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Interpretation of Ultrasonic Experiments on Finite-Amplitude Waves

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An analysis was made for the purpose of relating higher-order elastic constants to the growth of harmonics in an initially sinusoidal wave. It is supposed that the face of a semiinfinite nondissipative elastic medium—for which the equation of one-dimensional motion is $\ddot{u}=(\partial^2u/\partial x^2)g(\partial u/\partial x)$—is subjected to the sinusoidal displacement $u=u_0\cos(\omega t-\varphi)$. The function $g(\partial u/\partial x)$ is given as $g(\xi)=(1/\rho_0)[M_2+M_3\xi+M_4\xi^2+\cdots+M_i\xi^{i-2}]$, where M_i depends on elastic constants up to and including order i. A solution, including the effects due to terms as high as M_5 in the function $g(\xi)$, is expressed as $u=u_0\Sigma_n\{D_n\sin[n(\omega t-Kx)]+E_n\cos[n(\omega t-Kx)]\}+F(x)$, where the summation over n goes from 1 to ∞. D_n and E_n are expanded in powers of a Mach number M and a dimensionless distance X ($M=\omega u_0/c_0$, $X=\omega x/c_0$, $c_0{}^2=M_2/\rho_0$). It is found that as $u_0\rightarrow 0$, the limiting value of the ratio of the *second*-harmonic amplitude, $H_2=u_0(D_2{}^2+E_2{}^2)^{\frac{1}{2}}$, to u_0M is given by $-M_3X/8M_2$, independently of M_4, M_5, etc. Thus, with M_2 already known, M_3 can be calculated from an experimental determination of this ratio. Similarly, it is found that the limiting value of the ratio of the *third* harmonic amplitude to u_0M^2 depends on elastic constants only up through order four.

[Tutorial material is included to make the paper reasonably self-contained and to relate the present analysis to corresponding work for fluids. Careful derivations show that purely longitudinal waves in elastic solids and nondissipative fluids are governed by an equation of motion of precisely the same form. Hence, with appropriate choice of the parameters, an analysis of the one case applies equally well to the other.]

LIST OF SYMBOLS

A_n,B_n	coefficients in Fourier series for v/v_0
c_0	propagation velocity W, evaluated at $\partial u/\partial x=0$; this is the small-amplitude wave velocity in the unstrained medium
D_n,E_n	coefficients in Fourier series for u/u_0; $D_n=A_n/n$, $E_n=-B_n/n$
$e_0,e_1,e_2,\cdots$	defined by Eqs. 119–121
$F(x)$	added function in Eq. 102; its derivative $F'(x)$ is the time average of the extension $\partial u/\partial x$ (Sec. III-B9)
g	function defined in Eq. 47
G	function in Eqs. 104 and 105
k	constant in Eq. 55
K	constant term in expansion of ω/W as power series in $(v-V_0)$, Eqs. 76 and 77
l	characteristic length defined by Eqs. 76 and 78
L	"discontinuity" length, a characteristic length defined by Eqs. 63 or 64
m_3,m_4,m_5	dimensionless combinations of higher-order elastic constants defined by Eq. 103; m_j involves elastic constants up to and including order j
M	$\omega u_0/c_0=v_0/c_0$ equal to characteristic Mach number, Eq. 103
M_i	combination of elastic coefficients of order i and lower
r	$(k-V_0)/v_0$, Eq. 75
t	time
T	ωt equal to dimensionless "time"
u	particle displacement
u_0	particle-displacement amplitude of the sinusoidal driver (at $x=0$)
U	u/u_0 in Sec. III, internal energy in Secs. I and II
v	$\partial u/\partial t$ equal to particle velocity
v_0	ωu_0 equal to particle-velocity amplitude at $x=0$
V_0	constant velocity that can be added to driver's sinusoidal motion. Eq. 53

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w	$(v-k)/c_0$, Eq. 118
W	wavefront propagation velocity relative to reference coordinate x, the "natural" propagation speed, Eq. 56
W^+	value of W at the piston just after it starts
x	Lagrangian coordinate
X	$\omega x/c_0$
y	present position of particle x, $y=x+u$
α	m_3M
β	Kc_0/ω
ζ	see Eqs. 76 and 80
η	see Eqs. 76 and 79
ρ_0	mass density in the unstressed condition
σ	see Eqs. 71 and 72
τ	see Eqs. 71 and 73
ω	driving frequency

INTRODUCTION

THERE has recently been considerable interest in the use of ultrasonic waves to determine experimentally the nonlinear elastic properties of liquids and solids. Let us consider two types of experiment.

In the first, the time for small-amplitude waves to travel one round trip in the sample is observed as a function of an applied static stress. The analysis of Ref. 1 relates the second- and third-order elastic coefficients to the initial slope of the curve of repetition frequency (reciprocal of transit time) versus stress. Within the framework of elasticity theory, the analysis is exact, and the interpretation in terms of second- and third-order elastic coefficients is unambiguous. No assumptions need to be made concerning the fourth-order coefficients because they do not enter the formula for the initial slope. The fact that the measurement corresponds to the travel time of *small-amplitude* waves is easily confirmed experimentally by verifying that the transit time does not depend on the amplitude of the wave.

In another type of experiment, one face of the sample is given a sinusoidally oscillating displacement of sufficient amplitude to cause distortion of the wave as it propagates. The nonlinear elastic properties are inferred from the growth of the second harmonic (or some other chosen harmonic). The analysis of this type of experiment is complicated by the nonlinearity of the partial differential equation for the wavemotion. The existing analyses give the results in terms of second- and third-order elastic coefficients. These analyses seem plausible, but leave unanswered the following interesting question. Can *fourth*-order coefficients in principle affect the experimental results? In the present work, this question is answered definitively for purely longitudinal waves in elastic crystals. Let u_0 denote the displacement amplitude of the sinusoidal driver (at the "fundamental" or "first-harmonic" frequency) at the face of the crystal and $H_2(u_0^2,x)$ the amplitude of the second harmonic at some distance x from the driven face. Then the fourth-order coefficients do *not* enter the formula for the value of H_2/u_0^2 at $u_0=0$. Thus, when the "experimental results" correspond to the initial slope of the curve of second-harmonic amplitude versus (u_0^2), the interpretation in terms of second- and third-order elastic coefficients is unambiguous.

It can be seen that the relation of the data to nonlinear elastic properties is similar in the two types of experiment. As is clear from Ref. 1, the small-amplitude-wave transit time corresponding to any nonzero static stress depends in principle on elastic coefficients of all orders, but the initial derivative with respect to static stress is independent of coefficients of order higher than the third. Similarly, for any value of $u_0\neq 0$, $H_2(u_0^2,x)$ depends in principle on elastic coefficients of all orders, but the initial derivative with respect to u_0^2 does not depend on coefficients of order higher than the third. A major purpose of the present work was to establish this latter result. Having done so, we now have equally firm theoretical bases for interpreting the results of the two types of ultrasonic experiment.

An additional result of the present analysis is that the growth of the third harmonic depends on both third- and fourth-order elastic constants, even in the limit as $u_0 \to 0$.

The abovestated results were all obtained by solving the equation of motion for a nondissipative medium. The important question of how these results must be modified for a slightly dissipative medium is not examined in the present work.

I. EXPANSION OF THE STRAIN ENERGY

The adiabatic elastic coefficients of order n are by definition[2] the nth-order partial derivatives of the internal energy per unit of original volume with respect to the Lagrangian strain components, the differentiation being at constant entropy. Let x_i denote the Cartesian coordinates of a particle in the unstressed reference configuration and $y_i(x_i,t)$ the coordinates of the same particle in an arbitrary configuration. Then the displacement components u_i and Lagrangian strain components η_{ij} are given by

$$u_i=y_i-x_i, \tag{1}$$

$$\begin{aligned}\eta_{ij}&=\frac{1}{2}\left(\frac{\partial y_s}{\partial x_i}\frac{\partial y_s}{\partial x_j}-\delta_{ij}\right)\\&=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}+\frac{\partial u_s}{\partial x_j}\frac{\partial u_s}{\partial x_i}\right).\end{aligned} \tag{2}$$

[1] R. N. Thurston and K. Brugger, "Third-Order Elastic Constants and the Velocity of Small Amplitude Elastic Waves in Homogeneously Stressed Media," Phys. Rev. A133, 1604–1610 (1964); erratum, *ibid.* 135, No. AB7, 3 (1964).

[2] K. Brugger, "Thermodynamic Definition of Higher Order Elastic Coefficients," Phys. Rev. A133, 1611–1612 (1964).

