Theory of Mechanical Damping Due to Dislocations*

A. GRANATO† AND K. LÜCKE
Division of Applied Mathematics, Brown University, Providence, Rhode Island

(Received August 1, 1955; revised version received February 20, 1956)

A quantitative theory of damping and modulus changes due to dislocations is developed. It is found that the model used by Koehler of a pinned dislocation loop oscillating under the influence of an applied stress leads to two kinds of loss, one frequency dependent and the other not. The frequency dependent loss is found to have a maximum in the high megacycle range. The second type of loss is a hysteresis loss which proves to be independent of frequency over a wide frequency range which includes the kilocycle range. This loss has a strain-amplitude dependence of the type observed in the kilocycle range. The theory provides a quantitative interpretation of this loss.

1. INTRODUCTION

RECENTLY, considerable attention has been given to the problem of constructing a dislocation theory capable of quantitatively describing the part of the internal friction and modulus changes in metals which, as first suggested by T. A. Read,[1] is the result of the motion of dislocations. Theories and experiments have developed along two main lines, namely, the study of the influence of impurities and of cold work. Theories have been based upon two ideas, presented by J. S. Koehler[2] and A. S. Nowick.[3–5]

Koehler develops the analogy between the vibration under an alternating stress of a dislocation line segment pinned down by impurity particles and the problem of the forced damped vibration of a string. He solves the differential equation of motion by an approximate method which is good for frequencies in the kilocycle range, but not satisfactory for measurements in the megacycle region. Assuming that the impurity atoms are randomly distributed Koehler derives a distribution function for the lengths of the various dislocation loops which is exponentially decreasing with increasing loop length. In order to obtain a strain-amplitude dependence, as required by experimental evidence, Koehler derives the increase in the loss which attends the breaking away of the loops from the impurity particles which occurs when the stress is high enough so that the loop tension exceeds the Cottrell[6] binding force. He considers the increase of the loss with increasing strain-amplitude a result only of the fact that the average loop length has increased because of the breakaway. The theoretical basis for this is not very satisfactory, and therefore, another treatment is given in this paper. Also this loss should have, according to Koehler's theory, the same frequency dependence as the loss independent of the strain-amplitude, but there is some experimental evidence that the strain-amplitude dependent loss is independent of frequency in the kilocycle range.[5]

To account for this fact Nowick has suggested the qualitative theory that the loss mechanism is of a hysteresis type, in which the dislocations are moved from one potential minimum to another. J. Weertman and E. I. Salkovitz[7,8] construct specific hysteresis models using the theories of Mott and Nabarro[9–11] from which they are able to make semiquantitive calculations of the decrement and changes in the dynamic modulus to be expected in the strain-amplitude independent (small stress) region when negligible pinning takes place. In the development of the theory it is necessary to assume that a dislocation length under an applied stress will move over distances greater than $a/C^{\frac{1}{3}}$, where a is the Burgers vector and C the concentration of impurities. However, it can be shown that even if the longest possible dislocation length (the lengths of Frank-Read sources) are taken, the displacements obtained for relatively large applied stresses are at most of the order of a, so that the model no longer applies.

Weertman also solves the equations of Koehler for any frequency, but in a mathematical form which makes it difficult to describe the effect of the parameters (i.e., damping, loop length, etc.) and which prevents the study of the effect of distributions of loop lengths upon the losses. For purposes of discussion a simpler model due to Eshelby[12] is used by Weertman. However, this model cannot describe the pinning effects. Therefore, in the present paper, an exact solution for all

* This work was supported by the Thermionics Branch, U. S. Army Signal Corps under Department of the Army Project No. 3-99-13-0222, Signal Corps Project No. 11B, Contract No. DA-36-039-Sc 52623.

† Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Brown University.

1 T. A. Read, Trans. Am. Inst. Mining Met. Engrs. 143, 30 (1941).

2 J. S. Koehler, *Imperfections in Nearly Perfect Crystals* (John Wiley and Sons, Inc., New York, 1952), p. 197.

3 A. S. Nowick, Progr. Metal Phys. 4 (1954).

4 A. S. Nowick, Phys. Rev. 80, 249 (1950).

5 A. S. Nowick, J. Appl. Phys. 25, 1129 (1954).

6 A. H. Cottrell, *Report of a Conference on the Strength of Solids* (University of Bristol, England, Physical Society, London, 1948), p. 30.

7 J. Weertman and E. I. Salkovitz, Acta Metallurgica 3, 1 (1955).

8 J. Weertman, J. Appl. Phys. 26, 202 (1955).

9 F. R. N. Nabarro, Proc. Phys. Soc. (London) 58, 669 (1946).

10 N. F. Mott and F. R. N. Nabarro, *Report of a Conference on the Strength of Solids* (London Physical Society, 1948), p. 1.

11 N. F. Mott, *Imperfections in Nearly Perfect Crystals* (John Wiley and Sons, Inc., New York, 1953), p. 173.

12 J. D. Eshelby, Proc. Roy. Soc. (London) A197, 396 (1949).

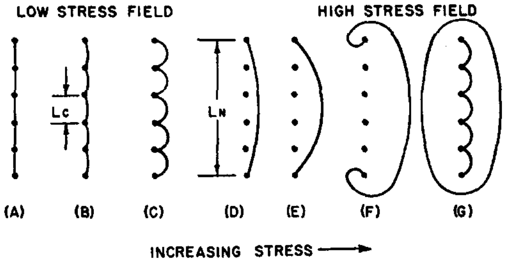

FIG. 1. The successive drawings indicate schematically the bowing out of a pinned dislocation line by an increasing applied stress. The length of loop determined by impurity pinning is denoted by L_c, and that determined by the network by L_N. As the stress increases, the loops L_c bow out until breakaway occurs. For very large stresses, the dislocations multiply according to the Frank-Read mechanism.

frequencies is derived in a form in which the dependence of the pertinent physical quantities on the loop length may be analyzed and applied.

It is furthermore found that Koehler's model contains the elements needed to describe both the frequency dependent loss (which should be appreciable in the megacycle range) and the strain-amplitude dependent frequency independent loss. In part I,‡ the mathematical consequences of the model are worked out by fixing attention on the stress-strain law, the distribution of loop lengths, and the breakaway stress. The theory is developed in this part, whereas part II is concerned with the application and comparison of the theory with experimental results. In part II the theory is severely tested quantitatively by a comparison with the available experimental data of the predicted dependences of the decrement and modulus changes on the principal variables and many other measurable parameters. The agreement between the theoretical and experimental results is surprisingly good and demonstrated in detail. Also, some disagreements are found and discussed in part II.

2. THE MODEL

It is assumed that a pure single-crystal contains, before deformation, a network of dislocations, such as Mott assumed in his paper on work-hardening.[13] For large enough concentrations of impurity atoms the length of loop determined by the intersection of the network loops is further pinned down by the impurity particles through the Cottrell mechanism. Such a model was used by Friedel[14] to describe the anomaly in the rigidity modulus of copper alloys for small concentrations found by Bradfield and Pursey.[15] There are, therefore, two characteristic lengths in the model: the network length L_N and the length L_c, determined by the impurities. The model is further modified in the mathematical treatment to take account of the fact that a distribution of lengths occurs.

