches of the W function, Johansson [35] contains an algorithm to approximate any branch by using complex interval arithmetic together with the Arb library.

Let us comment on the above formulae to motivate our work.

The recursion (3) is based on the functional Eq. (9), and has appeared many times in the literature. However, it is only linearly convergent.

The recursion (5) was devised in Iacono and Boyd [33, Section 3] (and see also Iacono and Boyd [33, Section 8] on D. J. Jeffrey's observation about interpreting (5) as a Newton's recursion). We only changed their notation from W_n to β_n, since W_n usually denotes the complex branches of the W function. The authors of Iacono and Boyd [33] mention that the convergence rate of (5) is quadratic, and it approximates $\mathrm{W}_0(x)$ for large x better than the standard—also quadratic—Newton iteration (4).

The Halley recursion (6) has third order of convergence. In general, Eqs. (4) and (6) both belong to the Schröder families of root-finding methods, see, e.g., Petković et al. [54]. The FSC scheme [24] converges at an even faster rate.

The pair of bounds (7)—based on the initial terms of the series (2)—appears in Hoofar and Hassani [31] (its weaker version is reproduced in our Lemma 1.5 below). For $x > e$, Iacono and Boyd [33, Section 4.3] describes some tighter, two-sided bounds for $\mathrm{W}_0(x)$, obtained by applying one step of (5) or (3) to a suitable initial function. These bounds contain more nested logarithms (hence, they are not of the form (2)).

When using any of the above expressions to approximate the real branches of the W function, a fundamental question is of course the quality of approximation.

- When dealing with various expansions (Taylor, Puiseux or asymptotic series in group (i) above), one can work only with their finite truncations in practice, so one also needs estimates of the remainder terms—estimates of this type were published only recently [35].
- From the inequalities in group (iii) above, one can obtain only very rough bounds for $\mathrm{W}_0(x)$ or $\mathrm{W}_{-1}(x)$.
- When one applies any of the recursive formulae (3)–(6), the following questions naturally arise. What starting value should one pick? Is the recursion well-defined then? Will it converge for a particular value of x? If yes, what is the error committed when n recursive steps are performed? How many steps to take to approximate $\mathrm{W}(x)$ to a given precision? How to tackle the difficulties when x is close to the branch point at $-1/e$, to the singularity of W_{-1} near $x < 0$, or when $x > 0$ is very large?

To the best of our knowledge, concrete answers to these questions with proofs have not yet been reported in the literature.

The aim of the present work is to partially fill this gap by analyzing the recursion (5). We chose this recursion because of its relatively simple structure and good global convergence properties. We will provide suitable starting values guaranteeing that (5) becomes a well-defined and monotonically convergent real sequence. More importantly, we also formulate explicit bounds on the approximation error committed in the n^{th} step. It is not our goal to develop an algorithm that approximates the W function more efficiently than many previously known methods—this was done earlier, see, e.g., Fukushima [25]. Rather, the main contribution of our work is to provide easily computable, theoretically guaranteed bounds on the number of iterations of (5) required to approximate the real branches of the W function to a desired, possibly very high (say, several ten thousand digits of) precision.

Remark 1.1. Assume one aims to approximate a particular W value to a certain precision in floating-point or interval arithmetic on a computer. Since the number of iterations to achieve this precision can easily be calculated in advance from our theorems, one can determine the necessary precision for the intermediate elementary function evaluations in (5) so that the target accuracy is still maintained after the possible loss of precision in each recursion step. However, these implementation details will not be discussed here.

Remark 1.2. We remark that some further analysis (e.g., on the recursion (3) or on some refinements of the bounds (7)) can be found in the document [42].

Remark 1.3. In [1,4,33,62,63], for example, various expressions—consisting of only elementary functions—are provided to approximate the W function on some intervals, and the maximum error committed is also measured. Let us compare numerically the quality of approximation: on the one hand, by considering the functions from, say, the above-mentioned recent paper [1], and, on the other hand, by considering a function $x \mapsto \beta_M(x)$ defined by (5) (where M is a small, fixed positive integer, and the starting values $\beta_0(x)$ are defined by (13), (21), (24), (27) in our Section 2).

To this end, let $\alpha^{(1)}(x), \alpha^{(2)}(x), \ldots, \alpha^{(8)}(x)$ denote the right-hand side of formulae (26), (27), (28), (A1), (31), (32), (35) and (36) of Álvarez [1], respectively. That is,

$$\alpha^{(1)}(x) := x\exp(0.71116x^2 - 0.98639x) \quad \text{for} \quad x \in [2 \cdot 10^{-16}, 0.2];$$

$$\alpha^{(2)}(x) := -1.6579 + 0.1396 \cdot \sqrt[4]{291790 - (x - 22.8345)^4} \quad \text{for} \quad x \in [0.2, 1.2];$$

$$\vdots$$

$$\alpha^{(8)}(x) := 0.14279 \cdot \ln^3(-x) + 1.04416 \cdot \ln^2(-x) + 3.92 \cdot \ln(-x) + 1.65795 \quad \text{for} \quad x \in [-0.27, -0.0732]$$

(for brevity, we do not reproduce the rest of these functions here). The expressions $\alpha^{(m)}(x)$ for $m = 1, \ldots, 4$ have been designed to approximate $W_0(x)$ on the specified intervals, while for $m = 5, \ldots, 8$ to approximate $W_{-1}(x)$.

The nine figures in our Fig. 2 depict the approximation errors as follows. The differences $\alpha^{(m)}(x) - W_0(x)$ ($m = 1, \ldots, 4$) and $\alpha^{(m)}(x) - W_{-1}(x)$ ($m = 5, \ldots, 8$) always correspond to the dashed red curves, while the differences $\beta_M(x) - W_0(x)$ or $\beta_M(x) - W_{-1}(x)$ correspond to the solid orange curves, with x on the horizontal axis. Fig. 2a and b correspond to $(m, M) = (1, 3)$ and $(m, M) = (2, 3)$, respectively; Fig. 2c and d correspond to $(m, M) = (3, 2)$ (here we need two cases because $\beta_M(x)$ is defined differently for $0 < x < e$ and for $x > e$); finally, Fig. 2e–i correspond to the cases $(m, M) = (4, 1)$, $(m, M) = (5, 1)$, $(m, M) = (6, 1)$, $(m, M) = (7, 2)$, and $(m, M) = (8, 2)$, respectively.

From these numerical results we can conclude that

- carrying out only a small number of iterations ($M = 1, 2,$ or 3) of (5) is sufficient to yield approximations which are overall as good as the approximations obtained from the expressions $\alpha^{(m)}$;
- by increasing the iteration number even by 1 (that is, if we used $\beta_{M+1}(x)$ instead of $\beta_M(x)$), then the corresponding (solid orange) error curves would almost be indistinguishable from the horizontal axis in the given plot window (due to the fact that the convergence rate of the sequence β_n is quadratic, as shown by the theorems in our Section 2);
- unlike the dashed red curves, the $\beta_M(x) - W_0(x)$ and $\beta_M(x) - W_{-1}(x)$ error functions always have the same sign on the given intervals (see Lemma 2.1 and its counterparts later in Section 2);
- near the branch point $x = -1/e$ or near the singularity $x = 0^-$, the approximations $\alpha^{(m)}(x) - W_{-1}(x)$ produce relatively large errors, and (as pointed out also in Álvarez [1]) no approximation is provided for $W_{-1}(x)$ within the intervals $[-1/e, -0.36785)$ and $(-10^{-40}, 0)$—the functions $\beta_M(x)$ do not suffer from these drawbacks.

Remark 1.4. According to some numerical tests, the recursion (5) can be used to approximate the complex branches W_k ($k \in \mathbb{Z}$) of the Lambert function as well. As a particular example, suppose that we are looking for all solutions to the equation $ze^z = z^*$ with, say, $z^* := \imath$ (where $\imath^2 = -1$), that is, we seek to approximate $W_k(\imath)$ for any fixed $k \in \mathbb{Z}$. Let us pick a starting value $\beta_0 \in \mathbb{C}$, and consider a branch of the complex logarithm appearing in (5)—that is, fix some $\ell \in \mathbb{Z}$ and consider the function $z \mapsto \ln(z) + 2\pi\ell\imath$, where ln is the principal branch of the logarithm whose imaginary part is in the interval $(-\pi, \pi]$. Let us denote the resulting sequence by $\beta_{n,\ell}$. Then we have observed the following interesting phenomenon.

