Photoacoustic effect in a sinusoidally modulated structure

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We derive solutions to an inhomogeneous Mathieu equation that describes the photoacoustic effect in a one-dimensional phononic structure whose acoustic properties vary sinusoidally in space. Solutions show splitting of resonances, the space equivalent of subharmonic generation, and spatial confinement. Properties of the photoacoustic effect including the damping of waves inside the band gaps, the dispersion relation, the positions and widths of the gaps, the frequencies of resonances, and the space dependence of the acoustic waves can be found in closed form from known properties of Mathieu functions. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.3703606]

Characteristics common to wave propagation in periodic structures[1–4] are forbidden regions of wave motion known as stop bands or band gaps and dispersion relations that deviate from linearity. In the case of the photoacoustic effect in periodic structures, the same kinds of band gaps and dispersion relations must obtain. However, since the excitation of photoacoustic waves is governed primarily by the temporal and spatial characteristics of the optical source, the photoacoustic effect takes on properties not commonly seen in other forms of wave motion in periodic structures, in particular, when the optical source forces excitation of waves within a band gap.

To date, photoacoustic experiments with layers[5] or patterned structures[6–8] have shown the existence of band gaps and determined dispersion relations for surface acoustic waves; theoretical work on the properties of phononic crystals[9–11] that parallels that of photonic crystals has appeared as well. Here, we investigate the properties of the photoacoustic effect in a one-dimensional structure whose acoustic properties are taken to vary sinusoidally in space. Solutions to an inhomogeneous Mathieu equation for the photoacoustic pressure are found for optical excitation repetitive in space, and excitation confined to a fixed spatial region; the latter being described in terms of traveling wave Mathieu functions.

The wave equation[12–15] for the photoacoustic pressure p is

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathrm{p}(x,t) = -\frac{\beta}{C_P}\frac{\partial H(\mathbf{x},t)}{\partial t}, \tag{1}$$

where c is the sound speed, β is the thermal expansion coefficient, C_P is the specific heat capacity, H is the energy per unit volume and time deposited by the optical beam, and t is the time. Consider a one-dimensional structure where the sound speed c is described by $c^{-2} = [1-\hat{n}\cos(\frac{2\pi x}{\bar{a}})]/c_0^2$, where the parameter $\hat{n}$ describes the modulation depth of the sound speed variation in the x direction, $\bar{a}$ is the lattice spacing, and c_0 is the sound speed when $\hat{n}=0$. Since $c_0 = (1/\rho\kappa_s)^{1/2}$, where ρ is the density and κ_s is the compressibility, the structure could vary either in its density or in its compressibility to produce the required modulation. Substitution of the expression for the sound speed into Eq. (1) and transformation of the resulting equation into the frequency domain according to $H(x,t) = \bar{\alpha}I(x)e^{-i\omega t}$ and $\mathrm{p}(x,t) = p(x)e^{-i\omega t}$, where ω is the modulation frequency, $\bar{\alpha}$ is the optical absorption coefficient, and $I(x)$ is the intensity profile of the optical radiation gives

$$\frac{d^2}{dz^2}p + [a - 2q\cos(2z)]p = f(z), \tag{2}$$

where the following dimensionless quantities have been defined:

$$z = \frac{\pi}{\bar{a}}x, \quad a = \hat{\omega}^2, \quad \hat{\omega} = \left(\frac{\omega\bar{a}}{\pi c_0}\right), \quad \gamma = \frac{1}{2}\hat{n}$$
$$q = \gamma a, \quad \text{and} \quad f(z) = \frac{i\hat{\omega}\bar{\alpha}\beta\bar{a}c_0}{\pi C_P}I(z). \tag{3}$$

Equation (2) without the forcing term is known as the Mathieu equation[1,16–19] whose solutions referred to as cosine elliptic $ce(a,q,z)$ and sine elliptic $se(a,q,z)$ functions can be periodic, aperiodic, or unbounded. Figure 1 shows the stability plot for Mathieu functions. Solutions for values of q and a that lie inside the regions marked "unstable" are