If an external stress is now applied, there will be, in addition to the elastic strain, an additional strain due to the dislocations called the dislocation strain. The stress-dislocation strain law is, in general, a function of frequency. However, as will be shown, it is independent of frequency for lower frequencies including the kilocycle range (quasi-static behavior). Qualitatively the action of a dislocation length under the influence of an increasing external stress can be seen from Fig. 1. For zero applied stress the length L_N is pinned down by the impurity particles (A). For a very small stress (B), the loops (L_c) bow out and continue to bow out until the breakaway stress is reached. The effective modulus of the stress-dislocation strain curve is determined by L_c in this range. At the breakaway stress, a large increase in the dislocation strain occurs for no increase in the stress (C–D). Now, for further increases in the stress the loop length (L_N) bows out (D–E) until the stress required to activate the Frank-Read source[16] (L_N) is reached. It is assumed that the network pinning is so strong that no breakaway of network lengths occurs. In this interval (D–E), the effective modulus is determined by the length L_N. Further increases in the applied stress lead to creation and expansion of new closed dislocation loops (F–G). The dislocation strain due to this process is known to be irreversible and shall be called plastic strain. From this picture, and the analysis with its modifications which follow, a qualitative picture of the stress-dislocation strain law may be obtained.

In Fig. 2 the stress-dislocation strain law corresponding to the above model is shown. Now it is apparent that the losses are made up of two different types.

The first loss is due to the fact that the measurement is a dynamic one. Because the motion forced by the external stress is opposed by some damping mechanism, there is a phase lag for an oscillating stress, and hence a decrement and change of modulus. This type of loss

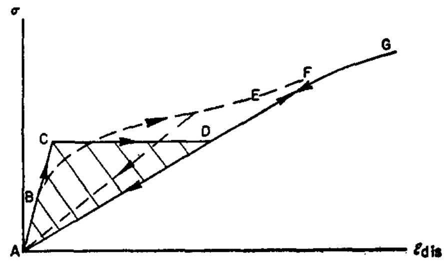

FIG. 2. The solid line drawing shows the stress strain law that results for the model shown in figure one. The elastic strain has been subtracted out so that only the dislocation strain is shown. The path $ABCDEF$ is followed for increasing stress, while the path FA is followed for decreasing stress. The dashed line curve is that which would result if not all of the loops have the same length, but there is a distribution of lengths L_c.

‡ The present paper is hereafter referred to as part I, and a subsequent paper on the application of dislocation theory to internal friction phenomena as part II.

[13] N. F. Mott, Phil. Mag. 43, 1151 (1952).

[14] J. Friedel, Phil. Mag. 44, 444 (1953).

[15] G. Bradfield and J. Pursey, Phil. Mag. 44, 437 (1953).

[16] F. C. Frank and W. T. Read, *Carnegie Institute of Technology Symposium on the Plastic Deformation of Crystalline Solids* (Office of Naval Research, Washington, D. C., 1950), p. 44.

is frequency dependent, since it has a resonance type character. It is largest near the resonant frequency determined by the loop length, and goes to zero for very low and very high frequencies.

The second type loss is due to the fact that, during the unloading part of the stress cycle (D–A), the long loops collapse elastically along a path determined by the long loop length, thus giving a hysteresis loop. When the loops have completely collapsed they are again pinned by the impurity particles and the same type of path is followed in the other half-cycle. This loss is simply proportional to the area enclosed by the stress-dislocation strain loop. For small enough stresses, it does not occur. Because of the fact that, in the frequency range considered, the stress-dislocation strain law is independent of frequency, this loss is frequency independent.

Some rather serious criticisms of this model can be made. The assumption that the dissipative mechanism can be adequately described by assuming a dissipative term proportional to the velocity is perhaps one such weak feature of the model, for little can be said at the moment with certainty as to the exact nature of the dissipative mechanisms.

Another question which must wait on further experimental work is that of determining the bounds of the "low" and "high" temperature regions. It is not expected that dislocation lines will be pinned by impurity particles at high enough temperatures, but just what temperatures are involved remains to a certain degree undetermined.

The value of the concept of breakaway becomes questionable when one examines this process from the point of view of the magnitudes of the displacements involved. The largest possible displacements allowed the dislocation line under the most generous possible conditions are still of the order only of a few lattice spacings. The interaction energy between an impurity and a segment of dislocation line varies only little then during the breakaway process as computed by the elasticity formulas. Of course, these formulas are not valid in the region very near impurity particles but still the concept becomes questionable under such considerations. The model by Weertman and Salkovitz suffers even more severely when examined in terms of the displacements involved, as already mentioned.

One may be tempted therefore to think of quite different models. As an example, one might consider the forced oscillations of a dislocation in a Peierls-Nabarro well. This potential well is the result of the slight difference in configuration of nearby positions. If the stress is large enough, the dislocation may move through these positions of maximum and minimum energy states, making a jump of one glide step distance into the neighboring well, thus giving a hysteresis loss. However, this model also seems to lead to serious difficulties, and moreover the consequences of the model have not been developed to the point where they may be compared with the available data. Therefore, in spite of the above remarks, since it is not likely that such questions can be resolved by purely theoretical arguments at the present time, it is felt that progress in this field might best be made by working out the consequences of each proposed model and comparing the predicted results with the experimental ones. But the model proposed here is so far the only one which can be checked through completely.

3. FREQUENCY-DEPENDENT LOSS

To find the decrement and apparent modulus change felt by a stress wave traveling through a solid which contains pinned dislocations, we start by applying Newton's law to the system, obtaining the usual equation of motion:

$$\frac{\partial^2\sigma}{\partial x^2}-\rho\frac{\partial^2\epsilon}{\partial t^2}=0. \tag{3.1}$$

The strain ϵ is made up of two kinds, the elastic strain ϵ_{el} and in addition a dislocation strain ϵ_{dis} due to the motion of the dislocations under the influence of the applied stress σ, i.e.,

$$\epsilon=\epsilon_{\mathrm{el}}+\epsilon_{\mathrm{dis}}. \tag{3.2}$$

The elastic strain is given by elasticity theory:

$$\epsilon=\sigma/G. \tag{3.3}$$

The dislocation strain produced by a loop of length l in a cube of unit dimensions is usually given by $\bar{\xi}la$ where $\bar{\xi}$ is the average displacement of a dislocation of length l and is given by

$$\bar{\xi}=\frac{1}{l}\int_0^l\xi(y)dy,$$

where y is the coordinate on the dislocation line. Thus, if Λ is the total length of movable dislocation line,

$$\epsilon_{\mathrm{dis}}=\frac{\Lambda a}{l}\int_0^l\xi(y)dy. \tag{3.4}$$

The displacement of the dislocation under the influence of an applied stress is given by the particular model chosen. The mathematical model for the equation of motion of a pinned down dislocation loop is taken to be that used by Koehler[2]:

$$A\frac{\partial^2\xi}{\partial t^2}+B\frac{\partial\xi}{\partial t}-C\frac{\partial^2\xi}{\partial y^2}=a\sigma, \tag{3.5}$$

where $\xi=\xi(x,y,t)$ and the boundary conditions are $\xi(x,o,t)=\xi(x,l,t)=0$. This is as shown in Fig. 3, where the plane of the paper is the slip plane. ξ is the displacement of an element of the dislocation loop from its equilibrium position, and y is the space coordinate of the element. A is the effective mass per unit length,

