

Application of Dislocation Theory to Internal Friction Phenomena at High Frequencies*

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A detailed discussion of data obtained over the past 15 years concerning the damping of mechanical vibrations in the kilocycle and megacycle range is given. The dependence of the decrement and modulus change on the variables of frequency and strain-amplitude and many other parameters is compared with predictions of the dislocation theory developed in an earlier paper. Although general agreement is obtained, and many interesting quantitative results are found, it is not possible to say that the theory agrees everywhere since not all the necessary parameters are known well enough theoretically. A number of new experiments are suggested which may permit stronger conclusions to be made. This part may be read independently of the earlier paper by the reader who does not wish to follow the development of the theory in detail.

1. INTRODUCTION

IN an earlier paper¹ (hereafter called part I), a theory of energy loss and modulus changes due to dislocation damping in undeformed and slightly deformed crystals was developed on a basis of the model of a pinned dislocation loop used by Koehler oscillating under the influence of an applied stress.

In the present paper, the available data are compared and interpreted insofar as is possible with the theory. The decrement (Δ) [or attenuation (α)], modulus change ($\Delta E/E$) and ratio of both ($r = \Delta/\Delta E/E$) are considered to be functions of the main variables frequency (ω) and strain amplitude (ϵ_0). In addition, these quantities depend also on the properties of the material used. The parameters needed in the theory are the following: impurity concentration (C), fractional difference in the size of the solute and solvent atoms (ϵ'), loop length determined by impurities (L_c), loop length determined by the dislocation network (L_N), Young's modulus (E), shear modulus (G), resolved shear stress factor (R), total length of dislocation line per unit volume (A), and damping constant for dislocation movement (B). One sees that these parameters in some cases depend on other quantities, such as temperature, orientation, annealing temperatures and times, and plastic deformation.

In no case has an experiment been made in which all of the measurable parameters are reported. A complete comparison of results with theory is therefore not possible. However, in many experiments, only a few of the parameters are varied, and these variations have been analyzed.

Although there are points in the development of the theory which may be questioned, the fact that such questions are not likely to be resolved at the moment on purely theoretical grounds leads us to believe that progress might best be made in this field by working

out all the consequences of each proposed model, and comparing these with the experimental results.

2. THE MODEL

It is assumed that an undeformed single crystal contains a network of dislocations. For large enough concentrations of impurity atoms the length of loop determined by the intersection of the network loops is further shortened by the impurity particles through the Cottrell pinning mechanism. There are, therefore, two characteristic lengths in the model: The network length L_N , and the length L_c , determined by the impurities. The model is further modified in the mathematical treatment to take account of the fact that a distribution of lengths (L_c) occurs.

If an external stress is now applied, there will be, in addition to the elastic strain, a strain due to the motion of the dislocations called the dislocation strain. For small applied stresses, the loops L_c bow out and continue to bow out until the stress becomes large enough for the loop lengths to break away from the impurity particles (see Fig. 1, part I). This picture leads to two types of loss, one a dynamic loss and the other a static loss.

The first loss is due to the fact that the measurement is a dynamic one. Under the action of the applied external stress, motion occurs. Because the motion is opposed by some damping mechanism there is a phase lag for an oscillating stress, and hence a decrement and change of modulus effect. This type of loss is frequency dependent, since it has a resonance type character. For low frequencies, the motion is in phase with the applied stress and the decrement is small. Near the resonant frequency determined by the loop length, the motion is out of phase by ninety degrees and the decrement is a maximum. For very large frequencies the motion is a half-cycle behind the applied stress and the decrement again becomes small.

The second loss arises because of the fact that, during the unloading part of the stress cycle, the long loops collapse elastically along a path determined by the long loop length, thus giving a static hysteresis loop. When the loops have completely collapsed they are again

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¹ A. Granato and K. Lücke, *J. Appl. Phys.* **27**, 583 (1956).

pinned by the impurity particles and the same type of path is followed in the other half-cycle. The loss is simply proportional to the area enclosed by the stress-dislocation strain loop. For low frequencies, which still cover the kilocycle range, it turns out that the stress-dislocation strain law is independent of frequency, and hence this loss is frequency independent.

As shown in part I, for low frequencies (kilocycle range), these two losses are given by

$$\Delta_H = \frac{\Omega \Delta_0 \Lambda L_N^3 K \epsilon' A}{\pi L_c} \frac{1}{L_c \epsilon_0} \exp\left(-\frac{K \epsilon' a}{L_c \epsilon_0}\right), \quad (2.1)$$

and

$$\Delta_I = \frac{\Omega \Delta_0 \Lambda L_c^4 B \omega}{\pi^2 c}. \quad (2.2)$$

(The subscript H denotes the strain amplitude dependent loss, and I the frequency dependent loss.)

L_e is an effective loop length which should be given by 3.3 times L_c . As is easily seen, with each loss mechanism, there is associated a modulus change, which is given by

$$\left(\frac{\Delta E}{E}\right)_H = \frac{\Omega \Delta_0 \Lambda L_N^3 K \epsilon' A}{\pi L_c} \frac{1}{L_c \epsilon_0} \exp\left(-\frac{K \epsilon' a}{L_c \epsilon_0}\right), \quad (2.3)$$

and

$$\left(\frac{\Delta E}{E}\right)_I = \frac{\Omega \Delta_0 \Lambda L_c^2}{t \pi}, \quad (2.4)$$

where t is a constant whose value is one for a delta function distribution of loop lengths and $\frac{1}{6}$ for an exponential distribution.

Also, for the frequency independent, strain-amplitude dependent loss, $r = \Delta_H / (\Delta E/E)_H$ is a constant of order unity. For strain amplitudes low enough so that there is no breakaway, and for all frequencies,

$$\Delta = \Omega \Delta_0 \Lambda \eta^2 \frac{\omega d}{[(\omega_0^2 - \omega^2)^2 + (\omega d)^2]}, \quad (2.5)$$

and

$$\frac{\Delta E}{E} = \frac{\Omega \Delta_0 \Lambda \eta^2}{\pi} \frac{(\omega_0^2 - \omega^2)}{[(\omega_0^2 - \omega^2)^2 + (\omega d)^2]}. \quad (2.6)$$

In the above formulas, the symbols have the following meanings[†]: Mechanical energy lost per cycle by the above mechanism divided by twice the total vibrational energy- Δ , percentage difference between the true and measured Young's modulus- $\Delta E/E$, ratio of the decrement to the modulus change- r , total length of movable dislocation line per unit volume- Λ , lattice parameter- a , strain amplitude- ϵ_0 , circular frequency- ω , Poisson's

[†] The symbols used in this part have the same meaning as in part I, with the following exceptions: The symbols T , Ω , and f in part I denote the orientation factor, the frequency normalized to the resonant frequency, and the Cottrell force, respectively, whereas in part II they denote the temperature, orientation factor, and frequency, respectively.

ratio- ν , $\Delta_0 = 4(1-\nu)/\pi^2$, $d = B/A$, $A = \pi \rho a^2$, density- ρ , Burger's vector- a' , $\eta = \pi(C/A)^{1/2}$, $C = 2Ga^2/\pi(1-\nu)$, $\omega_0 = \eta/L_e$, effective length of dislocation loop (as defined in part I, Sec. 3) L_e , unknown parameter which is a function of orientation and anisotropy- K .

The shape of the stress-dislocation strain curve obtained from this model for a complete cycle of stress is quite different from those which have so far been proposed.^{2,3} This is shown in Fig. 7 of part I. The curves already proposed are such that a complete cycle of stress is needed before the hysteresis loop is closed. The curve obtained from this model is one in which the loop closes after each half-cycle. Slow speed tension tests made on zinc single crystals⁴ give stress-strain curves of the type derived from this model.[§]

3. KILOCYCLE RANGE DATA

A. Data Reviewed

The data here reviewed is that listed in Table I, taken from the papers in the list of references²⁻⁹. These papers are believed to contain most of the main results in this field. All data known to us for which it was reasonable to attempt a comparison, were checked with the theory. Some data had to be left out because, for example, too many parameters were varied simultaneously, or the range of variations of the variables was too small to be checked successfully, or the printed figures were too small to obtain the values accurately enough. However, the table does contain enough examples so that changes due to variations of each of the parameters may be examined.

The data was interpreted in the following way. In Fig. 1, a curve of the decrement *versus* strain amplitude for aluminum (solid line) is shown. The decrement at a given strain amplitude is thought of as being made up of two parts. One part is the value obtained on extrapolation to zero strain amplitude and the second part is associated with the static hysteresis loss. It would be most natural to associate the first part with the strain amplitude independent term (2.2) and the second part with Eq. (2.1). However, it is shown later that Δ_I cannot be reasonably attributed to the frequency dependent loss due to the motion of the same impurity pinned dislocation loops causing the loss (2.1). If Δ_I is

² T. A. Read, Trans. Am. Inst. Mining Met. Engrs. 143, 30 (1941). See also T. A. Read, Phys. Rev. 58, 371 (1940).

³ A. S. Nowick, Carnegie Institute of Technology Symposium on the Plastic Deformation of Crystalline Solids, ONR (1950), p. 155.

⁴ T. A. Read and E. P. T. Tyndall, J. Appl. Phys. 17, 713 (1946).

[§] The value of the decrement obtained by measuring the area enclosed by these stress-strain curves is, however, much larger than values usually measured in the kilocycle range.

⁵ J. Weertman, and E. I. Salkovitz, Acta Met. 3, (1955), 1.

⁶ J. W. Marx, and J. S. Koehler, Carnegie Institute of Technology Symposium on the Plastic Deformation of Crystalline Solids, ONR, (1950) 171.

⁷ T. Noggle and G. Baker (unpublished results). See reference 19.

⁸ J. H. Swift and J. E. Richardson, J. Appl. Phys. 18, 417 (1947).

⁹ D. R. Frankl, Phys. Rev. 92, 573 (1953).

TABLE I. Analysis of kilocycle range decrement measurements by means of the relations,

$$\Delta = \Delta_H + \Delta_I \quad \text{and} \quad r = \Delta_H / (\Delta E / E)_H.$$

Here Δ_H is the strain amplitude dependent decrement given by $\Delta_H = C_1/\epsilon_0 \exp(-C_2/\epsilon_0)$, where $C_1 = \Omega_0 \Delta L_N^3 K \epsilon' a / \pi L_c^2$ and $C_2 K \epsilon' a / L_c$. Δ_I is given by $\Omega_0 \Delta L_c^4 B \omega / \pi^2 C$ and $(\Delta E / E)_H$ is the amplitude dependent modulus change. ϵ_0 is the strain-amplitude, Δ the dislocation density, L the dislocation loop length, and the remaining symbols are defined in the text.