Summation is understood to be carried out over repeated subscripts. δ_{ij} denotes the Kronecker delta, 1 if $i=j$ and otherwise zero. Let $U(\eta_{ij},S)$ denote the internal energy per unit mass, expressed as a function of the strain components η_{ij} and the entropy S. Then, at constant entropy, the internal energy change per unit of original unstressed volume can be expressed as

$$\begin{aligned}\rho_0\Delta U &= \rho_0[U(\eta_{ij},S)-U(0,S)]\\ &= \tfrac{1}{2}c_{ijkm}\eta_{ij}\eta_{km}+\tfrac{1}{6}c_{ijkmpq}\eta_{ij}\eta_{km}\eta_{pq}+\cdots\\ &= \tfrac{1}{2}c_{\lambda\mu}\eta_\lambda\eta_\mu+\tfrac{1}{6}c_{\lambda\mu\nu}\eta_\lambda\eta_\mu\eta_\nu+\cdots\\ &= \tfrac{1}{2}\sum_\lambda c_{\lambda\lambda}\eta_\lambda{}^2+\sum_{\lambda<\mu}c_{\lambda\mu}\eta_\lambda\eta_\mu+\tfrac{1}{6}\sum_\lambda c_{\lambda\lambda\lambda}\eta_\lambda{}^3\\ &\quad+\tfrac{1}{2}\sum_{\lambda\neq\mu}c_{\lambda\lambda\mu}\eta_\lambda\eta_\mu+\sum_{\lambda<\mu<\nu}c_{\lambda\mu\nu}\eta_\lambda\eta_\mu\eta_\nu+\cdots,\end{aligned}\tag{3}$$

where the c's are the previously defined adiabatic elastic coefficients, evaluated at zero strain, and ρ_0 is the density in the unstressed reference configuration. The second line of this equation is in tensor notation and the next line is in the abbreviated Voigt notation, in which the single-subscript normal strains equal the corresponding tensor components, but the single-subscript shear strains equal twice the corresponding tensor components. Greek-letter subscripts run from 1 to 6. We follow Brugger's conventions[2] on the abbreviated notation for higher-order elastic coefficients. The last equality in Eq. 3 illustrates Brugger's coefficient rule: in this form, the numerical factor before each summation symbol equals $(1/n!)$, where n is the number of equal indices of the strains.

II. EXPANSION OF THE EQUATION OF MOTION

A. General Equation for an Elastic Solid

Standard derivations such as that given in Ref. 3 show that the stress T_{km} is given by

$$T_{km}=\rho(\partial y_k/\partial x_p)(\partial y_m/\partial x_q)(\partial U/\partial\eta_{pq})_S.\tag{4}$$

With the body force omitted, the general equation of motion is

$$\rho\ddot{u}_i=\partial T_{ki}/\partial y_k.\tag{5}$$

The ratio of reference density to actual density must satisfy

$$\rho_0/\rho=\det[\partial y_i/\partial x_j].\tag{6}$$

In view of Eq. 6, the identity of Euler, Piola, and Jacobi[4] yields the relation

$$\frac{\partial}{\partial y_k}\left(\frac{\rho}{\rho_0}\frac{\partial y_k}{\partial x_j}\right)=0,\quad j=1,2,3.\tag{7}$$

By substituting from Eq. 4 into Eq. 5 and making use of Eq. 7, we transform the general equation of motion to

$$\rho_0\ddot{u}_r=\frac{\partial}{\partial x_p}\left[\frac{\partial y_r}{\partial x_q}\left(\rho_0\frac{\partial U}{\partial\eta_{pq}}\right)_S\right].\tag{8}$$

It is now a straightforward but tedious matter to expand the equation of motion in powers of the displacement gradients $\partial u_p/\partial x_q$. From Eqs. 1 and 3, the quantity in square brackets in Eq. 8 is

$$\left[\left(\delta_{rq}+\frac{\partial u_r}{\partial x_q}\right)(c_{ijpq}\eta_{ij}+\tfrac{1}{2}c_{ijkmpq}\eta_{ij}\eta_{km}+\cdots)\right].\tag{9}$$

Differentiating with respect to x_p, making use of the definition Eq. 2, and collecting the coefficients of like powers of the displacement gradients, we find that the equation of motion can be written as

$$\rho_0\ddot{u}_i=\frac{\partial^2u_k}{\partial x_j\partial x_m}\left[c_{ijkm}+\frac{\partial u_p}{\partial x_q}\left(M_{ijkmpq}+\frac{\partial u_r}{\partial x_s}M_{ijkmpqrs}+\cdots\right)\right],\tag{10}$$

where

$$M_{ijkmpq}=c_{ijkmpq}+\delta_{kp}c_{ijmq}+\delta_{ik}c_{jmpq}+\delta_{ip}c_{jkmq},\tag{11}$$

$$\begin{aligned}M_{ijkmpqrs}&=\tfrac{1}{2}c_{ijkmpqrs}+\delta_{ir}c_{sjkmpq}+\delta_{kp}c_{ijmqrs}\\ &\quad+\tfrac{1}{2}\delta_{pr}c_{ijkmqs}+\tfrac{1}{2}\delta_{ik}c_{jmpqrs}\\ &\quad+\delta_{kp}\delta_{ir}c_{jqms}+\tfrac{1}{2}\delta_{pr}\delta_{ik}c_{jmqs}.\end{aligned}\tag{12}$$

It has been assumed that the deformation takes place at constant entropy.

B. One-Dimensional Motion of a Solid

Our application of Eq. 10 is to purely longitudinal motion. In anisotropic media, such motion can occur only along certain special directions. If x (without a subscript) is the coordinate in one of these directions and u the displacement, then the one-dimensional form of Eq. 10 is

$$\rho_0\ddot{u}=\frac{\partial^2u}{\partial x^2}\left[M_2+M_3\frac{\partial u}{\partial x}+M_4\left(\frac{\partial u}{\partial x}\right)^2+\cdots\right],\tag{13}$$

where M_2 is a linear combination of second-order coefficients, M_3 a linear combination of second- and third-order coefficients, and M_4 a linear combination of second-, third-, and fourth-order coefficients. Explicitly, if $\mathbf{N}$ denotes a unit vector in the direction of the one-dimensional motion under consideration, then

$$\begin{aligned}M_2&=c_{ijkm}N_iN_jN_kN_m,\\ M_3&=M_{ijkmpq}N_iN_jN_kN_mN_pN_q,\\ M_4&=M_{ijkmpqrs}N_iN_jN_kN_mN_pN_qN_rN_s.\end{aligned}\tag{14}$$

It is interesting to note that the equation of motion would be nonlinear without higher-order terms in the strain-energy expansion. However, even the first nonlinear term involves third-order elastic coefficients, as can be seen from Eq. 11.

[3] R. N. Thurston, "Wave Propagation in Fluids and Normal Solids," in *Physical Acoustics—Principles and Methods*, W. P. Mason, Ed. (Academic Press Inc., New York, 1964–), Vol. 1: *Methods and Devices*, Pt. A, Chap. 1, pp. 1–110.

[4] C. Truesdell and R. Toupin, in *Handbuch der Physik*, S. Flügge, Ed. (Springer-Verlag, Berlin, 1960), Vol. 3/1, p. 226. The identity is also demonstrated in Ref. 3, pp. 91–92.

C. One-Dimensional Motion of a Fluid

In the present context, a *fluid* is a nondissipative material in which the stress is always isotropic:

$$T_{ij}=-p\delta_{ij}. \tag{15}$$

Substitution into Eq. 5 gives the equation of motion as

$$\rho\ddot{u}_i=-(\partial p/\partial y_i), \tag{16}$$

which can obviously be rewritten as

$$\rho\ddot{u}_i=-(\partial x_j/\partial y_i)(\partial p/\partial x_j). \tag{17}$$

For *one-dimensional* motion in the direction indicated by the coordinate x (without a subscript), we have

$$\rho_0/\rho=\partial y/\partial x=1+(\partial u/\partial x), \tag{18}$$

and hence an appropriate one-dimensional equation is

$$\rho_0\ddot{u}=-(\partial p/\partial x). \tag{19}$$

Subject to the additional assumption that the pressure is a function of the density, we have

$$\frac{\partial p}{\partial x}=\frac{dp}{d\rho}\frac{\partial\rho}{\partial x}=-\frac{\rho^2}{\rho_0}\frac{dp}{d\rho}\frac{\partial}{\partial x}\left(\frac{\rho_0}{\rho}\right)=-\frac{\rho^2}{\rho_0}\frac{dp}{d\rho}\frac{\partial^2 u}{\partial x^2}, \tag{20}$$

where Eq. 18 has been used in evaluating $(\partial/\partial x)(\rho_0/\rho)$. Finally, from Eqs. 19 and 20,

$$\ddot{u}=(\rho^2/\rho_0^2)(dp/d\rho)(\partial^2u/\partial x^2). \tag{21}$$

This equation appears in the work of Lamb[5] and (with ρ_0/ρ from Eq. 18) Rayleigh.[6] The derivative $dp/d\rho$ may be interpreted as a partial derivative at constant entropy, consistent with the assumption of no dissipation.

1. *Hypothetical Linear Fluid*

It may be noted that a linear equation with constant coefficients is obtained if

$$(\rho^2/\rho_0^2)(dp/d\rho)=K_0/\rho_0=\text{const.} \tag{22}$$

Equation 22 can be integrated to obtain

$$p-p_0=K_0\left(1-\frac{\rho_0}{\rho}\right),\quad\text{or}\quad\frac{\rho_0}{\rho}=1-\frac{p-p_0}{K_0}. \tag{23}$$

This hypothetical pressure–density relation for linear-wave propagation has been given by Earnshaw,[7] Rayleigh,[6] Lamb,[5] and many others. It corresponds to a bulk modulus equal to

$$K\equiv\rho(dp/d\rho)=K_0\rho_0/\rho=K_0-(p-p_0). \tag{24}$$

The value of the bulk modulus at $p=p_0$ may be chosen arbitrarily, but the pressure derivative of the bulk modulus is equal to minus one:

$$\frac{dK}{dp}=\frac{d}{dp}\left(\rho\frac{dp}{d\rho}\right)=\frac{d}{dp}\left(\frac{K_0\rho_0}{\rho}\right)=-1. \tag{25}$$

2. *Special Case of Isentropic Motion of a Perfect Gas*

The pressure of a perfect gas undergoing isentropic changes of state is related to the density by

$$p=p_0(\rho/\rho_0)^\gamma, \tag{26}$$

where γ is the ratio of specific heats. It follows that

$$K=\rho(dp/d\rho)=\gamma p, \tag{27}$$

and, in view of Eq. 18, the equation of motion Eq. 21 becomes

$$\ddot{u}=\frac{\partial^2u}{\partial t^2}=\gamma\frac{p_0}{\rho_0}\left[\left(1+\frac{\partial u}{\partial x}\right)^{-(\gamma+1)}\right]\frac{\partial^2u}{\partial x^2}$$

$$=\frac{\gamma p_0}{\rho_0}\frac{\partial^2u}{\partial x^2}\left[1-(\gamma+1)\frac{\partial u}{\partial x}+\cdots\right]. \tag{28}$$