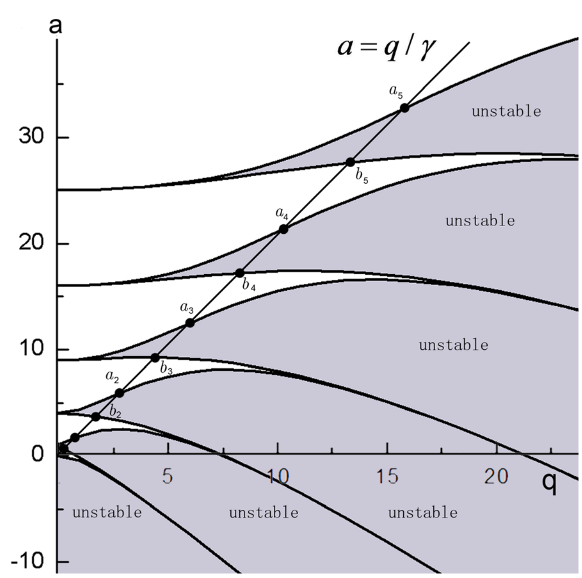

FIG. 1. Stability plot a versus q for Mathieu functions. The unstable regions correspond to band gaps. Integer order Mathieu functions are found at the borders of the white and gray regions. The points correspond to values of $q_m^{(c)}$ and $q_m^{(s)}$ satisfying Eq. (5).

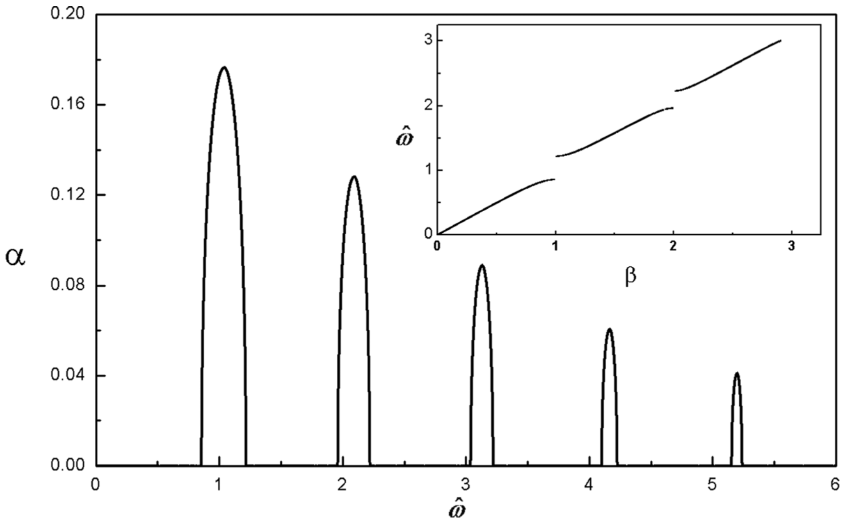

FIG. 2. Magnitude of the damping constant versus dimensionless frequency $\hat{\omega}$ for a structure with $\gamma = 0.35$ showing the damping constant α to be largest at the center of the gaps. As γ approaches zero, the damping becomes small, and the band gaps narrow and become centered close to integer values of $\hat{\omega}$. Outside the band gap, where $\alpha = 0$, solutions are periodic and a damping constant is not defined. Inset: Dispersion relation $\hat{\omega}$ versus β as determined by the Mathieu characteristic exponent for $\gamma = 0.35$.

unbounded and correspond to band gaps in the acoustic dispersion relation. Solutions to the Mathieu equation can also be expressed in Floquet form as

$$p(z) = Ae^{\mu z}\phi(z) + Be^{-\mu z}\phi(-z), \tag{4}$$

where ϕ is a periodic function, A and B are arbitrary constants, and μ is the Mathieu characteristic exponent, which is a function of q and a, and which is written, in general, as $\mu = \alpha + i\beta$ where α and β are real numbers. The dispersion relation for acoustic waves, see the inset to Fig. 2, is the imaginary component of the Mathieu characteristic exponent β wherever $\alpha = 0$.