For the case of the lead alloy data only, the misfit parameter ϵ' is known to be 0.062 for lines 1-4 and 0.018 for lines 5-7. Unless otherwise indicated, the measurements are at room temperature, Temperature (T), orientation (θ), and applied external stress (p) data are found in the remarks column.

Row	Sample & Reference	Impurity (%)	ω ($\times 10^3$)	Annealing		Remark	r	C_1 ($\times 10^7$)	$C_2 \log e$ ($\times 10^6$)	Δ_I ($\times 10^4$)	$1/K$	L_c/K ($\times 10^4$)	$\frac{\Omega \Delta L_c^3}{K}$ ($\times 10^5$)	$\Omega \Delta L_c^2 B$ ($\times 10^{12}$)	L_c ($\times 10^3$)	$\Delta \times 10^{-5}$ (for $L_N = 10^{-4}$)	$C \times 10^3$
				Hours	Temp. °C												
1	Pb ⁽⁵⁾	Sn (.058)	1.9			T=25°C		260	2.0	25	53	0.665	19	14.5	0.125	880	4.0
2						T=152°C		7.5	6.2	100	53	2.2	5.35	58	0.41	250	12
3						T=306°C		28.5	2.6	1500	53	5.1	120	870	0.97	5700	5.1
4	Pb ⁽⁵⁾	Sn (.016)	1.9				1.5	31	1.6	80	50	8.5	365	46	1.7	16,000	2.9
5	Pb ⁽⁵⁾	Bi (.035)	1.9					.051	0.885	7.0	50	4.4	0.54	4.1	0.88	2.7	5.6
6		Bi (.053)						0.16	2.8	6.0	50	1.4	0.17	3.5	0.28	8.5	1.8
7		Bi (.65)						60.5	57	2.0	50	.068	.066	11.6	.014	3.3	370
8	Pb ⁽⁶⁾	10 ⁻³	2.3	None				0.77	0.42	37	50	51	200	18	10.1	9,950	0.49
9	Cu ⁽⁶⁾	10 ⁻³	2.3	None	500			.056	0.45	41	50	35	8.45	72	6.9	420	0.505
10				4	500			.035	0.90	20	50	17	1.3	35	3.5	66.5	1.0
11				8	500					9.5	50			17			
12				12	500					6.3				11			
13				15	500					5.9				10			
14				+3hr H ₂	500					3.0				5.3			
15	Cu ⁽²⁾	2 $\times 10^{-3}$	2.1			p=0psi		.089	2.9	0.8	50	5.4	0.33	1.5	1.1	16	3.2
16						p=60		0.12	3.0	0.8	50	5.2	0.39	1.5	1.0	20	3.4
17						p=120		0.315	3.0	1.0	50	5.2	1.1	1.9	1.0	53	3.4
18						p=150		0.73	3.0	1.8	50	5.3	2.5	3.5	1.1	130	3.3
19	Cu ⁽²⁾	2 $\times 10^{-3}$	2.1				4.3	.049	0.975	1.4	50	16	1.6	2.7	3.2	78.5	1.1
20								.0053	0.96		50	16		3.3		1.1	
21	Cu ⁽⁶⁾	Al & Fe <.01 Ag (.0003)	2.45	4	600	T=33°C	2.0	.026	0.54	3.7	50	29	2.7	6.1	5.8	130	0.61
22						T=49°C		.01	0.55	2.5	50	28	0.98	4.1	5.7	49	0.62
23	Al ⁽⁷⁾	3 $\times 10^{-3}$		Before				0.705	6.25	0.16	50	2.8	1.2	0.23	0.56	60.5	7.2
24				After						0.135			0.20				
25	Zn ⁽⁴⁾	10 ⁻²	2.1	Many	Room	$\theta=70^\circ$		0.37	7.55	1.1	54	1.5	.099	1.3	0.28	1.8	9.4
26				1	330			58	65	0.28	54	0.18	0.21	0.33	.033	3.7	81
27	Zn ⁽⁸⁾	10 ⁻² (min.)	2.8	None	Room	$\theta=80^\circ$		600	8.8	1.5	30	1.3	120	1.3	0.44	14,000	6.0
28				53	Room			10	20	1.5	30	0.58	0.39	1.3	0.20	45	14
29				98	Room			50	25	1.5	30	0.47	1.25	1.3	0.16	140	17
30	Zn ⁽²⁾	Fe(.0004-9) Pb(.0002) Cd(.00005)	2.4	2		$\theta=20^\circ$	1.4	1.1	9.6	0.12	20	1.2	0.52	0.345	0.61	25	4.4
31				2		$\theta=61.3^\circ$	5.3	2.0	0.97	0.12	67.5	12	35	0.13	1.8	290	1.5
32				2		$\theta=74.5^\circ$	4.5	0.90	2.8	0.12	4.4	4.1	1.7	0.12	0.94	58	2.8
33				2		$\theta=88^\circ$	0.55	0.21	13.5	0.12	5.7	0.81	.018	0.13	1.5	260	1.8
34	Zn ⁽²⁾	Same	2.45					0.58	2.2	0.35	50	5.2	2.4	0.48	1.0	120	2.6
35			4.9					0.24	2.2	0.35	50	5.2	1.0	0.24	1.0	50	2.6
36	NaCl ⁽⁹⁾	Al & Fe (2 max)	4.4		600			7.2	1.0	5.8	50	1.7	2.25	4.0	0.35	110	11.5
37					200			0.12	6.2	2.4	50	2.8	.09	1.6	0.56	4.5	7.15

to be associated with a frequency dependent term, it must be due to the motion of different dislocations, as, for example, network pinned screw dislocation loops. There is, then, no connection between L_e and L_c .

B. Comparison of the Dependence of the Decrement and Modulus Change on the Principal Variables

1. Frequency Dependence

According to Eq. (2.1), Δ_H is independent of frequency. Nowick³ has reviewed the evidence on the frequency dependence of the strain amplitude dependent loss and concludes that it is frequency independent. Recently, Marx¹⁰ has obtained measurements of the decrement from about 40 kc/sec to 260 kc/sec on a copper single crystal, and he finds that the strain amplitude dependent loss is nearly frequency independent.

At present, the measured frequency dependence of the strain amplitude independent loss (Δ_I) in the kilocycle region is not definitely established. Measurements show examples of both increasing and decreasing Δ_I with frequency. Nowick³ has pointed out that the uncertainty here can be due to the fact that measurements made at different frequencies excite different parts of the specimen. Since the measurement is structure sensitive, variations by factors of two or three in dislocation density in the specimen could account for the discrepancies. It should be possible to check this possibility. Since both the decrement and the modulus change are proportional to the dislocation density, measurements made at several harmonics of both the decrement and absolute value of the modulus should give more information than decrement measurements alone. According to the theory, one would expect the decrement to increase and the change in modulus to stay fixed for an increase in frequency. If, however,

¹⁰ J. W. Marx (unpublished data). See reference 20.

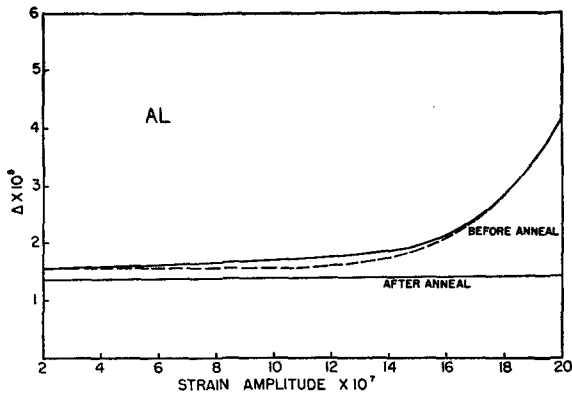


FIG. 1. The decrement as a function of strain amplitude of an aluminum crystal before and after annealing. Data of Noggle and Baker. The dashed line shows the theoretical curve fitted to the data at large strain amplitudes.

part of the decrement change is due to a change in the "effective dislocation density," then the modulus should change also by a corresponding amount. Thus modulus measurements might help to determine what part of the decrement change is due to the frequency change alone. Of course, there will also be a slight change in modulus due to geometrical dispersion, but this can be corrected for.

The static hysteresis modulus change $(\Delta E/E)_H$ is found experimentally^{2,3,5} to be linearly related to Δ_H . That is, the ratio is a constant. This fact is in accord with the theory. The constant is furthermore found to be of order unity, again in accord with the theory.

There are really no measurements of the relative difference between the true modulus and the measured modulus, $(\Delta E/E)_I$ since the true modulus cannot be measured. However, there are some measurements which throw light on this variation and these will be discussed later.

2. Strain Amplitude Dependence

The dependence of the decrement (Δ_H) on the strain amplitude has been described in some cases as parabolic^{3,11} and in others as being zero for low strain amplitudes, but abruptly increasing at a certain strain amplitude.⁵ According to the above theory, all of the strain amplitude data must be describable in terms of a single function of the form

$$\Delta_H = \frac{C_1}{\epsilon_0} \exp(-C_2/\epsilon_0). \quad (3.1)$$

If this is so, then the experimental data when plotted in the form

$$\log \epsilon_0 \Delta_H \text{ versus } 1/\epsilon_0$$

should lie on a straight line. Furthermore, the slope of

¹¹ J. S. Koehler, *Imperfections in Nearly Perfect Crystals* (John Wiley and Sons, Inc., New York, 1952), p. 197.

this straight line should be given by

$$C_2 = K\epsilon'a/L_c \quad (3.2)$$

and the intercept by

$$C_1 = \Omega \Delta_0 \frac{\Delta L_N^3 K \epsilon a}{\pi L_c L_c}. \quad (3.3)$$

All the data considered was plotted in this way and gave rather good straight lines. Examples are given in Figs. 4 and 5. Only for very low strain amplitudes does the decrement sometimes appear to be a little too large. Figures 1 and 2 show an extreme example of this effect. However, the deviation between the measured (solid line) and the theoretical (dashed line) is highly exaggerated in the semilog plot as can be easily seen in Fig. 1. Moreover, the errors of measurement are strongly magnified for small values of Δ_H , since Δ_H is found by subtracting the two large quantities Δ and Δ_I . In spite of this small deviation, the theoretical relation (3.1) may be considered to be confirmed, especially since all the data (which includes curves which vary in form from roughly parabolic to curves with sharp breaks) is fitted to the same function.

Because $(\Delta E/E)_H$ is known to be linearly related to Δ_H , the measured values therefore have the same strain amplitude dependence as Δ_H . The same remarks made for the strain amplitude agreement of Δ_H therefore apply as well to $(\Delta E/E)_H$.