3. *Fluid with Constant Pressure Derivative of the Bulk Modulus*

It is of considerable interest to record the exact equation of motion for a fluid with a constant pressure derivative of the bulk modulus, i.e., with

$$(d/dp)[\rho(dp/d\rho)]=K_0'=\text{const.} \tag{29}$$

Both the hypothetical linear fluid and the perfect gas may be regarded as special cases of this somewhat more general (but still very special) fluid. A first integration gives the bulk modulus as

$$K\equiv\rho(dp/d\rho)=K_0+K_0'(p-p_0), \tag{30}$$

where K_0 is the value of the bulk modulus at $p=p_0$, while a second integration yields

$$\frac{\rho}{\rho_0}=\left[1+\frac{K_0'(p-p_0)}{K_0}\right]^{1/K_0'} \tag{31}$$

or

$$p-p_0=\frac{K_0}{K_0'}\left[\left(\frac{\rho}{\rho_0}\right)^{K_0'}-1\right]. \tag{32}$$

In this case, the factor $(\rho^2/\rho_0^2)(dp/d\rho)$ in Eq. 21 becomes

$$\left(\frac{\rho}{\rho_0}\right)^2\frac{dp}{d\rho}=\frac{K_0}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^{K_0'+1}=\frac{K_0}{\rho_0}\left(1+\frac{\partial u}{\partial x}\right)^{-(K_0'+1)} \tag{33}$$

and the exact equation of motion is

$$\ddot{u}=\frac{K_0}{\rho_0}\frac{\partial^2u}{\partial x^2}\left(1+\frac{\partial u}{\partial x}\right)^{-(K_0'+1)}. \tag{34}$$

[5] H. Lamb, *Hydrodynamics* (Dover Publications, Inc., New York, 1945), p. 481.

[6] J. W. Strutt Lord Rayleigh, *Theory of Sound* (Dover Publications, Inc., New York, 1945), Vol. 2, p. 32.

[7] S. Earnshaw, "On the Mathematical Theory of Sound," Phil. Trans. Roy. Soc. (London) **150**, 133–148 (1860), p. 147.

The hypothetical linear fluid of Sec. II-C1 is obtained by setting $K_0'=-1$. The perfect gas is a special case with $K_0'=\gamma=K_0/p_0$. It has sometimes been observed that the equation of motion is linear for a perfect gas with $\gamma=-1$. If taken literally, this would require a negative bulk modulus and there could be no waves because Eq. 28 would not be a wave equation, but an equation of elliptic rather than hyperbolic type. A more sensible embodiment of the essential idea inherent in "$\gamma=-1$" is a fluid with positive bulk modulus but pressure derivative of the bulk modulus equal to -1. The equation of state for such a fluid is most emphatically *not* Eq. 26, but Eq. 23, which is the same as Eq. 31 or Eq. 32 with $K_0'=-1$.

When Eq. 34 is expanded as a power series in the displacement gradient $\partial u/\partial x$, one obtains

$$\ddot{u}=\frac{K_0}{\rho_0}\frac{\partial^2 u}{\partial x^2}\left[1-(K_0'+1)\frac{\partial u}{\partial x}+\cdots\right]. \quad (35)$$

4. Other Fluids

For fluids other than a perfect gas, it has been customary to expand the pressure p in the form

$$p=p_0+As+\tfrac{1}{2}Bs^2+\frac{Cs^3}{3!}+\frac{Ds^4}{4!}+\cdots, \quad (36)$$

where

$$s=(\rho-\rho_0)/\rho_0. \quad (37)$$

Clearly,

$$A=\rho_0(dp/d\rho)_{\rho=\rho_0}, \quad (38)$$

$$B=\rho_0^2(d^2p/d\rho^2)_{\rho=\rho_0}. \quad (39)$$

In conformity with the assumption of no dissipation, the above derivatives are understood to be partial derivatives at constant entropy.

With this notation, the factor $(\rho^2/\rho_0^2)(dp/d\rho)$ in Eq. 21 can be written as

$$\frac{\rho^2}{\rho_0^2}\frac{dp}{ds}\frac{ds}{d\rho}=\left(1+\frac{\partial u}{\partial x}\right)^{-2}(A+Bs+\cdots)/\rho_0. \quad (40)$$

But from Eqs. 37 and 18,

$$s=\frac{\rho}{\rho_0}-1=\left(1+\frac{\partial u}{\partial x}\right)^{-1}-1=-\frac{\partial u}{\partial x}\left(1+\frac{\partial u}{\partial x}\right)^{-1}. \quad (41)$$

By substituting for s from Eq. 41 and applying the binomial theorem to the negative powers of $[1+(\partial u/\partial x)]$, the expression in Eq. 40 can be expanded as a power series in $\partial u/\partial x$. The equation of motion, Eq. 21, then becomes

$$\ddot{u}=\frac{A}{\rho_0}\frac{\partial^2 u}{\partial x^2}\left[1-\left(2+\frac{B}{A}\right)\frac{\partial u}{\partial x}+\cdots\right]. \quad (42)$$

The results of studies on the nonlinear behavior of liquids are commonly expressed in terms of the nonlinearity parameter B/A (Refs. 8, 9).

D. Summary of One-Dimensional Results

By comparing the expansions of the equation of motion for the various cases discussed above, one can establish the associations

$$-\frac{M_3}{M_2}=2+\frac{B}{A}=\gamma+1,$$
$$\frac{M_4}{M_2}=3+\frac{3B}{A}+\frac{C}{2A}=\tfrac{1}{2}(\gamma+1)(\gamma+2), \quad (43)$$
$$-\frac{M_5}{M_2}=4+\frac{6B}{A}+\frac{2C}{A}+\frac{D}{6A}=\tfrac{1}{6}(\gamma+1)(\gamma+2)(\gamma+3).$$

The results for a fluid with constant pressure derivative of the bulk modulus are obtained by substituting K_0' for γ. These associations are helpful in correlating the treatments of one-dimensional waves of finite amplitude in solids, perfect gases, and other fluids.

Because of the way in which both second- and third-order elastic coefficients enter the constant M_3, it is frequently[10] broken down into

$$M_3=K_3+3K_2, \quad (44)$$

where

$$K_2\equiv M_2=c_{ijkm}N_iN_jN_kN_m \quad (45)$$

and, by comparison with Eqs. 11 and 14,

$$K_3=c_{ijkmpq}N_iN_jN_kN_mN_pN_q. \quad (46)$$

III. SOLUTION OF THE DIFFERENTIAL EQUATION GOVERNING THE PROPAGATION OF FINITE-AMPLITUDE PLANE SOUND WAVES IN A NONDISSIPATIVE SOLID

A. Introduction and Historical Background

Reviews of the propagation of plane sound waves of finite amplitude have been given by Blackstock[11] and by Beyer.[12] The historical information in this paper is meant only to supply some more recent references and to serve as an introduction to the subject matter for the reader.

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[8] R. T. Beyer, "Parameter of Nonlinearity in Fluids," J. Acoust. Soc. Am. **32**, 719–721 (1960).

[9] A. B. Coppens, R. T. Beyer, M. B. Seiden, J. Donohue, F. Guepin, R. H. Hodson, and C. Townsend, "Parameter of Nonlinearity in Fluids. II," J. Acoust. Soc. Am. **38**, 797–804 (1965).

[10] M. A. Breazeale and J. Ford, "Ultrasonic Studies of the Nonlinear Behavior of Solids," J. Appl. Phys. **36**, 3486–3490 (Nov. 1965).

[11] D. T. Blackstock, "Propagation of Plane Sound Waves of Finite Amplitude in Nondissipative Fluids," J. Acoust. Soc. Am. **34**, 9–30 (1962).

[12] R. T. Beyer, "Nonlinear Acoustics," in *Physical Acoustics—Principles and Methods*, W. P. Mason, Ed. (Academic Press Inc., New York, 1964–), Vol. 2, Pt. B (1965): *Properties of Polymers and Nonlinear Acoustics*, Chap. 10, pp. 231–264.

The partial differential equation governing the propagation of purely longitudinal finite-amplitude plane sound waves in solids, given in its exact form in Eq. 13, has been considered by recent investigators in various forms. Thus, the so-called piston problem described briefly in the Introduction was considered by Keck and Beyer,[13] who rederived the now classical solution of Fubini-Ghiron[14] for finite-amplitude waves in fluids. Melngailis *et al.*[15] wrote the differential equation for solids as Eq. 13, but truncated the power series in $\partial u/\partial x$ with M_3; i.e., they considered $M_i=0$, $i>3$. In their solution, they followed a perturbation approach and concluded that the coefficient of the second harmonic grew linearly with x. Breazeale and Ford[10] adapted the solution of Fubini-Ghiron, intended for fluids, to the corresponding problem in solids.

Characteristic of the work done to date on the solution of the piston problem for solids is the exclusion of effects due to fourth-order elastic coefficients. The analysis that follows indicates how these coefficients will affect the solution to the differential equation and also suggests means for including effects due to elastic coefficients higher than order four.

