Use of Artificial Bee Colonies Algorithm as Numerical Approximation of Differential Equations Solution

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Abstract. The differential equation is one of the branches in mathematics which is closely related to human life problems. Some problems that occur in our life can be modeled into differential equations as well as systems of differential equations such as the *Lotka-Volterra* model and SIR model. Therefore, solving a problem of differential equations is very important. Some differential equations are difficult to solve, so numerical methods are needed to solve that problems. Some numerical methods for solving differential equations that have been widely used are Euler Method, Heun Method, Runge-Kutta and others. However, some of these methods still have some restrictions that cause the method cannot be used to solve more complex problems such as an evaluation interval that we cannot change freely. New methods are needed to improve that problems. One of the method that can be used is the artificial bees colony algorithm. This algorithm is one of metaheuristic algorithm method, which can come out from local search space and do exploration in solution search space so that will get better solution than other method.

INTRODUCTION

In a formalist view, "Mathematics is the study of abstract structures defined axiomatically by symbolic logic and mathematical nitation". Whereas in general, "Mathematics is asserted as a pattern study of a structure, change and space". The specific structures investigated by mathematics often come from natural sciences including biology, but the most common comes from physics.

In its development, advanced mathematics has used as a tool for studying various complex physical phenomena, especially the various observed natural phenomena so that the structural patterns, changes in space and the properties of the phenomena can be approached or expressed in a systematic and full formulation of conventions, symbols and notations. The results of the formulation describing the behavior and the process of physical phenomena scientifically led to the relationship between the quantity and the rate of change, then built the calculus, which is specifically the topic discussed in differential equations.

The differential equation was originally called "derived equation" which is the equation introduced by Leibniz in 1676. By definition, "differential equations are equations concerning derivatives of one or more independent variables to one or more independent variables". Furthermore known system of differential equations which is a combination of n pieces of differential equations.

The growing knowledge of mathematics causes the problems to be solved more complex. Some problems are difficult to solve by exact methods, so we need to develop some numerical methods to approach the solution of the problem. With so many numerical methods to approach differential equations that we already know, the authors are inspired to look for new methods that are expected to produce smaller error values and can use for more general problems. In this case, the author tries to use Artificial Bee Colonies Algorithm that has been proven to use in solving the problem of system optimization of linear equations and approaching the roots of complex functions. In addition, the author tries to use this algorithm in solve optimization of differential equations, looking for solutions of differential equations and solutions of some problems that are modeling on the system of differential equations. The author's interest in Artificial Bee Colonies algorithm is because this method classified as a new metaheuristic method that developed in the last 10 years. Metaheuristic methods generally perform better than simple heuristic methods.

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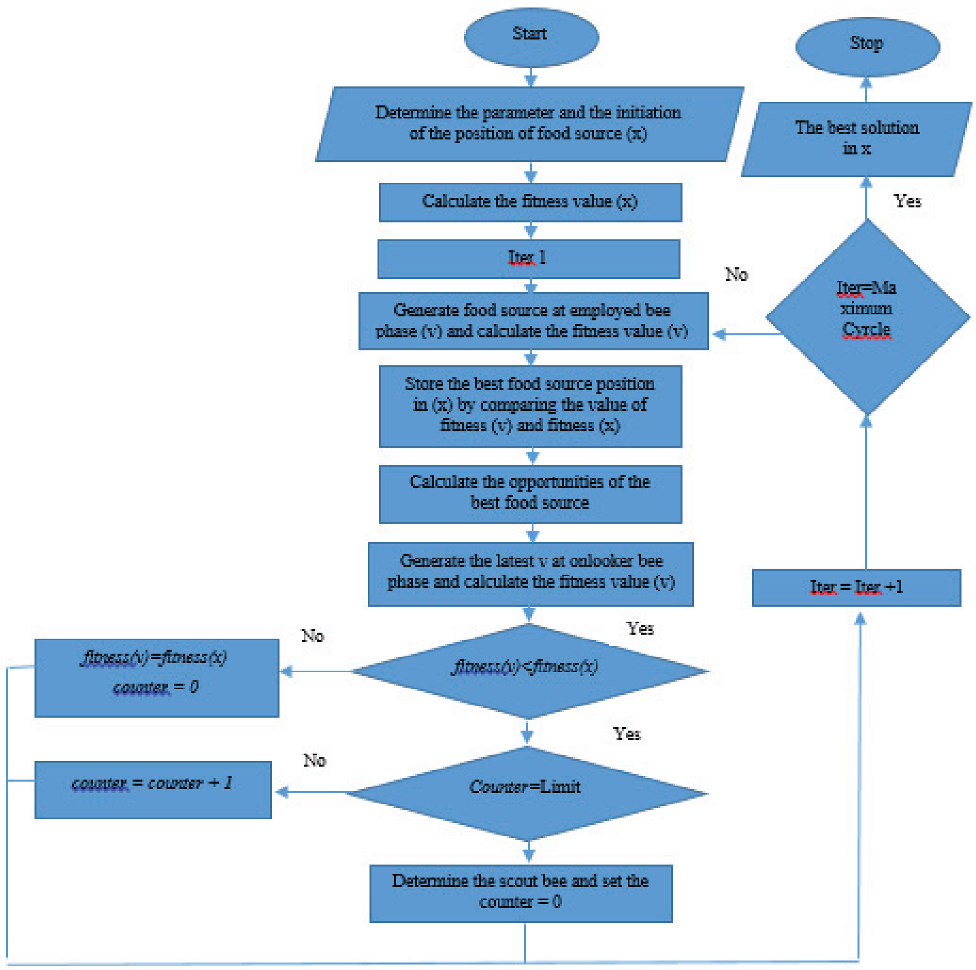
ARTIFICIAL BEE COLONIES ALGORITHM