Solution of Eq. (2) can be found by noting first that since the Mathieu equation is of the form of a Sturm-Liouville equation, the integer order cosine elliptic $ce_m(z, q)$ and sine elliptic $se_m(z, q)$ Mathieu functions obey

$$\frac{d^2}{dz^2}ce_m(z, q_m^{(c)}) = -a_m(q_m^{(c)})[1 - 2\gamma\cos(2z)]ce_m(z, q_m^{(c)}),$$

$$\frac{d^2}{dz^2}se_m(z, q_m^{(s)}) = -b_m(q_m^{(s)})[1 - 2\gamma\cos(2z)]se_m(z, q_m^{(s)}),$$

where a_m and b_m are the Mathieu characteristic values, lying along the line $a = q/\gamma$, found from solution to

$$q_m^{(c)} = \gamma a_m(q_m^{(c)}) \quad \text{and} \quad q_m^{(s)} = \gamma b_m(q_m^{(s)}). \tag{5}$$

The orthogonality relation,

$$\int_0^{2\pi}[1 - 2\gamma\cos(2z)]ce_m(z, q_m^{(c)})ce_n(z, q_n^{(c)})dz = \bar{\pi}_m^{(c)}\delta_{m,n}, \tag{6}$$

also follows from the Sturm-Liouville equation, where $\bar{\pi}_m^{(c)}$ is a constant. An analogous relation is valid for the se_m, as well, where the constants are designated $\bar{\pi}_m^{(s)}$. It also follows from the Sturm-Liouville equation that the functions ce_m and se_m are orthogonal with respect to the factor $[1 - 2\gamma\cos(2z)]$,

$$\int_0^{2\pi}[1 - 2\gamma\cos(2z)]ce_m(z, q_m^{(c)})se_n(z, q_n^{(c)})dz = 0. \tag{7}$$

The photoacoustic pressure can thus be written as expansions in $ce_m(z, q_m^{(c)})$ and $se_m(z, q_m^{(s)})$ along the line $a = q/\gamma$ as

$$p(z) = \sum_{m=0}^{\infty}A_m ce_m(z, q_m^{(c)}) + \sum_{m=1}^{\infty}B_m se_m(z, q_m^{(s)}), \tag{8}$$

where the A_m and B_m are constants. Following substitution of Eq. (8) for p into Eq. (2) and application of the orthogonality relations, $p(z)$ is found to be

$$p(z) = \sum_0^{\infty}\frac{ce_m(z, q_m^{(c)})}{\bar{\pi}_m^{(c)}[a - a_m(q_m^{(c)})]}\int_0^{2\pi}ce_m(z', q_m^{(c)})f(z')dz' + \sum_1^{\infty}\frac{se_m(z, q_m^{(s)})}{\bar{\pi}_m^{(s)}[a - b_m(q_m^{(s)})]}\int_0^{2\pi}se_m(z', q_m^{(s)})f(z')dz'. \tag{9}$$

For a fixed value of γ, the only frequency dependent quantity in Eq. (9) is a, so that the positions of the resonances are given by $\hat{\omega}_m^{(c)} = [a_m(q_m^{(c)})]^{1/2}$ or $\hat{\omega}_m^{(s)} = [b_m(q_m^{(s)})]^{1/2}$. The frequency dependence of the magnitude of the photoacoustic pressure, as shown in Fig. 3 exhibits a series of resonances, but, notably, not simply at frequencies corresponding to the period of modulation of the structure. Splittings between the pairs of resonances, i.e., those with same values of m, decrease as γ approaches zero.