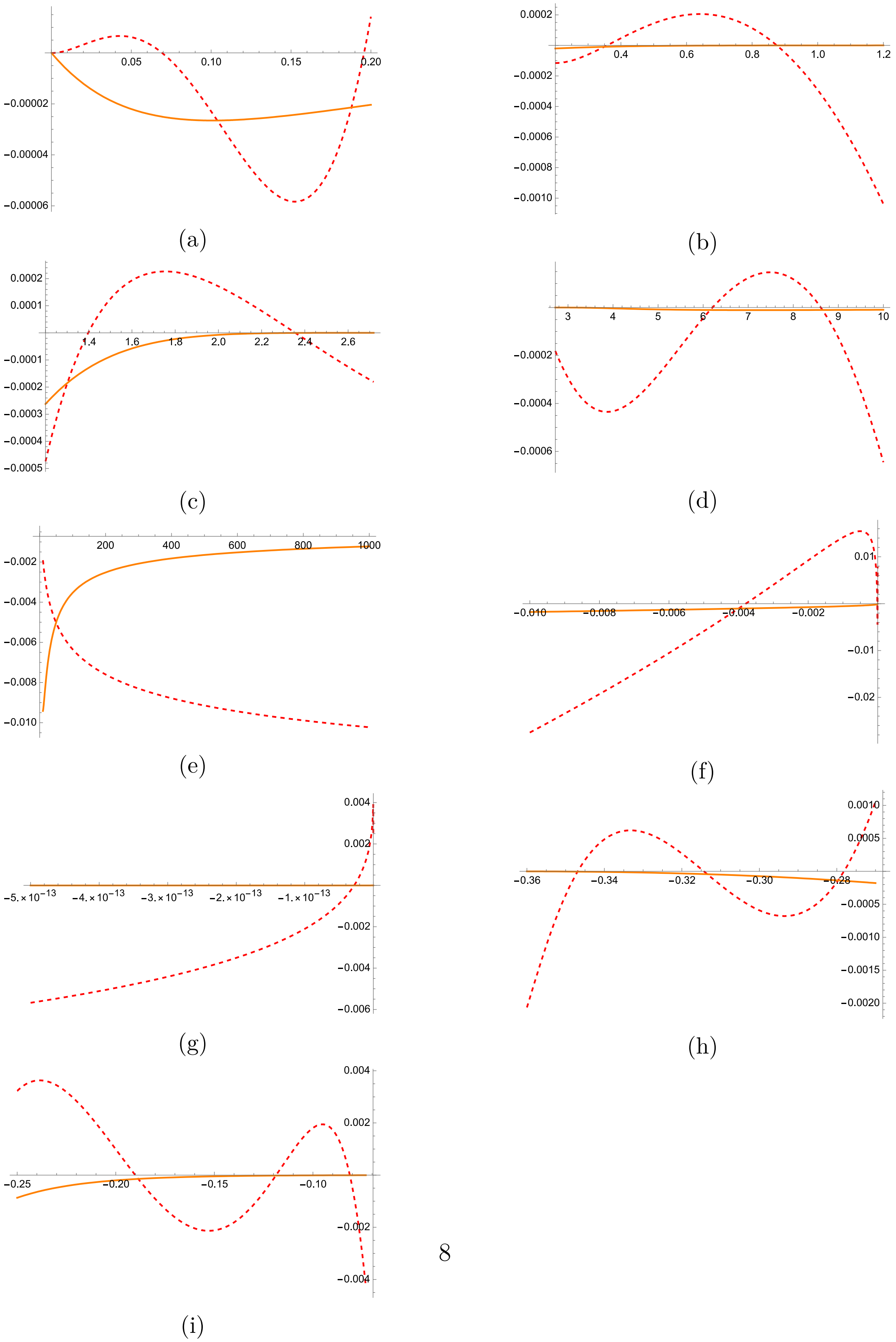


Fig. 2. These graphs show the error between W_0 or W_{-1} and their various approximations—see Remark 1.3.

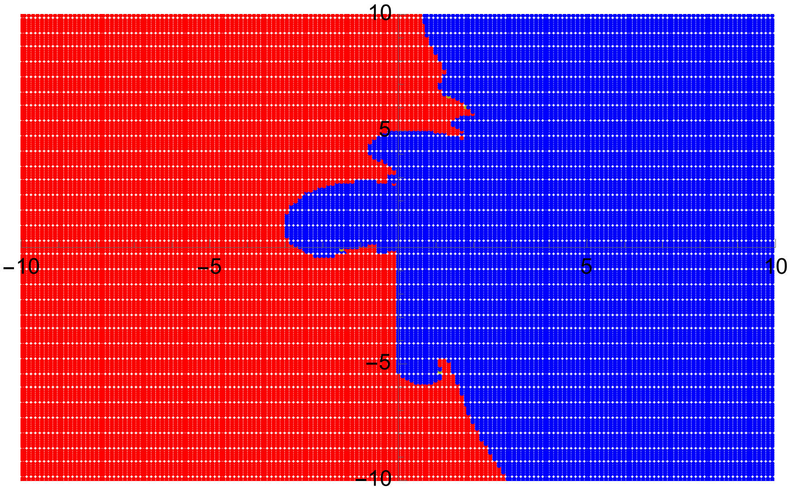


Fig. 3. Illustration for Remark 1.4.

- For any fixed $\ell \neq 0$, the sequence $\beta_{n,\ell}$ converges to $W_k(\imath)$ (as $n \to \infty$) for some k, and each possible $W_k(\imath)$ value *except two* can be found this way by using a suitable logarithm branch $\ell \neq 0$. For $z^* = \imath$, these exceptional values are $w_r \approx -1.0896 - 2.7663\imath$ and $w_b \approx 0.3746 + 0.5764\imath$. For fixed ℓ, the $n \to \infty$ limit of the sequence $\beta_{n,\ell}$ does not depend on the starting value β_0.
- For $\ell = 0$, we can recover the missing two values w_r and w_b by using a *suitable* starting value β_0 for the recursion $\beta_{n,0}$, see Fig. 3. In this figure, the complex numbers, as β_0 starting values, have been colored "red" if $\beta_{n,0}$ converges to w_r, and "blue" if $\beta_{n,0}$ converges to w_b as $n \to \infty$.The above properties seem to hold not just for the equation $ze^z = z^*$ with $z^* = \imath$, but for a general complex z^* as well. The verification of these observations together with the construction of suitable starting values (also to make the recursion (5) well-defined) and some convergence estimates could be the subject of a future study.

1.1. Summary of the results and structure of the paper

In Section 2, a complete analysis of the recursion (5) is given.

We propose simple and suitable starting values (consisting of the basic operations, logarithms, or square roots) that guarantee monotone convergence on the full domain of definition of both real branches:

- for the branch W_0 on $(e, +\infty)$, $(0, e)$, and $(-1/e, 0)$, as well as
- for the branch W_{-1} on $(-1/e, 0)$.

The essential feature of these theorems is that the quadratic rate of convergence of (5) is proved via explicit and uniform error estimates. Thanks to their simplicity, the maximum number of iteration steps needed to achieve a desired precision can easily be determined in advance.

We also reproduce some guaranteed, high-precision approximations of W_0 in Wolfram *Mathematica* that were computed in a different software environment and reported in Johansson [35]—for arguments very close to the branch point $x = -1/e$, or for very large arguments (say, for $x \approx 10^{10^{20}}$, being so large that their direct evaluation in *Mathematica* via its built-in function `ProductLog` is not possible). These huge W_0 values may have significance in analytic number theory, because some estimates of the non-trivial roots of the Riemann ζ function [23], or estimates of the prime counting function [65] have been expressed in terms of the W_0 function.

We highlight the fact that our analysis covers the unbounded branch W_{-1} as well—this branch is more difficult to treat due to the presence of the branch point at $-1/e$ and the singularity at 0.

To make the presentation of our results easier, all technical proofs of the theorems and lemmas are collected in Appendix A. The proofs are almost entirely of symbolic character. As for the techniques, monotonicity arguments are typical. To handle transcendental inequalities (e.g., ones with roots, exponential functions and logarithms simultaneously), repeated differentiation and various substitutions are used to convert them to inequalities containing only polynomials, whose behavior is easier to analyze.

1.2. Acknowledgements

The author is indebted to the anonymous referees of the manuscript for their suggestions that helped improving the presentation and arrangement of the material.

1.3. Notation and some preliminary results

The set of natural numbers and positive integers are denoted by $\mathbb{N} := \{0, 1, 2, \ldots\}$ and $\mathbb{N}^+$, respectively, and the abbreviations

$$L_1 := \ln \quad \text{and} \quad L_2 := \ln \circ \ln$$

will occasionally appear. Auxiliary functions in the proofs will sometimes carry subscripts referring to the section in which they appear (for example, the function $f_{A.5}$ appears in Section A.5).

Next, we collect some elementary results which will also be used later.

(i) From the definition of W_0 and W_{-1}, it is easily seen that the following identities are satisfied:

$$\mathrm{W}_0(x) = \ln(x) - \ln(\mathrm{W}_0(x)) \quad \text{for } x \in (0, +\infty); \tag{9}$$

$$\mathrm{W}_0(x) = -\ln\left(\frac{\mathrm{W}_0(x)}{x}\right) \quad \text{for } x \in (-1/e, 0); \tag{10}$$

$$\mathrm{W}_{-1}(x) = -\ln\left(\frac{\mathrm{W}_{-1}(x)}{x}\right) \quad \text{for } x \in (-1/e, 0). \tag{11}$$

(ii) The strict monotonicity of the function $[-1, +\infty) \ni x \mapsto xe^x$ implies that for any $\alpha, \beta \in [-1, +\infty)$ we have

$$\alpha \boxed{\substack{\leq \\ >}} \beta \text{ if and only if } \alpha e^\alpha \boxed{\substack{\leq \\ >}} \beta e^\beta, \tag{12}$$

where $\boxed{\substack{\leq \\ >}}$ is either "<", or " = ", or ">".

(iii) The following auxiliary inequality appears in Hoofar and Hassani [31]; for the sake of completeness, we reprove it in Section A.1.

Lemma 1.5. *On* $(e, +\infty)$ *we have* $L_1 - L_2 < \mathrm{W}_0 < L_1$, *and* $L_1(e) - L_2(e) = \mathrm{W}_0(e) = L_1(e) = 1$.

2. A quadratically convergent recursion for W_0 and W_{-1} on their full domains of definition

In this section, we analyze the recursion (5) by proposing some starting values on each subinterval, then prove explicit, quadratic convergence estimates.

2.1. Convergence to W_0 *on the interval* $(e, +\infty)$

Due to $\mathrm{W}_0(e) = 1$, let us fix an arbitrary $x > e$ in this section. Here we propose the following starting value:

$$\begin{cases} \beta_0(x) := \ln(x) - \ln(\ln(x)), \\ \beta_{n+1}(x) := \frac{\beta_n(x)}{1+\beta_n(x)}\left(1 + \ln\left(\frac{x}{\beta_n(x)}\right)\right) \quad (n \in \mathbb{N}). \end{cases} \tag{13}$$

Lemma 2.1. *For any* $x > e$, *the recursion (13) satisfies*

$$0 < \beta_n(x) < \beta_{n+1}(x) < \mathrm{W}_0(x) \quad (n \in \mathbb{N}).$$

The proof of this lemma is found in Section A.2. The lemma says, in particular, that the recursion (13) is well-defined and real-valued. In the remainder of Section 2.1, we show that

$$\lim_{n \to +\infty} \beta_n(x) = \mathrm{W}_0(x). \tag{14}$$

We prove the convergence by giving some explicit error estimates as follows.

We start with the inductive step. The proof of the following lemma is given in Section A.3.

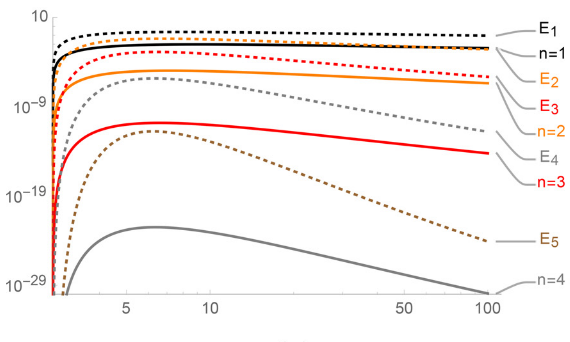
Lemma 2.2. *For any* $x > e$ *and* $n \in \mathbb{N}$, *we have*

$$0 < \mathrm{W}_0(x) - \beta_{n+1}(x) < \frac{(\mathrm{W}_0(x) - \beta_n(x))^2}{(1 + \beta_n(x))\mathrm{W}_0(x)}. \tag{15}$$

The next lemma describes some simple estimates for the starting value. Its proof is found in Section A.4.