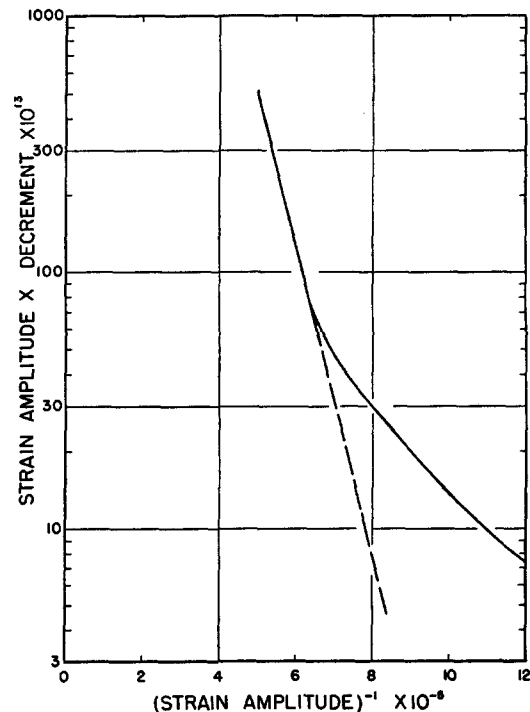


FIG. 2. Decrement versus strain amplitude data, when plotted as in this figure, should lie on a straight line according to the present theory. Here, an extreme case of deviation from the theoretical curve at low strain amplitudes is shown. The curves are those of Fig. 1 before anneal. It is readily seen that the discrepancy in Fig. 2 between the theory and the measurements is greatly exaggerated.

On the basis of the above comparisons, it is therefore concluded that the theory is in reasonable agreement with the observed dependence of the decrement and modulus change on the principal variables of frequency and strain amplitude.

C. Table I

In Table I all information supplied by the author about the parameters is listed. The four metals, lead, copper, aluminum, and zinc are found in the rows arranged so that the most isotropic (zinc)^{||} is at the bottom. In addition, measurements made on rocksalt are listed. In the columns are given the material, reference, purity, solute, frequency (ω), temperature at which measurement was made (T), amount of external stress applied, annealing treatment, and orientation of the sample. The only information available on orientation is that in the case of some of the zinc samples, where the angle between the specimen axis and the normal to the slip plane (θ), is given.

In addition, the results derived from the graphical analysis of the data described by Eqs. (2.2), (3.2), and (3.3) and the values for $r = \Delta_H/(\Delta E/E)_H$ are listed. The values of the slope (3.2) are obtained more accurately graphically than values of the intercept (3.3), since very slight variations of the slope can lead to large variations in the value obtained for the intercept. Therefore, the dislocation densities which are derived from the intercepts are considered to have a rather large inaccuracy.

Next using all the known quantities, columns 11, 12, and 13 headed by L_c/K , $\Omega\Delta L_N^3/K$, and $\Omega\Delta L_c^4$ were computed from columns 7, 8, and 9. Since the value of ϵ' is only known in the case of the lead data, a value of $\epsilon' = 0.1$ was used for the rest of the data.

Finally, a value of K is determined in a way to be described in the next section. From the L_c/K column the impurity length (L_c) and concentration (c) are determined. In the cases where the orientation of the specimen is known, the appropriate orientation factor Ω is used. In those cases where it is not known, an average value of $\Omega = 1/25$ estimated by Weertman¹² is used. From the $\Omega\Delta L_N^3/K$ column, the product ΔL_N^3 is found. Then the values of Λ corresponding to a network length of L_N of 10^{-4} cm are given in column 15. Finally, from column 13, the magnitudes of the initial decrement are inspected to see if values of the damping constant in accord with present theoretical estimates are obtained.

D. Comparison of the Dependence of the Decrement and Modulus on the Parameters

1. General Remarks

It may be quickly seen from the table that a qualitative agreement is obtained from the dependence of the

^{||} In the case of zinc, only anisotropy in the basal plane is of interest.

¹² J. Weertman, J. Appl. Phys. 26, 202 (1955).

strain amplitude dependent decrement on the concentration of impurities at the dislocation line. This is observed by noting that the slope (column 8) which is inversely proportional to L_c [Eq. (3.2)] or proportional to C is generally smaller for the purer specimens.

In order to determine the magnitude of L_c , it is necessary to know the value of K . The value of K obtained when the pinning force is assumed to be that due to the Cottrell interaction force between an impurity particle and a dislocation line in an isotropic medium is given by (see Part I)

$$K = \frac{32}{\pi^2 p^2} \frac{G}{RE}, \quad (3.4)$$

where E is Young's modulus, G is the Shear modulus, R the resolved shear stress factor, and p is a constant with value of about 2 or 3. Under these circumstances K would vary with orientation and material and would be expected to be not much larger than $\frac{1}{2}$. This would mean that the impurity loop lengths would be given nearly by the values in the column headed by L_c/K . But this leads to rather large values of L_c ; larger, in many cases than the value of 10^{-4} cm which is commonly considered to be a reasonable value for the network length. If K were given by a factor more like $1/50$, then reasonable values of L_c are obtained, such as are shown in column 14. As shown in the next section, such a value is also suggested by using the data giving the temperature dependence of the decrement in lead-tin, where the concentration of the impurity is known. Since, however, it is not known to what extent Cottrell's formula, which is used in deriving 3.4, is valid for the interaction between dislocations and substitutionally dissolved atoms, the disagreement might not be too serious.

In some cases the value found for L_c (column 14) is almost as large as L_N . In such cases deviations of the curves from the theoretical one could be expected since the approximations used (for the theoretical derivations) would no longer hold.

The range of Λ for all the data listed obtained from column 12 for $L_N = 10^{-4}$ cm is 1.8×10^5 to 1.6×10^9 , as shown in column 15. This means a reasonable range of values is obtained (discussed later). For $L_N = 10^{-3}$ cm, all these values are smaller by a factor of 10^3 . If we assume there is a dislocation network, then there must be a connection between L_N and Λ^* (total length of dislocation line), which depends on the geometry of the network. For a cubic array, this relation is

$$\Lambda^* L_N^2 = 3. \quad (3.5)$$

Thus, one obtains

$$\Lambda^* = 3/L_N^2 = 3 \times 10^8 \text{ cm/cm}^2,$$

or generally, a value larger than the listed values of Λ . But this is to be expected, since only the nonsessile, edge-type dislocation segments which lie on favorably

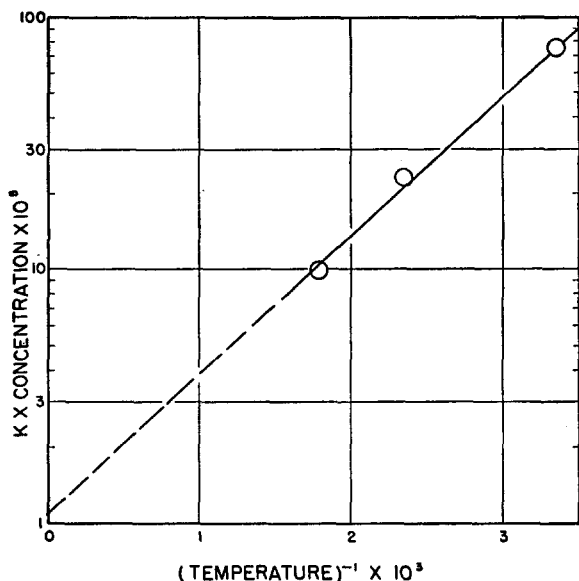


FIG. 3. The slopes of the lines found by plotting the strain amplitude dependent decrement in the manner shown in Fig. 2 should be proportional to the concentration of pinning points on the dislocation line. If the pinning is of the Cottrell type, then a plot of the logarithm of the derived slopes against the inverse temperature should give a straight line. The points used here are derived from the data of Weertman and Salkovitz taken on a single crystal of lead with 0.058% tin.

disposed slip planes participate in the breakaway process.

2. Detailed Discussion of the Strain-Amplitude Dependent Loss

For the PbSn data (rows 1, 2, 3), only the temperature is varied. The only two parameters considered to be temperature sensitive are the concentration C and the active dislocation length Λ . Since Λ does not appear in the expression for the slope, the variation in the slope is considered to be all due to the changing concentration on the dislocation line due to the changing temperature. If now, the value KC , obtained from column 11 is plotted as a function of the inverse temperature, the points in Fig. 3 are obtained. Thus we find that, the higher the temperature, the smaller the concentration of impurities on the dislocation line, and the larger the loop length.

If the impurities are attracted to the dislocation line through the Cottrell mechanism, then the points would be expected to lie on a straight line whose slope is the activation energy and whose intercept is the impurity concentration. That is, the concentration c on the dislocation line at the temperature T is given by

$$c = c_0 \exp(Q/kT), \quad (3.6)$$

where c_0 is the impurity concentration and Q the interaction energy. The points do seem to lie on a straight line with measured slope of 0.11 electron volt and intercept of $Kc_0 = 1.1 \times 10^{-5}$. The energy as calculated by

Weertman and Salkovitz⁵ from Cottrell's formula is 0.044 ev. According to this interpretation, the value of K can be determined, since c_0 is known to be $c_0 = 5.8 \times 10^{-4}$. One obtains in this way $K = 1/53$. As already mentioned, the value of K determined in this manner, also gives reasonable quantitative agreement with all other cases for the concentration and impurity loop length. At high temperatures, it appears from the data on PbSn that the total active length of dislocation line is larger.

Considering the PbSn samples in lines 1 and 4, the only difference here is in concentration and possibly orientation. Since the orientation is not known it is not possible to say how much of a change should be expected in K , but since in line 4 the concentration is smaller, the slope would be expected to be smaller due to this effect, as is the case.