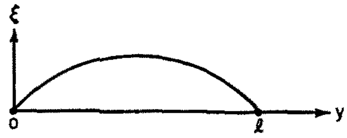

FIG. 3. A bowed out dislocation of length l is shown. The displacement of the dislocation from its equilibrium position is given by ξ, while y denotes the coordinate of an element of the dislocation line.

the term in B is the damping force per unit length, the term in C gives the force per unit length due to the effective tension in a bowed-out dislocation, and the term on the right is the force per unit length exerted on the dislocation by the external shearing stress. The constants are given by $A=\pi\rho a^2$; $C=2Ga^2/\pi(1-\nu)$, where ρ is the density of the material, a the Burger's vector, G the shear modulus, and ν is Poisson's ratio.

The foregoing set of Eqs. (3.1) to (3.5) may be condensed somewhat to the following system of two simultaneous partial differential integral equations

$$\begin{cases} \dfrac{\partial^2\sigma}{\partial x^2}-\dfrac{\rho}{G}\dfrac{\partial^2\sigma}{\partial t^2}=\dfrac{\Lambda\rho a}{l}\dfrac{\partial^2}{\partial t^2}\displaystyle\int_0^l \xi dy, \\ A\dfrac{\partial^2\xi}{\partial t^2}+B\dfrac{\partial\xi}{\partial t}-C\dfrac{\partial^2\xi}{\partial y^2}=a\sigma, \end{cases} \tag{3.6}$$

plus the boundary conditions for ξ. The first equation is a perturbation of the wave equation. Of all the possible sets of solutions: $\sigma=\sigma(x,y,t)$; $\xi=\xi(x,y,t)$ to the system, we are most interested in that for which σ is periodic in time and independent of y. If the dislocation lines are considered to be normal to the wave propagation direction, the results hold good for all frequencies.

The trial solution,

$$\sigma=\sigma_0 \exp[-\alpha x] \exp[i\omega(t-x/v)], \tag{3.7}$$

leads to

$$\xi=4a\sigma\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\sin\frac{(2n+1)\pi y}{l}\frac{\exp[i(\omega t-\delta_n)]}{[(\omega_n^2-\omega^2)^2+(\omega d)^2]^{\frac{1}{2}}},$$

with the substitutions

$$d=\frac{B}{A},\quad \omega_n=(2n+1)\frac{\pi}{l}\left(\frac{C}{A}\right)^{\frac{1}{2}},\quad\text{and}\quad \delta_n=\tan^{-1}\frac{\omega d}{\omega_n^2-\omega^2}.$$

The above form for $\xi=\xi(x,y,t)$ is to be preferred over a closed form, which can also be obtained, because of its simplicity and the fact that the first term of the series already gives a very good representation of the function for cases of interest. That is, it develops that for low frequencies, the contributions of the higher order terms in n to the decrement as defined in 3.9 go down like $1/(2n+1)^6$. Even when the driving frequency is equal to one of the odd harmonics of the fundamental resonant frequency, the contributions of the higher order terms can be neglected if the damping is not small. Hence, the first term of the series only will be used in what follows.

The above listed ξ and σ satisfy the system, Eq. (3.6), if

$$\alpha(\omega)=\frac{\omega}{2v}\frac{\Lambda\Delta_0\eta^2}{\pi}\frac{\omega d}{[(\omega_0^2-\omega^2)^2+(\omega d)^2]},$$

and

$$v(\omega)=v_0\left[1-\frac{\Lambda\Delta_0\eta^2}{2\pi}\frac{(\omega_0^2-\omega^2)}{[(\omega_0^2-\omega^2)^2+(\omega d)^2]}\right], \tag{3.8}$$

where $v_0=(G/\rho)^{\frac{1}{2}}$, $\Delta_0=8Ga^2/\pi^3C$, and $\eta^2=\pi^2C/A$.

The attenuation and velocity of the stress wave are therefore frequency dependent. It is further seen that only the component of the dislocation displacement which is in phase with the applied stress contributes to the change in velocity $v(\omega)-v_0$ where $v(\omega)$ is the measured velocity and v_0 is the velocity corresponding to the true elastic modulus. The component of the displacement which is out of phase with the applied stress leads to the attenuation $\alpha(\omega)$. The decrement Δ is defined by

$$\Delta=\Delta W/2W, \tag{3.9}$$

where ΔW is the energy lost per cycle and W is the total vibrational energy of the specimen. If

$$\Delta(\omega)=\alpha(\omega)2\pi v/\omega, \tag{3.10}$$

is taken to be the relation between the decrement and the attenuation (and if his expression for the total vibrational energy of the specimen is corrected§), then one obtains the same results as would be obtained by computing the energy lost per cycle as is done for a restricted frequency range in Koehler's paper.

Thus one finds that

$$\Delta=\frac{\Delta_0\Lambda L^2}{D}\left[\frac{\Omega}{(1-\Omega^2)^2+\Omega^2/D^2}\right],$$

and

$$\frac{\Delta G}{G}=\frac{\Delta_0\Lambda L^2}{\pi}\left[\frac{(1-\Omega^2)}{(1-\Omega^2)^2+\Omega^2/D^2}\right], \tag{3.11}$$

with the substitutions

$$D=\omega_0/d\quad\text{and}\quad\Omega=\omega/\omega_0. \tag{3.12}$$

In Fig. 4 and Fig. 5 the dependence of the decrement and change in modulus on frequency is shown, and in Table I the resonant frequency ω_0, the frequency ω_m at which the maximum decrement occurs, and the value $\Delta(\omega_m)$ of the maximum decrement for copper are given for values of $B=5\times10^{-3}$, 5×10^{-5}, and values of L ranging from 10^{-6} to 10^{-3} cm, and $\Lambda=10^7$.

Several interesting features of the loss are already apparent.

(1) The frequency response has two main branches depending, for a fixed L, upon whether the damping is large or small. (Large and small damping are defined by $D\ll1$ and $D\gg1$, respectively.) For very small

§ Koehler has used $\sigma_0^2/4G$ for the total vibrational energy of the specimen. This should be $\sigma_0^2/2G$.

TABLE I. Range of the resonant frequency and maximum decrement for a given range of damping and loop length for copper with a dislocation density of 10^7 cm/cm^3.