Lemma 2.3. *For any* $x > e$, *one has*

$$0 < \mathrm{W}_0(x) - \beta_0(x) < \frac{e}{e-1} \frac{\ln(\ln(x))}{\ln(x)}. \tag{16}$$



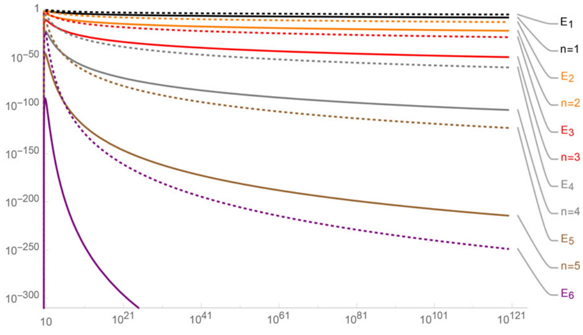


Fig. 4. The log-log plot in figure (a) illustrates the quantities in (18) with x on the horizontal axis (see Theorem 2.4). The continuous curves correspond to the actual differences $\mathrm{W}_0(x) - \beta_n(x)$ for various values of n, whereas the dotted curves E_n depict the right-hand side of the estimate (18). Figure (b) shows the same expressions but for an extended range of x values.

In particular, with $\kappa_1 := \ln(1+1/e) \in (0.31, 0.32)$ *and for any* $x > e$

$$0 < \mathrm{W}_0(x) - \beta_0(x) \le \kappa_1. \tag{17}$$

Now we can state the main result of this section.

Theorem 2.4. *For* $n \in \mathbb{N}^+$ *and for any* $x > e$*, the recursion (13) satisfies*

$$0 < \mathrm{W}_0(x) - \beta_n(x) < \frac{\left(\frac{e}{e-1}\frac{\ln(\ln(x))}{\ln(x)}\right)^{2^n}}{(\ln(x) - \ln(\ln(x)))^{-1+2^n}}, \tag{18}$$

and also the uniform estimate

$$0 < \mathrm{W}_0(x) - \beta_n(x) < \kappa_1^{2^n} < \left(\frac{32}{100}\right)^{2^n}. \tag{19}$$

Proof. To prove (18), one drops the factor $1 + \beta_n(x) > 1$ from the denominator of the upper estimate in (15), then applies it recursively to get

$$0 < \mathrm{W}_0(x) - \beta_n(x) < \frac{(\mathrm{W}_0(x) - \beta_0(x))^{2^n}}{(\mathrm{W}_0(x))^{-1+2^n}}. \tag{20}$$

Then we use (16) in the numerator and Lemma 1.5 in the denominator. To prove (19), due to $\mathrm{W}_0(x) > 1$, we drop the denominator of the upper estimate in (20) and use (17). □

The above theorem of course also proves (14). Regarding the estimate (18), due to Lemma A.2, we have $\frac{e}{e-1}\frac{\ln(\ln(x))}{\ln(x)} \in (0, 1)$ and $\ln(x) - \ln(\ln(x)) > 1$ for $x > e$. Moreover, (18) shows that the convergence of (13) becomes faster for larger and larger values of x. The quality of approximations appearing in Theorem 2.4 can be observed in Fig. 4.

Remark 2.5. According to (19), we have the following uniform estimates for any $x > e$:

$$0 < \mathrm{W}_0(x) - \beta_5(x) < 8 \cdot 10^{-17},$$

$$0 < \mathrm{W}_0(x) - \beta_{10}(x) < 7 \cdot 10^{-517},$$

$$0 < \mathrm{W}_0(x) - \beta_{15}(x) < 8 \cdot 10^{-16519}.$$

Remark 2.6. In *Mathematica* (version 11), a direct evaluation of $\mathrm{W}_0\left(10^{10^3}\right)$ with its command ProductLog is not possible: although the number 10^{1000} itself can easily be represented in this computer system, its internal algorithms cannot handle $\mathrm{W}_0(10^{1000})$. (Based on the error messages, the reason is probably the following: *Mathematica* uses (a variant) of the recursion (4), which contains the expression $xe^{-\nu_n(x)}$, and here $x > 0$ is large, while $e^{-\nu_n(x)}$ is too close to 0. Indeed, it seems that this particular piece of code tries to represent $e^{-\nu_n(x)}$ as a "machine number", even if high-precision computation is requested.)

Now with the recursion (13), it is straightforward to estimate even $W_0\left(10^{10^{20}}\right)$ in *Mathematica* by taking advantage of the logarithms appearing in the starting value β_0 and rewriting $\ln\left(10^{10^{20}}\right)$ as $10^{20}\cdot\ln(10)$. In particular, due to Theorem 2.4 we have

$$0 < W_0\left(10^{10^{20}}\right) - \beta_9\left(10^{10^{20}}\right) < 10^{-10000}.$$

In fact, the difference above is even smaller than $2\cdot 10^{-19873}$, and the computation of $\beta_9\left(10^{10^{20}}\right)$ to the desired precision took less than 0.33 seconds in *Mathematica* on a standard laptop.

The approximation of the quantity $W_0\left(10^{10^{20}}\right)$ to 10,000 digits of precision appears in [35, Section 6]; it is implemented in the Arb library. We found that all the displayed digits of this number are in perfect agreement with the corresponding digits of our quantity $\beta_9\left(10^{10^{20}}\right)$ computed in *Mathematica*.

2.2. Convergence to W_0 on the interval $(0, e)$

Let us fix an arbitrary $0 < x < e$ in this section. On this interval, we propose the following simple starting value:

$$\begin{cases} \beta_0(x) := & x/e, \\ \beta_{n+1}(x) := & \frac{\beta_n(x)}{1+\beta_n(x)}\left(1+\ln\left(\frac{x}{\beta_n(x)}\right)\right) \quad (n\in\mathbb{N}). \end{cases} \tag{21}$$

By using the formula for the derivative of the inverse function, we have

$$W_0''(x) = -\frac{(W_0(x))^2(W_0(x)+2)}{x^2(W_0(x)+1)^3} < 0,$$

so W_0 is strictly concave on $(0, e)$, and $W_0(x) = x/e$ holds at $x = 0$ and $x = e$, hence $0 < \beta_0(x) < W_0(x)$ on this interval. But this means that Lemmas 2.1 and 2.2 and their proofs remain valid also for $x \in (0, e)$. Therefore, we can repeat the first few steps of the proof of Theorem 2.4 to arrive at the inequality

$$0 < W_0(x) - \beta_n(x) < \frac{(W_0(x)-\beta_0(x))^{2^n}}{(W_0(x))^{-1+2^n}} \tag{22}$$

again ($n \in \mathbb{N}^+$). However, unlike on the interval $(e, +\infty)$ in the previous section, now the denominator of (22) can get arbitrarily close to 0 on $(0, e)$, so some care must be taken. First, we state the following lemma, whose proof is given in Section A.5.

Lemma 2.7. *For any $x \in (0, e)$ we have*

$$0 < W_0(x) - \beta_0(x) < \frac{1}{5}.$$

Remark 2.8. There is no simple formula for the global maximum of the function $W_0 - \beta_0$ on $(0, e)$ (with β_0 defined in (21)). Nevertheless, the value $1/5$ given above is close to the actual global maximum (which is approximately 0.1993)—cf. Lemma 2.3, with β_0 defined in (13), where the global maximum on $(e, +\infty)$ is exactly κ_1.

The following uniform upper estimate is the main result of this section, also proving $\lim_{n\to+\infty}\beta_n(x) = W_0(x)$ for $0 < x < e$.

Theorem 2.9. *With $\kappa_2 := 1 - 1/e$ and for any $n \in \mathbb{N}^+$ and $0 < x < e$, the recursion (21) satisfies*

$$0 < W_0(x) - \beta_n(x) < \frac{1}{5}\cdot\kappa_2^{-1+2^n} < \frac{1}{5}\cdot\left(\frac{633}{1000}\right)^{-1+2^n}. \tag{23}$$

Proof. We give a simple upper estimate of the rightmost fraction in (22). Let us set $m := 2^n - 1 \in \mathbb{N}^+$ and consider the decomposition

$$\frac{(W_0(x)-\beta_0(x))^{2^n}}{(W_0(x))^{-1+2^n}} = (W_0(x)-\beta_0(x))\cdot\left(1-\frac{\beta_0(x)}{W_0(x)}\right)^m.$$

The first factor is upper estimated by using Lemma 2.7. As for the second one, notice that

$$\left(1-\frac{\beta_0}{W_0}\right)'(x) = -\frac{1}{e(W_0(x)+1)} < 0,$$

hence, for $0 < x < e$,

$$0 < 1-\frac{\beta_0(x)}{W_0(x)} < \lim_{x\to 0^+}\left(1-\frac{\beta_0(x)}{W_0(x)}\right) = \lim_{x\to 0^+}\left(1-\frac{x/e}{W_0(x)}\right).$$

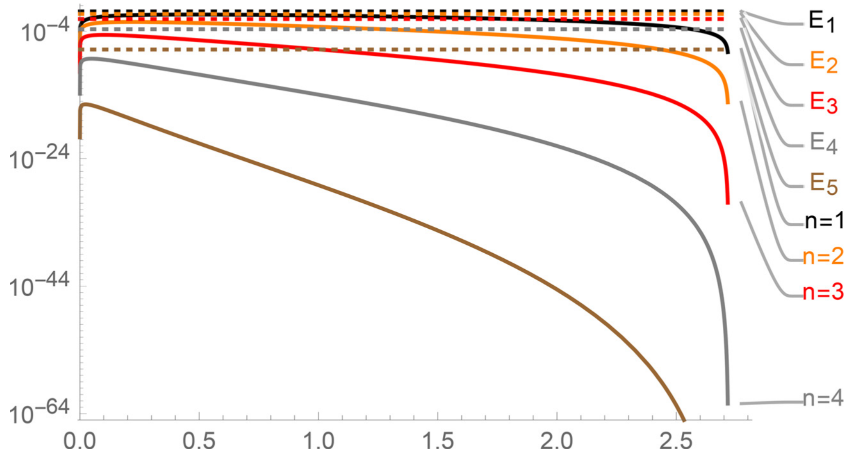


Fig. 5. A semi-log plot illustrating Theorem 2.9 with x on the horizontal axis. The continuous curves correspond to the actual differences $\mathrm{W}_0(x) - \beta_n(x)$ for various values of n (and they tend to 0 as x converges to any of the endpoints of the interval $(0, e)$), whereas the dotted lines E_n depict the uniform estimates $\frac{1}{5} \cdot \kappa_2^{-1+2^n}$ in (23).