For the next sequence of measurements on PbBi (lines 5, 6, 7) the concentration and orientation are varied. The variation in slopes is shown in Fig. 4. It may be seen that the derived concentration varies by about the expected ratios from sample to sample. Also, as a group, the PbBi derived values for c are in general

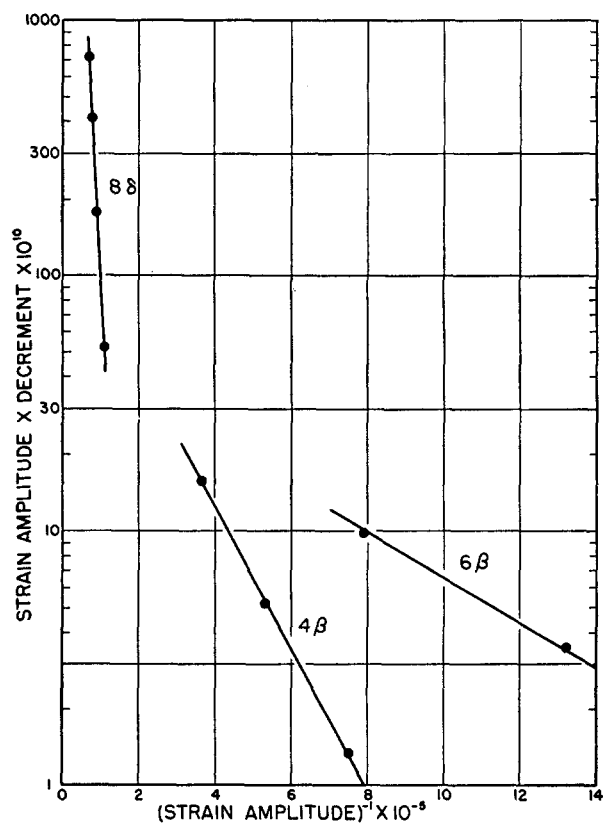


FIG. 4. The data of Weertman and Salkovitz are here plotted for lead with three different concentrations of added bismuth impurity. According to the theory, the points should lie on straight lines whose slopes are proportional to the concentration of impurity on the dislocation line. The deliberately added concentrations of bismuth in the specimens as given by Weertman and Salkovitz are 0.035, 0.053, and 0.65 atomic percent for samples 6β, 4β, and 8δ, respectively.

agreement with those found for PbSn. The reason for the big difference in Λ between PbSn and PbBi is not known. Very little is known about the origin of dislocations.

Generally speaking, all of the lead and lead alloy data fit the straight line plots rather well compared to the other materials. The deviations from the straight line at low strain amplitude mentioned earlier was very slight in the case of the lead data.

Measurements were also made on lead by Marx and Koehler (line 8). In this case the solute was not known, but the lead was of extremely high purity. From the table it is seen that in agreement with the above fact, the derived concentration is the smallest obtained for all the lead samples, and the loop length L_c the largest.

Next, (lines 10, 11) the effect of annealing at 500°C on the decrement of very pure copper is analyzed. First, the value of L_c obtained is large, in agreement with the fact that the specimen is very pure. Secondly, the effect of the annealing treatment from 4 to 8 hours at 500°C in vacuum is to double the concentration of impurities on the dislocation line. Although all of the curves were not analyzed, they all follow the same qualitative trend. This result is consistent with the idea that for lower temperatures the equilibrium concentration on the dislocation is larger according to the Cottrell expression (3.6) but that the time taken to attain equilibrium is determined by diffusion. Presumably in copper, the concentration value to be expected for no annealing is one associated with a very high temperature. Even though the specimen is then measured at room temperature the concentration does not change immediately to the new equilibrium value because the diffusion is too slow. If, however, the specimen is annealed at a higher temperature, diffusion is more rapid, and a higher concentration is attained. It would be expected then, that continued annealing at the same temperature, should eventually lead to no further changes in the decrement *vs* strain amplitude curve. This is what is shown in the curves corresponding to lines 12 and 13. In addition, it would be expected that a further anneal at a more elevated temperature would reverse the direction of the change in the curve shape. This could easily be checked experimentally. In the case of lead and zinc, equilibrium would not be delayed so long because of the low melting points of the metals.

The total length of dislocation line seems to be decreased by the annealing treatment. This is also understandable since some of the dislocations could be expected to diffuse out as a result of the annealing treatment. It would be of interest to know if the derived Λ would continue to decrease if the trend of the curve is reversed by annealing at higher temperatures.

The fact that the derived concentration has changed by a factor of two between 4 and 8 hour anneals will be of use later in discussing the ΔI loss and its dependence on concentration.

The next group of measurements (lines 15 to 18)

shows the effect of small compressive stresses applied for one minute on the decrement of a pure copper specimen. The copper is not quite as pure as that discussed in the preceding section, and the derived concentration value is in agreement with this fact. It would be expected that the main effect of the compressive stress would be to increase the dislocation density. Inspection of the semilog curves (Fig. 5) shows that they are all parallel and therefore have the same derived concentration C . However, the intercepts increase with the amount of cold work. Thus Λ increases with cold work, in agreement with the expected behavior. In Fig. 6, the derived Λ is plotted as a function of the applied load. It is seen that for stresses even below the yield stress (≈ 210 psi)^{13,¶} the derived dislocation density increases rapidly, by a factor of ten or more. This shows that ultrasonic measurements are much more sensitive to the earlier stages of plastic deformation than are stress-strain measurements. This is understandable, since in stress-strain curves, plastic deformation can be observed only if the plastic strain is not

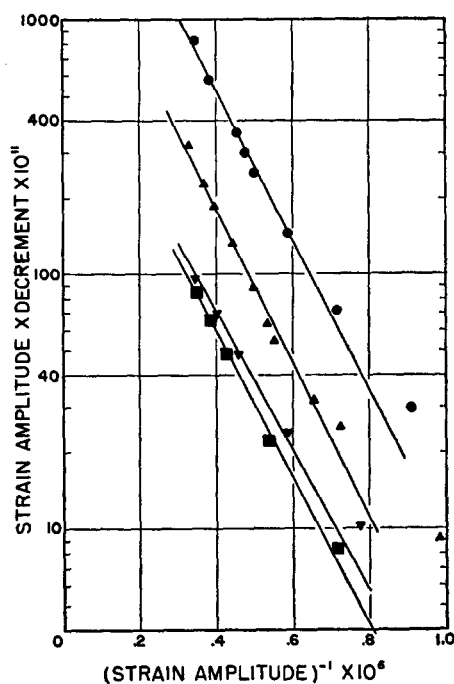


FIG. 5. Measurements made by T. A. Read of the decrement of 99.998% pure copper single crystal after applied compressive loads of zero, 60, 120, and 150 psi are here plotted by the method of the present theory. The lowest curve is that for no applied stress, while the higher curves correspond to the successively larger loads. According to the theory, it would be expected in this case that the slopes of the successive curves would not change much, but that the intercepts which are proportional to the dislocation density, would increase.

¹³ F. D. Rosi, Trans. Am. Inst. Mining Met. Engrs. **200**, 1009 (1954).

¶ The yield stress for this purity specimen as determined by Rosi is about 96 psi. Using a typical value of 0.46 for the resolved shear stress factor one obtains 208 psi for the unresolved yield stress.

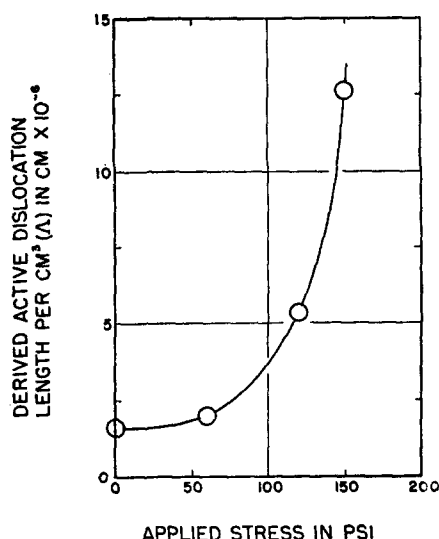


FIG. 6. Here, the dislocation densities derived from Fig. 5 are plotted as a function of the applied load. The derived dislocation density increases very rapidly before any measurable strain occurs.

too much smaller than the elastic strain, whereas the damping is a measure of the plastic strain alone.

In line 19 the results are given for the decrement of a high purity copper specimen. To check the theoretical finding that the modulus change should have the same functional relationship with the same slope as that for the decrement [Eqs. (2.1) and (2.3)], the data for the frequency change is given in line 20. The fact that the derived slope is the same for both kinds of measurements and that the intercept is smaller for the frequency change shows that the relationship is the same for the dependence of both quantities on the strain amplitude and that the ratio is independent of strain amplitude and of magnitude 4.3. (The ratio of 8.6 obtained from the table becomes 4.3 when one converts frequency changes to modulus changes.)

In lines 21 and 22, data obtained on high-purity copper at 33°C and -49°C is used. The range of strain amplitude used in this experiment is very small and corresponds with only the very early stages of the curves on copper used in lines 15-18 and 19 and 20. Thus it could be expected that the points would deviate from a straight line, and this is the case. Fitting the points, however, with a straight line yields very low values for the concentration, in accord with the measured purity. The fact that the sample is a high-purity one also means that deviations could be expected from the straight line plot, since approximations used for the theory (see part I) do not hold in the limit of pure materials. The concentration on the dislocation line which should be higher at the lower temperature, turns out to be nearly independent of temperature (column 16) probably because there is not enough time for diffusion at these low temperatures.

In lines 23 and 24 measurements showing the effect

of annealing on aluminum are given (see also Figs. 1 and 2). Before annealing, the decrement was nearly constant until a strain amplitude of about 15×10^{-7} was reached. After this it rose swiftly. After annealing, the decrement was constant over the entire measured range so that an analysis of the strain amplitude data was not possible. However, one can deduce that the concentration on the dislocation lines must have increased. This is the only case for which the dislocation density was measured. As determined by x-ray line width measurements, the dislocation density was found to be $5 \times 10^6 \text{ cm}^{-2}$ or less. The value derived from the data in column 21 is $6 \times 10^6 \text{ cm}^{-2}$. The closeness of the agreement obtained in this case is remarkable but not necessarily significant in view of the fact that the intercepts, and consequently the derived dislocation densities, are the least accurately determined quantities from the analysis.

For a 99.99% pure zinc specimen, the results of an anneal for one hour at 330°C are shown in lines 25 and 26. It may be seen first, that the derived values of the concentration of impurities on the dislocation lines are an order of magnitude higher than those found for the 99.999% samples, as would be expected simply from the impurity content. Further, the effect of the anneal has been to increase the concentration of impurities at the dislocations by almost a factor of nine, while the derived dislocation density has not changed significantly. In zinc, decrement measurements are not reversible with respect to strain amplitude even for moderate strain amplitudes. This indicates that the stress amplitude of the sound wave is already large enough to create dislocation lines. In fact it was shown in the paper by Swift and Richardson that a static stress of 1.8 g per mm² applied for one minute is large enough to produce an appreciable effect on the measured decrement. The stress amplitudes used in this experiment were up to 200 times this value. This effect complicates the measurements, since part of the loss is due to the motion of the newly created loops. No amount of care apparently is sufficient to avoid cold working the specimens through handling. However, after mounting a specimen, a recovery effect is observed. That is, the decrement usually decreases with time as the zinc anneals itself at room temperature.

In lines 27, 28, and 29, the effect of annealing at room temperature is listed. Line 27 gives the results of measurements obtained immediately after the specimen was mounted. Usually, for measurements in zinc, the sample is allowed to stand at room temperature for a couple of days before measurements are taken. The derived concentration corresponds to that found in line 25 for a sample of similar purity with no annealing at high temperature. However the derived dislocation density is huge and may be due to the handling. After 53 hours at room temperature, the derived concentration has more than doubled. After 98 hours, it has increased a little more, and no further change was ob-

served for longer times. The dislocation density appears to have decreased by a large amount as a result of the standing at room temperature.