For optical excitation confined to a single region of space, waves outside the excitation region must be traveling waves. Two linearly independent traveling wave solutions to the Mathieu equation $he^{(1)}(a, q, z)$ and $he^{(2)}(a, q, z)$ can be defined according to

$$\begin{aligned} he^{(1)}(a, q, z) &= ce(a, q, z) + i\ se(a, q, z), \\ he^{(2)}(a, q, z) &= ce(a, q, z) - i\ se(a, q, z). \end{aligned} \tag{10}$$

Solution to the inhomogeneous wave equation, Eq. (2), for values of $\hat{\omega}$ either within or outside the band gap, can be found from the method of variation of parameters.[20] If the region along the x axis where optical energy is deposited is taken to extend from –L to L, the spatial component of the photoacoustic pressure is found to be

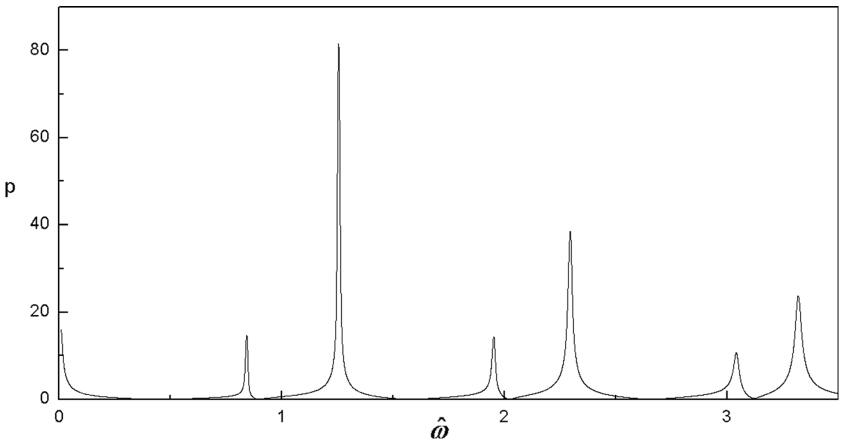

FIG. 3. Photoacoustic pressure versus dimensionless frequency $\hat{\omega}$ for delta function at $z = \pi/12$ with $\gamma = 0.4$. For small γ, the lowest frequency resonances near $\hat{\omega} = 1$ corresponding to a pressure that varies approximately as either $\cos z$ or $\sin z$. In order to increase the visibility of the peaks in the plot, a damping term proportional to $i\hat{\omega}^3$ was added to the factors in brackets in Eq. (9).

$$p(z) = -\frac{he^{(1)}(z)}{\bar{W}}\int_{-\hat{L}}^{z} he^{(2)}(z')f(z')dz' - \frac{he^{(2)}(z)}{\bar{W}}\int_{z}^{\hat{L}} he^{(1)}(z')f(z')dz', \tag{11}$$

where $\hat{L} = L/\bar{a}$. The Wronskian $\bar{W}$ is related to the Wronskian W for Mathieu functions through $\bar{W}\{he^{(1)}, he^{(2)}\} = -2iW\{ce, se\}$ and as such depends on q and a but not z. In the derivation of Eq. (11), there are two arbitrary constants of integration, depending on whether the solution to the right or the left of the origin is bounded; the sign of L is reversed in Eq. (11) to give bounded solutions.

The positions of the band gaps for a given value of γ are those values of $\hat{\omega}$ that give μ with a nonzero real component. Solutions to the Mathieu equation can be written either in the form of Eq. (4) or (10). As well, it can be shown that[21]

$$\begin{aligned} he^{(1)}(a,q,z) &= e^{\mu z}\phi(z), \\ he^{(2)}(a,q,z) &= e^{-\mu z}\phi(-z), \end{aligned} \tag{12}$$

where ϕ is a periodic function, and the signs are chosen according to whether the solution is taken along the positive or negative z axis. The damping constant for waves inside the band gap is thus given by α, the real part of μ. Fig. 2 shows the positions of the band gaps and the spatial damping constant as a function of $\hat{\omega}$ for frequencies within the band gaps. Photoacoustic waveforms calculated from Eq. (11) for waves inside and outside the first band gap are shown in Figs. 4 and 5, respectively. Waves generated inside of the band gaps show a kind of "photoacoustic confinement" as can be seen in Fig. 4. The confinement, which is governed by the magnitude of α is strongest at the center of the gaps, where the damping is greatest. The acoustic amplitude, on the other hand, is largest near the edges of the gaps. Note that the model chosen here does not include acoustic damping, which would influence the amplitudes of the waves.