	$B=5\times10^{-5}$				$B=5\times10^{-3}$			
l(cm)	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
ω_0(cycles/sec)	3.9×10^8	3.9×10^9	3.9×10^{10}	3.9×10^{11}	3.9×10^8	3.9×10^9	3.9×10^{10}	3.9×10^{11}
ω_m(cycles/sec)	5.6×10^7	3.9×10^9	3.9×10^{10}	3.9×10^{11}	5.6×10^7	5.6×10^8	5.6×10^9	3.9×10^{11}
$\Delta(\omega_m)$[a]	3.5	0.1	10^{-2}	10^{-3}	5.3	0.035	3.5×10^{-4}	10^{-5}

[a] The magnitude of $\Delta(\omega_m)$ obtained is for the case where the resolved shear stress on the slip planes in the slip direction is equal to the applied stress. For applied longitudinal stresses, the resolved shear stress will be, in general, smaller by a factor near 1/25, as is discussed later.

damping the response is linear for frequencies up nearly to the resonant frequency, passes through a maximum whose sharpness depends on the smallness of the damping, and then decreases like the inverse third power of the frequency. For large damping, the initial response is linear up to a maximum value which occurs at an earlier frequency than the resonance frequency. It then decreases like the inverse first power of the frequency through the resonant frequency range and finally decreases like the inverse third power of the frequency.

(2) The resonant frequency depends only on A, C, and L, i.e.,

$$\omega_0^2=\pi^2C/L^2A. \tag{3.13}$$

(3) The maximum loss occurs at $\omega=\omega_0$ for small damping, and at $\omega=\omega_0^2/d$ for large damping.

(4) Near the resonant frequency, the loss is inversely proportional to the damping.

(5) Considering the values of L and B encountered, it is apparent that losses in solids due to this mechanism should be looked for in the megacycle region.

It must be expected that in a solid, not all dislocations have the same loop length, but rather, there exists a distribution of loop lengths. Since the decrement depends on the fourth power of the loop length, one may suspect that the final results might be sensitive to the distribution. To obtain a measure of the sensitivity of the decrement and modulus change to the distribution, various distributions have been considered and investigated in detail.|| It is found that the qualitative results are not changed very much except for the fact that the very large increase in damping at the resonant frequency for small damping does not occur for a distribution of loop lengths. In fact, the detailed comparison of the results shows that a good description of the results for a distribution of loop lengths is obtained by using the results for a single loop length, where the loop length is replaced by a new "effective loop length." The effective loop length is always larger than the average loop length since the decrement depends on the fourth power of the loop length and is therefore more sensitive to the long loops. It is found, for example, that the effective loop length L_e for an exponential and for a rectangular distribution is 3.3 and 1.5, respectively, times the average loop length. This result is very useful, since it allows one to think of the effects of different distributions in terms only of the differences of the effective lengths. Also, the mathematical analysis is greatly simplified.

|| This investigation may be found in a report submitted to the U. S. Signal Corps under the contract numbers previously mentioned. See also the thesis mentioned earlier, which is available from University Microfilms, Ann Arbor, Michigan, Publication No. 13,1712.

The distribution function which Koehler derives for randomly arranged solvent and solute atoms along the dislocation line is

$$N(l)dl=\frac{\Lambda}{L^2}\exp(-l/L)dl, \tag{3.14}$$

where $N(l)dl$ is the number of loops which have lengths between l and $l+dl$, and L is the average loop length.

The decrement resulting from this distribution is then

$$\Delta=\int_0^\infty l\Delta(l)N(l)dl. \tag{3.15}$$

The result is found in terms of the Exponential Integral function of a complex argument¶ in the appendix and plotted in Fig. 6.

The low-frequency behavior of the decrement may be obtained by suitable expansions of this function

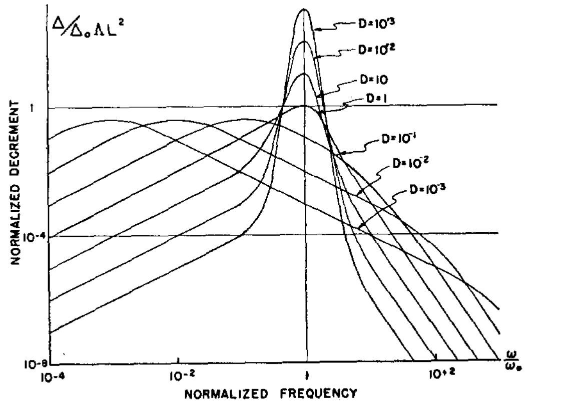

FIG. 4. The various curves show the frequency dependence of the decrement for various values of the damping constant for a delta function distribution of loop lengths. The frequency has been normalized to the resonant frequency and the decrement has been normalized by the factor $\Delta_0\Lambda L^2$, where Δ_0 is a constant of order one, Λ is the dislocation density, and L is the loop length.

¶ Tables of this function have been prepared and are to be published soon by the National Bureau of Standards.

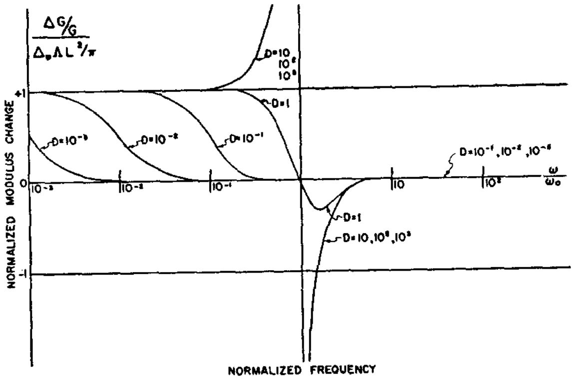

FIG. 5. The frequency dependence of the percentage change in modulus is here shown. The normalization factors are the same as those used in Fig. 4.

(see Appendix). For $\omega/d \ll 1$,

$$\Delta_L = \frac{8Ga^2}{\pi^5 C^2}\Lambda L^4 5! B\omega \times\left[1+\frac{2\cdot 6\cdot 7L^2A\omega^2}{\pi^2 C}-\frac{6\cdot 7\cdot 8\cdot 9L^4B^2\omega^2}{\pi^4 C^2}\right], \quad (3.16)$$

$$\frac{\Delta G}{G}=\frac{8Ga^2}{\pi^4 C}3!\Lambda L^2.$$

The leading term and the damping-independent correction term of the decrement agrees with that of Koehler's if his expression for the total vibrational energy is corrected. However, the damping correction has a different magnitude and even a different sign, and in this respect, the result is quite different.

The fact that the correction term depending on the damping constant is negative shows that, for large enough damping, the frequency dependence of the decrement becomes less than linear at high frequencies, as would be expected by inspection of Fig. 4.

A special case of interest is obtained by letting the loop length l become infinite; i.e., the dislocation line becomes straight. Such a model can be used for the motion of screw dislocations for which impurities do not restrict the motion.

By letting $l\rightarrow\infty$ in the general solution, Eq. (3.7), one obtains

$$\Delta=\frac{\pi^2}{8}\Lambda\Delta_0\eta^2\frac{1}{\omega(\omega^2+d^2)}, \quad (3.17)$$

which is the same expression as that obtained for the case of finite loop lengths when the frequency is far above the resonant frequency. The decrement becomes infinite for small frequencies because the displacement

$$\xi=\frac{a\sigma_0 \sin\omega t}{a\omega[\omega^2+d^2]^{\frac{1}{2}}}\simeq\frac{a\sigma_0}{B\omega}\sin\omega t, \quad (3.18)$$

of the dislocation becomes very large. Even if the displacement is restricted to distances of order 10^{-4} cm, as is done by Koehler, the predicted decrement is many times larger than that which is observed, as one can show easily.