The general scheme of the Artificial Bee Colonies Algorithm described in the following stages:



Scout Bees will search for a random food source with this formula:

$$x_{mi} = l_i + rand(0,1) * (u_i - l_i) \quad (1)$$

m =1,2,…,CS/2, with CS representing population size
i = 1,2,…,n with n denoting the number of variables
l_i = the lower limit of an interval of the i-th variable
u_i = the upper limit of an interval of the i-th variable

Employed Bees is looking for new food sources that have more nectar around the food source obtained in the early stages following this formula:

$$v_{mi} = x_{mi} + \phi_{mi} * (x_{mi} - x_{ki}) \quad (2)$$

v = new food sources
ϕ = random value in hose $[-1,1]$
x_k= random food sources

After finding new food sources, we will compare the fitness values between x_{ij} and v_{ij}. With the following fitness value formula:

$$fit_m(x_m) = \begin{cases} \frac{1}{1+f_m(x_m)} & f_m(x_m) \geq 0 \\ 1 + |f_m(x_m)| & f_m(x_m) < 0 \end{cases} \quad (3)$$

After that, we will calculate the value of p_m (m-th individual opportunity) to find the best food sources.

$$p_m = \frac{fit_m(x_m)}{\sum_{n=1}^{CS/2} fit_n(x_n)} \tag{4}$$

Unemployed Bees in charge of looking for food sources called Scout Bees. At this stage Scout Bees acting looking for food sources will look for new food sources to reach the limit or "abandonment criteria" where the fitness value of the Employed Bees does not change again on each iteration.

ABC ALGORITHM FOR DIFFERENTIAL EQUATIONS

To solve differential equations, the authors emphasize the initial value problems in differential equations. The initial value problem is a differential problem that comes with one or more conditions at a particular point. Examples of initial value problems are as follows:

- $y' = f(t,y)$ at $[a,b]$, $y(a) = \alpha$
- $y'' = f(t,x,y)$ at $[a,b]$, $y(a) = \alpha$, $y'(a) = \beta$

The author is inspired by the Taylor series in modifying the existing ABC algorithm in solving the initial value problem in this differential equation. The Taylor Series description *y(t)* around t_0 is as follows:

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + \frac{y''(t_0)}{2!}(t - t_0)^2 + \cdots \tag{5}$$

By assuming that $h = t_1 - t_0$ and $y'(t_0) = f(t,y)$ so the equation will be

$$y(t_1) = y(t_0) + hf(t_0, y_0) + \frac{y''(t_0)}{2!}h^2 + \cdots \tag{6}$$

For h that is small enough then it will be

$$y(t_1) \approx y_1 = y_0 + hf(t_0, y_0) \tag{7}$$

$$y(t_0) \approx \hat{y}_0 = y_1 - hf(t_0, y_0) \tag{8}$$

From the Taylor Series, the result of the food source search formula at the Employed Bees stage in ABC Algorithm will fill in place y_1 in equation (8) and the value of $f(t_0, y_0)$ will be filled with the value of the derived function under initial conditions. The next step of this method is to calculate the value of $F(y_1)$ which will be useful in finding the fitness value of each food source. The formula of $F(y_1)$ is as follows:

$$F(y_1) = \hat{y}_0 - y(t_0) \tag{9}$$

Where the value of $y(t_0)$ is known as the initial condition.