Now (1)—the Taylor expansion of W_0 about the origin, with positive radius of convergence—implies that $\lim_{x\to 0} \frac{\mathrm{W}_0(x)}{x} = 1$, so the above limit is κ_2, completing the proof. □

Theorem 2.9 is illustrated by Fig. 5.

2.3. *Convergence to* W_0 *on the interval* $(-1/e, 0)$

Let us fix any $x \in (-1/e, 0)$ in this section. On this interval, we make the following choice for the starting value:

$$\begin{cases} \beta_0(x) := & \frac{ex\ln(1+\sqrt{1+ex})}{\sqrt{1+ex}\,(1+\sqrt{1+ex})}, \\ \beta_{n+1}(x) := & \frac{\beta_n(x)}{1+\beta_n(x)}\left(1+\ln\left(\frac{x}{\beta_n(x)}\right)\right) \quad (n \in \mathbb{N}). \end{cases} \tag{24}$$

The following lemma gives a two-sided initial estimate of W_0; its proof is given in Section A.6.

Lemma 2.10. *For any* $-1/e < x < 0$ *we have*

$$-1 < -1 + \sqrt{1+ex} < \mathrm{W}_0(x) < \beta_0(x) < 0.$$

Remark 2.11. The choice of the lower bound $-1+\sqrt{1+ex}$ in Lemma 2.10 is motivated by the Puiseux expansion of W about the branch point $-1/e$, while $\beta_0(x)$ is the result of a single iteration step of (5) applied to $-1+\sqrt{1+ex}$.

The lemma below establishes the monotonicity and boundedness properties of the sequence (24), and shows that it is well-defined and real-valued. Its proof—found in Section A.7—is analogous to that of Lemma 2.1.

Lemma 2.12. *For any* $x \in (-1/e, 0)$, *the recursion (24) satisfies*

$$-1 < \mathrm{W}_0(x) < \beta_{n+1}(x) < \beta_n(x) < 0 \quad (n \in \mathbb{N}).$$

The error estimate in Theorem 2.17 will be based on the following inequality (cf. Lemma 2.2), whose proof is found in Section A.8.

Lemma 2.13. *For any* $-1/e < x < 0$ *and* $n \in \mathbb{N}$, *we have*

$$0 < \beta_{n+1}(x) - \mathrm{W}_0(x) < \frac{(\beta_n(x) - \mathrm{W}_0(x))^2}{-\mathrm{W}_0(x)(1+\beta_n(x))}. \tag{25}$$

Regarding the above upper estimate, note that this time the denominator of the fraction in (25) can get arbitrarily close to 0 near *both* endpoints of the interval $(-1/e, 0)$.

The following two lemmas constitute the final building blocks in the proof of Theorem 2.17, with proofs in Sections A.9 and A.10, respectively.

Lemma 2.14. *For any* $-1/e < x < 0$, *we have*

$$0 < \beta_0(x) - \mathrm{W}_0(x) < \frac{1}{10}.$$

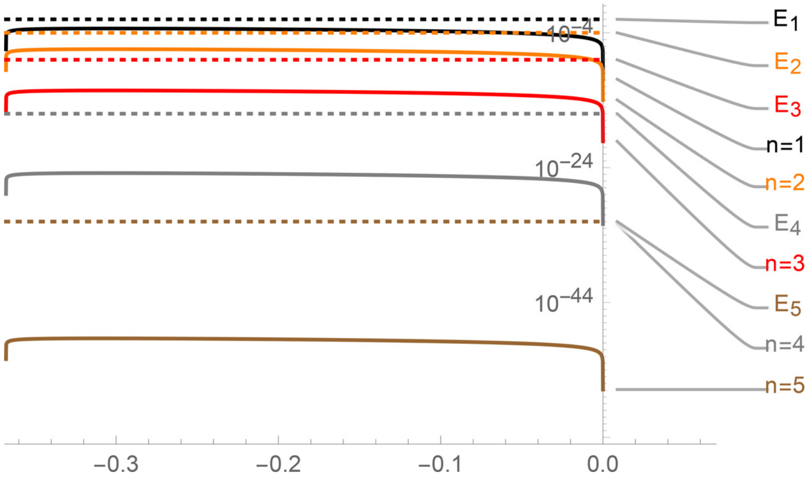


Fig. 6. A semi-log plot illustrating Theorem 2.17 with x on the horizontal axis. The continuous curves correspond to the actual differences $\beta_n(x) - \mathrm{W}_0(x)$ for various values of n (and they tend to 0 as x converges to any of the endpoints of the interval $(-1/e, 0)$), whereas the dotted lines E_n depict the uniform estimates $1/10^{2^n}$.

Remark 2.15. The upper bound $1/10$ in Lemma 2.14 could be replaced by, say, 0.015, but the proof of that inequality would require more effort.

Lemma 2.16. *For any $-1/e < x < 0$, we have*

$$0 < \frac{\beta_0(x) - \mathrm{W}_0(x)}{-\mathrm{W}_0(x)\sqrt{1+ex}} < \frac{1}{10}.$$

The main result of this section is given below, also proving convergence of the recursion (24) on $(-1/e, 0)$.

Theorem 2.17. *For any $n \in \mathbb{N}^+$ and $x \in (-1/e, 0)$, the recursion (24) satisfies the uniform estimate*

$$0 < \beta_n(x) - \mathrm{W}_0(x) < \left(\frac{1}{10}\right)^{2^n}.$$

Proof. Due to (25) and Lemmas 2.10 and 2.12, we have

$$0 < \beta_n(x) - \mathrm{W}_0(x) < \frac{(\beta_{n-1}(x) - \mathrm{W}_0(x))^2}{-\mathrm{W}_0(x)(1+\beta_{n-1}(x))} < \frac{(\beta_{n-1}(x) - \mathrm{W}_0(x))^2}{-\mathrm{W}_0(x)\sqrt{1+ex}},$$

so, recursively, we get

$$0 < \beta_n(x) - \mathrm{W}_0(x) < (\beta_0(x) - \mathrm{W}_0(x)) \cdot \left(\frac{\beta_0(x) - \mathrm{W}_0(x)}{-\mathrm{W}_0(x)\sqrt{1+ex}}\right)^{-1+2^n}.$$

Now Lemmas 2.14 and 2.16 finish the proof. □

Theorem 2.17 is illustrated by Fig. 6.

Remark 2.18. In Johansson [35, Section 6], the first 9950 digits of the quantity $\mathrm{W}_0\left(-\frac{1}{e} + 10^{-100}\right)$ near the branch point are computed. We computed $\beta_{14}\left(-\frac{1}{e} + 10^{-100}\right)$ by using *Mathematica* ($2^{14} > 9950$), and found that all the first and last few digits displayed in Johansson [35] are again in agreement—the computations within two different systems yielded the same result.

2.4. Convergence to W_{-1} on the interval $(-1/e, 0)$

In this section we propose suitable starting values for the recursion (5) to converge to $\mathrm{W}_{-1}(x)$ for any $x \in (-1/e, 0)$. The convergence is again proved via simple (uniform) error estimates.

Although the statements and proofs are similar to those in Sections 2.1–2.3, let us highlight some differences, including

- the branch W_{-1} over the bounded interval $(-1/e, 0)$ is unbounded—with a branch point at the left endpoint, and a singularity at the right endpoint—hence we will split $(-1/e, 0)$ when defining the recursion starting values $\beta_0(x)$;
- when using the bijective reparametrization $x = ye^y$ in the proofs of transcendental inequalities to eliminate W_{-1}, this time $\mathrm{W}_{-1}(ye^y) = y$ will hold for $y < -1$ (cf. the identity $\mathrm{W}_0(ye^y) = y$ for $-1 < y < 0$ used earlier);

- the branch W_{-1} is strictly decreasing, so instead of (12) we now have

$$\alpha \boxed{\substack{\leq\\=\\>}} \beta \text{ if and only if } \alpha e^{\alpha} \boxed{\substack{\geq\\=\\<}} \beta e^{\beta}, \tag{26}$$

for any $\alpha, \beta \in (-\infty, -1]$.

Due to the above reasons, the proofs will be presented in detail.

For $x \in (-1/e, 0)$, we define the recursion as follows:

$$\begin{cases} \beta_0(x) := & -1 - \sqrt{2}\sqrt{1+ex} \quad \text{for } -1/e < x \le -1/4, \\ \beta_0(x) := & \ln(-x) - \ln(-\ln(-x)) \quad \text{for } -1/4 < x < 0, \\ \beta_{n+1}(x) := & \frac{\beta_n(x)}{1+\beta_n(x)}\left(1 + \ln\left(\frac{x}{\beta_n(x)}\right)\right) \quad (n \in \mathbb{N}). \end{cases} \tag{27}$$

Remark 2.19.

(i) The point $-1/4$ to split the interval $(-1/e, 0)$ in the definition of β_0 in (27) is somewhat arbitrary; it has been chosen to make the constants in the estimates of this section simple, small positive numbers.

(ii) With the above definition, β_0 is a piecewise continuous function. In fact, it is possible to construct a function that is continuous over the *whole* interval $(-1/e, 0)$ and approximates W_{-1} so well that all the lemmas and the theorem below would remain true (with slightly different constants, of course). The choice $\widetilde{\beta}_0(x) := \ln(-x) - \ln(-\ln(-x))$, for example, would *not* be an appropriate one on the interval $(-1/e, 0)$, as it would result in some singular estimates near $x = -1/e$. One suitable choice for the starting value of (27) could be

$$\widetilde{\beta}_0(x) := \frac{ex\ln\left(1-\sqrt{1+ex}\right)\ln\left(\frac{1+ex-\sqrt{1+ex}}{\ln\left(1-\sqrt{1+ex}\right)}\right)}{1+ex-\sqrt{1+ex}+ex\ln\left(1-\sqrt{1+ex}\right)} \qquad (x \in (-1/e, 0)),$$

but with this formula the proofs of the estimates would become more involved. We remark that the difference $\mathrm{W}_{-1} - \widetilde{\beta}_0$ is strictly increasing and satisfies

$$0 < \mathrm{W}_{-1}(x) - \widetilde{\beta}_0(x) < \lim_{x\to 0^-}\left(\mathrm{W}_{-1}(x) - \widetilde{\beta}_0(x)\right) = \ln(2) - \frac{1}{2} \approx 0.193$$

for any $x \in (-1/e, 0)$. The expression for $\widetilde{\beta}_0(x)$ has been obtained by taking *two* iteration steps with (5) started from $-1-\sqrt{1+ex}$ (cf. Remark 2.11).