Strain amplitude measurements made as a function of annealing time (lines 10, 11, 23, 24, 27, 28, 29) suggest a way in which damping measurements may be connected with diffusion measurements. Although the equilibrium of atoms on a dislocation line should be determined by the Cottrell force, the rate at which they migrate to the dislocation should be controlled by the diffusion rate. For atoms interacting elastically with an edge dislocation, Cottrell and Bilby¹⁴ showed that the number $c(t)$ which migrate to unit length of it in time t , from a solution containing initially c_0 solute atoms per unit volume, is given by

$$c(t) = 3 \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \left[\frac{(4Gae'r_0^3)}{kT} D_1 t \right]^{\frac{1}{2}} c_0, \quad (3.7)$$

where r_0 is the radius of the solvent atom, D_1 the diffusion coefficient, kT has its usual meaning, and the rest have already been defined. Since the slopes found from the data analysis are directly proportional to the concentration, measurements taken from the slopes should follow the $t^{\frac{1}{2}}$ law. It should be possible to arrange experimental conditions so that this connection could be quantitatively checked by decrement measurements.

Next, the effect of orientation is investigated in lines 30, 31, 32, and 33. Four zinc crystals of 99.999% purity with four different orientations were prepared in the same way. Measurements were made of the decrement and frequency change as a function of strain amplitude. The ratios of the decrement to the frequency change was found to be a constant independent of strain amplitude. Since it was inconvenient to replot the decrement curves, the slope and intercept of the frequency change curves were found. Because the ratio is constant, the slope must be the same as that for the decrement data. The corresponding intercepts for the decrement curves must be given by the constant times the value of the intercept obtained from the frequency change curves. The data obtained in this way are listed in the table. Since the samples have all been grown from the same material and have been given similar treatment, one would expect that the concentration of impurities and the dislocation density might not vary by too much from sample to sample. Then the differences from specimen to specimen should be mostly determined by the orientation, which enters through the resolved shear stress factor R and the Young's modulus E . With respect to the slope measurements, the only factor which is orientation dependent is K which is inversely proportional to $R(\theta) \times E(\theta)$. Here θ is the angle between the specimen axis and the hexagonal axis. The resolved shear stress factor $R(\theta) = \sin\theta \cos\theta$ gives the resolved shear stress on the slip plane corresponding to a given

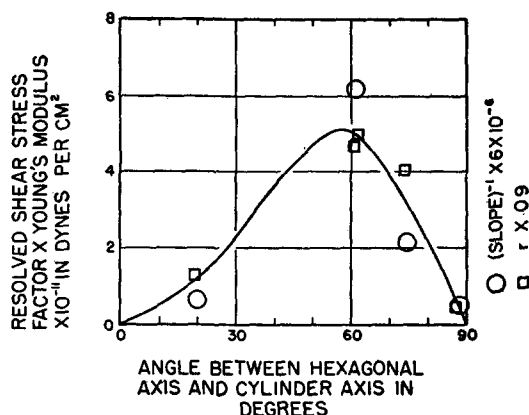


FIG. 7. The solid curve shows the dependence of the product of the resolved shear stress factor and the Young's modulus of zinc as a function of angle. The slopes found by the method of the present theory when applied to the data of T. A. Read are shown by circled points, while the ratios of the decrement to the modulus change are shown by the squares.

applied longitudinal stress. Further resolution into the slip direction has not been made but this only yields a factor between 0.866 and one. In any case, only the angle θ is reported in the experiments. When the product of the resolved shear stress factor and the Young's modulus is plotted against θ , the curve of Fig. 7 is obtained. If the inverse of the slopes is plotted on this same scale, the points shown in the figure are obtained. The agreement between the shape of the curve and the location of the points is considered to be good, considering the fact that some variation in the concentration of impurity could be expected from sample to sample. If now, the value of K calculated from the relation $K = G/[50R(\theta)E(\theta)]$ is used, the values of concentration found in column 22 are obtained. These are seen to vary by a factor of less than three. Since the slopes varied by a factor of fourteen before the orientation was taken into account, there can be little doubt that the orientation dependence of the slopes is confirmed. For example, for the 61.3° and 88° specimens, the discrepancy in the slopes is a factor of fourteen, while the discrepancy in the derived concentrations is a factor of 1.2. The derived values of the dislocation density vary by larger amounts, but here again, the discrepancy in the intercepts is greatly reduced by taking into account the orientation effects. In the case of the 61.3° and 88° specimens, the discrepancy is reduced by a factor of nearly two thousand.

When we look at the orientation dependence of the ratio

$$r = \Delta/\Delta E/E,$$

a difficulty arises. As has been mentioned earlier, no dependence of r on orientation is expected. If, however, the ratio r is plotted against orientation, we find that the experimental points follow the shape of the curve for the resolved shear stress factor times the Young's modulus even better than do the inverse

¹⁴ A. H. Cottrell and B. A. Bilby, Proc. Phys. Soc. (London) **A62**, 49 (1949).

slopes. This is shown in Fig. 7. The explanation for this discrepancy has not been found. It should be noticed that the expected result of r independent of orientation does not depend on the details of the model, but only on the assumption that the elastic after-effect which produces the damping is caused only by slip in the base plane. This makes the discrepancy much more serious. Since there is only one set of measurements available, additional measurements of this type should be made.

Lines 34 and 35 show, that for a high-purity zinc specimen at two different frequencies the same slopes, but different intercepts (and therefore different dislocation densities) result. This is evidence for the suggestion of Nowick already mentioned that since different parts of the specimen are excited at different frequencies, the measured values might correspond to slightly different dislocation densities. The derived value of the concentration for this specimen is comparable with that found for the high purity specimens in lines 30–33, and an order of magnitude lower than the values found for 99.99% pure specimens in lines 25, 26, and 27, 28, and 29, as would be expected.

Finally, some measurements made on rock salt are shown in lines 36 and 37 as an example of measurements in ionic crystals. In these experiments Frankl found that the internal friction of rock salt crystals is markedly decreased by mild x-irradiation. This he interprets as the pinning of dislocations by cation vacancies liberated from vacancy pairs on the trapping of photoelectrons. It is not known to what extent the theory could be expected to be applicable for ionic crystals. Just as a matter of interest, however, the rock salt data is treated in the same way as the metal data to see what results are obtained. For a sample which was measured just after being air-cooled from 600°C and then again after a 200°C anneal, the derived slope and therefore the concentration are seen to decrease. This is the opposite from what has been found for the case of metals when annealed. However, if the pinning is done by vacancies, then this result can be understood since the number of vacancies in the lattice at 200°C should be less than the number at 600°C.

3. Discussion of the Strain Amplitude Independent Loss

Turning our attention now to column 13, we consider the possibility of accounting for the initial loss Δ_I by attributing it to the frictional loss associated with the dynamic motion of the dislocation loops. If in Eq. (2.2) the known values of Δ_0 , ω , d , and η are now used, then the values found in column 13 are obtained and the column should be headed by $\Delta B(5/L_e)^4$, or ΔBL_e^4 where L_e , the effective length, is in this case $(3.3L_e)$. It would be tempting to use here the values of Λ and L_e derived from the strain amplitude dependent data, but doing

this leads to difficulties or contradictions,** two of which are mentioned here.

First, one obtains values of B which seem to be in most cases much too large. Leibfried¹⁵ and Eshelby¹⁶ have tried to estimate the value of the damping constant. The largest values (of order of magnitude of 10^{-4}) are given by Leibfried's theory. But it is still not clear whether or not the mechanisms so far proposed are really the main sources of damping. The values of B derived from column 19 using the above Λ and L_e are in almost all cases too large by several powers of ten. In addition, the value of B found should always be the same for a given metal.

Also, when the above values of Λ and L_e are used a discrepancy between this theory and experiment shows up in the annealing experiments. For example, the loop length is derived to decrease by a factor of two between line 10 and 11, and Λ decreases by a factor of 6.3. Thus Δ_I would be expected to decrease by $2^4 \times 6.3$ or about a hundred if Δ_I were determined by the motion of the same dislocation loops which contribute to the strain amplitude dependent loss. But Δ_I decreases only by a factor of two.

It is therefore concluded that the initial loss Δ_I cannot be ascribed to the frictional loss associated with the motion of the same loops which contribute to the strain amplitude dependent loss.

There are also some difficulties with Δ_I from the experimental point of view. The measurements are not always consistent. For example, while it is generally believed that the purest specimens give the largest decrements, and this behavior appears to be borne out in most cases by the results in column 9, still, there are examples for which this is not true. For the 99.999% zinc in lines 30–33, the decrement is an order of magnitude less than the 99.99% zinc in lines 27–29.

Moreover, it is not clear how well the vibration of the specimen can be described by the ordinarily assumed simple one-dimensional longitudinal standing wave. J. Adem¹⁷ has analyzed the wave propagation in an isotropic elastic circular semi-infinite bar on the end of which a harmonically varying displacement is prescribed. He finds that only in the case of very low frequencies is a single longitudinal wave propagated. With higher frequency an increasing number of waves have to be considered, and the displacement vector of each wave varies in amplitude, direction, and sense over a given cross section of the specimen. The results indicate that the stress distribution in the specimens may be more complex than that which has been assumed. The stress distribution found by Adem for the special case of 10 Mc/sec is extremely complicated and

** Other difficulties may be found in the report or thesis mentioned earlier.

¹⁵ G. Leibfried, *Z. Physik* **127**, 344 (1950).

¹⁶ J. D. Eshelby, *Proc. Roy. Soc. (London)* **A197**, 396 (1949).

¹⁷ J. Adem, *Quart. Appl. Math.* **12**, 261 (1954).

shows that such measurements¹⁸ should be interpreted with caution. In addition, for a nonisotropic specimen, other modes of vibration will appear on reflection of the waves from free surfaces. A more detailed analysis of the vibration behavior of the specimens of the type used in the experiments is badly needed. However it is not expected that these difficulties will be sufficient to account for all the discrepancies already cited.

We next consider some possibilities of accounting for the initial loss. The fact that the initial loss is not to be associated with the dynamic loss for only the dislocation loops which participate in the breakaway process suggests that this loss may be due to the motion of screw type dislocations. A dislocation which is part screw type would be difficult to pin and for a pure screw, no pinning by impurities would take place. However, the measurements of column 9 suggests that impurities are influential in determining Δ_I because, for most cases, the highest purity samples have the highest initial decrements. Still, it is not possible to say from these measurements how effective the impurity particles are since, in addition to the discrepancies of the type already mentioned for zinc, the dislocation density varies at the same time, as for example in the annealing experiments. Evidence that screw dislocation motion cannot be the main part of this loss is perhaps more clearly given by the experiments of Bradfield and Pursey¹⁹ and Marx and Koehler.⁸ Bradfield and Pursey have shown that small amounts of impurities in copper can produce significant changes in the dynamically measured modulus. Marx and Koehler show that even when prolonged annealing in a vacuum of a copper specimen produces no further decreases in the decrement, further annealing in a hydrogen atmosphere leads to an additional reduction in the initial decrement.