B. Solution of the Equation

1. Differential Equation and Its Characteristics

The partial differential equation satisfied by purely longitudinal motion at constant entropy is Eq. 13, which we rewrite as

$$\frac{\partial^2 u}{\partial t^2}=g\left(\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2},$$

$$g\left(\frac{\partial u}{\partial x}\right)=\frac{1}{\rho_0}\left[M_2+M_3\left(\frac{\partial u}{\partial x}\right)+M_4\left(\frac{\partial u}{\partial x}\right)^2+\cdots+M_{n+2}\left(\frac{\partial u}{\partial x}\right)^n+\cdots\right]. \tag{47}$$

We assume $g>0$. The characteristic curves of Eq. 47 are readily shown to have slope $dx/dt=\pm g^{\frac{1}{2}}$ in the (t,x) plane, while the changes of $\partial u/\partial t$ and $\partial u/\partial x$ along a characteristic must obey

$$d(\partial u/\partial t)=\pm g^{\frac{1}{2}}d(\partial u/\partial x), \tag{48}$$

where the $+$ sign pertains to the integration along characteristics with slope $+g^{\frac{1}{2}}$, and the $-$ sign with slope $-g^{\frac{1}{2}}$. Defining

$$\lambda\equiv-\int_{\xi_0}^{\partial u/\partial x}[g(\xi)]^{\frac{1}{2}}d\xi, \tag{49}$$

and noting that the particle velocity is

$$v\equiv\partial u/\partial t, \tag{50}$$

we see that Eq. 48 becomes

$$d(v\pm\lambda)=0. \tag{51}$$

A pair of "Riemann invariants"[16] (r,s) can be defined by

$$2r\equiv v+\lambda,\quad -2s\equiv v-\lambda. \tag{52}$$

Then, by integration of Eq. 48 along the characteristics, we see that r is constant along the characteristics of positive slope and s is constant along the characteristics of negative slope. If either r or s is the same constant *throughout a region* of the (t,x) plane, the solution in that region is called a *simple wave*.[16,11]

2. Boundary and Initial Conditions

The motion that results when a prescribed voltage is applied to a transducer attached to the end face of a specimen depends on the properties of the transducer–specimen system. The details are undoubtedly rather complicated and not strictly one-dimensional. However, it is expected that, to a very good approximation, the motion resulting from a sinusoidal voltage will be one-dimensional and sinusoidal.

To investigate the growth of harmonics, we may suppose that the half-space $x\geq 0$ is unstrained and at rest at $t=t_0$, after which the motion of the face $x=0$ is prescribed to be sinusoidal. It is well known[16] that this problem has no solution with a continuous velocity field in the region $(0\leq x\leq\infty,\ t_0\leq t<\infty)$, except for a hypothetical linear medium having $g(\partial u/\partial x)=\text{const}$. However, a simple-wave solution that is valid for short times can be found,[11–14] and the predicted harmonic growth near the piston presumably has relevance to the experimental situation.

If a small constant velocity V_0 is added to the piston's sinusoidal motion, some additional interesting features are revealed. Thus, we consider the boundary and initial conditions implied by

$$u(t,0)=(t-t_0)V_0+\bar{u}+u_0\cos(\omega t-\varphi),\quad t\geq t_0, \tag{53}$$

$$u(t,x)=\text{const}=\bar{u}+u_0\cos(\omega t_0-\varphi),\quad t\leq t_0. \tag{54}$$

The purely sinusoidal case can always be obtained by setting $V_0=0$. In the sinusoidal case, the constant $\bar{u}$ represents the piston's mean position, which need not be the same as its position at the time t_0. [For example, in order to start the piston with zero velocity, Fubini-Ghiron[14] and Blackstock[11] have considered the case $u(t,0)=A(1-\cos\omega t)$, $t>0$.]

With the usual type of nonlinearity, in which $g'<0$, it is well known that the piston initiates a shock at $t=t_0$ unless $v(t_0^+,0)\leq 0$.

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[13] W. Keck and R. Beyer, "Frequency Spectrum of Finite Amplitude Ultrasonic Waves in Liquids," Phys. Fluids 3, 346–352 (1960).

[14] E. Fubini-Ghiron, "Anomalie nella propagazione di onde acustiche di grande ampiezza," Alta Frequenza 4, 530–581 (1935).

[15] J. Melngailis, A. A. Maradudin, and A. Seeger, "Diffraction of Light by Ultrasound in Anharmonic Crystals," Phys. Rev. 131, 1972–1975 (1963).

[16] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves* (Interscience Publishers, Inc., New York, 1948).

3. *Simple-Wave Solution*

A solution describing a continuous transition of velocity from a constant initial state necessarily contains a simple-wave region.[16] In the present case, this is the region of the (t,x) plane covered by those negative-sloping characteristics that lead back to the zone of constant initial conditions (constant v and λ) without crossing any discontinuities in v or λ. We shall be concerned with the solution only in this region. Clearly, s is constant throughout this region, and each characteristic of positive slope carries constant values of v and λ, and therefore also a constant value of $\partial u/\partial x$. But $\partial u/\partial x$ determines the slope of the characteristics through the function $g(\partial u/\partial x)$. Since a line of constant slope is straight, the positive-sloping characteristics are straight lines. Because $(v-\lambda)$ is constant throughout the simple wave, we may write

$$v=\partial u/\partial t=k-\int_0^{\partial u/\partial x}[g(\xi)]^{\frac{1}{2}}d\xi. \tag{55}$$

The constant k is the value of v when $\partial u/\partial x=0$. If the medium is initially unstrained and at rest, then $k=0$. However, we carry along the k in order to make the formulas applicable to other initial conditions. The important properties of the simple wave are Eq. 55 and the fact that both v and $\partial u/\partial x$ are constant along straight lines of slope $dx/dt=[g(\partial u/\partial x)]^{\frac{1}{2}}$.

This last property can be described by saying that surfaces of constant $\partial u/\partial t$ or constant $\partial u/\partial x$ are *propagated* with the speed $|\delta x/\delta t|=g^{\frac{1}{2}}$. We denote this quantity by W:

$$W=g^{\frac{1}{2}}. \tag{56}$$

In the present notation, x is a Lagrangian or material coordinate. Hence, W is the speed of propagation in the medium relative to the *reference positions* of the particles, i.e., relative to their positions in a configuration of constant density ρ_0. The corresponding speed *in space* may be obtained as $V=v+c$, where $c=W\ \partial y/\partial x=W(1+\partial u/\partial x)$. This quantity c may be interpreted as a local propagation speed relative to the moving medium. W and c correspond, respectively, to the natural and actual speeds of propagation of small-amplitude waves in a stressed medium.[1]

Because the particle velocity is constant along a straight line of slope $W=g^{\frac{1}{2}}$, the velocity at any point (t,x) is equal to the piston velocity at the earlier time $\tau=t-x/W$. With $\partial u/\partial t$ denoted by $v(t,x)$, we have

$$v(t,x)=v(\tau,0), \tag{57}$$

$$\tau=t-x/W=t-xg^{-\frac{1}{2}}. \tag{58}$$

By differentiation of the given conditions, Eqs. 53 and 54, we obtain

$$\begin{aligned} v(\tau,0)&=V_0-v_0\sin(\omega\tau-\varphi), & \tau&>t_0,\\ v(\tau,0)&=0, & \tau&<t_0, \end{aligned} \tag{59}$$

where

$$v_0=\omega u_0 \tag{60}$$

is the velocity amplitude of the piston. Finally, by substituting for τ and $v(\tau,0)$, Eq. 57 becomes

$$\begin{aligned} v(t,x)&=V_0-v_0\sin[\omega(t-xg^{-\frac{1}{2}})-\varphi], & t-t_0&>xg^{-\frac{1}{2}},\\ v(t,x)&=0, & t-t_0&<xg^{-\frac{1}{2}}. \end{aligned} \tag{61}$$

In Eqs. 61 and 58, the function $g(\partial u/\partial x)$ is to be evaluated at the value of $\partial u/\partial x$ that corresponds, through Eq. 55, to the v under consideration. One need not be overly impressed with Eqs. 61, for they say only what the preceding equations have said: namely, that the piston velocity at the time τ, given by Eqs. 59, is the velocity all along the characteristic, Eq. 58.

The essential features of simple-wave solutions such as that described here have been known for a long time. References 11 and 16 contain clear up-to-date treatments with references to the early work of Poisson, Stokes, Earnshaw, Riemann, and others.

In Sec. III-B5, we use Eq. 55 to expand $g^{-\frac{1}{2}}(\partial u/\partial x)$ as a power series in $v-k$. When this power series is substituted into Eq. 61, there results an implicit relation for v. Following essentially the procedure of Fubini-Ghiron, we can then expand $v-V_0$ in a Fourier series and eventually obtain $v(t,x)$. Integration with respect to time will then give $u(t,x)$, except for an added function $F(x)$, which can be determined by requiring compatibility with Eqs. 54 and 55.

4. *Discontinuity Distance*

Many workers have calculated the distance at which the simple-wave solution breaks down. Unless g is constant, the waveform becomes distorted as it progresses through the medium. In a hypothetical lossless nonlinear medium such as that described by the present equations, the waveform becomes distorted to such an extent that the velocity–distance curve, plotted for some constant time, acquires a vertical tangent; i.e., $\partial v/\partial x\rightarrow\infty$. The solution cannot be continued analytically beyond the corresponding value of x. In terms of characteristics, the simple wave cannot persist indefinitely because positive-sloping characteristics corresponding to different values of v would intersect. The vertical tangent $\partial v/\partial x\rightarrow\infty$ signals the onset of a nonphysical multivaluedness that can be associated with the intersection of characteristics carrying different values of v and $\partial u/\partial x$. Hence, we are interested in the shortest distance x at which positive-sloping characteristics intersect.