(iii) Regarding the factor $\sqrt{2}$ in the definition of β_0 in (27), it directly appears in the Puiseux expansion of W about $x = -1/e$, and it gives a better approximation for W_{-1} close to $-1/e$. However, the constant $\sqrt{2}$ was *not* included in the starting value of the recursion (24), because this way that β_0 yields an overall better estimate for W_0 on $(-1/e, 0)$.

(iv) The choice for the other starting value in (27) is motivated by the estimate (8). This estimate appears in Alzahrani and Salem [2] (but by using a different—equivalent—parametrization).

(v) As we will see, the sequence β_n is only monotone for $n \in \mathbb{N}^+$ (and not for $n \in \mathbb{N}$). Again, this is a consequence of the trade-off between simple proofs and good uniform error estimates.

The first lemma estimates the initial difference; its proof is found in Section A.11.

Lemma 2.20. *For any $x \in (-1/e, 0)$, the starting value in (27) satisfies the estimates*

$$0 < \beta_0(x) - \mathrm{W}_{-1}(x) < 1/2. \tag{28}$$

The well-definedness and monotonicity properties of the sequence β_n, and the inductive part of the error estimates are summarized next. The proof of the lemma is given is Section A.12.

Lemma 2.21. *For any $x \in (-1/e, 0)$ and $n \in \mathbb{N}^+$, the recursion (27) is well-defined, real-valued, and satisfies the following:*

$$\beta_n(x) < \beta_{n+1}(x) < \mathrm{W}_{-1}(x) < \beta_0(x) < -1, \tag{29}$$

and

$$0 < \mathrm{W}_{-1}(x) - \beta_n(x) < (\beta_0(x) - \mathrm{W}_{-1}(x)) \cdot \left(\frac{\beta_0(x) - \mathrm{W}_{-1}(x)}{|\mathrm{W}_{-1}(x)| \cdot |1+\beta_0(x)|}\right)^{-1+2^n}. \tag{30}$$

Regarding the upper estimate (30), note that this time its denominator can get arbitrarily close to 0 near the left endpoint of the interval $(-1/e, 0)$, and both terms in its numerator are singular as $x \to 0^-$. The following lemma yields suitable upper estimates of this fraction. Its proof is found in Section A.13.

Lemma 2.22. *For any $x \in (-1/e, 0)$, the starting value in (27) satisfies the estimates*

$$0 < \frac{\beta_0(x) - \mathrm{W}_{-1}(x)}{|\mathrm{W}_{-1}(x)| \cdot |1+\beta_0(x)|} < \frac{1}{2}. \tag{31}$$

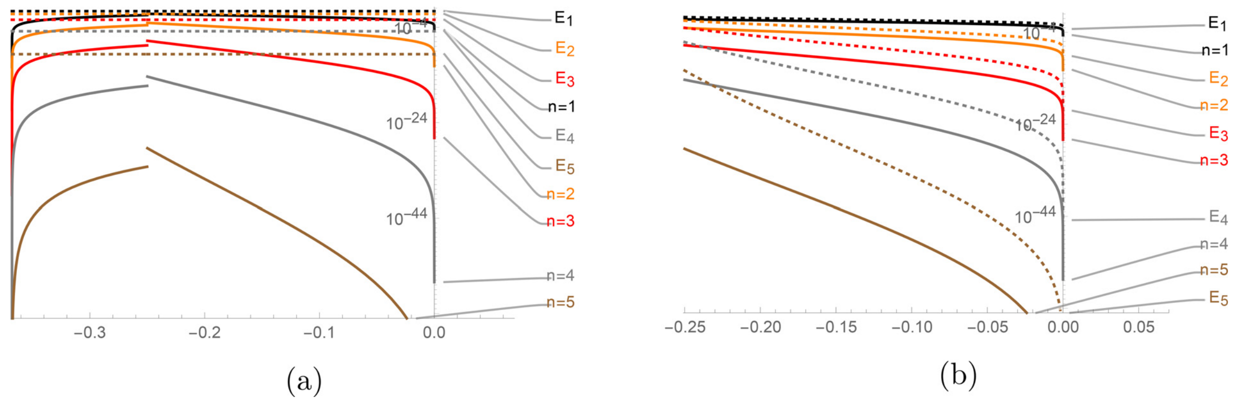


Fig. 7. Semi-log plots illustrating Theorem 2.23 with x on the horizontal axis. The piecewise continuous curves in figure (a) (reflecting the different definitions for $x \in (-1/e, -1/4]$ and for $x \in (-1/4, 0)$ in (27)) correspond to the actual differences $\mathrm{W}_{-1}(x) - \beta_n(x)$ for various values of n, whereas the dotted lines E_n depict the uniform estimates $1/2^{2^n}$. In figure (b), the dotted curves E_n correspond to the right-hand side of the "sharper estimate" in the theorem for $x \in (-1/4, 0)$.

Moreover, for $-1/4 < x < 0$ we also have

$$0 < \frac{\beta_0(x) - \mathrm{W}_{-1}(x)}{|\mathrm{W}_{-1}(x)| \cdot |1 + \beta_0(x)|} < \frac{1/2}{|\ln(-x) - \ln(-\ln(-x))| \cdot |1 + \ln(-x) - \ln(-\ln(-x))|}. \tag{32}$$

Summarizing the above, we have the following main result.

Theorem 2.23. *For any $-1/e < x < 0$ and $n \in \mathbb{N}^+$, the recursion (27) satisfies*

$$0 < \mathrm{W}_{-1}(x) - \beta_n(x) < \left(\frac{1}{2}\right)^{2^n}.$$

In particular, for $-1/4 < x < 0$, the sharper estimate

$$\mathrm{W}_{-1}(x) - \beta_n(x) < \left(\frac{1}{2}\right)^{2^n} \left(\frac{1}{|\ln(-x) - \ln(-\ln(-x))| \cdot |1 + \ln(-x) - \ln(-\ln(-x))|}\right)^{-1+2^n}$$

also holds.

Proof. The proof directly follows by combining Lemma 2.21 with Lemmas 2.20 and 2.22. □

Theorem 2.23 is illustrated by Fig. 7.

Data Availability

No data was used for the research described in the article.

Appendix A. The proofs of the lemmas and theorems

A1. The proof of Lemma 1.5

The function W_0 is strictly increasing on $[e, +\infty)$ and $\mathrm{W}_0(e) = 1$, so by using (12) we see that $\mathrm{W}_0(x) < L_1(x)$ for $x > e$ is equivalent to $x = \mathrm{W}_0(x)e^{\mathrm{W}_0(x)} < \ln(x)e^{\ln(x)} = x\ln(x)$. Similarly, for the lower estimate, (49) says that $L_1(x) - L_2(x) > 0$ for $x \geq e$, hence by (12) again, $L_1(x) - L_2(x) < \mathrm{W}_0(x)$ is equivalent to

$$x\left(1 - \frac{\ln(\ln(x))}{\ln(x)}\right) = (L_1(x) - L_2(x))e^{L_1(x) - L_2(x)} < \mathrm{W}_0(x)e^{\mathrm{W}_0(x)} = x,$$

but $1 - \frac{\ln(\ln(x))}{\ln(x)} < 1$ for $x \in (e, +\infty)$, so the proof is complete.

A2. The proof of Lemma 2.1

Lemma 1.5 shows that $0 < \beta_0(x) < \mathrm{W}_0(x)$.

Step 1. Suppose that $0 < \beta_n(x) < \mathrm{W}_0(x)$ for some $n \in \mathbb{N}$. Then, due to (12), we have $\beta_n(x)e^{\beta_n(x)} < \mathrm{W}_0(x)e^{\mathrm{W}_0(x)} = x$, so $1 + \beta_n(x) < 1 + \ln\left(\frac{x}{\beta_n(x)}\right)$, implying

$$\beta_n(x) < \frac{\beta_n(x)}{1+\beta_n(x)}\left(1 + \ln\left(\frac{x}{\beta_n(x)}\right)\right) = \beta_{n+1}(x).$$

Step 2. Suppose that $0 < \beta_n(x) < \mathrm{W}_0(x)$ for some $n \in \mathbb{N}$. Then—by using (9) in the brackets [...] below—we have

$$\begin{aligned}
&\mathrm{W}_0(x) - \beta_{n+1}(x) = \mathrm{W}_0(x) - \frac{\beta_n(x)}{1+\beta_n(x)}\left(1 + \ln\left(\frac{x}{\beta_n(x)}\right)\right) = \\
&\frac{1}{1+\beta_n(x)}\left(\beta_n(x)\left[\mathrm{W}_0(x) - \ln\left(\frac{x}{\beta_n(x)}\right)\right] + \mathrm{W}_0(x) - \beta_n(x)\right) = \\
&\frac{1}{1+\beta_n(x)}\left(\beta_n(x)\left[\ln(x) - \ln(\mathrm{W}_0(x)) - \ln\left(\frac{x}{\beta_n(x)}\right)\right] + \mathrm{W}_0(x) - \beta_n(x)\right) = \\
&\frac{1}{1+\beta_n(x)}\left(\beta_n(x)\ln\left(\frac{\beta_n(x)}{\mathrm{W}_0(x)}\right) + \mathrm{W}_0(x) - \beta_n(x)\right) = \\
&\frac{\mathrm{W}_0(x)}{1+\beta_n(x)}(y\ln(y) + 1 - y)
\end{aligned} \tag{33}$$

with $y := \beta_n(x)/\mathrm{W}_0(x) \in (0, 1)$. But, due to Lemma A.1, $y\ln(y) + 1 - y > 0$, so $\beta_{n+1}(x) < \mathrm{W}_0(x)$.

Step 3. The recursive application of Steps 1 and 2 completes the proof.

A3. The proof of Lemma 2.2

First we use Lemma 2.1 and the identity (33), then set $z := \frac{\mathrm{W}_0(x)-\beta_n(x)}{\mathrm{W}_0(x)} \in (0, 1)$ and use the elementary estimate $\ln(1 - z) < -z$ for $z \in (0, 1)$ to get

$$\begin{aligned}
&0 < \mathrm{W}_0(x) - \beta_{n+1}(x) = \frac{\beta_n(x)}{1+\beta_n(x)}\ln\left(1 - \frac{\mathrm{W}_0(x)-\beta_n(x)}{\mathrm{W}_0(x)}\right) + \frac{\mathrm{W}_0(x)-\beta_n(x)}{1+\beta_n(x)} < \\
&\frac{\beta_n(x)}{1+\beta_n(x)}\cdot\frac{-(\mathrm{W}_0(x)-\beta_n(x))}{\mathrm{W}_0(x)} + \frac{\mathrm{W}_0(x)-\beta_n(x)}{1+\beta_n(x)} = \frac{(\mathrm{W}_0(x)-\beta_n(x))^2}{(1+\beta_n(x))\mathrm{W}_0(x)}.
\end{aligned}$$

A4. The proof of Lemma 2.3

The estimate (16) simply follows from the definition of $\beta_0(x)$ in (13) and the earlier estimate (7) (given in Hoofar and Hassani [31]).