The etching experiments of Marx and Koehler⁶ are of interest in a discussion of the initial loss. They found that with no exception, etching of unannealed specimens of copper and lead reduced the initial loss by amounts comparable to those obtained by annealing. The major part of this loss occurs even before 5% of the mass of the specimen is etched away. They point out that the surface of the specimen may be deformed. The copper crystals used were grown in graphite crucibles by the Bridgman technique. On cooling from the melting temperature, the copper contracts more than the graphite. Thus small pits in the crucible which are filled with molten copper provide a means for cold-working the specimen. If such effects are possible, then quantitative agreements would not be expected. The effect of such cold-working should be to produce many long loops, since under stress, the longest Frank-Read sources operate first.

It should next be asked, if using the different values

of Λ and L_s from those used in the strain amplitude dependent loss, reasonable values of the damping constant can be attained. To see this, we write

$$\Delta_I^* = \Omega \Delta_0 \Lambda B (120 L^4),$$

$$\text{and} \quad \left(\frac{\Delta E}{E} \right)_I = \Omega \Delta_0 \Lambda \frac{(6 L^2)}{\pi}, \quad (3.8)$$

where an exponential distribution of loop lengths with average length L has been assumed. Δ_I^* is the reduced value of the decrement in column 13. Combining these equations, we may write

$$BL^2 = \frac{\Delta_I^*}{(\Delta E/E)_I} \cdot \frac{1}{20\pi}.$$

For high-purity copper, with the typical values of $\Delta_I^* = 2 \times 10^{-11}$ and $(\Delta E/E) = 6 \times 10^{-2}$,¹¹ one obtains for $L = 10^{-5}$, 3.2×10^{-5} , 10^{-4} , 3.2×10^{-4} , $B = 5 \times 10^{-2}$, 5×10^{-3} , 5×10^{-4} , or 5×10^{-5} , respectively. Loop lengths of order 10^{-4} and larger are not inconsistent with yield stress data, and so it cannot be concluded from the data so far available that there is no possibility of associating the initial loss with the dynamic motion of dislocation loops.

Measurements in the megacycle range for extremely high purity materials could throw light on this question. For the values $L = 10^{-4}$ and $B = 10^{-4}$ used above, a maximum in the internal friction *versus* frequency near 40 megacycles per sec would be expected. The existence or nonexistence of such a maximum would yield information on this question. Such measurements have not yet been reported.

Since it cannot be concluded on the basis of existing measurements that the initial loss is not due to the frictional dynamic motion of dislocations, it will be for the moment assumed that the initial loss is to be accounted for in column 13 by a term $\Lambda L_s^4 B$, where the parameters Λ and L_s are different from those used for the static loss.

Although it is not possible to check this formula quantitatively with present experiments, some qualitative features will be noted.

As was mentioned earlier, reliable frequency measurements cannot be made at present in the kilocycle region. The decrement in the megacycle region increases with frequency (see next section) as would be expected from Eqs. (2.2) or (2.5). It should be mentioned, however, that measurements made in the kilocycle region, when extrapolated by means of a linear frequency relation, seem to give much larger decrements than those which are observed in the megacycle region. In the megacycle region measurements have not yet been made on very high purity specimens, except for germanium. Unfortunately, in this case, no kilocycle results have been reported. To make progress on this question, measurements should be made on the same specimens in both the kilocycle and megacycle regions.

¹⁸ W. P. Mason and H. J. McSkimin, J. Acoust. Soc. Am. **19**, 464 (1947).

¹⁹ G. Bradfield and H. Pursey, Phil. Mag. **44**, 437 (1953).

In lines 1–3 for lead and lines 21–22 for copper it may be seen that the initial decrement is temperature dependent. Several reasons are possible for it; for example, evaporation of impurities from dislocations, sensitivity of the constant B to temperature, or annealing out of dislocation lines.

The decrement appears to decrease in all cases on annealing. In the case of high purity copper (lines 9–13), the decrement seems to approach a limiting value after a vacuum anneal of about 15 hours at 500°C. We are not aware of experiments in which the annealing effects have been reversed, that is, of experiments for which later anneals were at higher temperatures. Both Δ and the loop length L_e are expected to be affected by annealing times and temperatures.

For small amounts of cold work (lines 15–18) the initial value of the decrement increases. Here Δ is expected to increase and the average length of loop may also change.

For all other parameters held constant, Δ_I should be strongly dependent on orientation. For the case of Zn (lines 30–33), Δ_I should be proportional to $R^2(\theta) \times E(\theta)$. Since this variation is not observed, the suspicion is increased that in this case Δ_I is not due to dislocation motion. It should be pointed out here that this orientation dependence would be expected whatever the model is, so long as the nonelastic shear strain is proportional to the resolved shear stress in the slip plane. However, for these measurements, Δ_I is so small it is difficult to come to any definite conclusions.

In addition, there are some other observations which can be made from column 9 which may have a meaning. Koehler²⁰ has already pointed out that the most anisotropic materials seem to have the largest initial decrements. Inspection of column 9 bears this out and reveals in this connection another property. The most anisotropic materials also show the widest variations of the initial value of the decrement.

4. MEGACYCLE REGION

Recently, in a number of papers, results of damping measurements in the megacycle region have been reported using the pulse method.²¹ The possibility of obtaining useful information concerning dislocation be-

TABLE II. Expected values for an effective loop length (L_e) of 10^{-4} cm and for a damping constant (B) of 5×10^{-4} gram $\text{sec}^{-1} \text{cm}^{-1}$.

Material	Resonant frequency f_0 (Mc/sec)	Frequency for maximum decrement f_m (Mc/sec)	Attenuation at f_m α' (decibels/microsecond)
Lead	200	22	1.9
Copper	625	89	7.8
Aluminum	875	66.5	5.8
Zinc	640	81	7.05
Germanium	640	51	4.5

²⁰ J. S. Koehler, Tech. Rept. No. 2, Ct. DA-11-022-ORD-212, Project No. TB2-0001(756), 12/1/53.

²¹ R. L. Roderick and R. Truell, J. Appl. Phys. **23**, 267 (1952).

havior from these measurements seems largely to have been overlooked, in spite of some noteworthy advantages which these measurements have over kilocycle measurements. For example measurements of frequency dependence, which can be crucial in distinguishing between possible theories, are now easily made in the range from 3 to 300 megacycles/sec. Also, studies of the influence of orientation by the use of shear waves with various polarizations are easily made.

A. Expected Results

For high-frequency measurements, formulas (2.5) and (2.6) apply. These values of Δ and $\Delta E/E$ as functions of frequency are discussed and shown in part one (Figs. 4 and 5). The decrement has a maximum at $\omega = \omega_0$ for small damping, and at $\omega = \omega_0^2/d$ for large damping (i.e., $d/\omega_0 > 1$). The modulus is constant for low frequencies but goes to zero as the frequency approaches the resonant frequency ω_0 . For extremely pure materials the loop lengths of dislocations should be given by the network lengths (lengths of the Frank Read source). In Table II the expected resonant frequency $f_0 = \omega_0/2\pi$ is given for several materials if for this length a value of 10^{-4} cm is chosen. The value of f_0 for lead is considerably lower than those for the others and is in the presently measurable range. Thus, megacycle measurements in high purity lead should reveal the maximum in the decrement *versus* frequency relation.^{††}

The frequency f_m for which the decrement is a maximum may be smaller if the damping constant is large enough. For a damping constant $B = 5 \times 10^{-4}$ g $\text{cm}^{-1} \text{sec}^{-1}$, calculated values of f_m are given in the table. The value of the decrement at the maximum given by (2.2) is

$$\Delta(f_m) = \frac{\Omega \Delta_0 \Delta L_e^2}{2} \quad (4.1)$$

which corresponds to the attenuation $\alpha' = f\Delta$ given in Table II, if one uses the values $B = 5 \cdot 10^{-4}$, $R^2 \Delta_0 = 0.2$, $\Delta = 10^7$ cm/cm², and $L_e = 10^4$ cm. These attenuations are not so large that they cannot be measured by the pulse technique. In the case of germanium, which is discussed in the next section, measurements made on extremely high purity specimens reveal no maximum up to 300 Mc/sec. This can mean that the value of $L_e = 10^{-4}$ cm is too large, or that if $L_e = 10^{-4}$ cm, the damping value of 5×10^{-4} is too large.

Contributions to the expected damping constant from two sources have been calculated by Eshelby¹⁶ and Leibfried.¹⁵ Eshelby considers the loss due to the irreversible heat flow which occurs in the neighborhood of an oscillating dislocation. Leibfried estimates the loss associated with the interaction of a moving dislocation with thermally excited sound waves. Nabarro²² has

^{††} Since in the formulas the elastic anisotropy is not considered, these values are only approximate.

²² F. R. N. Nabarro, Proc. Roy. Soc. (London) **209A**, 278 (1951).

shown that Leibfried's result is correct in form and order of magnitude, but numerically too high.

According to Eshelby's theory, the damping constant B should be given by

$$B = \beta G a^2 \quad (4.2)$$

where

$$\beta = \frac{1}{10D_2} \frac{C_p - C_v}{C_p} \log \frac{D_2}{\omega l^2},$$

and D_2 is the thermal diffusivity, C_p the specific heat, and l the "cut-off" length ($\approx 10^{-7}$ cm).

Leibfried's formula for the damping constant may be written as

$$B = \frac{a}{10C_T} \frac{3kT}{a^3} Z, \quad (4.3)$$

where a is the lattice constant, C_T is the shear wave velocity, k is Boltzmann's constant, T is the temperature, and Z is the number of atoms per elementary cell.

Values of B found from each of these formulas are given in Table III for several metals, where the frequency used for Eshelby's formula is 30 Mc/sec (for 30 kc/sec. Eshelby's result would be very slightly larger.) Leibfried's formula gives larger values than Eshelby's.^{††}

In addition the value of the parameter,

$$D = \omega_0 / B / A, \quad (4.4)$$

used for the curves in Fig. 4, part I, for both formulas is found. The calculated value of ω_0 used is that corresponding to an effective loop length of 10^{-4} cm. It should be noticed that the value D obtained from Eq. (4.4) and (4.3) is always smaller than one. Therefore, no strong resonance effects would be expected. It should be mentioned again that the assumptions made which lead to the result of Leibfried and Nabarro are still somewhat questionable, and the possibility that different mechanisms not considered here will be the main cause of the damping cannot be excluded.