Consider the positive-sloping characteristics through $(t_1,0)$ and $(t_2,0)$. When $t_2\rightarrow t_1$, the value of x at which these characteristics intersect tends to

$$x_1=2g^{\frac{3}{2}}/(g'\partial v/\partial x)=-[2g^2/g'\ddot{u}(t_1,0)], \tag{62}$$

where g and g' are to be evaluated on the characteristic through $(t_1,0)$. When $g'<0$, which is the usual case, the positive solutions of Eq. 62 correspond to positive piston acceleration $\ddot{u}$. From Eqs. 55 and 47, $\partial v/\partial x$ is a negative quantity $(-1/W)$ times the acceleration. Hence, the growth of $\partial v/\partial x$ (and of the acceleration) corresponds in this case to the familiar situation in which the velocity crests gain on the troughs so that the portions of the waveprofile for which $\partial v/\partial x<0$ become progressively steeper. If there were a medium with $g'>0$, the troughs would gain on the crests, resulting in a steepening of the portion of the profile with $\partial v/\partial x>0$. We assume that $g'<0$.

If the piston starts with a velocity jump, positive-sloping characteristics intersect already at the origin. If $v(t_0^+,0)>0$, the jump propagates as a shock with a speed $>c_0$, and there is strictly no simple-wave region. If $v(t_0^+,0)<0$, the corresponding propagation speed $<c_0$, and the velocity jump gets immediately smoothed out into a continuous rarefaction.

An approximation to the minimum positive value of x_1 satisfying Eq. 62 can be obtained by evaluating g and g' at $\partial u/\partial x=0$ and using the maximum piston acceleration ωv_0. The value of x_1 so obtained is called the discontinuity length L:

$$L=-\{2[g(0)]^2/\omega v_0 g'(0)\}. \tag{63}$$

Equation 63 may be rewritten as

$$L=-(2c_0^4\rho_0/\omega v_0 M_3)=-(2c_0^2 M_2/\omega v_0 M_3) \tag{64}$$

because, from Eq. 47,

$$g(0)=c_0^2=M_2/\rho_0, \quad g'(0)=M_3/\rho_0. \tag{65}$$

Other forms of Eq. 64 may be obtained by replacing M_3/M_2 as indicated in Eq. 43. In the case defined by Eqs. 26 and 36, the well-known results (given, e.g., in Refs. 11 and 12) are

$$L=2c_0^2/[\omega v_0(\gamma+1)], \tag{66}$$

$$L=2c_0^2/[\omega v_0(2+B/A)]. \tag{67}$$

Equation 63 must be regarded as an *approximation* (however good) to the exact minimum of Eq. 62. However, if the piston starts with zero initial velocity and positive initial acceleration ωv_0, it can be shown[11,16] that the point at which the solution first breaks down is given exactly by $x=L$, $t=L/c_0$. Blackstock[11] has discussed the effect of the piston's initial phase on the time and place where $\partial v/\partial x$ first becomes infinite, and has also calculated, for perfect gases, the distance that corresponds to the exact minimum of Eq. 62.

When the piston starts with zero velocity but nonzero initial acceleration, a surface of discontinuity in the acceleration (and in $\partial v/\partial x$) is initiated. It is interesting that Eq. 63 can be connected with a corresponding result in the theory of propagation of singular surfaces across which there is a jump in the acceleration of initial magnitude ωv_0. The magnitude of a plane jump in the acceleration propagating into a homogeneous region at rest has been shown to decay if the jump has the same sign as $g'(0)$, but to become infinite in the finite time L/c_0 if the jump and $g'(0)$ are of opposite sign.[17,18]

The blowing-up of $\partial v/\partial x$ limits the region of validity of the simple-wave solution. It is tempting to conjecture, as did Stokes,[19] that a solution can be continued by postulating that velocity jumps (shock waves) originate at the places where $\partial v/\partial x$ becomes infinite. The problem would need to be reset, however, because constant-entropy conditions do not hold across the velocity jumps. The jump conditions would relate the strengths of the jumps to their propagation speeds, but their histories would still represent unknowns that would have to be determined by considering the compatibility of the over-all solution with the boundary conditions and basic equations. The first jump would affect the time and place of formation of the next, and so on. It is not known whether there is a unique solution.[16] The problem would be formidable. Stokes wrote, "The full discussion of the motion which would take place$\cdots$, if possible at all, would probably require more pains than the result would be worth."[19] Stokes's opinion seems just as valid today, as far as an "exact" treatment is concerned. However, this emphasis on the expected "pains" is not intended to minimize the important insights that have been and can perhaps still be gained by clever approaches to the problem.

After a certain time, the simple-wave solution breaks down even near $x=0$, for it may not be continued into the range of influence of the shock that presumably originates where $\partial v/\partial x\rightarrow\infty$. Thus, with a shock forming at $x\doteq L$, $t\doteq L/c_0$, the solution near $x=0$ would be expected to break down for t greater than about $2L/c_0$, the time for reflections off the shock to arrive back at the piston.

Reasonably accurate approximate solutions in the presence of shocks are suggested by the fact that the change of entropy and the change of one of the Riemann invariants through a shock are of third order in the shock strength.[11,16] Thus, to a good approximation, the problem can be simplified by assuming that the wave behind a weak shock remains simple, and that the conditions across the shock are isentropic. Additional discussion may be found in Refs. 11 and 16 and the literature cited there.

5. *Implicit Relation for the Particle Velocity v*

Equation 61 contains $g^{-\frac{1}{2}}=1/W$, which we now wish to express in terms of v. To do this, we expand the function $1/W$ in a Maclaurin's series in the variable $(v-k)$

[17] T. Y. Thomas, "The Growth and Decay of Sonic Discontinuties in Ideal Gases," J. Math. Mech. 6, 455–469 (1957), p. 464.

[18] B. D. Coleman and M. E. Gurtin, "Waves in Materials with Memory II. On the Growth and Decay of One-Dimensional Acceleration Waves," Arch. Rat. Mech. Anal. 19, 239–265 (1965), pp. 251–252.

[19] G. G. Stokes, "On a Difficulty in the Theory of Sound," Phil. Mag. 33, 349–356 (1848), pp. 352–353.

and evaluate the derivatives in the series by means of Eqs. 55 and 56. To facilitate the calculation, we note that

$$dW/dv=[d(\partial u/\partial x)/dv][dW/d(\partial u/\partial x)] \qquad (68)$$

and, from Eqs. 56 and 55,

$$dW/d(\partial u/\partial x)=\tfrac{1}{2}g'g^{-\frac{1}{2}}=g'/2W, \qquad (69)$$

$$d(\partial u/\partial x)/dv=-g^{-\frac{1}{2}}=-1/W. \qquad (70)$$

It would be insufficient for our purpose to truncate the series with the linear term in $(v-k)$ because we wish to show explicitly that some results depend on fourth-order elastic coefficients and some do not. In general, it may be shown that the nth derivative of $(1/W)$ with respect to $(v-k)$ will contain elastic coefficients up to order $n+2$. It follows that we will have to include at least the quadratic term to account for any of the effects arising from fourth-order elastic coefficients. With just a little extra effort, we can carry along the cubic term as well. Thus, including explicitly the cubic term in $(v-k)$, we obtain

$$\frac{1}{W}=\frac{1}{c_0}-\frac{v-k}{\omega v_0 L}-\frac{\sigma(v-k)^2}{\omega v_0^2}-\frac{\tau(v-k)^3}{\omega v_0^3}+\cdots, \qquad (71)$$

where L is given by Eq. 64 and σ and τ are reciprocal lengths defined by

$$\sigma=-\frac{\omega v_0^2}{2}\left[\frac{d^2(1/W)}{dv^2}\right]_{v=k}=-\frac{\omega v_0^2}{2c_0^7}\left[\left(\frac{M_3}{\rho_0}\right)^2-c_0^2\frac{M_4}{\rho_0}\right], \qquad (72)$$

$$\tau=-\frac{\omega v_0^3}{6}\left[\frac{d^3(1/W)}{dv^3}\right]_{v=k}=-\frac{\omega v_0^3}{6c_0^4}$$

$$\times\left[\frac{3M_5}{M_2}+\frac{7}{2}\left(\frac{M_3}{M_2}\right)^3-\frac{13}{2}\frac{M_3M_4}{M_2^2}\right]. \qquad (73)$$

It suits our purpose to define a new variable

$$z=(v-V_0)/v_0 \qquad (74)$$

and a new constant

$$r=(k-V_0)/v_0. \qquad (75)$$

Then Eq. 71 can be rewritten in the form

$$\omega/W=K-(z/l)-\eta z^2-\zeta z^3+\cdots, \qquad (76)$$

where

$$K=(\omega/c_0)+(r/L)-\sigma r^2+\tau r^3+\cdots, \qquad (77)$$

$$1/l=(1/L)-2\sigma r+3\tau r^2+\cdots, \qquad (78)$$

$$\eta=\sigma-3\tau r+\cdots, \qquad (79)$$

$$\zeta=\tau+\cdots. \qquad (80)$$

By substituting Eq. 76 into Eq. 61, we obtain

$$z=-\sin\left(\omega t-\varphi-Kx+\frac{xz}{l}+\eta xz^2+\zeta xz^3+\cdots\right), \qquad (81)$$
$$t>t_0+x/W^+.$$

Here, W^+ is the value of W that corresponds to the initial velocity of the piston $v(t_0{}^+,0)=V_0-v_0\sin(\omega t_0-\varphi)$.