The value of *F(y)* will be used in the calculation of fitness value with the following formula:

$$fit(y_1) = \frac{1}{1 + F(y_1)} \tag{10}$$

For steps in the Onlooker Bees and Scout Bees stages will follow the usual ABC Algorithm stage.

- Test Problems

  In this case, the authors use the following differential equations to be the Test Problem of Artificial Bee Colonies Algorithm:

  - *Test Problem 1*:

$$y' = y - t^2 + 1,\ \ y(0) = 0{,}5 \qquad 0 \le t \le 2 \tag{11}$$

➨ *Test Problem 2*:

$$y' = y + t,\ \ y(0) = 0 \qquad (12)$$
$$0 \leq t \leq 1$$

➨ *Test Problem 3:*

$$y' = -2t^3 + 12t^2 - 20t + 8{,}5,\ \ y(0) = 1 \qquad (13)$$
$$0 \leq t \leq 4$$

The Exact solution of the above differential equation is

➨ *Test Problem 1*:

$$y(t) = (t+1)^2 - 0{,}5e^t \qquad (14)$$

➨ *Test Problem 2:*

$$y(t) = e^t\text{-}t\text{-}1 \qquad (15)$$

➨ *Test Problem 3:*

$$y(t) = -0{,}5t^4 + 4t^3 - 10t^2 + 8{,}5t + 1 \qquad (16)$$

The results of the Artificial Bee Colonies Algorithm will compare with the Euler Method.

➢ Result of the test problems

In this section we will show the results of numerical solutions using the ABC Algorithm and use Euler's Method of each problem in the test problem. Both numerical methods will compare the value of his error. On the use of ABC Algorithm will be running running program as much as 20 times and selected which has the smallest error. The solution of each problem on the test problem as follows:

Test Problem 1: The following parameters will be used to search for almost a solution of differential equations on problem 1.

Table 1. Parameters for problem 1

Parameter	Value
Colony Size	100
Maximum Cyrcle	100
Limit	100
Epsilon	0.01

The result of the exact solution, ABC algorithm and Euler method is Figure 1.

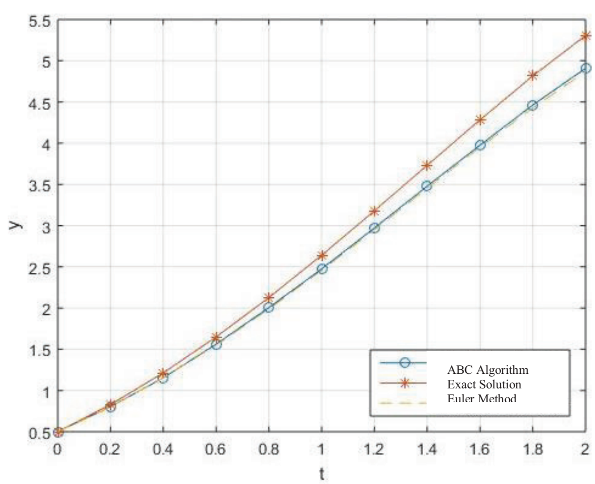


Figure 1. Graph of the result of problem 1

- Test Problem 2: The following parameters will be used to search for almost a solution of differential equations on problem 2.

Table 2. Parameters for problem 2

Parameter	Value
Colony Size	100
Maximum Cyrcle	100
Limit	100
Epsilon	0.01

The result of the exact solution, ABC algorithm and Euler method is Figure 2.

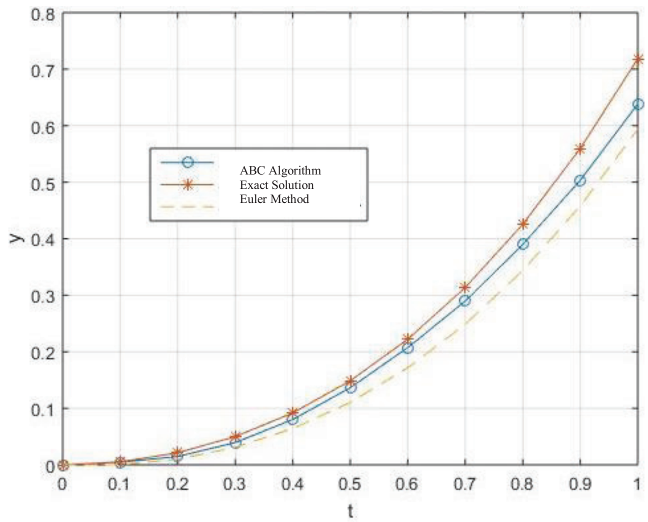


Figure 2. Graph of the result of problem 2

- Test Problem 3: The following parameters will be used to search for almost a solution of differential equations on problem 1.