The expected values of the resonant frequency and damping indicate that presently available megacycle measurements should be near but on the low-frequency side of the maximum in the decrement *versus* frequency relation. For frequencies much less than that at which the maximum occurs, one may use Eqs. (2.2) and (2.4).

Since decrement is related to attenuation by means of the relation

$$\alpha = \frac{f}{c} \Delta, \quad (4.5)$$

and the decrement increases with the first power of the frequency, the attenuation would be expected to increase with the square of the frequency in this range.

^{††} Koehler has stated¹⁰ that both theories give a value of B of about $5 \cdot 10^{-6}$ gram sec $^{-1}$ cm $^{-1}$ in copper.

TABLE III. Some calculated damping constants.

Material	Eshelby		Leibfried	
	$B \times 10^4$	D	$B \times 10^4$	D
Lead	0.033	17	2.8	0.20
Copper	0.014	50	1.6	0.44
Aluminum	0.98	0.39
Zinc	0.93	0.69
Germanium	0.0034	120	1.3	0.31

Expected orientation effects are especially interesting for megacycle measurements. With the techniques used for these high-frequency measurements, as distinguished from those used in the kilocycle range, it is possible to propagate pure modes of vibration in several directions. For example in cubic crystals, one can propagate pure longitudinal waves in each of the 100, 110, and 111 directions, pure shear waves with arbitrary polarizations in each of the 100 and 111 directions, and pure shear waves in the 110 direction with polarizations parallel to the 100 or another 110 direction.²³ The decrement and change in modulus for a given orientation depend on the value of the appropriate modulus for that direction and polarization, the resolved shear stress on each of the slip systems corresponding to the stress distribution in the specimen, and on the distribution of the dislocations over the various slip systems. Thus the orientation dependence is, in general, different for the different types of waves. This provides a means for directly checking the theory. Results of such considerations will be presented at a later date.

Effects due to dislocation motion, while directly affecting the attenuation of a wave, produce only second-order changes in the velocities, or differences between the velocities due to elastic strains and velocities due to elastic and dislocation strains. Nevertheless, the accuracies claimed for velocity measurements are such that effects due to dislocation motion should be observable in many cases. It has already been mentioned, that, in the case of cubic crystals, seven pure modes of vibration can be propagated. The velocities are directly related to the elastic constants C_{11} , C_{12} , and C_{44} . Thus one can obtain seven equations to determine the three elastic constants. The values of the elastic constants determined using different sets of velocity measurements should differ by an amount corresponding to the dislocation strain. Since the orientation factor can be calculated, and the only unknown factor is ΔL^2 which is common to all of the measurements, it should be possible to determine this factor using high accuracy velocity measurements and more than three measured values.

In this connection a general criticism of most velocity measurements made so far seems to be necessary. The pulse technique for measuring velocities in metals in the megacycle range has been used extensively in recent

²³ Sakadi, Proc. Phys. Math. Soc. Japan 23, 539 (1941). See also S. Levy and R. Truell, Revs. Modern Phys. 25, 145 (1953).

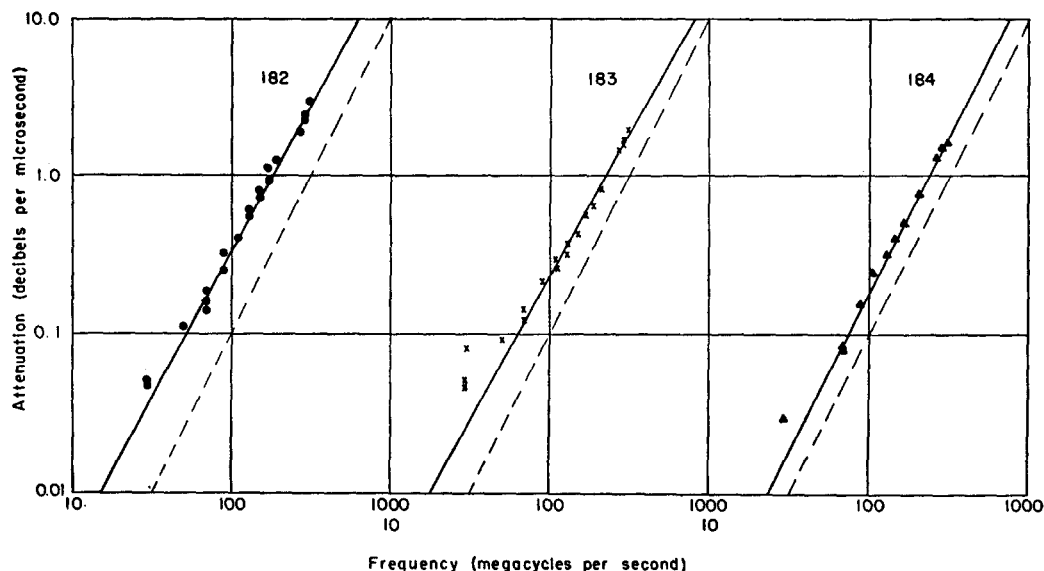


Fig. 8. The attenuation of sound for three germanium specimens²⁶ is here shown as a function of frequency. The dashed line is a reference line with a slope of two. According to the present theory, the slope should be two for frequencies much below the frequency at which the decrement is a maximum, and become less than two as the maximum is approached.

years. As has just been pointed out, however, elastic constant measurements made in this way may not necessarily be comparable unless the wave propagation directions and polarizations are the same. In addition, other conditions describing the state of the specimens which should be described are usually not mentioned in reporting such work. The experiments of Bradfield and Pursey¹⁹ show that in very pure unannealed copper, an anomaly in the measured modulus exists such that the measured value may be a couple of percent less than the expected value. Also, measurements made by Marx and Koehler⁶ of the modulus of copper as a function of annealing temperature and duration indicate that variations of nearly ten percent can result. Thus impurity content, orientation, and annealing treatment of specimens used for these measurements should be specified.

B. Data Reviewed

A great deal of what has been said in the previous section about expected results for the decrement and modulus changes in the megacycle region anticipates results of experiments now underway but in most cases still incomplete. Although velocity measurements in single crystals have been widely reported, attenuation measurements have not. This is partly due to the difficulty in evaluating sources of errors in these measurements. Results for which some of the difficulties have been overcome have recently been made, however, and will be discussed here.

Attenuation measurements were made by Alers²⁴ at 7.8 megacycles per second on zinc single crystals as a

²⁴ G. Alers, *Phys. Rev.* **97**, 863 (1955).

function of time while the crystals were being sheared. The propagation direction was normal to the slip planes. It was found that compressional waves, which have no resolved shear stress component in the slip planes, were not affected by the deformation, whereas the attenuation of the shear waves was markedly affected. When a longitudinal wave was propagated in a direction making a small angle with the normal to the slip planes, the attenuation was slightly affected by the deformation. These results show conclusively that dislocation motion produces a considerable damping in the megacycle region. Recovery effects were observed after the shearing load was removed. The frequency dependence of the loss was not measured.

Additional evidence which shows that megacycle sound absorption is sensitive to dislocation motion is provided by the measurements of Hikata,²⁵ who showed that the attenuation at 5 Mc/sec in mild steel and copper increases sharply when the specimens are stressed in the neighborhood of the yield point.

Frequency dependence measurements have been made on high purity germanium samples in this laboratory²⁶ over the range from 10 to 300 megacycles per second. Measurements for 3 specimens are shown in Fig. 8. It was found that the attenuation could be represented by a relation of the form

$$\alpha = \text{const} \times f^n,$$

where n had a value close to the value of 2 as predicted by Eqs. (2.2) and (4.5). In some specimens values for n

²⁵ A. Hikata, *Govt. Mech. Lab. Bull.* **I**, 1 (1954).

²⁶ Granato, Lücke, and Truell, Fourth Quarterly Report, Contract No. DA-36-039 SC-5263, Department of the Army Project No. 3-99-022, Signal Corps Project No. 112 B, June, 1954.

TABLE IV. Megacycle attenuation data.

Material	Reference	Frequency Mc/sec	Impurity concentration	Attenuation db/ μ sec	Decrement $\times 10^4$	Type of wave	Direction of propagation	$\Delta L/B \times 10^{18}$
Cu	30	50	10^{-2}	0.3	6.9	compr.	110	0.62
Cu	30	45	2×10^{-4}	0.50 ^a	13	compr.	100	1.3
				0.17 ^b	4.35	compr.	100	0.43
Al	31	50	10^{-4}	0.13	3	compr.	110	0.20
Zn	24	7.8	10^{-4}	0.16 ^c	24	shear	hex axis	2.5
Zn	32	50		1.8	42	shear	hex axis	3.4
NaCl	32	70		0.3	4.9	compr.	100	0.26
KCl	32	100		0.4	4.6	compr.	100	0.17
Ge	26	100	$< 10^{-5}$	0.30	3.5	compr.	100	0.09
		100	$< 10^{-9}$	0.20	2.3	compr.	100	0.06
Si	33	100		0.2	2.3	compr.	100	0.06
		100		0.15	1.7	compr.	110	0.04

^a As grown.^b Annealed.^c Only the increase in decrement due to an applied load of 37 g/mm² is given.

slightly smaller than 2 have been found. According to the present theory this would indicate that the maximum of the decrement is being approached. No variation in the velocity as large as one percent was found either from sample to sample, or over the frequency range in any given sample.

There are three unknowns in the theory, ΔL , and B . The dislocation density could be measured directly. Thus, if velocity measurements of sufficient accuracy could be made, the velocity and attenuation measurements should be sufficient to determine the damping constant B for the germanium specimens just mentioned. Such measurements are planned. A knowledge of the value of the damping constant for dislocation motion is necessary for questions in dislocation dynamics. Thus far it has not been measured directly in any way.

Although the measured loss in germanium may contain components due to other presently unknown mechanisms, if we assume for the moment that it is all due to dislocation motion, then it is possible to find bounds for the damping constant with the information now available. Because the samples are of extremely high purity, the effective loop length would be expected to be about 10^{-4} cm. Measurements of dislocation densities made on many germanium specimens by both x-ray and etch-pit counting techniques give values which lie between 10^6 and 10^8 per cm².^{27,28}

The available data, in fact, provide rather narrow bounds for the constant B . The fact that no maximum is observed in germanium up to nearly 300 megacycles means that the resonant frequency is larger than, say, 340 Mc/sec which implies $L_e < 2 \cdot 10^{-4}$ cm.