For waves outside of the band gaps, the characteristic exponent does not have a real component. Thus, the relationship between the solutions $he^{(1)}$ and $he^{(2)}$ and the Floquet form of the solutions to the Mathieu equation can be written

$$\begin{aligned} he^{(1)}(a,q,z) &= e^{i\beta z}\phi(z), \\ he^{(2)}(a,q,z) &= e^{-i\beta z}\phi(-z). \end{aligned} \tag{13}$$

From the theory of Mathieu functions, if β is of the form quotient of two integers, then the solutions for the pressure are periodic; if β is an irrational number, the solutions are not periodic.[16] When β is a rational number, it can be seen that $he^{(1)}$ and $he^{(1)}$ are products of two periodic functions and, thus, appear visually as amplitude modulated carrier waves, as shown in Fig. 5.

For the periodically excited infinite structure, both the distribution of acoustic amplitudes and the frequencies at which the resonances are found change with the magnitude of γ. For small γ the positions of the resonances are found immediately to the right and left of integer values of $\hat{\omega}$. As γ increases, the resonance frequencies, as shown in Fig. 2, are found at values somewhat higher than integer values. If the resonances for small γ are considered to lie roughly at $n\hat{\omega}$ for integer values of n $(n = 1, 2, 3, \ldots)$, then the acoustic

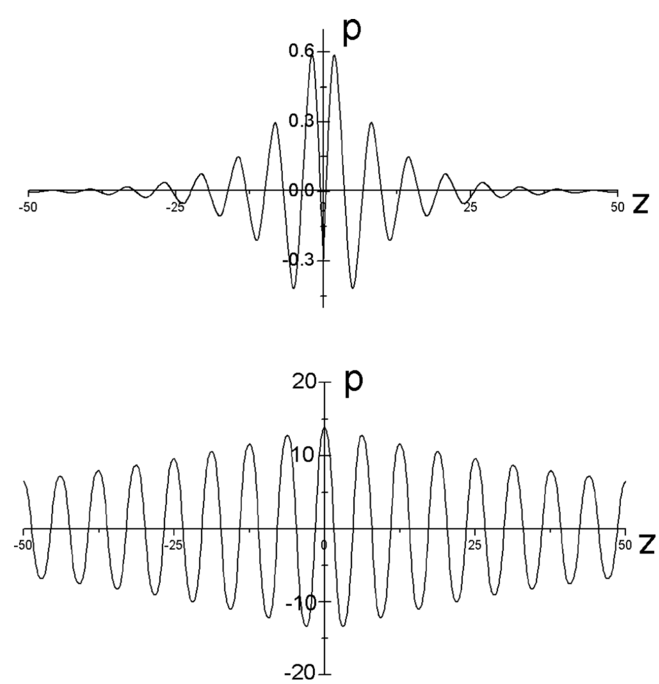

FIG. 4. Photoacoustic pressure versus dimensionless coordinate z from Eq. (11) with delta function optical excitation at the origin for two values of $\hat{\omega}$ immediately inside the edge of the first band gap at (a) $\hat{\omega} = 0.8956$ and (b) $\hat{\omega} = 1.2172$.