As for (17), one could find the global maximum of $\frac{e}{e-1}\frac{\ln(\ln(x))}{\ln(x)}$ for $x > e$. It can be easily shown via differentiation that for $x > e$ we have

$$\frac{e}{e-1}\frac{\ln(\ln(x))}{\ln(x)} \le \frac{1}{e-1} \approx 0.582,$$

with equality exactly for $x = e^e$. However, we maximize the quantity $\mathrm{W}_0(x) - \beta_0(x)$ directly to get the sharper upper bound $\kappa_1 \approx 0.3133$. By the formula for the derivative of the inverse function we have $\mathrm{W}_0'(x) = \frac{\mathrm{W}_0(x)}{x(\mathrm{W}_0(x)+1)}$, so

$$(\mathrm{W}_0 - \beta_0)'(x) = -\frac{\ln(x) - \mathrm{W}_0(x) - 1}{x\ln(x)\cdot(\mathrm{W}_0(x)+1)}.$$

Here the denominator is positive because $x > e$. As for the numerator, its derivative is

$$(\ln - \mathrm{W}_0 - 1)'(x) = \frac{1}{x(\mathrm{W}_0(x)+1)} > 0,$$

and

$$\ln(e) - \mathrm{W}_0(e) - 1 = -1, \quad \ln\left(e^{e+1}\right) - \mathrm{W}_0\left(e^{e+1}\right) - 1 = e - \mathrm{W}_0(e\cdot e^e) = 0.$$

This means that $\ln(x) - \mathrm{W}_0(x) - 1$ is negative for $x \in \left(e, e^{e+1}\right)$, zero at $x = e^{e+1}$, and positive for $x \in \left(e^{e+1}, +\infty\right)$. That is, the function $\mathrm{W}_0 - \beta_0$ is strictly increasing on $\left(e, e^{e+1}\right)$ and decreasing on $\left(e^{e+1}, +\infty\right)$, hence it has a global maximum at $x = e^{e+1}$, and $\mathrm{W}_0\left(e^{e+1}\right) - \beta_0\left(e^{e+1}\right) = \ln(1 + 1/e) = \kappa_1$. The proof is complete.

A5. The proof of Lemma 2.7

We need to prove that $0 < \mathrm{W}_0(x) < 1/5 + x/e$ holds for any $0 < x < e$. Due to (12), this is equivalent to

$$x = \mathrm{W}_0(x) e^{\mathrm{W}_0(x)} < \left(\frac{1}{5} + \frac{x}{e}\right) e^{1/5 + x/e}.$$

Let us set

$$f_{A.5}(x) := \left(\frac{1}{5} + \frac{x}{e}\right) e^{1/5 + x/e} - x,$$

and notice that $f''_{A.5}(x) > 0$, $f_{A.5}(0) > 0$, $f_{A.5}(e) > 0$, so the strictly convex function $f_{A.5}$ is positive at both endpoints of the interval. By solving $f'_{A.5}(x) = 0$ symbolically, we find that this equation has a unique root at $x^* = e\left(\mathrm{W}_0\left(e^2\right) - 6/5\right) \in (0, e)$, corresponding to the global minimum of $f_{A.5}$ on $(0, e)$. After some simplification, we get that

$$f_{A.5}(x^*) = -\frac{e}{\mathrm{W}_0\left(e^2\right)} - e\mathrm{W}_0\left(e^2\right) + \frac{11e}{5},$$

and verify (for example, by using the recursion of Section 2.1) that the right-hand side above is positive (> 0.0017). This means that $f_{A.5} > 0$ on $(0, e)$, completing the proof.

A6. The proof of Lemma 2.10

The leftmost and rightmost inequalities are obvious.

Step 1. We prove the second inequality first. Since now $-1 < \mathrm{W}_0(x)$ is also true, we have, due to (12), that $-1 + \sqrt{1 + ex} < \mathrm{W}_0(x)$ is equivalent to

$$\left(-1 + \sqrt{1 + ex}\right) e^{-1 + \sqrt{1 + ex}} < \mathrm{W}_0(x) e^{\mathrm{W}_0(x)} = x.$$

After introducing the new variable $z := \sqrt{1 + ex} \in (0, 1)$, the above inequality becomes the obvious one

$$(z - 1) e^{z - 1} < \frac{z^2 - 1}{e}.$$

Step 2. We now prove $\mathrm{W}_0(x) < \beta_0(x)$ by using the following bijective reparametrization: for any $-1/e < x < 0$ there is a unique $y \in (-1, 0)$ such that $ye^y = x$, namely, $y = \mathrm{W}_0(x)$. So the inequality $\mathrm{W}_0(x) < \beta_0(x)$ becomes

$$y < \frac{ye^{y+1} \ln\left(1 + \sqrt{1 + ye^{y+1}}\right)}{1 + ye^{y+1} + \sqrt{1 + ye^{y+1}}}.$$

The denominator of this fraction is positive, but $y < 0$, so the above is equivalent to

$$1 + ye^{y+1} + \sqrt{1 + ye^{y+1}} - e^{y+1} \ln\left(1 + \sqrt{1 + ye^{y+1}}\right) > 0.$$

This left-hand side vanishes at $y = -1$, so it is enough to prove that its derivative (no longer containing a logarithm) is positive for any $-1 < y < 0$, that is

$$\frac{\sqrt{1 + ye^{y+1}}\left(3 - y - 4e^{-y-1}\right) - 4e^{-y-1} - e^{y+1} + ye^{y+1} - 3y + 3}{2\sqrt{1 + ye^{y+1}}\left(1 + \sqrt{1 + ye^{y+1}}\right)} > 0.$$

Here again, the denominator is positive, and, unexpectedly, the numerator can be factorized to yield

$$e^{-y-1}\left(e^{y+1} - 1 - \sqrt{1 + ye^{y+1}}\right)\left(ye^{y+1} - e^{y+1} + 2\sqrt{1 + ye^{y+1}} + 2\right).$$

After some elementary manipulations, we see that each of the three factors above are positive for any $-1 < y < 0$, completing the proof.

A7. The proof of Lemma 2.12

We prove the lemma by induction. Lemma 2.10 shows that $-1 < \mathrm{W}_0(x) < \beta_0(x) < 0$.

Step 1. Suppose that $-1 < \mathrm{W}_0(x) < \beta_n(x) < 0$ for some $n \in \mathbb{N}$. Then (12) implies $\beta_n(x) e^{\beta_n(x)} > \mathrm{W}_0(x) e^{\mathrm{W}_0(x)} = x$, so—by carefully noting that now $x, \beta_n(x), \mathrm{W}_0(x) \in (-1, 0)$—we get $1 + \beta_n(x) < 1 + \ln\left(\frac{x}{\beta_n(x)}\right)$, therefore

$$\beta_n(x) > \frac{\beta_n(x)}{1 + \beta_n(x)}\left(1 + \ln\left(\frac{x}{\beta_n(x)}\right)\right) = \beta_{n+1}(x).$$

Step 2. Suppose that $-1 < \mathrm{W}_0(x) < \beta_n(x) < 0$ for some $n \in \mathbb{N}$. Then

$$\beta_{n+1}(x) - \mathrm{W}_0(x) = \frac{1}{1+\beta_n(x)}\left(\beta_n(x)\left[\ln\left(\frac{x}{\beta_n(x)}\right) - \mathrm{W}_0(x)\right] + \beta_n(x) - \mathrm{W}_0(x)\right),$$

and now (10) is used in the brackets $[\ldots]$ above to get

$$\frac{1}{1+\beta_n(x)}\left(\beta_n(x)\ln\left(\frac{\mathrm{W}_0(x)}{\beta_n(x)}\right) + \beta_n(x) - \mathrm{W}_0(x)\right) = \tag{34}$$

$$\frac{-\mathrm{W}_0(x)}{1+\beta_n(x)}(y\ln(y) - y + 1)$$

with $y := \beta_n(x)/\mathrm{W}_0(x) \in (0,1)$. Finally, due to Lemma A.1, $y\ln(y) + 1 - y > 0$, so $\mathrm{W}_0(x) < \beta_{n+1}(x)$.

A8. The proof of Lemma 2.13

The proof is analogous to that of Lemma 2.2. This time we use Lemma 2.12, the identity (34) with $z := \frac{\beta_n(x)-\mathrm{W}_0(x)}{\mathrm{W}_0(x)} \in (-1,0)$, and the estimate $\ln(1+z) < z$ for $z \in (-1,0)$ to get

$$0 < \beta_{n+1}(x) - \mathrm{W}_0(x) = \frac{-\beta_n(x)}{1+\beta_n(x)}\ln(1+z) + \frac{\beta_n(x) - \mathrm{W}_0(x)}{1+\beta_n(x)} <$$

$$\frac{-\beta_n(x)}{1+\beta_n(x)}\cdot\frac{\beta_n(x) - \mathrm{W}_0(x)}{\mathrm{W}_0(x)} + \frac{\beta_n(x) - \mathrm{W}_0(x)}{1+\beta_n(x)} = \frac{(\beta_n(x) - \mathrm{W}_0(x))^2}{-\mathrm{W}_0(x)(1+\beta_n(x))}.$$

A9. The proof of Lemma 2.14

The first two inequalities below follow from Lemma 2.10:

$$0 < \beta_0(x) - \mathrm{W}_0(x) < \beta_0(x) - (-1 + \sqrt{1+ex}) =$$

$$\frac{ex\ln(1+\sqrt{1+ex})}{\sqrt{1+ex}(1+\sqrt{1+ex})} + 1 - \sqrt{1+ex}. \tag{35}$$

It is thus sufficient to upper estimate (35). Since $\ln(1+z) > z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} > 0$ for $z \in (0,1)$, and we have $x < 0$, by denoting $z := \sqrt{1+ex} \in (0,1)$ we see that (35) is further increased by

$$\frac{ex\left(z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4}\right)}{z(1+z)} + 1 - z. \tag{36}$$

But now $x = (z^2-1)/e$, so (36) can be rewritten as

$$-\frac{z(z-1)}{12}\left(3z^2 - 4z + 6\right),$$

and one checks that the global maximum of this quartic polynomial for $z \in (0,1)$ is less than $1/10$ (in fact, it is approximately 0.09928).