6. *Writing z as a Fourier Series*

We assume that (as in the solution given by Fubini-Ghiron,[14] reviewed by Blackstock[11] and by Beyer[12]) $(v-V_0)/v_0$ for $t>t_0+x/W^+$ can be written as a Fourier series:

$$z\equiv\frac{v-V_0}{v_0}=\tfrac{1}{2}A_0+\sum_{n=1}^{\infty}[A_n\cos n(\omega t-Kx)$$
$$+B_n\sin n(\omega t-Kx)],\quad t>t_0+x/W^+. \qquad (82)$$

The coefficients A_n and B_n are given by the standard formulas

$$A_n=\frac{1}{\pi}\int_0^{2\pi}z\cos n(\omega t-Kx)d(\omega t-Kx), \qquad (83)$$

$$B_n=\frac{1}{\pi}\int_0^{2\pi}z\sin n(\omega t-Kx)d(\omega t-Kx). \qquad (84)$$

If we make the change in variable

$$z=\sin\xi, \qquad (85)$$

then Eq. 81 shows that we may take

$$\omega t-Kx=\xi-\frac{x}{l}\sin\xi-\eta x\sin^2\xi-\zeta x\sin^3\xi+\varphi-\pi, \qquad (86)$$

and, by differentiating Eq. 86 at constant x,

$$d(\omega t-Kx)=\left[1-\left(\frac{x}{l}+2\eta x\sin\xi+3\zeta x\sin^2\xi\right)\cos\xi\right]d\xi. \qquad (87)$$

We evaluate the coefficients A_n and B_n only for the case $\varphi=\pi$. The simple transformations indicated in Appendix A relate these coefficients to those for other values of φ. Equations 85–87 used in Eqs. 83 and 84 yield, with $\varphi=\pi$ as stated above,

$$A_n=\frac{1}{\pi}\int_0^{2\pi}\sin\xi\cos\left[n\xi-\frac{nx}{l}\sin\xi-n\eta x\sin^2\xi-n\zeta x\sin^3\xi\right]$$
$$\times\left[1-\left(\frac{x}{l}+2\eta x\sin\xi+3\zeta x\sin^2\xi\right)\cos\xi\right]d\xi, \qquad (88)$$

$$B_n=\frac{1}{\pi}\int_0^{2\pi}\sin\xi\sin\left[n\xi-\frac{nx}{l}\sin\xi-n\eta x\sin^2\xi-n\zeta x\sin^3\xi\right]$$
$$\times\left[1-\left(\frac{x}{l}+2\eta x\sin\xi+3\zeta x\sin^2\xi\right)\cos\xi\right]d\xi. \qquad (89)$$

For an assigned n, A_n and B_n can be expanded as power series in the variables x/l, ηx, and ζx. We omit the details and list the leading terms in Eqs. 90–98 below. The terms explicitly written include all those that make any contribution up through the order M^3 when the coefficients are expanded in powers of the Mach number $M=v_0/c_0$, as is done in Sec. III-B8. It will be seen that, for $n\leq 4$, the lowest-order contribution to A_n or B_n—i.e., to the nth harmonic of $(v-V_0)/v_0$—is of order M^{n-1}. We have also expressed the results for general n (but $\zeta=0$) as an infinite series over two indices. This work, too detailed to publish here, can be made available on request.

The Fourier coefficients are, thus,

$$A_0=0. \tag{90}$$

$$A_1=(x\eta/4)-\cdots. \tag{91}$$

$$A_2=\tfrac{1}{2}(\eta/l)x^2-\cdots. \tag{92}$$

$$A_3=-(x\eta/4)+\cdots. \tag{93}$$

$$A_4=-\tfrac{1}{2}(\eta x^2/l)+\cdots. \tag{94}$$

$$B_1=1-\tfrac{1}{8}(x^2/l^2)-\cdots. \tag{95}$$

$$B_2=(x/2l)+\tfrac{1}{4}\zeta x-\tfrac{1}{6}(x^3/l^3)-\cdots. \tag{96}$$

$$B_3=\tfrac{3}{8}(x^2/l^2)+\cdots. \tag{97}$$

$$B_4=-(\zeta x/8)+\tfrac{1}{3}(x^3/l^3)+\cdots. \tag{98}$$

It is of interest to note that the integral of Fubini-Ghiron (Eqs. 88 and 89 with $\eta=\zeta=0$) yields

$$A_n\equiv 0, \tag{99}$$

$$B_n=2J_n(nx/l)/(nx/l), \tag{100}$$

in agreement with Eqs. 90–98 when η and ζ are set equal to zero.

7. *Obtaining an Explicit Expression for the Particle Displacement u in Terms of x and t*

From Eq. 82, it is merely a simple integration with respect to time that yields an expression for the particle displacement u. Thus,

$$u=u_0\sum_{n=1}^{\infty}[D_n\sin n(\omega t-Kx)+E_n\cos n(\omega t-Kx)] +(t-t_0)V_0+F(x),\quad t>t_0+x/W^+, \tag{101}$$

$$D_n=A_n/n,\quad E_n=-B_n/n. \tag{102}$$

8. *Expansion in Powers of v_0/c_0.*

The Fourier coefficients contain various powers of $(1/l)$, η, and ζ which, in turn, contain various powers of r and v_0/c_0, the latter being the ratio of the velocity amplitude of the piston to the sound speed when $\partial u/\partial x=0$. We now wish to render explicit the dependence of the Fourier coefficients on v_0/c_0 and r. We also put the whole problem in a convenient dimensionless setting by introducing the following additional notation:

$$\begin{aligned}
&G[\alpha(\partial U/\partial X)]=(1/c_0^2)g(\partial u/\partial x),\\
&M=\omega u_0/c_0=v_0/c_0,\\
&M_3/M_2=-m_3,\\
&M_4/M_2=m_3^2(1-m_4),\\
&M_5/M_2=m_3^3[-m_5+\tfrac{1}{2}(-6+13m_4)]/3,\\
&T=\omega t,\\
&U=u/u_0,\\
&X=\omega x/c_0,\\
&\alpha=m_3M,\\
&\beta=Kc_0/\omega.
\end{aligned} \tag{103}$$

It may be noted that $\partial u/\partial x=M\partial U/\partial X$. The differential equation (Eq. 47) now takes the form

$$\partial^2U/\partial T^2=G[\alpha(\partial U/\partial X)](\partial^2U/\partial X^2), \tag{104}$$

where the function $G(\alpha\partial U/\partial X)$ is

$$G(\xi)=1-\xi+(1-m_4)\xi^2 +\tfrac{1}{3}[-m_5+\tfrac{1}{2}(-6+13m_4)]\xi^3+\cdots. \tag{105}$$

The conditions Eqs. 53 and 54 imply (with $\varphi=\pi$) that

$$\begin{aligned}
U(T,0)&=(T-T_0)V_0/v_0-\cos T+\bar{U}, & T&\geqslant T_0,\\
&=\bar{U}-\cos T_0, & T&\leq T_0.
\end{aligned} \tag{106}$$

The expansions of $1/W$ in Eqs. 71 and 76 can be put in the form

$$\frac{c_0}{W}=1-\frac{c_0}{\omega L}\left(\frac{v-k}{v_0}\right)-\frac{c_0\sigma}{\omega}\left(\frac{v-k}{v_0}\right)^2-\frac{c_0\tau}{\omega}\left(\frac{v-k}{v_0}\right)^3-\cdots$$

$$=\beta-c_0z/\omega l-c_0\eta z^2/\omega-c_0\zeta z^3/\omega-\cdots. \tag{107}$$

In terms of the notation introduced in Eq. 103, the coefficients appearing in these expansions are

$$c_0/\omega L=\alpha/2, \tag{108}$$

$$c_0\sigma/\omega=-\alpha^2m_4/2, \tag{109}$$

$$c_0\tau/\omega=\alpha^3m_5/6, \tag{110}$$

$$\beta=1+\tfrac{1}{2}\alpha r+\tfrac{1}{2}m_4\alpha^2r^2+\tfrac{1}{6}m_5\alpha^3r^3+\cdots, \tag{111}$$

$$c_0/\omega l=\tfrac{1}{2}\alpha+m_4\alpha^2r+\tfrac{1}{2}m_5\alpha^3r^2+\cdots, \tag{112}$$

$$c_0\eta/\omega=-\tfrac{1}{2}\alpha^2(m_4+m_5\alpha r+\cdots), \tag{113}$$

$$c_0\zeta/\omega=\alpha^3m_5/6+\cdots. \tag{114}$$

The Fourier coefficients listed in Eqs. 90–98 now take the form of expansions in the parameter α:

$$\begin{aligned}
&A_0=0,\\
&A_1=-\tfrac{1}{8}\alpha^2X(m_4+\alpha rm_5)+\cdots,\\
&A_2=-\tfrac{1}{8}\alpha^3m_4X^2+\cdots,\\
&A_3=\tfrac{1}{8}\alpha^2X(m_4+\alpha rm_5)+\cdots,\\
&A_4=\tfrac{1}{8}\alpha^3m_4X^2+\cdots,\\
&B_1=1-\tfrac{1}{32}\alpha^2(1+4\alpha m_4r)X^2+\cdots,\\
&B_2=\tfrac{1}{4}\alpha[1+2\alpha m_4r+m_5\alpha^2(r^2+\tfrac{1}{6})]X-(1/48)\alpha^3X^3+\cdots,\\
&B_3=\tfrac{3}{32}\alpha^2(1+4\alpha m_4r)X^2+\cdots,\\
&B_4=-(1/48)\alpha^3m_5X+(1/24)\alpha^3X^3+\cdots.
\end{aligned}\tag{115}$$

9. *Evaluation of* $F(x)$

$F(x)$ may be found by matching Eq. 101 to the value along the bounding characteristic $t=t_0+x/W^+$. However, the most interesting thing about $F(x)$ is its derivative, $F'(x)$, which is the time average of the extension, $\partial u/\partial x$ at the particle x, averaged over a complete period. To obtain an expression for $F'(x)$, we first use Eq. 55 to expand $\partial u/\partial x$ as a power series in $(v-k)$. Then, taking v from Eq. 82, we can use this power series to compute an expression for $\partial u/\partial x$ in which $F'(x)$ does not appear. Comparison with the derivative of Eq. 101 then serves to determine $F'(x)$.