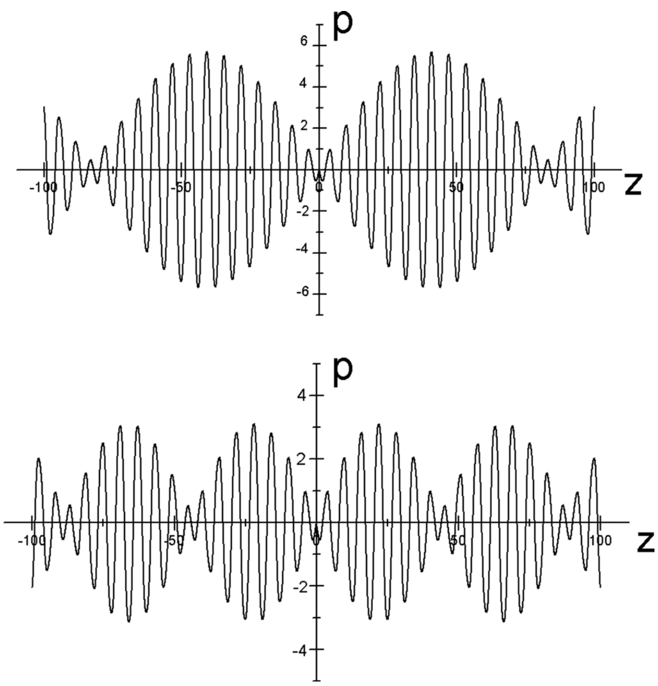

FIG. 5. Photoacoustic pressure versus dimensionless coordinate z from Eq. (11) with delta function optical excitation at the origin for two values of $\hat{\omega}$ immediately outside the edge of the first band gap at (a) $\hat{\omega} = 1.2220$ and (b) $\hat{\omega} = 1.2320$.

wavelength for the lowest frequency resonance is not at the expected value of $\hat{a}$, but rather at $2\bar{a}$. This phenomenon, which is a characteristic of solutions to the Mathieu equation, is a direct analog of what is referred to in the time domain as subharmonic resonance.[16] Its origin can be, at least, qualitatively understood by considering the cos2z term for small γ to be a perturbation in the homogeneous Mathieu equation. The solutions for p without the perturbation are either $\sin\sqrt{a}z$ or $\cos\sqrt{a}z$, where $a \simeq m^2$ for small γ. A perturbation solution is then obtained by considering $p\cos 2z$ as a source term. Thus, a sinusoidal term multiplies $\cos 2z$, which gives sum and difference wave numbers, one of which is a "sub" wave number at z.

Plots of the acoustic amplitude as the frequency approaches the edges of the band gap, irrespective of the direction of approach, show the amplitude of the photoacoustic pressure to become arbitrarily large, which comes as a result of constructive interferences within the structure. A further conclusion from examination of Eq. (9) is that in the case of uniform optical excitation of a structure, the overlap integrals in the first term of Eq. (9) do not vanish, giving the result that photoacoustic waves can be generated even when the density, specific heat capacity, and the thermal expansion coefficient are uniform in space—only κ_s need vary in order for sound to be generated, or, equivalently, if κ_s is constant then ρ can vary. This result is somewhat unexpected in that uniform excitation of an object unbounded along the z axis does not yield a photoacoustic effect. If there are density or compressibility variations, the present work indicates that sound will be generated solely as a result of variations in acoustic impedance—an effect that, heretofore, has not been demonstrated in experiment. A further conclusion is that for repetitive excitation in space, there is no damping at frequencies within the band gap, as attenuation in the present model can only be a consequence of destructive interference. The results given here have been found for the simple case of sinusoidal variations in the material properties of the irradiated structure. If more complicated variations in the sound speed are considered, Hill's equation, which notably is related to the Mathieu equation,[16] can be used to determine the character of the photoacoustic effect. It should be pointed out that the development given here is rigorously valid for fluids and one-dimensional solids, both of which obey wave equations of the same form;[22] extension of the present work to solids where heat deposition varies in space is not straightforward.

A feature of the sinusoidally modulated structure is that the dispersion relation, the positions of the band gaps, the frequencies of resonances, and solutions for the wave motion, both inside and outside the band gaps, can be found immediately in closed form from the theory of Mathieu functions without resort to numerical integration of a differential equation.[23] In addition, for other kinds of waves in sinusoidally modulated structures where the governing wave equation can be reduced to a scalar wave equation, the results given here for photoacoustic waves are valid as well.

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