Since neither these magnitudes nor this frequency dependence of Δ is observed, it is concluded that there must be some restrictive mechanism which prevents this type of motion. This is assumed to be the network pinning.

4. STRESS-DISLOCATION STRAIN LAW

To compute the strain-amplitude dependent loss, we first need to find the stress-dislocation strain law as a function of the frequency. Using Eqs. (3.4), (3.7), and (3.12) one obtains for the shearing strain produced by dislocation loops of length l

$$\epsilon_{\text{dis}}=\frac{\sigma_0 \exp[-\alpha x]\Lambda 8a^2l^2}{\pi^4 C}[\cos(\omega t-kx-\delta)]F(\Omega), \quad (4.1)$$

where

$$F(\Omega)=[(1-\Omega^2)^2+\Omega^2/D^2]^{-\frac{1}{2}}.$$

$F(\Omega)=1$ for frequencies much less than the frequency at which the frequency-dependent decrement is a maximum.

Thus for low frequencies (which embrace the kilocycle range), the stress-dislocation strain law is independent of frequency and the phase angle δ is very small. The dislocation strain is then directly proportional to the applied stress and also is a function of the loop length. As the breakaway proceeds, the average loop length will change as a function of the stress, and then the stress-dislocation strain law will no longer be linear.

Since breakaway occurs when the dislocation-strain achieves a certain given value, the breakaway stress σ_0 will have according to Eq. (4.1) a frequency dependence which is like $1/F(\Omega)$. Thus for frequencies much larger than the resonant frequency, the breakaway stress increases like the square of the frequency, but is constant for low frequencies.

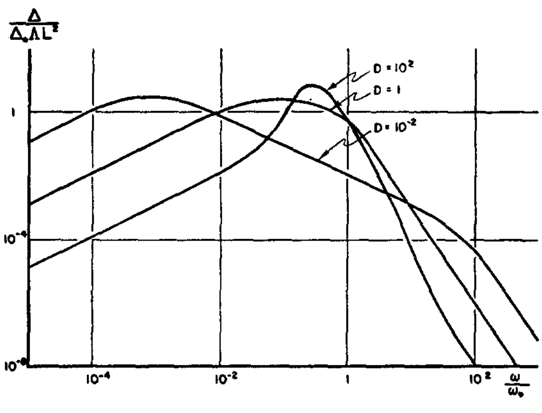

FIG. 6. Frequency dependence of the decrement for an exponential distribution of loop lengths.

5. STRAIN-AMPLITUDE DEPENDENT FREQUENCY INDEPENDENT LOSS

A. Distribution of Loop Lengths as a Function of Stress

It is assumed that the initial distribution of loop lengths is exponential. As the breakaway process proceeds the distribution changes and so is a function of the applied stress.

For this (low) frequency range the force exerted by the dislocation line on an impurity at any instant during a cycle is given by

$$\pi C(\phi_1-\phi_2)=4a\sigma(l_1+l_2)/\pi, \tag{5.1}$$

where πC is the loop tension, and ϕ_1 is the angle made by a loop length l_1 at the impurity when the displacement is a maximum. Breakaway occurs when this force is larger than the binding force which is given in Cottrell's theory[6] by

$$f\approx 4G\epsilon' a^4/Z^2,$$

where ϵ' is the difference in atomic radii divided by the atomic radius of the solvent atom, and Z is the distance of the impurity from the dislocation axis.

This means breakaway occurs if l_1+l_2 is greater than the breakaway length

$$\mathfrak{L}=\pi f_m/4a\sigma, \tag{5.2}$$

where f_m is the maximum value of the binding force obtained. Using these ideas Koehler derives a distribution function for the loop length as a function of the strain amplitude. But this function does not seem to have much physical sense, because the distribution should be a function of the instantaneous strain rather than the strain amplitude.

In addition, the loops do not return to their original positions in the same manner in which they left (Figs. 1 and 2). Therefore the distribution of loop lengths should be different for the increasing part of the stress cycle from that which occurs for the decreasing part.

Further, Koehler has not taken account of the fact that the breakaway process is a catastrophic one within the network length. That is, if two adjoining lengths l_1 and l_2 have the breakaway length $\mathfrak{L}$, then as the two lengths break away to one new large length l_1+l_2, now, the adjoining lengths all break away since $(l_1+l_2)+l_3>\mathfrak{L}$, etc. This continues until the whole dislocation line between the two strongly pinned network points has broken away. Taking into consideration the above arguments, a new distribution and a theory of dislocation movement including breakaway can be derived.

The distribution function is assumed to change from the initial exponential distribution, Eq. (3.14), at zero strain to a delta function distribution (all lengths having the same network length) when the strain is very large and all the loops have broken away:

$$N_f(l)dl=\Lambda\frac{\delta(l-L_N)dl}{L_N}. \tag{5.3}$$

For intermediate strains the distribution function, which will be denoted by $N'(l)dl$, is assumed to consist of two parts: an exponential part due to loops in networks which have not yet broken away, and a delta function part due to the broken away loops, which may be written as:

$$N'(l)dl=\begin{cases}(\Lambda/L_c^2)\exp(-l/L_c)J(\mathfrak{L},L_c,L_N)dl, & 0\leq l<\mathfrak{L}\\ (\Lambda/L_N)\delta(l-L_N)Mdl, & \mathfrak{L}\leq l<\infty,\end{cases}$$

where M is the fraction of network lengths in which the catastrophic breakaway process has occurred and J is determined by the condition that the length of line lost from the exponential distribution is gained by the delta distribution.

The number of times a loop of length l_1 is found next to a loop of length l_2 is given by the number of loops having lengths between l_1 and l_1+dl_1 times the probability that a loop has a length between l_2 and l_2+dl_2. This is

$$\frac{\Lambda J}{L_c^2}\exp\left(-\frac{l_1}{L_c}\right)dl_1\cdot\frac{1}{L_c}\exp\left(-\frac{l_2}{L_c}\right)dl_2' =\frac{\Lambda J}{L_c^3}\exp\left[-\frac{(l_1+l_2)}{L_c}\right]dl_1dl_2.$$