The expansion of $\partial u/\partial x$ as a power series in $(v-k)$ may be conveniently obtained from Eqs. 69 and 70. Since

$$d^{n+1}(\partial u/\partial x)/dv^{n+1}=-(d^n/dv^n)(1/W),\tag{116}$$

we have

$$\frac{\partial u}{\partial x}=-w+\tfrac{1}{4}m_3w^2-\tfrac{1}{6}m_3{}^2m_4w^3+(1/24)m_3{}^3m_5w^4+\cdots,\tag{117}$$

where

$$w\equiv(v-k)/c_0.\tag{118}$$

The corresponding expansion of $\partial U/\partial X$ in powers of $(v-V_0)/v_0$ $(=z)$ is

$$\frac{\partial U}{\partial X}=\frac{1}{M}\frac{\partial u}{\partial x}=e_0+e_1z+e_2z^2+e_3z^3+e_4z^4+\cdots,\tag{119}$$

where

$$e_0=r+\tfrac{1}{4}\alpha r^2+\tfrac{1}{6}m_4\alpha^2r^3+(1/24)m_5\alpha^3r^4+\cdots,\tag{120}$$

$$\begin{aligned}
&e_1=-\beta,\\
&e_2=\tfrac{1}{2}(c_0/\omega l),\\
&e_3=\tfrac{1}{3}(c_0\eta/\omega),\\
&e_4=\tfrac{1}{4}(c_0\zeta/\omega).
\end{aligned}\tag{121}$$

The quantities β, $c_0/\omega l$, $c_0\eta/\omega$, and $c_0\zeta/\omega$ have already been expanded in Eqs. 111–114.

Now $F'(x)$, being the time average of $\partial u/\partial x$ averaged over a complete period, can be obtained by taking the time average of Eq. 119. Since z itself has zero time average $(A_0=0)$, the result is

$$\frac{1}{M}F'(x)=e_0+\sum_{j=2}^{\infty}e_j\langle z^j\rangle,\tag{122}$$

where the angular brackets denote the time average. Since e_j is of order α^{j-1} for $j\geq 1$, and e_0 has been expressed only up to order α^3, we need $\langle z^j\rangle$ only to order α^{4-j}. To the required order in α, denoting

$$\theta\equiv\omega t-Kx=T-\beta X,\tag{123}$$

we find, from Eqs. 82 and 115,

$$\langle z^4\rangle=\langle\sin^4\theta\rangle=\tfrac{3}{8},\tag{124}$$

$$\langle z^3\rangle=\langle(\sin\theta+\tfrac{1}{4}\alpha X\sin2\theta)^3\rangle=0,\tag{125}$$

while in general, since $A_0=0$,

$$\langle z^2\rangle=\tfrac{1}{2}\sum_{n=1}^{\infty}(A_n{}^2+B_n{}^2).\tag{126}$$

Now all the Fourier coefficients except B_1 and B_2 have their squares of order α^4 or higher. Thus, to order α^2,

$$\langle z^2\rangle=\tfrac{1}{2}(B_1{}^2+B_2{}^2)=\tfrac{1}{2}(1-\tfrac{1}{16}\alpha^2X^2+\tfrac{1}{16}\alpha^2X^2)=\tfrac{1}{2}.\tag{127}$$

Finally, the time average of the extension $\partial u/\partial x$ is

$$\begin{aligned}
F'(x)&=M[e_0+\tfrac{1}{2}e_2+\tfrac{3}{8}e_4+\cdots]\\
&=M[r+\tfrac{1}{8}\alpha(1+2r^2)+\tfrac{1}{4}m_4\alpha^2r(1+\tfrac{2}{3}r^2)\\
&\quad+\tfrac{1}{64}m_5\alpha^3(1+8r^2+(8/3)r^4)+\cdots].
\end{aligned}\tag{128}$$

To order α^3, $F'(x)/M$ is a constant, independent of x. This constant must bear the given relation to $r=(k-V_0)/v_0$.

If $F'(x)$ is a constant, then $F(x)$ itself is $xF'(x)+C$. But the boundary condition Eq. 106 requires that $C=\bar{U}u_0=\bar{u}$. Hence, to the order of terms retained in the present paper,

$$F(x)=xF'(x)+\bar{u},\tag{129}$$

where $F'(x)$ is the *constant* in Eq. 128.

The solution obtained may be expressed as

$$\begin{aligned}
U-\bar{U}&=\sum_{n=1}^{\infty}[D_n\sin n(T-\beta X)+E_n\cos n(T-\beta X)]\\
&\quad+\frac{V_0(T-T_0)}{v_0}+\frac{XF'(x)}{M},\\
&\qquad T\geqslant T_0+Xc_0/W^+.
\end{aligned}\tag{130}$$

C. Discussion of the Solution

We suppose that the medium (with $g'<0$) is at rest and unstrained until some initial time t_0, at which the piston begins the motion (Eq. 53). Then $k=0$ ($r=-V_0/v_0$) in the simple wave. If the piston starts with positive initial velocity $v(t_0^+,0)>0$, a shock wave forms immediately at the piston and there is strictly no simple-wave region. We do not discuss this case, although the approximations mentioned at the end of Sec. III-B4 are applicable. If the piston starts with zero or negative initial velocity, there is no motion before $t=t_0+x/c_0$ ($T=T_0+X$). If the initial velocity $v(t_0^+,0)$ is zero, then $W^+=c_0$ and the solution Eq. 130 joins onto the region of constant conditions across the characteristic $t=t_0+x/c_0$. If $v(t_0^+,0)<0$, then $W^+<c_0$ and there is a "centered simple wave"[16] between the characteristic $t=t_0+x/c_0$ and the characteristic $t=t_0+x/W^+$ ($T=T_0+Xc_0/W^+$) that marks the beginning of Eq. 130. In the centered wave, all the positive-sloping characteristics in the (t,x) plane intersect at $(t_0,0)$, where the velocity jump occurs.

The velocity at any point is related to the slope of the characteristic through that point. Specifically, the value of c_0/W given by Eq. 107 is the reciprocal of the slope dX/dT of the straight-line characteristic along which the velocity has any arbitrary constant value v. From Eqs. 107–110,

$$\frac{c_0}{W}=1-\tfrac{1}{2}\alpha\frac{v}{v_0}+\tfrac{1}{2}\alpha^2m_4\left(\frac{v}{v_0}\right)^2-\tfrac{1}{16}\alpha^3m_5\left(\frac{v}{v_0}\right)^3+\cdots. \quad (131)$$

As an example, consider the case $V_0=0$, $\varphi=\pi$, $T_0=-\pi/2$. Then the piston starts suddenly at $T=T_0$ with initial velocity $v=-v_0$. In view of Eq. 131, we can see that the velocity $v=-v_0$ is first attained at the time satisfying

$$T-T_0=(1+\tfrac{1}{2}\alpha+\tfrac{1}{2}\alpha^2m_4+\tfrac{1}{16}\alpha^3m_5+\cdots)X. \quad (132)$$

The course of events can be described as follows. The medium at point X is initially unstressed, unstrained, and at rest. For $X>0$, a smooth expansion starts at the time $(T-T_0)=X$ and continues until the time given in Eq. 132, after which U has the time-periodic form given in Eq. 130 with $V_0=0$. The region of validity of the solution is further limited by the blowing up of $\partial v/\partial x$ at a distance $x\doteq L$ ($X\doteq 2/\alpha$), as discussed in Sec. III-B4.

Whenever $k=0$, different values of V_0 correspond to different values of r and hence, through Eq. 128, to different values of $F'(x)$. The case $k=V_0=0$ ($r=0$) corresponds to a nonzero $F'(x)$ given by

$$\begin{aligned}F'(x)&=\tfrac{1}{8}M\alpha(1+m_5\alpha^2+\cdots) \quad (133)\\&=\tfrac{1}{8}m_3M^2(1+\tfrac{1}{8}m_5m_3^2M^2+\cdots),\end{aligned}$$

while $F'(x)=0$ requires

$$r=-\tfrac{1}{8}\alpha+(\tfrac{1}{32}m_4-\tfrac{1}{64}m_5-1/256)\alpha^3+\cdots. \quad (134)$$

These relations illustrate the fact that the velocity and the extension at the particle x in a nonlinear medium do not both have zero time averages if they are simultaneously zero. Equation 133 gives the time average of the extension that results when the time average of the velocity is zero, while r in Eq. 134 is the value of $-V_0/v_0$ needed to bring the time average of the extension to zero.

The solution in the form given, with $F'(x)$ from Eq. 128, can be applied to a medium with constant initial strain by appropriate choice of k.