Table 3. Parameters for problem 3

Parameter	Value
Colony Size	100
Maximum Cyrcle	100
Limit	100
Epsilon	0.01

The result of the exact solution, ABC algorithm and Euler method is Figure 3.

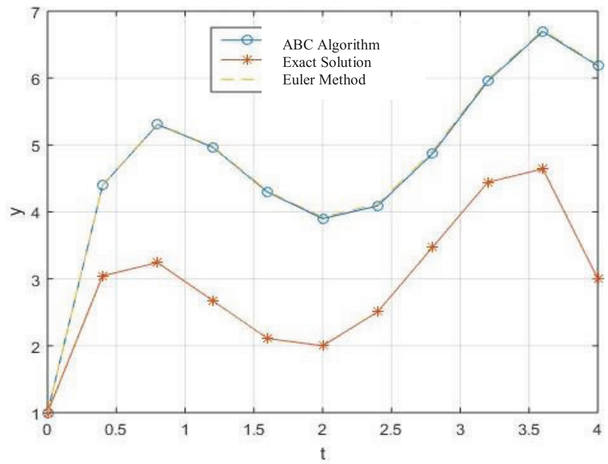


Figure 3. Graph of the result of problem 3

From the result of three test problem above we can see that the result of solution of differential equation with ABC Algorithm have error value smaller than in result of by using Euler Method.

ABC ALGORITHM FOR APPROACHING THE OPTIMUM SOLUTION OF DIFFERENTIAL EQUATIONS

After in the previous sections described by the author on the solution of differential equations using ABC Algorithm, the author tries to deepen the usefulness of the ABC Algorithm by trying to use the algorithm in solving the optimization problem related to the system of differential equations. In everyday life, so many problems that can model into an optimization model related to differential equations. One example of such problems is to optimize the number of a product with the process of making a product that depends on the speed of the process of producing the product and the constraints of the existing environment. From the case it can model into an optimization problem with an objective function that depends on the derivative of a function and other constraints that may affect it.

The idea in solving the problem of optimization of differential equations is by first changing the objective function containing derivative from from a function to an equation that does not contain derivative or in the sense of first solving the first differential equation with ABC Algorithm. Furthermore, the usual ABC algorithm used to find the optimum solution.

Using that idea, the author use some of the following test problems to test it:

- *Test problem* 1 : Minimize $y = y' - t^2 + 1,\ \ y(0) = 0.5$ (17)
  $0 \le t \le 2$
- *Test problem* 2 : Minimize $y' = -2t^3 + 12t^2 - 20t + 8{,}5,\ \ y(0) = 1$ (18)
  $1 \le t \le 4$

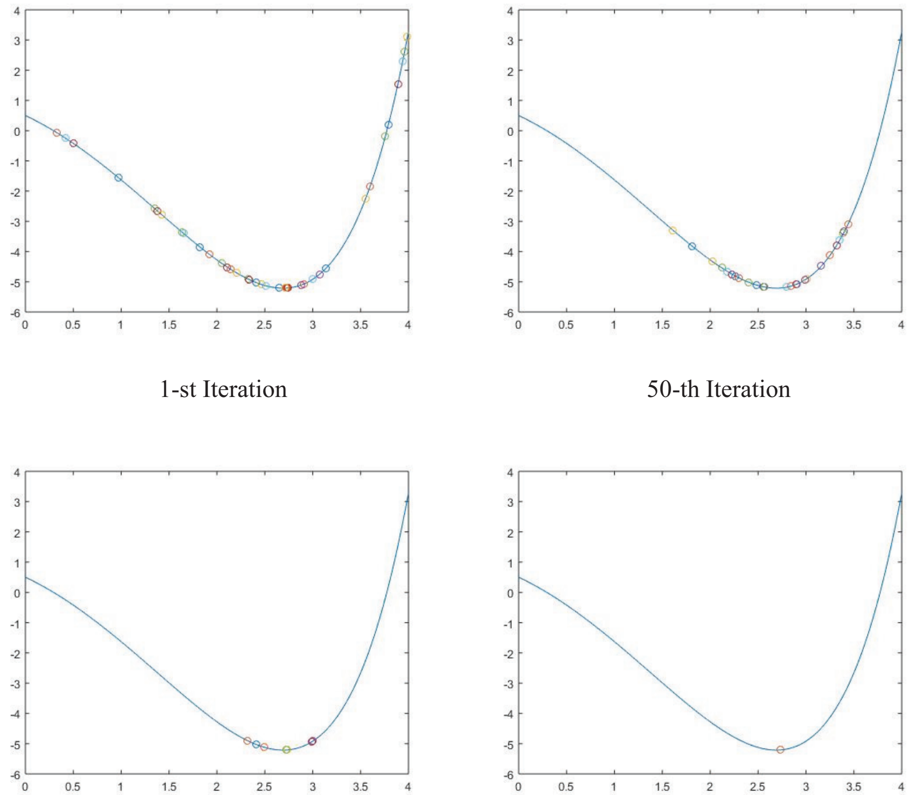
In the test problem, the minimized value is the value of y. Some of the above test problems will search by using the parameters of ABC algorithm as follows:

Table 4. Parameters for ABC algorithm for differential equation

Parameter	Value
Colony Size	100
Maximum Cyrcle	100
Limit	100
Epsilon	0.01

The result of numerical simulation of test problem by using ABC Algorithm is as follows:

- *Test problem* 1: Illustration of the results of its ABC algorithm is Figure 4.



1-st Iteration

50-th Iteration

75-th Iteration

100-th Iteration

Figure 4. Illustration of test problem 1

With the optimum value occurs when:

Table 5. Result of test problem 1

t	y(t)
2.7300	-5.2084

- *Test problem* 2: Illustration of the results of its ABC algorithm is Figure 5.

The optimum value occurs when:

Table 6. Result of test problem 2

T	y(t)
1.8200	1.9679

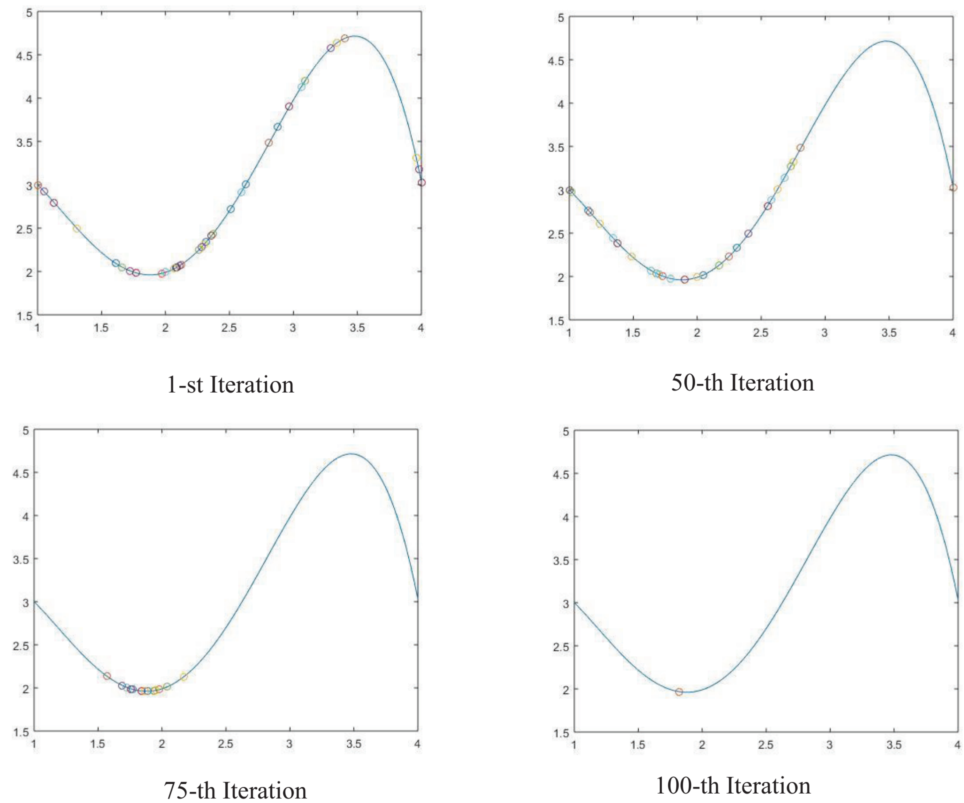


Figure 5. Illustration of test problem 2

ABC ALGORITHM FOR FINDING DIFFERENTIAL EQUATION SYSTEM SOLUTIONS

In this section will be discussed two systems of differential equations to be solved with ABC Algorithm: Lotka-Volterra and SIR Problem. The motivation from the athor to try ABC Algorithm in solving the system of differential equations above is caused by the still limited ode45 algorithm that we know to approach the solution of the system of differential equations. These limitations include the evaluation time interval *(h)* that we cannot determine freely and the coefficients of the parameters must be homogeneous.