Using the measured attenuation values, and taking $\Omega\Delta_0$ to be 0.2, one can construct the normalized decrement *versus* normalized frequency curves (as in Fig. 4, part I) for different assumed values of Λ and L_e within the above limits. Comparison with the set of curves given by the theory shows that a fit is only possible for

a restricted range of damping constants. One finds in this way that $2 \times 10^{-5} \leq B \leq 15 \times 10^{-5}$ in surprisingly good agreement with the value taken from the Leibfried-Nabarro formula.

If the values of the decrement are extrapolated to 30 kc/sec, by means of a first power frequency law, then values of about 10^{-7} §§ are obtained. Measurements in the kilocycle range have not yet been reported, but preliminary results found by Kessler²⁹ indicate that the decrement for a high purity germanium specimen in the kilocycle region is less than 5×10^{-6} . It is planned to obtain both kilocycle and megacycle measurements on the same specimens.

Although germanium is so far the only material for which measurements have been made over a wide enough frequency range and with sufficient reproducibility to really determine frequency dependence, attenuation measurements have also been made on other materials, such as copper,³⁰ aluminum,³¹ zinc,³² silicon,³³ and alkali halides.³² Generally one finds that the attenuation increases with frequency. However, for these materials, one cannot use the present measured frequency dependences for checking the theory for two reasons. The first is that, in some cases, the measurements are not sufficiently reliable. The second is that it is not known to what extent other sources of loss contribute to the measured attenuation. For instance, for some materials which are much less pure than the germanium specimens mentioned, scattering by impurity particles may occur.³⁴ Also, because of scattering by the grain structure, measurements on polycrystalline materials cannot be used in most cases. For copper, aluminum, and zinc, losses of thermoelastic origin

§§ Since the measured frequency dependence of the decrement for the germanium specimens was in some cases less than a first power frequency law, extrapolation along the measured curve would lead to values larger than 10^{-7} in these cases.

²⁹ J. Kessler, private communication.

³⁰ C. F. Ying and R. Truell (private communication).

³¹ N. Pitula and K. Lücke (unpublished measurements).

³² P. Waterman (private communication).

³³ R. Truell (private communication).

³⁴ K. Lücke (unpublished).

²⁷ Vogel, Pfann, Corey, and Thomas, Phys. Rev. **90**, 489 (1953).

²⁸ S. S. Kulin and A. D. Kurtz, Acta Metallurgica **2**, 354 (1954).

TABLE V. $\Delta L^4 B = 10^{-16}$. Derived dislocation densities (Λ) for various combinations of average loop length (L) and damping constant (B).

L (cm)	Λ (cm ⁻²) for $B = 10^{-6}$	Λ (cm ⁻²) for $B = 10^{-5}$	Λ (cm ⁻²) for $B = 10^{-4}$	Λ (cm ⁻²) for $B = 10^{-3}$	Remarks
10^{-3}	10^3	10^2	10	1	Unreasonable
10^{-4}	10^7	10^6	10^5	10^4	
10^{-5}	10^{11}	10^{10}	10^9	10^8	
10^{-6}	10^{15}	10^{14}	10^{13}	10^{12}	Unreasonable

become important.³⁵ Also it is difficult to separate out the diffraction loss except in the case of the germanium measurements, where this component is easily identified from the shape of the curve. The measured attenuations are nevertheless listed in Table IV, for these materials so that an idea of the order of magnitude of the loss may be obtained.

If in Eq. (3.2), we substitute all the known quantities, then we may determine what the unknown factor $\Delta L_e^4 B$ should be for the theory to agree with these measurements.

For this an approximate value of 0.1 has been used for $\Omega\Delta_0$ except in the case of a shear wave propagating along the hexagonal axis of zinc, for which 0.5 has been used. The unknown factor $\Delta L_e^4 B$ for the metals cited in the table is generally near 10^{-18} . If the loop lengths are exponentially distributed, then the effective length L_e would be 3.3 times the average loop length. This would mean that a typical value of the factor $\Delta L^4 B$ would be near 10^{-16} . To see if this corresponds to reasonable values of the three unknown parameters, Table V has been constructed.

In this table, the value of Λ which would be necessary for various combinations of the average loop length and damping constant in order to obtain $\Delta L^4 B = 10^{-16}$ is tabulated. One sees immediately, that for the most probable values of $B(10^{-4})$ and $L(10^{-5} \text{ to } 10^{-4})$ the most reasonable values of $\Lambda(10^5 \text{ to } 10^9)$ are obtained.

Finally, although the amount of data available is quite limited, some qualitative agreements can be seen between these results and those found in the kilocycle range. The annealed copper specimen has a lower decrement than does the as received sample. Also the purest specimen seems to have a larger decrement than the less pure as received sample.

5. CONCLUSION

Theories so far proposed to account for sound damping measurements in terms of dislocation behavior have not been developed to the point where they apply for all the forms of damping behavior observed. Even in the ranges for which the proposed theories are supposed to apply, it has not been possible to check them quantitatively with the available data.

³⁵ K. Lücke, "Ultrasonic attenuation in crystals due to the thermoelastic effect," report prepared for Watertown Arsenal under Contract DA-19-020-ORD-1579, Report WAL 143/14-46 (February, 1956).

Here a theory, based on a model used by Koehler, is developed to a point where quantitative checks are possible. It is found that the same model leads to both the kind of loss found in the kilocycle region and that found in the megacycle region. The predicted results of the theory are compared in so far as is possible with the available data with respect to the dependence of the loss on the principal variables, and on many of the parameters. No serious contradictions are found, but because not all of the parameters in the theory are well known theoretically, it is not possible to say firmly that the theory agrees everywhere. It seems to be especially difficult to come to any conclusions about the initial loss in the kilocycle region. Where conclusions are possible, it is evident that a great deal of quantitative information about dislocation behavior in metals can be obtained from damping measurements.

A number of experiments are suggested by the theory which may permit stronger conclusions to be made. Many of these have been discussed in the text, but will be listed here, with a reference given to the more detailed discussion.

(a) Megacycle attenuation measurements on high purity lead should show the location and magnitude of the maximum in the decrement *versus* frequency relation (Sec. 4.A).

(b) Attenuation measurements made on 99.999% pure copper should make some conclusions possible about the initial loss in the kilocycle region (Sec. 3.D.3).

(c) Decrement measurements in both the kilocycle and megacycle regions for the same specimen are needed (Sec. 3.D.3).

(d) Orientation measurements in the megacycle region should give information useful for separating out dislocation losses from other possible losses. Also information about the dislocation distribution on various slip systems in zinc, say, might be obtained by taking measurements before and after deformation as a function of orientation (Sec. 4.A).

(e) Measurements of the absolute value of the modulus as a function of frequency may help to determine the frequency dependence of the initial loss in the kilocycle region (Sec. 4.B.1).

(f) High accuracy measurements of the elastic constants for many directions and polarizations should be checked for consistency and the possibility of determining the unknown quantity ΔL^2 (Sec. 4.A).

(g) All the measurable parameters needed in the theory should be reported for experiments so that the data may be checked more completely (Sec. 1).

(h) Orientation and deformation effects should be measured for germanium to determine what part of the loss is due to dislocation motion (Sec. 4.A).

(i) Temperature variations of decrement data are needed to give information about the damping constant (Sec. 4.A).

(j) Annealing experiments at successively higher

temperatures would check a prediction of the theory (Sec. 3.D.2).

(k) Annealing experiments at a given temperature as a function of time should provide quantitative information about the diffusion of impurities to dislocation lines (Sec. 3.D.2).

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APPENDIX. INTEGRATION OF Δ_H OVER THE SPECIMEN LENGTH

The strain amplitude dependent loss has the form

$$\Delta_H = \frac{A}{\epsilon} \exp\left[-\frac{B}{\epsilon}\right], \quad \text{where} \quad \epsilon = \epsilon_0 \cos\left(\frac{\pi x}{2L}\right),$$

the specimen is of length L , x is a coordinate measured along the length, ϵ_0 is the maximum strain amplitude in the specimen, and A and B are constants. Thus the strain amplitude for a standing wave varies along the length of the specimen, and the measured value should be the value averaged over the length that is,

$$\Delta_H = \frac{1}{L} \int_0^L \frac{A}{\epsilon_0 \cos(\pi x/2L)} \exp\left[\frac{-B}{\epsilon_0 \cos(\pi x/2L)}\right] dx.$$

Making the substitution,

$$u = 1/\cos(\pi x/2L),$$

the integral may be written in the form,

$$\Delta_H = \frac{2A}{\pi \epsilon_0} \int_1^\infty \frac{\exp(-au)}{(u^2-1)^{1/2}} du,$$

where $a = B/\epsilon_0$, or

$$\Delta_H = \frac{2A}{\pi \epsilon_0} I(a), \quad \text{where} \quad I(a) = \int_1^\infty \frac{\exp(-au)}{(u^2-1)^{1/2}} du.$$

$I(a)$ may be evaluated by differentiation with respect to the parameter a and integrating by parts to give the differential equation

$$I''(a) + (1/a)I'(a) - I(a) = 0,$$

where the primes denote differentiation with respect to a . The solution of this equation satisfying the boundary condition $I(\infty) = 0$ is the zero-order modified Bessel function of the second kind,

$$I(a) = K_0(a).$$

Thus,

$$\Delta_H = \frac{2A}{\pi \epsilon_0} K_0\left(\frac{B}{\epsilon_0}\right).$$

For the strain amplitudes used in the kilocycle measurements, the asymptotic form of this function may be used. That is,

$$K_0\left(\frac{B}{\epsilon_0}\right) \sim \left(\frac{\pi \epsilon_0}{2B}\right)^{1/2} \exp\left(-\frac{B}{\epsilon_0}\right).$$

then

$$\Delta_H = \frac{2A}{\pi \epsilon_0} \left(\frac{\pi \epsilon_0}{2B}\right)^{1/2} \exp\left(-\frac{B}{\epsilon_0}\right),$$

or

$$\Delta_H = \left(\frac{2\epsilon_0}{\pi B}\right)^{1/2} \frac{A}{\epsilon_0} \exp\left(-\frac{B}{\epsilon_0}\right).$$

Thus the effect of the integration has been to give the result obtained for a uniform strain amplitude distribution ϵ_0 times the slowly varying factor

$$(2\epsilon_0/\pi B)^{1/2}.$$

For the range of strain amplitudes used in the tables this factor varies from about $\frac{1}{3}$ to $\frac{1}{2}$. This formula therefore gives the same derived loop lengths and concentrations as the formula used in the table, but a derived dislocation density of from two to three times that obtained in the table. Since the table was made up before this result was obtained, and since it only leads to a somewhat larger dislocation density, it was not felt worthwhile to replot all the data as a $\log \epsilon_0^{1/2} \Delta_H$ versus $1/\epsilon_0$ plot. The exponential factor is clearly the most influential, and this is not changed by the integration. Spot checks showed that such plots had negligible influence on the value of the derived slopes.