Without the specification of initial conditions, the constant k remains completely arbitrary, just as in the corresponding linear problem in which $g(\partial u/\partial x)$ is a constant. In the expression obtained by Melngailis, Maradudin, and Seeger,[15] who did not apply any initial conditions, velocity and extension are not simultaneously zero, but they both have zero time averages. To achieve this condition, we need $V_0=0$ and $k=rv_0$, where r is given by Eq. 134, in which case the constant β is

$$\beta=1+\tfrac{1}{2}\alpha r+\cdots=1-\tfrac{1}{16}\alpha^2+\cdots. \quad (135)$$

When our solution is adapted to the case treated in Ref. 15 ($M_4=M_5=0$, $V_0=0$, $\varphi=-\pi/2$, r given by Eq. 134), we find agreement to the order of the terms quoted there, i.e., α^2. In the comparison, it must be noted that our Fourier series is in the variable $(T-\beta X)$, whereas Ref. 15 used $(T-X)$. To order α^2, the only difference in the two sets of Fourier coefficients is in the coefficient of the α^2X term in the fundamental. This difference arises because, with β from Eq. 135,

$$\begin{aligned}\cos(T-\beta X)&=\cos[(T-X)-(\beta-1)X]\\&=\cos(T-X)-\tfrac{1}{16}\alpha^2X\sin(T-X)+\cdots.\end{aligned} \quad (136)$$

We have also compared our Fourier coefficients with those given for a perfect gas by Blackstock.[11] The higher-order elastic constants for a perfect gas can be expressed in terms of the single parameter γ by comparing Eq. 47 with Eq. 28. The combinations m_3, m_4, m_5 turn out to be

$$m_3=\gamma+1,\quad m_4=\frac{\gamma}{2(\gamma+1)},\quad m_5=\frac{\gamma(3\gamma-1)}{4(\gamma+1)^2}. \quad (137)$$

When these substitutions are made, our Fourier coefficients (with $r=0$) agree exactly with Eqs. 61a–61d in Ref. 11.

In accordance with Eqs. 43 and 103, the application to a fluid described by the equation of state Eq. 36 would be made by setting

$$\begin{aligned}&m_3=2+\frac{B}{A},\quad m_3^2m_4=1+\frac{B}{A}+\frac{B^2}{A^2}-\frac{C}{2A},\\&m_3^3m_5=1+\frac{3B}{2A}+\frac{3B^2}{2A^2}+\frac{7B^3}{2A^3}-\frac{C}{2A}-\frac{13BC}{4A^2}+\frac{D}{2A}.\end{aligned} \quad (138)$$

IV. INTERPRETATION AND CONCLUSIONS

The results obtained in Eq. 115 indicate the effect that higher-order elastic coefficients have in introducing distortion into an initially sinusoidal wave propagating into an elastic half-space.

We believe that the present analysis is adequate for the interpretation of ultrasonic experiments on finite-amplitude waves in media that are only very slightly dissipative. However, several questions can be raised that remain unanswered. In the first place, one may question the applicability of the boundary condition on the displacement. The waves are ordinarily excited by applying an electrical signal to a piezoelectric transducer that is bonded to one face of the specimen under test. Hence, a model in which an electrical signal is applied to a piezoelectric layer at the face of the half-space would better simulate the experiment. In this model, the outer face of the piezoelectric layer would be stressfree.

In the second place, one may question the simple-wave assumption that we have waves traveling in only one direction. A reflected wave can in principle be very nearly eliminated if detection of the waveform is accomplished by light diffraction, but additional questions concerning the planeness of the wave and its constancy across the active cross section then become more critical. Another highly successful method of detection of the harmonics[20] is by a capacitance probe at the face of the specimen opposite the transducer. In this case, there is a reflection at the stressfree surface, and one may question whether this changes the relation of the amplitudes of the harmonics to that of the fundamental. A similar question arises if the harmonics are detected by a second piezoelectric transducer.

In the third place, one may question the planeness and pure longitudinality of the wave. This question, which arises in all ultrasonic determinations of elastic properties, is usually disposed of by making the wavelength very short as compared with the transverse dimensions of the ultrasonic beam (and, of course, by using a longitudinal-wave transducer and choosing a crystal direction that permits the propagation of a purely longitudinal mode). Plane-wave analyses seem to work well when the ultrasonic beam does not intersect the lateral faces of the specimen and when the wavelength, compared with the transverse dimensions of the beam, is very very short—the shorter the better.

Finally, one may question the applicability of purely elastic theory to media that are observed to be dissipative, even if only slightly so. This last question is probably the most serious of all.

It is legitimate to raise all of the preceding questions. However, except for the question concerning dissipation, we expect them to have no effect on our conclusions. The basic phenomenon of progressive distortion is not expected to depend strongly on the detailed boundary condition at the excited face. Of course, care must be taken to avoid the generation of harmonics at the source. If the wave is detected at a reflecting face, it is presumed that experiments will be carried out using a pulse-modulated carrier with a pulse duration considerably shorter than the one-way travel time and with a pulse repetition rate slow enough that the sample reaches an essentially quiescent state before the beginning of each exciting pulse. We expect the conditions at the stressfree surface to be essentially the same as if the medium were linear, i.e., a doubling of the displacement amplitude without changing the relative amplitudes of the harmonics. The rate of harmonic growth may be changed in the region near the reflecting face where an incident pulse is overlapping its own reflection; it is for this reason that the pulse duration should be *considerably* shorter than the one-way travel time. (Otherwise, one could use a pulse that is almost as long as the roundtrip travel time.) The question of dissipation cannot be disposed of so easily, nor should it be. In highly or moderately dissipative media, the effect of dissipation must undoubtedly be taken into account.

Let it be desired to determine m_3, and ultimately the quantity M_3 in Eq. 47, from measurements on the second harmonic. We consider the driving amplitude u_0 as known, and suppose that the experimenter measures the amplitude of the second-harmonic component of the displacement

$$H_2 \equiv u_0(D_2^2+E_2^2)^{\frac{1}{2}}. \tag{139}$$

In terms of the variables defined in Eq. 103, we see from Eq. 115 that

$$\begin{aligned} D_2 &= \tfrac{1}{2}A_2 = -\tfrac{1}{16}m_3{}^3m_4X^2M^3+\cdots, \\ E_2 &= -\tfrac{1}{2}B_2 = -\tfrac{1}{8}m_3XM(1+2rm_3m_4M+\cdots)+\cdots. \end{aligned} \tag{140}$$

The terms not written explicitly in E_2 are of order M^3 and higher. For fixed X, m_3 may be determined independently of m_4 and r by finding the initial slope of the curve of (H_2/u_0) versus M, or, alternatively, by finding the intercept of the curve of (H_2/u_0M) versus M, for it follows from Eqs. 139 and 140 that

$$(d(H_2/u_0)/dM)_{M=0} = (H_2/u_0M)_{M=0} = \tfrac{1}{8}m_3X. \tag{141}$$

Thus, although for any finite driving amplitude, H_2 depends in principle on elastic constants of order higher than the third, the limiting values indicated in Eq. 141 do not. Hence, with second-order elastic constants already known, certain combinations of third-order constants can be determined by making measurements on the second harmonic as a function of driving amplitude and then making the indicated extrapolation to $M=0$.

[20] W. B. Gauster and M. A. Breazeale, "Detector for Measurement of Ultrasonic Strain Amplitudes in Solids," Rev. Sci. Instr. 37, 1544–1548 (Nov. 1966).

Let us now consider the third harmonic. We have

$$D_3=(1/24)m_3^2m_4XM^2+\cdots,\qquad E_3=-\tfrac{1}{32}m_3^2X^2M^2+\cdots. \tag{142}$$

The terms not written explicitly in Eq. 142 are of order M^3 and higher. Defining the amplitude of the third harmonic as

$$H_3=u_0(D_3^2+E_3^2)^{\frac{1}{2}}, \tag{143}$$

it can be seen that

$$\left(\frac{H_3}{u_0M^2}\right)_{M=0}=\tfrac{1}{32}m_3^2X^2\left(1+\frac{16m_4^2}{9X^2}\right)^{\frac{1}{2}}. \tag{144}$$

If X is sufficiently large that $(16m_4^2/9X^2)\ll 1$, then m_4 has negligible influence on the quantity in Eq. 144. In this case, measurement of the third-harmonic amplitude as a function of M would enable m_3 to be determined from Eq. 144, thus providing a check on the determination from the second harmonic in Eq. 141. (The quantity in Eq. 144 should, in this case, be just twice the square of the quantity in Eq. 141.) It is conceivable that sufficiently accurate measurements on both the second and third harmonics would enable some bounds to be placed on m_4.

It might be noted that the result of Eq. 141 could have been obtained from available approximate solutions to the differential equation (Eq. 47) with $M_4=0$. This is so because Eq. 140 shows the linear term in the driving amplitude M to be independent of fourth-order elastic coefficients. However, we have not found in the previous literature any proof of this independence or any formulas showing the actual variation of the various harmonics with fourth-order elastic coefficients as indicated by Eq. 115.

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Appendix A. Fourier Coefficients for Arbitrary φ

Since we have preempted A_n and B_n for the special case $\varphi=\pi$, let us rewrite the general case of Eqs. 82–84 as

$$z=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta);$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}z\cos n\theta d\theta,\quad b_n=\frac{1}{\pi}\int_0^{2\pi}z\sin n\theta d\theta;$$

$$\theta\equiv\omega t-Kx.$$

Then Eq. 86 has the form

$$\theta=\psi(\xi)+(\varphi-\pi)$$

where, from the context, $\psi(\xi)$ is the function

$$\psi(\xi)=\xi-\frac{x}{l}\sin\xi-\eta x\sin^2\xi-\zeta x\sin^3\xi.$$

By substitution, the formula for a_n becomes

$$a_n=\frac{\cos n(\varphi-\pi)}{\pi}\int_0^{2\pi}z\cos[n\psi(\xi)]\psi'(\xi)d\xi-\frac{\sin n(\varphi-\pi)}{\pi}\int_0^{2\pi}z\sin[n\psi(\xi)]\psi'(\xi)d\xi$$
$$=A_n\cos n(\varphi-\pi)-B_n\sin n(\varphi-\pi).$$

Similarly,

$$b_n=B_n\cos n(\varphi-\pi)+A_n\sin n(\varphi-\pi).$$