- Determination of Solutions for Differential Equations System of Lotka-Volterra

The Lotka–Volterra equations, also known as the predator-prey equations, are a pair of first-order, nonlinear, differential equations generally used to describe the biological systems in which two species interact, predator and prey. The populations change through time according to the pair of equations:

$$\frac{dx(t)}{dt} = \alpha.x(t) - \beta.x(t).y(t) \tag{19}$$

$$\frac{dy(t)}{dt} = -\gamma.y(t) + \delta.x(t).y(t) \tag{20}$$

In this section the author tries to apply the results of the ABC algorithm on the modified Lotka-Volterra model for the case of harvesting. The addition of the harvesting factor aims to find the harvesting term that will generate maximum profit.

The addition of such modifications will effect on the equations on the solution of the system of the Lotka-Volterra differential equations. For each harvesting period then the solution of the system of differential equations of its ABC Algorithm results will be:

$$x(t) = \hat{x}(t) - \theta_x . x(t) \tag{21}$$

$$y(t) = \hat{y}(t) - \theta_y . y(t) \tag{22}$$

with $\hat{x}(t)$ and $\hat{y}(t)$ each stating the results of the ABC Algorithm for prey and predator populations at time t. While θ_x and θ_y each represent the capture coefficient of the species of prey and predator.

Because of the purpose of this section is to find the harvesting period that produces the most optimum gain, an objective function make as follows:

$$G(t) = \mu_x . x_p + \mu_y . y_p - \rho \tag{23}$$

with μ_x and μ_y each stating the selling price of prey and predator species. While x_p and y_p each represent species of prey and predator caught on the harvesting process, and ρ represents the cost to each harvesting process.

The simulation will be use the value of each parameter and the coefficient as follows:

Table 7. Parameters for ABC algorithm for *Lotka-Volterra*

Parameter	Value	Parameter	Value
Colony Size	100	α	0.2
Maximum Cyrcle	100	β	0.005
Limit	100	γ	0.5
Epsilon	0.01	δ	0.01

Using the above parameters and taking the case that the harvesting every 4 time units and the capture coefficient of 0.25, the graphs of prey and predator populations as follows:

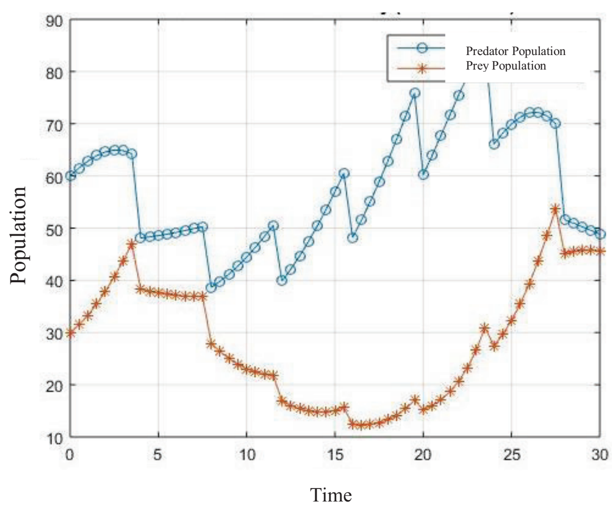


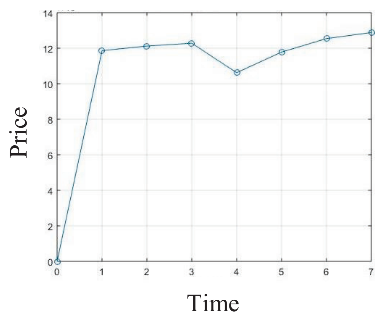
Figure 6. Graph of the result of ABC algorithm for *Lotka-Volterra*

Meanwhile, to find the maximum profit, the author uses 4 cases with each value of μ_x, μ_y, ρ, θ_x dan θ_y as follows:

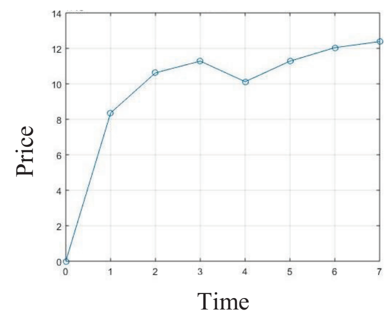
- Case 1: with the value $\mu_x = 200, \mu_y = 4000, \rho = 0, \theta_x = 0.5$ and $\theta_y = 0.5$

- Case 2: with the value $\mu_x = 200, \mu_y = 4000, \rho = 10000, \theta_x = 0.5$ and $\theta_y = 0.5$
- Case 3: with the value $\mu_x = 200, \mu_y = 4000, \rho = 5000, \theta_x = 0.25$ and $\theta_y = 0.25$
- Case 4: with the value $\mu_x = 200, \mu_y = 4000, \rho = 25000, \theta_x = 0.25$ and $\theta_y = 0.25$

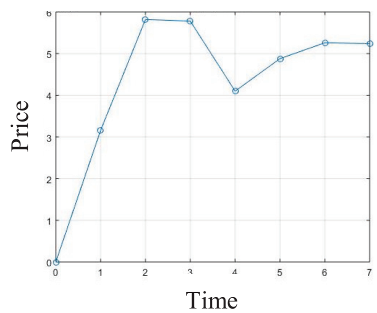
Numerical simulation results are showed in Figure 7.



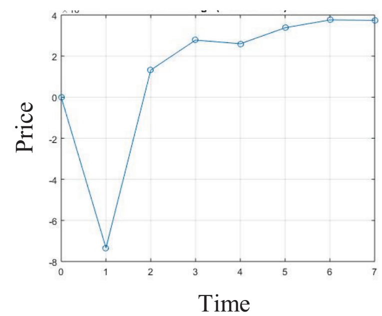
Case 1: The optimum solution by harvesting every 7 time units



Case 2: The optimum solution by harvesting every 7 time units



Case 3: The optimum solution by harvesting every 2 time units



Case 4: The optimum solution by harvesting every 6 time units

Figure 7. Result of optimum gain

With the success of the simulation results from several cases above it can conclude that the selection of harvesting period strongly influence by the values of each parameter in its Lotka-Volterra equation.

- Determination of SIR Differential Equation System Solutions

In this section it will show that ABC algorithm is also able to be used to determine the solution of system of differential equation of other model that is SIR. Taking a simple SIR differential equation system as described in the previous chapter, the following differential equations system is taken to check the ABC Algorithm's capabilities

$$\frac{dS}{dt} = N\beta - \left(\frac{\alpha}{N} I(t) + \beta\right) S(t) \tag{24}$$

$$\frac{dI}{dt} = \left(\frac{\alpha}{N} S(t) - \beta - \gamma\right) I(t) \tag{25}$$

$$\frac{dR}{dt} = -R(t)\beta + \gamma I(t) \tag{26}$$

Will be tested two test problem that is with each test problem will have value of each coefficient as follows:

- *Test problem* 1:
  S(0)=88, *I(0)*=12, *R(0)*=0, $\alpha = 0.6,\ \beta = 0.8, \gamma = 0.8$, *N* constant with *N=S(0)+R(0)+I(0).*
- *Test problem* 2:
  S(0)=88, *I(0)*=12, *R(0)*=0, $\alpha = 1.5,\ \beta = 0.3, \gamma = 0.45$, *N* constant with *N=S(0)+R(0)+I(0).*

The parameter values for the ABC Algorithm that will be used are as follows:

Table 8. Parameters for SIR

Parameter	Value
Colony Size	100
Maximum Cyrcle	100
Limit	100
Epsilon	0.01

By entering the value of each coefficient to the system of SIR differential equations that have been defined before the graph obtained a numerical approach from the solution of each test problem by using the method of ABC and ode45 Algorithm as follows:

- *Test problem* 1:
  The result of ABC Algorithm:

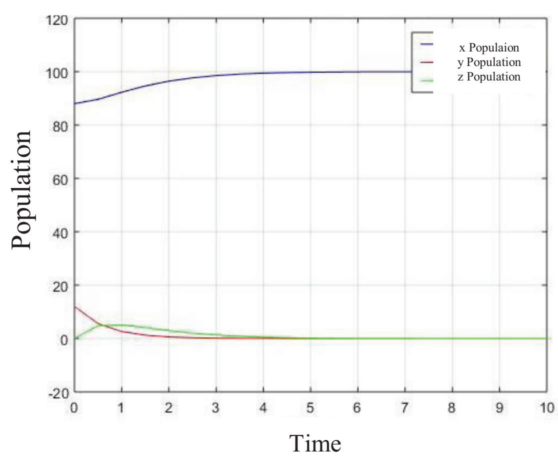


Figure 8. Result of ABC Algorithm for test problem 1

The result of Ode45:

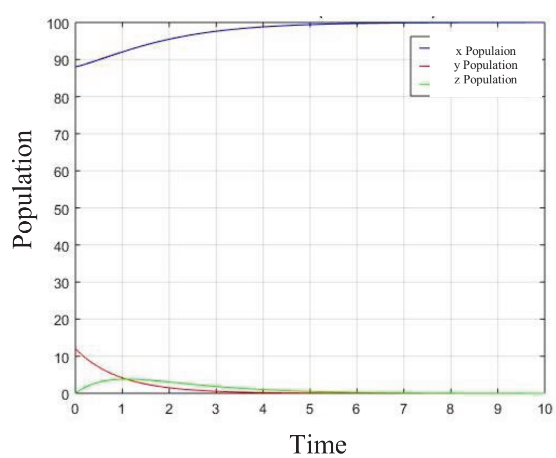


Figure 9. Result of ode45 for test problem 1

- *Test problem* 2:
  The result of ABC Algorithm:

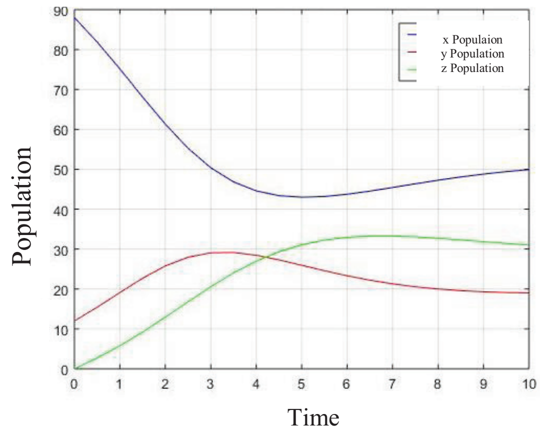


Figure 10. Result of ABC Algorithm for test problem 2

The result of Ode45:

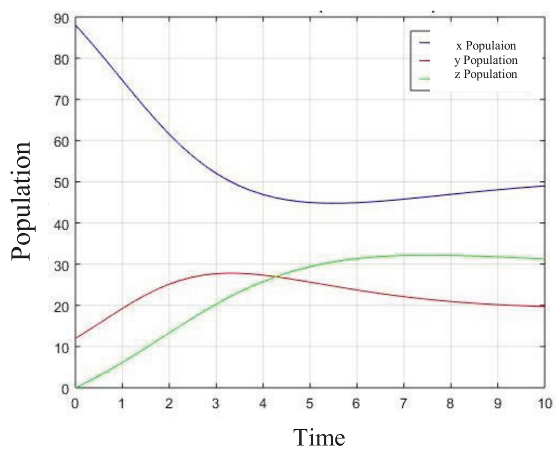


Figure 11. Result of ode45 for test problem 2

From the plot of both methods (ABC and ode45 algorithm) we can see the similarity of the two results, so we can conclude ABC algorithm can be used to solve SIR differential equation system. In addition, the ABC Algorithm method has advantages over the ode45 method that we can independently determine the time interval of evaluation *(h)* from the solution of the SIR differential equation system while for ode45 cannot be freely selected and the value of each coefficient in the differential equation can make to homogeneous.

CONCLUSION

ABC algorithm has computation performance which is good enough in determining solution initial value problem of differential equation proven with the error value of the result with ABC Algorithm is smaller than the error value of the result with Euler Method.

The ABC algorithm is also capable of obtaining solutions of optimization problems related to differential equations. In this case the results obtained are near-optimal solutions rather than the exact optimal solution.

The ABC algorithm can also be used to solve community problems that can model into the Lotka-Volterra Model or SIR Model. These models are related to the system of differential equations. ABC algorithm also has its own advantages in hose determination and coefficient values compared ode45 method. To find a solution of the models there is little modification in the existing ABC Algorithm by combining it with an equation inspired by the Taylor series.

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