A10. The proof of Lemma 2.16

Due to $-1 < \mathrm{W}_0(x) < \beta_0(x) < 0$, we have

$$0 < \frac{\beta_0(x) - \mathrm{W}_0(x)}{-\mathrm{W}_0(x)\sqrt{1+ex}} < \frac{\beta_0(x) - \mathrm{W}_0(x)}{-\beta_0(x)\sqrt{1+ex}},$$

so it is enough to prove that the rightmost expression above is less than $1/10$. This sufficient condition can be rearranged into the form

$$\frac{ex\left(10+\sqrt{1+ex}\right)\ln\left(1+\sqrt{1+ex}\right)}{10\left(1+ex+\sqrt{1+ex}\right)} < \mathrm{W}_0(x). \tag{37}$$

Let us use again the parametrization $ye^y = x$ with $y \in (-1,0)$ as in Step 2 of the proof of Lemma 2.10. Then (37) becomes

$$\frac{ye^{y+1}\left(10+\sqrt{1+ye^{y+1}}\right)\ln\left(1+\sqrt{1+ye^{y+1}}\right)}{10\left(1+ye^{y+1}+\sqrt{1+ye^{y+1}}\right)} < y,$$

or, since the denominator is positive for $0 < y < 1$,

$$\ln\left(1+\sqrt{1+ye^{y+1}}\right) - \frac{10\left(1+ye^{y+1}+\sqrt{1+ye^{y+1}}\right)}{e^{y+1}\left(10+\sqrt{1+ye^{y+1}}\right)} > 0. \tag{38}$$

The left-hand side of (38) vanishes at $y=-1$, so it is enough to prove that its derivative is positive for $-1 < y < 0$. This derivative can be written as

$$\frac{e^{-y-1}}{2\sqrt{1+ye^{y+1}}\left(1+\sqrt{1+ye^{y+1}}\right)\left(10+\sqrt{1+ye^{y+1}}\right)^2} \cdot f_{A.10}(y,z),$$

where

$$f_{A.10}(y,z) := 440 + 10\sqrt{1+yz}\left(\left(y^2+y+2\right)z^2 + (15y-31)z + 44\right) +$$

$$y^2z^2(z+30) + yz\left(z^2-109z+370\right) + 101z^2 - 310z$$

with $z := e^{y+1} \in (1,e)$. Then we also have $-1 < yz < 0$. Clearly, to finish the proof, it suffices to prove that $f_{A.10}(y,z) > 0$ for any $-1 < y < 0$, $1 < z < e$ and $-1 < yz < 0$. Now, by introducing the new variable $w := \sqrt{1+yz} \in (0,1)$, the expression $f_{A.10}(y,z)$ becomes

$$(w+10)^2z^2 + (w+1)(w^3+9w^2-120w-200)z + 10(w+1)^3(w^2+10),$$

so it is enough to prove that this bivariate polynomial is positive for any $1 < z < e$ and $0 < w < 1$. But its discriminant with respect to z, $w^2(w+1)^2(w^4-22w^3-999w^2-7760w-1600)$, is trivially negative for $w \in (0,1)$, completing the proof.

A11. The proof of Lemma 2.20

Step 1. First we prove

$$\mathrm{W}_{-1}(x) < \ln(-x) - \ln(-\ln(-x)) \tag{39}$$

for $x \in (-1/e, 0)$ (instead of only for $x \in (-1/4, 0)$). Although (39) is identical to (8), we provide a direct proof for the sake of completeness. By using the bijective reparametrization $x = ye^y$ mentioned in the beginning of Section 2.4, (39) is equivalent to

$$y < \ln\left(-ye^y\right) - \ln\left(-\ln\left(-ye^y\right)\right) \qquad \text{for} \quad y < -1,$$

that is, to $0 < \ln\left(\frac{-y}{-\ln(-ye^y)}\right)$, which reduces to the obvious inequality $-y > -\ln\left(-ye^y\right)$.

Step 2. Now we prove that

$$f_{A.11}(x) := \ln(-x) - \ln(-\ln(-x)) - \mathrm{W}_{-1}(x) < 1/2,$$

again, for any $x \in (-1/e, 0)$. We have

$$f'_{A.11}(x) = \frac{f_{A.11\,1}(x)}{x\ln(-x)(\mathrm{W}_{-1}(x)+1)}$$

with

$$f_{A.11\,1}(x) := -\mathrm{W}_{-1}(x) + \ln(-x) - 1.$$

The denominator of $f'_{A.11}$ is clearly negative. Moreover,

$$f'_{A.11\,1}(x) = \frac{1}{x(\mathrm{W}_{-1}(x)+1)} > 0,$$

so $f_{A.11\,1}$ is strictly increasing on $(-1/e, 0)$. But we notice that

$$f_{A.11\,1}(-e^{1-e}) = -\mathrm{W}_{-1}(-e^{1-e}) - e = 0,$$

so $f_{A.11\,1} < 0$ on $(-1/e, -e^{1-e})$ and $f_{A.11\,1} > 0$ on $(-e^{1-e}, 0)$.

This means that $f_{A.11}$ is strictly increasing on $(-1/e, -e^{1-e})$, strictly decreasing on $(-e^{1-e}, 0)$, and it has a global maximum at $x = -e^{1-e} \approx -0.179$ (we remark that $f_{A.11}(-1/e) = \lim_{0^-} f_{A.11} = 0$). Since $f_{A.11}(-e^{1-e}) = 1 - \ln(e-1) < 1/2$, the proof of Step 2 is complete.

Step 3. Next, we show that

$$f_{A.11\,2}(x) := -1 - \sqrt{2}\sqrt{1+ex} - \mathrm{W}_{-1}(x) > 0$$

holds for any $x \in (-1/e, 0)$. To this end, we first verify that $f_{A.11\,2}$ is strictly increasing on $(-1/e, 0)$.

After applying the reparametrization $x = ye^y$, and noticing that $y \mapsto ye^y$ is strictly decreasing on $(-\infty, -1)$, we need to verify that

$$f_{A.11\,3}(y) := -1 - \sqrt{2}\sqrt{1 + ye^{y+1}} - y$$

is strictly decreasing on $(-\infty, -1)$. But

$$f'_{A.11\,3}(y) = -1 + f_{A.11\,4}(y),$$

with

$$f_{A.11\,4}(y) := -\frac{e^{y+1}(y+1)}{\sqrt{2ye^{y+1} + 2}},$$

so it is enough to show that $f_{A.11\,4}(y) < 1$ for any $y < -1$. Clearly, $f_{A.11\,4}(y) < 1$ is equivalent to $-e^{y+1}(y+1) < \sqrt{2ye^{y+1} + 2}$, and here both sides are positive—so squaring the inequality is allowed, reducing it to $0 < -e^{2y+2}(y+1)^2 + 2ye^{y+1} + 2$. After the substitution $z := y + 1 < 0$, we are to show $-e^{2z}z^2 + 2e^z(z-1) + 2 > 0$. The left-hand side here vanishes at $z = 0$, and its derivative is $-2e^z z(e^z(z+1) - 1) < 0$, finishing the claim.

Now, as the strict monotonicity of $f_{A.11\,2}$ has been established, notice that $f_{A.11\,2}(-1/e) = 0$, so Step 3 is complete.

Step 4. Finally, we show that

$$-1 - \sqrt{2}\sqrt{1 + ex} - \mathrm{W}_{-1}(x) < 1/2$$

for any $-1/e < x \le -1/4$. In Step 3 we proved that the left-hand side, $f_{A.11\,2}$ is strictly increasing on $(-1/e, 0)$, so it is sufficient to show that $f_{A.11\,2}(-1/4) < 1/2$. But this last inequality is equivalent to $\mathrm{W}_{-1}(-1/4) > \left(-3 - \sqrt{8 - 2e}\right)/2$, being true due to (26), hence completing the proof of the lemma.

A12. The proof of Lemma 2.21

In (29), the inequality $\beta_0(x) < -1$ is elementary, and $\mathrm{W}_{-1}(x) < \beta_0(x)$ has been proved in Lemma 2.20, so we have

$$\mathrm{W}_{-1}(x) < \beta_0(x) < -1. \tag{40}$$

Now let us formulate two conditional statements in Steps 1a and 1b, to be used in Step 2.

Step 1a. We claim that if

$$\beta_n(x) < \mathrm{W}_{-1}(x) < -1 \tag{41}$$

for some $n \in \mathbb{N}^+$, then $\beta_n(x) < \beta_{n+1}(x)$.

Indeed, due to (26), the assumption (41) implies $\beta_n(x)e^{\beta_n(x)} > \mathrm{W}_{-1}(x)e^{\mathrm{W}_{-1}(x)} = x$. By taking into account $x < 0$, $\beta_n(x) < 0$, and $1 + \beta_n(x) < 0$, this leads to $1 + \beta_n(x) < 1 + \ln\left(\frac{x}{\beta_n(x)}\right)$, that is, to

$$\beta_n(x) < \frac{\beta_n(x)}{1 + \beta_n(x)}\left(1 + \ln\left(\frac{x}{\beta_n(x)}\right)\right) = \beta_{n+1}(x).$$

Step 1b. Assume in this step that we have $\beta_n(x) < -1$ for some $n \in \mathbb{N}$.

Then $\beta_{n+1}(x)$ is well-defined, real, and clearly satisfies

$$\mathrm{W}_{-1}(x) - \beta_{n+1}(x) = \frac{1}{1 + \beta_n(x)}\left(\mathrm{W}_{-1}(x) - \beta_n(x) + \beta_n(x)\left[\mathrm{W}_{-1}(x) - \ln\left(\frac{x}{\beta_n(x)}\right)\right]\right).$$

Now by using (11), $x < 0$, $\beta_n(x) < 0$, and $\mathrm{W}_{-1}(x) < 0$, the expression in $[\ldots]$ above is $\ln\left(\frac{\beta_n(x)}{\mathrm{W}_{-1}(x)}\right)$, hence

$$\mathrm{W}_{-1}(x) - \beta_{n+1}(x) = \frac{1}{1 + \beta_n(x)}\left(\mathrm{W}_{-1}(x) - \beta_n(x) + \beta_n(x)\ln\left(\frac{\beta_n(x)}{\mathrm{W}_{-1}(x)}\right)\right), \tag{42}$$

or, in other words,

$$\mathrm{W}_{-1}(x) - \beta_{n+1}(x) = \frac{\mathrm{W}_{-1}(x)}{1 + \beta_n(x)}\left(1 - \frac{\beta_n(x)}{\mathrm{W}_{-1}(x)} + \frac{\beta_n(x)}{\mathrm{W}_{-1}(x)}\ln\left(\frac{\beta_n(x)}{\mathrm{W}_{-1}(x)}\right)\right). \tag{43}$$

Step 2a. Since $\beta_0(x) < -1$ due to (40), we can consider (43) with $n = 0$ and with $y := \frac{\beta_0(x)}{\mathrm{W}_{-1}(x)}$. Then $y \in (0, 1)$ and $\frac{\mathrm{W}_{-1}(x)}{1+\beta_0(x)} > 0$, due to (40) again. From these, by using Lemma A.1, we conclude that $-1 > \mathrm{W}_{-1}(x) > \beta_1(x)$.