The probability that two adjoining loops whose sum is l_1+l_2 have not broken away is then

$$\frac{1}{L_c^2}\int_0^{\mathfrak{L}}dl_1\int_0^{\mathfrak{L}-l_1}\exp\left[-\frac{(l_1+l_2)}{L_c}\right]dl_2 =1-(q+1)\exp(-q),$$

where $q=\mathfrak{L}/L_c$. The probability that a catastrophic breakaway has not occurred in a network length is then

$$[1-(q+1)\exp(-q)]^n,$$

where n is the number of loop lengths in the network length. The average value of $n=L_N/L_c-1$. The probability that a catastrophy has occurred in the network length is therefore

$$M=1-[1-(q+1)\exp(-q)]^n.$$

Now, J is determined by requiring that

$$\int_0^\infty lN'(l)dl=\Lambda.$$

For the early stages of the breakaway process, a good approximation to M is

$$M=n(q+1)\exp(-q).$$

In the derivation just made, it has been assumed that the loop lengths are statistically independent. That this assumption is incorrect is easily noticed by considering the fact that any two adjoining pairs have one loop length in common. Furthermore, the fact that the network points also pin the dislocation line, has been neglected. However, a detailed analysis[17] shows that the result is a good approximation over the range considered if only the leading term is used. Up to this point, the breakaway length $\mathcal{L}$, and therefore the applied stress σ, has been considered constant. If now we consider σ to be a function of time, there should elapse a finite time before the distribution readjusts to a changed value of σ. However, it is easy to show that the distribution follows changes closely, since the relaxation time for transients is very small. This time is given by $2A/B$ which is of an order 10^{-11} to 10^{-9} sec. The following considerations are only made for frequencies small enough so that the distribution can be considered to follow a changing stress instantaneously.

The distribution function $N_1'(l)$ for the quarter cycle of increasing stress is then determined by the instantaneous value of the stress. For the quarter cycle when the stress is decreasing, the loops simply collapse elastically and so there is no change in the distribution function $[N_2'(l)]$. It simply is the distribution corresponding to the largest stress achieved during the first quarter cycle. Now, using Eq. (5.2), we may write q in terms of the stress as

$$q=\mathcal{L}/L_c=\pi f_m/4aL_c\sigma=\Gamma/\sigma. \tag{5.4}$$

The resulting distribution function is then

$$N'(l)dl=\begin{cases} N_1'(l)dl=\begin{cases}\dfrac{\Lambda}{L_c^2}\left[1-n\left(\dfrac{\Gamma}{\sigma}+1\right)\exp\left(\dfrac{-\Gamma}{\sigma}\right)\right]\exp\left(\dfrac{-l}{L_c}\right)dl, & 0\leq l\leq \mathcal{L}\\ \dfrac{\Lambda}{L_N}\delta(l-L_N)n\left(\dfrac{\Gamma}{\sigma}+1\right)\exp\left(\dfrac{-\Gamma}{\sigma}\right)dl, & \mathcal{L}\leq l<\infty\end{cases}\\ N_2'(l)dl=\begin{cases}\dfrac{\Lambda}{L_c^2}\left[1-n\left(\dfrac{\Gamma}{\sigma_0(x)}+1\right)\exp\left(\dfrac{-\Gamma}{\sigma_0(x)}\right)\right]\exp\left(\dfrac{-l}{L_c}\right)dl, & 0\leq l\leq \mathcal{L}\\ \dfrac{\Lambda}{L_N}\delta(l-L_N)n\left(\dfrac{\Gamma}{\sigma_0(x)}+1\right)\exp\left(\dfrac{-\Gamma}{\sigma_0(x)}\right)dl, & \mathcal{L}\leq l<\infty\end{cases}\end{cases} \tag{5.5}$$

where

$$\Gamma=\pi f_m/4aL_c \quad\text{and}\quad \sigma=\sigma_0(x)\cos(\omega t-kx).$$

It is readily seen that for small stresses $N'(l)dl$ approaches the original distribution.

B. Stress-Dislocation Strain Law for the General Distribution Function and Low Frequencies, Neglecting Dynamic Effects

Now, the stress-dislocation strain law may be computed taking the breakaway into account. For low frequencies, $F(\Omega)$ may be set $=1$ in Eq. (4.1). Also δ is small and if neglected, the dislocation strain is directly proportional to the stress. By neglecting δ, we exclude dynamic effects and consider only the frequency independent loss. If one also neglects the slight variation of the stress over the sample dimensions, one obtains for the contribution of a single loop of length l to the dislocation strain

$$\left[\frac{8a^2\sigma_0}{\pi^4C}\cos\omega t\right]l^3=\psi l^3.$$

Therefore, the dislocation strain for the distribution of lengths $N_1'(l)$ is

$$\epsilon_{1\text{ dis}}(\mathcal{L})=\int_0^\infty \psi l^3 N_1'(l)dl. \tag{5.6}$$

This integral may be expressed by

$$\epsilon_{1\text{ dis}}=\psi\Lambda L_c^2 3!\left[1+\exp(-q)\left\{\frac{\gamma^2(\gamma-1)(q+1)}{3!}-\left[\frac{q^3}{3!}+\frac{q^2}{2}+q+1+(\gamma-1)(q+1)\right]\right\}\right], \tag{5.7}$$

where $\gamma=L_N/L_c$ and $q=\mathcal{L}/L_c=\Gamma/\sigma_0\cos\omega t$. In this derivation terms of the order $\exp(-2q)$ have been neglected since for early stages of breakaway q is large. Since no breakaway occurs when $\mathcal{L}>L_N$, the range of q is taken to be $0\leq q\leq\gamma$. The negative term in the bracket is then small compared to the positive term, which fact should allow an estimate to be made of when the stress-dislocation strain law starts to become nonlinear.

If we define this point by the equation

$$\{\gamma^2(\gamma-1)(q+1)/3!\}\exp(-q)=0.1,$$

(i.e., 10% of ϵ_1'), then the following values are obtained for the breakaway length:

If

$$\gamma=10,\quad \gamma=100,$$

[17] G. F. Newell, "Some statistical problems encountered in a theory of pinning and breakaway of dislocations" (to be published).

then

$$\mathfrak{L}=9.7L_c,\quad \mathfrak{L}=19.6L_c.$$

An approximate formula for this measure of the breakaway length for $\gamma=L_N/L_c\geqslant 10$ is

$$\mathfrak{L}=4L_c\log(L_N/L_c), \tag{5.8}$$

or

$$\sigma=\Gamma/4\log\gamma.$$

Now, the stress-dislocation strain relation, Eq. (5.7) for the early stages of the breakaway process is given for the increasing stress quarter cycle approximately by the nonlinear relation

$$\epsilon_{1\ \mathrm{dis}}=Q\left[1+\frac{\gamma^3}{3!}\left(\frac{\Gamma}{\sigma}+1\right)\exp\left(\frac{-\Gamma}{\sigma}\right)\right]\sigma, \tag{5.9}$$

where

$$Q=8a^2\Lambda L_c^2 3!/\pi^4 C.$$

For the decreasing stress quarter cycle, it is given by the linear relation

$$\epsilon_{2\ \mathrm{dis}}=Q\left[1+\frac{\gamma^3}{3!}\left(\frac{\Gamma}{\sigma_0}+1\right)\exp\left(\frac{-\Gamma}{\sigma_0}\right)\right]\sigma,$$

and the coefficient of σ is the reciprocal of the effective dislocation modulus. $\epsilon_{1\ \mathrm{dis}}$ and $\epsilon_{2\ \mathrm{dis}}$ are plotted in Fig. 7.

C. Frequency-Independent Decrement

The energy lost per cycle (ΔW) is then the area between the curves in Fig. 7 and is given by

$$\Delta W=2\int_0^{\sigma_0}(\epsilon_{1\ \mathrm{dis}}-\epsilon_{2\ \mathrm{dis}})d\sigma. \tag{5.10}$$

The integration involves exponential integral functions. If these are expanded into an asymptotic series, and if the definition, Eq. (3.9), is used in the form

$$\Delta=\frac{\Delta W}{\sigma_0^2/G},$$

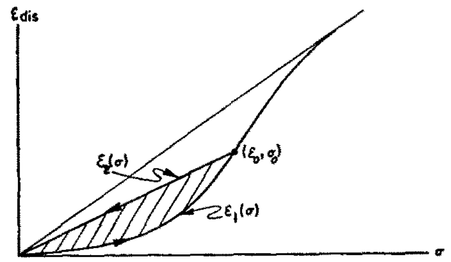

FIG. 7. The stress dislocation-strain path followed in the breakaway process when the strain amplitude is ϵ_0. The shaded area is proportional to the decrement.

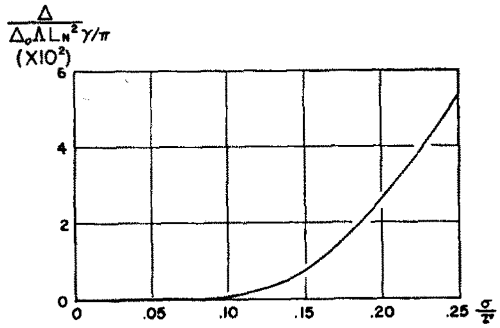

FIG. 8. The stress-amplitude dependence of the decrement. The decrement has been normalized by the dislocation density and the network loop length factors, while the stress amplitude has been normalized by the stress required for breakaway when all of the loops have a value equal to the average value of the distribution of loop lengths.

then one obtains

$$\Delta=GQ\frac{\gamma^3}{3!}\left(\frac{\Gamma}{\sigma_0}-1+\cdots\right)\exp\left(\frac{-\Gamma}{\sigma_0}\right),$$

or, by making previously indicated substitutions,

$$\Delta=\frac{\Delta_0\Lambda L_N^2}{\pi}\left(\frac{L_N}{L_c}\right)\left[\frac{\Gamma}{\sigma_0}-1+\cdots\right]\exp\left(\frac{-\Gamma}{\sigma_0}\right). \tag{5.11}$$

This, then, is the frequency independent, strain-amplitude dependent loss first mentioned in the introduction.

This dependence of the decrement on the strain amplitude is shown in Fig. 8. The stress corresponding to the applied strain amplitude has been normalized to the value the breakaway stress would have if all the loops had the same length. Although the calculation has been made only for the early stages of the breakaway process, it is readily seen from the model that for large strain amplitudes the decrement levels off. This behavior is experimentally observed. It is clear that the final result is sensitive to the distribution of loop lengths assumed.

In this way, it has been possible to derive the decrement by the use of simple considerations alone. In order to find the modulus change, a more complicated analysis of the wave propagation properties of a solid with dislocations has been made and may be found in the previously referred to report or thesis. The result found is that the modulus change is given by the same form as that for the decrement, and that the ratio r of the decrement to the modulus change is a constant of order unity, in agreement with the observed results.**

** In addition, the following terms which are both frequency and strain amplitude dependent are found in this way.

$$\Delta=\Delta_0\Lambda\frac{L_N^5}{L_c}\frac{\omega d}{n^2}\frac{\Gamma}{\sigma_0}\exp\left(\frac{-\Gamma}{\sigma_0}\right),$$

and

$$\frac{\Delta G}{G}=\Delta_0\Lambda L_c^4\frac{\omega d}{n^2}\frac{\Gamma}{\sigma_0}\exp\left(\frac{-\Gamma}{\sigma_0}\right).$$

However, it can be easily shown that these terms can be neglected in comparison with the others.

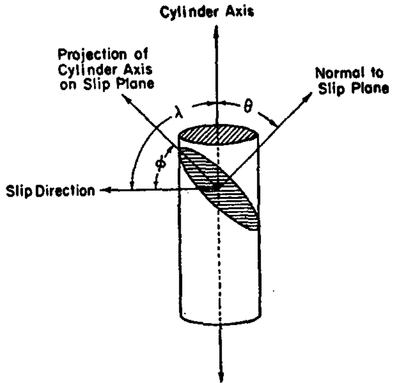

FIG. 9. Diagram defining angles used for calculations relating applied stresses to resolved shear stress in the slip plane and in the slip direction.

6. ADJUSTMENT OF THE THEORY TO EXPRESS RESULTS IN TERMS OF MEASURED QUANTITIES

In the theory as developed so far, the symbols σ and ϵ have the meaning of shear stress and shear strain. In most of the experiments, longitudinal stresses and strains are used. In order to differentiate between these, we now denote the former by τ and γ and reserve the symbols σ and ϵ for the corresponding longitudinal quantities.

Then, in a similar way as in Sec. 3, one obtains from Newton's law for the propagation of longitudinal waves:

$$\frac{\partial^2\sigma}{\partial x^2}-\frac{\rho}{E}\frac{\partial^2\sigma}{\partial t^2}-\rho\frac{\partial^2}{\partial t^2}\epsilon_{\text{dis}}=0. \tag{6.1}$$

Also, we write now

$$\tau=R\sigma, \tag{6.2}$$

$$\epsilon_{\text{dis}}=g\gamma_{\text{dis}}, \tag{6.3}$$

and

$$\gamma_{\text{dis}}=\tau/P, \tag{6.4}$$

where R and g are factors taking into account the orientation relations between the direction of propagation of the longitudinal wave and the slip plane and slip direction in the crystal. They turn out to be both given by

$$R=g=\cos\theta\,\sin\theta\,\cos\phi, \tag{6.5}$$

where θ and ϕ are as shown in Fig. 9. P has the nature of a modulus and is taken to be complex so that the shear strain may be out of phase with the shear stress. Its value can be easily found using Eqs. (3.4) and (3.7).

By introducing Eqs. (6.2), (6.3), and (6.4) in (6.1), and solving this equation, one obtains a damped or attenuated wave, as in Sec. 3. The resulting expressions for the decrement and modulus change for the dynamic loss differ from Eq. (3.11) by the factor R^2. In addition, the factor G which is contained in Δ_0 ($\Delta_0=8Ga^2/\pi^3c$) should be replaced by E. Since there are usually many slip systems in a metal, this product must be taken for all slip systems and summed. Also the actual number of dislocations Λ_i belonging to the ith slip system may in general be different. The orientation effects are then taken into account by multiplying the previously obtained expressions (3.11) by the orientation factor T, where

$$T=\frac{E(\theta,\phi)}{G}\sum_i\frac{\Lambda_i}{\Lambda}R_i^2(\theta,\phi) \tag{6.6}$$

and

$$\sum\Lambda_i=\Lambda.$$

For the strain amplitude dependent loss, there is one further change to be made. The quantity which is usually measured is the longitudinal strain amplitude, while the result derived here for the decrement is in terms of the shear stress in the slip plane. Using Eq. (6.2), the expression for the decrement becomes

$$\Delta=\frac{T\Delta_0L_N^3}{\pi L_c}\frac{\Gamma}{RE\epsilon_0}\exp\left[\frac{-\Gamma}{RE\epsilon_0}\right], \tag{6.7}$$

and $r=\Delta/\Delta E/E=$ a constant of order unity.