Step 2b. Assume (41), as an inductive hypothesis, for some $n \in \mathbb{N}^+$. For $n = 1$, this has been proved in Step 2a, so the induction can be started. Then Step 1a shows that $\beta_{n+1}(x)$ is well-defined, real, and satisfies $\beta_n(x) < \beta_{n+1}(x)$. Moreover—since the assumption of Step 1b is fulfilled—we can apply (43) with $y := \frac{\beta_n(x)}{W_{-1}(x)}$. Then $y > 1$ and $\frac{W_{-1}(x)}{1+\beta_n(x)} > 0$ are both consequences of the inductive hypothesis (41), so Lemma A.1 implies $\beta_{n+1}(x) < W_{-1}(x) < -1$.

By taking into account (40) also, the above induction verifies (29) and the left inequality of (30) for any $n \in \mathbb{N}^+$.

Step 2c. Let us show the second inequality in (30) for $n = 1$.

Notice that the assumption of Step 1b is fulfilled because of (40), so we apply (42) with $n = 0$ and get

$$W_{-1}(x) - \beta_1(x) = \frac{1}{1+\beta_0(x)}(W_{-1}(x) - \beta_0(x) + \beta_0(x)\ln(1-z)) \tag{44}$$

with $z := \frac{W_{-1}(x)-\beta_0(x)}{W_{-1}(x)}$. Due to (40) again, we have $z \in (0, 1)$, so we can use the elementary inequality $\ln(1-z) < -z$ (and $1 + \beta_0(x) < 0$) to estimate (44) as

$$W_{-1}(x) - \beta_1(x) < \frac{W_{-1}(x) - \beta_0(x)}{1+\beta_0(x)} - \frac{\beta_0(x)}{1+\beta_0(x)} \cdot \frac{W_{-1}(x) - \beta_0(x)}{W_{-1}(x)} =$$

$$\frac{(\beta_0(x) - W_{-1}(x))^2}{(1+\beta_0(x))W_{-1}(x)} = (\beta_0(x) - W_{-1}(x)) \cdot \left(\frac{\beta_0(x) - W_{-1}(x)}{|W_{-1}(x)| \cdot |1+\beta_0(x)|}\right),$$

completing Step 2c.

Step 2d. Finally, we prove the second inequality in (30) for any $n \geq 2$ by induction.

The induction can be started, since the second inequality in (30) for $n = 1$ is Step 2c. So let us suppose that

$$W_{-1}(x) - \beta_n(x) < (\beta_0(x) - W_{-1}(x)) \cdot \left(\frac{\beta_0(x) - W_{-1}(x)}{|W_{-1}(x)| \cdot |1+\beta_0(x)|}\right)^{-1+2^n} \tag{45}$$

holds for some $n \geq 1$. Then we can apply (42) (since the assumption of Step 1b is satisfied due to (29) we already know), hence

$$W_{-1}(x) - \beta_{n+1}(x) = \frac{1}{1+\beta_n(x)}(W_{-1}(x) - \beta_n(x) + \beta_n(x)\ln(1-z)),$$

with $z := \frac{W_{-1}(x)-\beta_n(x)}{W_{-1}(x)}$. This time, however, we have $z < 0$ due to (29). Nevertheless, the inequality $\ln(1-z) < -z$ still holds, so

$$W_{-1}(x) - \beta_{n+1}(x) < \frac{W_{-1}(x) - \beta_n(x)}{1+\beta_n(x)} - \frac{\beta_n(x)}{1+\beta_n(x)} \cdot \frac{W_{-1}(x) - \beta_n(x)}{W_{-1}(x)}, \tag{46}$$

where we have also taken into account that $\frac{\beta_n(x)}{1+\beta_n(x)} > 0$ (being a consequence (29)). But the right-hand side of (46) is equal to $\frac{(W_{-1}(x)-\beta_n(x))^2}{|W_{-1}(x)|\cdot|1+\beta_n(x)|}$, so we proved

$$W_{-1}(x) - \beta_{n+1}(x) < \frac{(W_{-1}(x) - \beta_n(x))^2}{|W_{-1}(x)| \cdot |1+\beta_n(x)|}.$$

Notice now that—due to (29)—we have $\frac{1}{|1+\beta_n(x)|} < \frac{1}{|1+\beta_0(x)|}$, therefore

$$W_{-1}(x) - \beta_{n+1}(x) < \frac{(W_{-1}(x) - \beta_n(x))^2}{|W_{-1}(x)| \cdot |1+\beta_0(x)|}. \tag{47}$$

The left-hand side of (45) is positive (due to (29)), so we can combine (47) and (45) to get

$$W_{-1}(x) - \beta_{n+1}(x) < \frac{1}{|W_{-1}(x)| \cdot |1+\beta_0(x)|} \cdot \frac{\Big(\beta_0(x) - W_{-1}(x)\Big)^{2^{n+1}}}{\Big(|W_{-1}(x)| \cdot |1+\beta_0(x)|\Big)^{-2+2^{n+1}}} =$$

$$(\beta_0(x) - W_{-1}(x)) \cdot \left(\frac{\beta_0(x) - W_{-1}(x)}{|W_{-1}(x)| \cdot |1+\beta_0(x)|}\right)^{-1+2^{n+1}},$$

completing the induction, and the proof of the lemma.

A13. The proof of Lemma 2.22

Step 1. Let us first consider the case $-1/4 < x < 0$. Then, due to Lemma 2.20 and (29), we have

$$0 < \frac{\beta_0(x) - W_{-1}(x)}{|W_{-1}(x)| \cdot |1+\beta_0(x)|} < \frac{1/2}{|W_{-1}(x)| \cdot |1+\beta_0(x)|} < \frac{1/2}{|\beta_0(x)| \cdot |1+\beta_0(x)|},$$

proving (32). Moreover, it is elementary to check that both $x \mapsto |\ln(-x) - \ln(-\ln(-x))|$ and $x \mapsto |1 + \ln(-x) - \ln(-\ln(-x))|$ are strictly increasing for $-1/4 < x < 0$, and their product satisfies $|\beta_0(-1/4)| \cdot |1 + \beta_0(-1/4)| > 1$, so (32) implies (31) for $-1/4 < x < 0$.

Step 2. Let us consider now the case $-1/e < x \le -1/4$. Then (29) yields

$$0 < \frac{\beta_0(x) - \mathrm{W}_{-1}(x)}{|\mathrm{W}_{-1}(x)| \cdot |1 + \beta_0(x)|} < \frac{\beta_0(x) - \mathrm{W}_{-1}(x)}{|1 + \beta_0(x)|} = \frac{-1 - \sqrt{2}\sqrt{1+ex} - \mathrm{W}_{-1}(x)}{\sqrt{2}\sqrt{1+ex}},$$

so to prove (31), it is sufficient to show that the right-hand side above is less than $1/2$. This last sufficient condition (RHS $< 1/2$) is equivalent to

$$2\mathrm{W}_{-1}(x) + 3\sqrt{2 + 2ex} + 2 > 0,$$

which, after the reparametrization $x = ye^y$, becomes

$$2y + 3\sqrt{2 + 2ye^{y+1}} + 2 > 0. \tag{48}$$

It is enough to prove (48) for $-2.2 < y < -1$, because the range of the function $(-2.2, -1) \ni y \mapsto ye^y$ includes the interval $(-1/e, -1/4]$. But for $-2.2 < y < -1$, (48) is equivalent to $0 < 2 + 2ye^{y+1} - \left(\frac{2y+2}{3}\right)^2$, or, to $9 - 2z^2 + 9e^z(z-1) > 0$ after the shift $z := y + 1 \in (-1.2, 0)$. On this interval, the degree-6 Taylor polynomial of the exponential function about the origin is greater than e^z, so $9 - 2z^2 + 9e^z(z-1)$ is decreased by replacing e^z with its degree-6 Taylor approximation.

This way we get $\frac{1}{80}z^2\left(z^2 + 5z + 10\right)\left(z^3 + 14z + 20\right)$, which is clearly positive for $z \in (-1.2, 0)$, completing the proof.

A14. Auxiliary estimates

Here we state some auxiliary inequalities used in the earlier sections.

Lemma A.1. *For $y \in (0,1) \cup (1, +\infty)$ one has $y\ln(y) + 1 - y > 0$.*

Proof. We set $g(y) := y\ln(y) + 1 - y$. Since $g'(y) = \ln(y) < 0$ for $0 < y < 1$, $g'(y) > 0$ for $y > 1$, and $g(1) = 0$, the proof is complete. □

Lemma A.2. *On the interval $(e, +\infty)$, the following inequalities hold:*

$$L_1 - L_2 + \frac{L_2}{L_1} > L_1 - L_2 > 1, \tag{49}$$

$$-\frac{4}{5} < \frac{L_2}{L_1^2} - \frac{L_2}{L_1} < 0, \tag{50}$$

$$0 < \frac{e}{e-1}\frac{L_2}{L_1} < 1. \tag{51}$$

Proof. We have $(L_1 - L_2)'(x) = \frac{\ln(x)-1}{x\ln(x)} > 0$ and $(L_1 - L_2)(1) = 1$, proving (49).

The upper bound in (50) is just $\left(\frac{L_2}{L_1^2} - \frac{L_2}{L_1}\right)(x) = -\frac{(\ln(x)-1)\ln(\ln(x))}{\ln^2(x)} < 0$, while the lower bound in (50) is equivalent to the elementary inequality $0 < 4y^2 - 5(y-1)\ln(y)$ with $y := \ln(x)$ for $y > 1$.

The lower bound in (51) is trivial. With $y := \ln(x)$ again, the upper bound is the elementary inequality $\ln(y) < \frac{(e-1)y}{e}$. □

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