7. ACKNOWLEDGMENTS

We thank Professor R. Truell for his continuous interest and encouragement in the course of this work, Professor G. Newell and Mr. P. Waterman for many interesting discussions, and Miss I. Stegun of the National Bureau of Standards for providing us with tables of the exponential integral function of a complex variable before publication. We are also indebted to Mr. Rothstein, Mr. Greenberg, Mr. LaMorte, and Mr. Epstein of the U. S. Signal Corps Laboratories for their help and encouragement.

APPENDIX

1. Exponential Distribution

$$\Delta=\int_0^\infty l\Delta(l)N(l)dl.$$

If the substitutions $x=\omega/\pi(A/C)^{\frac{1}{2}}l$, $\beta=d/\omega$, $\eta^2=\pi^2C/A$, and $D=\eta/Ld$ are made, then one obtains

$$\Delta=\frac{\Lambda 8Ga^2}{\pi A^2}\frac{Bn^2}{L^2\omega^5}\int_0^\infty\frac{x^5\exp[-D\beta x]dx}{[(1-x^2)^2+\beta^2x^4]}.$$

The integrand, apart from the exponential factor, may be expanded into partial fractions as

$$H(x)=\frac{x^5}{(1-x^2)^2+\beta^2x^4}=\frac{x}{1+\beta^2}+\frac{1}{(1+\beta^2)^2}\sum_{n=1}^{4}\frac{A_n}{(x-x_n)}.$$

The four finite poles lie on a circle in the complex plane and have the modulus $(1+\beta)^{-\frac{1}{4}}$. In this expression

$$x_1=\frac{2^{\frac{1}{2}}}{2}\frac{1}{(1+\beta^2)^{\frac{1}{2}}}\{[(1+\beta^2)^{\frac{1}{2}}+1]^{\frac{1}{2}}+i[(1+\beta^2)^{\frac{1}{2}}-1]^{\frac{1}{2}}\}$$

$$x_2=-x_1,\quad x_3=\bar{x}_1,\quad\text{and}\quad x_4=-\bar{x}_1.$$

Also

$$A_1=\tfrac{1}{2}+i(-1+\beta^2)/4\beta,$$
$$A_2=A_1,\quad A_3=\bar{A}_1,\quad\text{and}\quad A_4=\bar{A}_1,$$

where $\bar{A}$ denotes the complex conjugate of A. The integration may now be performed to give

$$I=\int_0^\infty H(x)\exp[-D\beta x]dx$$
$$=\frac{1}{(1+\beta^2)}\frac{1}{D^2\beta^2}+\frac{1}{(1+\beta^2)^2}\sum_{n=1}^4 A_n\int_0^\infty\frac{\exp[-D\beta x]}{(x-x_n)}dx.$$

Now by a change of variable, the integral may be expressed in terms of exponential integral functions of a complex variable:

$$I=\frac{1}{(1+\beta^2)}\frac{1}{D^2\beta^2}+\frac{1}{(1+\beta^2)^2}\sum_{n=1}^4 A_n \times\exp[-D\beta x_n](-E_i[D\beta x_n]).$$

The result is then

$$\Delta=\Delta_0\Lambda L^2\left\{\frac{D^2\beta^3}{1+\beta^2}+\frac{D^4\beta^5}{(1+\beta^2)^2}\sum_{n=1}^4 A_n \times\exp[-D\beta x_n](-E_i[D\beta x_n]).\right.$$

Similarly, the modulus change for the exponential distribution is found from

$$\frac{\Delta G}{G}=\frac{\Lambda}{L^2}\int_0^\infty\left(\frac{\Delta G}{G}\right)_l\exp\left(\frac{-l}{L_c}\right)dl,$$

where the expression for $\Delta G/G$ obtained in Sec. 3 is used under the integral. One obtains

$$\frac{\Delta G}{G}=\frac{\Lambda 8Ga^2}{\pi^2L^2A}\int_0^\infty\left\{l\left[\frac{(n^2/l^2-\omega^2)}{(n^2/l^2-\omega^2)^2+(\omega d)^2}\right]\exp\left(\frac{-l}{L}\right)dl.\right.$$

If now, we let $x=\omega l/n$, $\beta=d/\omega$, and $D=n/Ld$, this may be written in the form

$$\frac{\Delta G}{G}=\frac{\Lambda\Delta_0L^2(D\beta)^4}{\pi}\int_0^\infty\frac{x^3(1-x^2)\exp(-D\beta x)}{[(1-x^2)^2+\beta^2x^4]}dx$$

or

$$\Lambda\Delta_0L^2(D\beta)^4I/\pi,$$

where $I=I_1-I_2$ and I_2 has been evaluated in finding the decrement. I_1 is found in the same way as was I_2 to be

$$\frac{1}{(1+\beta^2)^2}\sum_{n=1}^4 C_n\exp(-D\beta x_n)[-E_i(D\beta x_n)],$$

where the x_n are as before and $C_1=\frac{1}{4}(1-i/\beta)$, $C_2=C_1$, $C_3=\bar{C}_1$, $C_4=\bar{C}_1$. Then

$$\frac{\Delta G}{G}=\frac{\Lambda\Delta_0L^2}{\pi}(D\beta)^4\left\{\frac{1}{1+\beta^2}\sum_{n=1}^4 C_n \times\exp(-D\beta x_n)[-E_i(D\beta x_n)]-\frac{1}{(1+\beta^2)}\frac{1}{D^2\beta^2} -\frac{1}{(1+\beta^2)^2}\sum_{n=1}^4 A_n\exp(-D\beta x_n)[-E_i(D\beta x_n)]\right\}.$$

2. Low-Frequency Expansion of the Exponential Distribution Decrement

If $\beta\rightarrow\infty$, then, successive integration by parts gives

$$-E_i(-z)=\int_z^\infty\frac{\exp(-y)}{y}dy \sim\left[\sum_{K=0}^\infty(-1)^K\frac{K!}{Z^{K+1}}\right]\exp(-z).$$

The terms $K=0$, 1, 3, 4, 5, 7, 9 give zero contribution to the summation. The term $K=2$ gives a contribution which exactly cancels the $1/(1+\beta^2)D^2\beta^2)$ term. The $K=6$ term gives the leading low-frequency term. The $K=8$ term gives a first correction term which is independent of the damping constant B; $K=10$ gives a term which depends on the damping constant B and is of the same order but opposite in sign to the $K=8$ term.

3. High-Frequency Approximation

For the approximation as $\beta\rightarrow 0$, the exponential integral functions are of higher order than the $1/(1+\beta^2)D^2\beta^2$ term. Therefore, the leading term is $1/D^2\beta^2$.