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Optimal pre- and post-selections of weak measurements for precision parameter estimation

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Weak measurements, which offer a significant signal enhancement with pre- and post-selections, have proven to be powerful for precision metrology. However, the measured precision of the parameter of interest is always limited by the post-selection-induced probe loss, especially in the presence of technical noise. In this paper, we investigate the optimal pre- and post-selections of weak measurements for precision metrology in the presence of technical noise. While weak measurements apply more generally, we experimentally measure the spin splitting induced by the spin Hall effect of light (SHEL) to approach this issue. The highest measured precision of the spin splitting can be obtained with the post-selected state in the nonlinear intermediate regime. Surprisingly, the pre- and post-selections for the largest weak-value amplification are incapable of estimation of the spin splitting, since the amplified shift is independent of the spin splitting. Our results are not restricted to the SHEL and could be applied to other parameter estimations with weak measurements. *Published by AIP Publishing*. [http://dx.doi.org/10.1063/1.4984264]

I. INTRODUCTION

In 1988, Aharonov *et al*. proposed weak measurements[1] based on the pre- and post-selected weak value

$$A_w = \frac{\langle \Psi_{post} | \hat{A} | \Psi_{pre} \rangle}{\langle \Psi_{post} | \Psi_{pre} \rangle}, \tag{1}$$

where Ψ_{pre} and Ψ_{post} are the pre- and post-selected states, respectively, and $\hat{A}$ denotes the observable of the system. The weak value can be arbitrarily large and even complex by choosing appropriate pre- and post-selected states. However, weak measurements have become a reliable tool for exploring the foundations of quantum mechanics, such as Hardy's paradox,[2,3] superluminal light propagation in dispersive materials,[4] and direct measurements of the quantum wave-function.[5] Recently, weak measurements have been widely used for precision estimation of the parameter of interest, such as beam shifts,[6,7] phase shifts,[8–11] frequency shifts,[12] angular rotations,[13] thickness of nanometal film,[14] magneto-optical constant,[15] and optical rotation of chiral molecules.[16]

As we all know, the measured precision is one of the most important qualities for a metrology technique. In most of these metrological experiments, weak measurements meet and even surpass the measurement precision of standard techniques limited by technical noise,[6,17–20] since weak-value amplification (WVA) helps by enhancing the signal while retaining or even suppressing the technical noise. The WVA is closely related to the pre- and post-selections [Eq. (1)]. Previous researches showed that the maximum WVA and pointer shift can be obtained by optimizing the overlap of pre- and post-selections.[21–23] However, in the presence of technical noise (such as environmental noise and the noise induced by imperfect experimental devices), the optimal pre- and post-selections for precision parameter estimation and whether the maximum WVA corresponds to the highest measured precision remain open questions.

In this paper, we investigate the measured precision of weak measurements for different post-selected states in the presence of technical noise. The spin splitting of the spin Hall effect of light (SHEL) is detected via weak measurements. The standard deviation of the amplified shift obtained from the statistics of measurements is used for estimating the measured precision of the original shift. Our results suggest that the pre- and post-selections for the largest WVA and amplified shift are incapable of estimating the original shift. The highest measured precision of the original shift can be obtained by choosing a relatively large post-selected angle in the nonlinear intermediate regime.

II. THEORETICAL ANALYSIS

We consider a two-level physical system prepared in an initial state: $|\Psi_{pre}\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$, $|0\rangle$ and $|1\rangle$ represent the eigenstates of the observable $\hat{A}$. In a quantum measurement, the system is first coupled to an ancillary meter. The interaction between the system and the meter can be represented by a unitary operation $U = \exp(-igp\hat{A})$ and then the state of the whole system is given by

$$|\Phi\rangle = \exp(-igp\hat{A})|\Psi_{pre}\rangle\phi(p). \tag{2}$$

Here, $\phi(p)$ is the wave-function of the meter and p denotes the degree of freedom used for experimental readout from the meter (pointer shift). For simplicity, assume that $\langle p\rangle = 0$. g denotes the coupling strength which is the physical

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parameter of interest. The information of the parameter can be obtained by measuring the meter distribution (i.e., the original shift)

$$\delta_{ori} = \langle\Phi|q|\Phi\rangle \leq g|a_{max}|, \tag{3}$$

here, q and p are canonically conjugate variables, and a_{max} denotes the maximum eigenvalue of $\hat{A}$. In the case of weak coupling ($g \ll 1$), the original shift is too small to be measured with standard techniques. However, after the post-selection procedure is introduced, the state of the whole system turns into

$$\begin{aligned}|\Phi'\rangle &= \langle\Psi_{post}|\exp(-igp\hat{A})|\Psi_{pre}\rangle\phi(p)\\ &\simeq \sqrt{P_{post}}\exp(-igA_w p)\phi(p).\end{aligned} \tag{4}$$

Here, $|\Psi_{post}\rangle = \mu|0\rangle + \nu|1\rangle$ with $|\mu|^2 + |\nu|^2 = 1$ is the post-selected state which is almost orthogonal to the pre-selected one, and $P_{post} = |\langle\Psi_{post}|\Psi_{pre}\rangle|^2$ is the post-selected probability. The approximation is only feasible with $g|A_w|_\Delta p \ll 1$ which is the so-called linear regime (Δp denotes the uncertainty of p). Moreover, $g|A_w|_\Delta p \gg 1$ and $g|A_w|_\Delta p \sim 1$ correspond to the inverted and nonlinear intermediate regimes, respectively.[24] Equation (4) shows that the pointer shift in q-space and p-space is, respectively, proportional to the real and imaginary parts of weak value in the linear regime, i.e., $\delta^q_{amp} \approx \mathrm{Re}(A_w)$ and $\delta^p_{amp} \approx 2g\mathrm{Im}(A_w)_\Delta p^2$, respectively.[25] However, the post-selected amplified pointer shift without linear approximation can be calculated as

$$\delta_{amp} = \frac{\langle\Phi'|M|\Phi'\rangle}{\langle\Phi'|\Phi'\rangle}, \tag{5}$$

where $M = p, q$. When weak measurements are used for precision metrology, we are always concerned about the measured precision of the parameter of interest. The standard deviation on the estimation of δ_{ori} is given by[26,27]

$$\sigma_{ori} = \frac{\sigma_{amp}}{\left|\frac{\partial\delta_{amp}}{\partial\delta_{ori}}\right|}, \tag{6}$$

here, $\partial\delta_{amp}/\partial\delta_{ori}$ depending on the pre- and post-selections indicates the sensitivity of δ_{amp} to δ_{ori}. The standard deviation σ_{amp} of the amplified shift can be obtained from the statistical results in experiments. Obviously, σ_{amp} is associated with not only the metrology method itself but also with the technical noise in experimental implementation.

To clarify the confusion mentioned above, we experimentally measure the spin splitting induced by the SHEL using weak measurements with different post-selections. Consider a Gaussian light beam [$\phi(k_y)$, the transverse distribution plays the role of meter] pre-selected in the horizontal polarization state

$$\begin{aligned}|\Psi_{pre}\rangle &= |H\rangle\\ &= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle),\end{aligned} \tag{7}$$

where $\pm$ represents the right and left circular polarizations, i.e., the eigenstates of the spin operator $\hat{\sigma}_3$. Upon reflection at an air-glass interface, the SHEL takes place as a consequence of the spin-orbital coupling, $U = \exp(-i\delta_{ori}k_{ry}\hat{\sigma}_3)$, here, $\delta_{ori} = (1 + r_s/r_p)\cot\theta_i/k_0$ denotes the spin splitting which is the parameter to be measured. Finally, the light beam is post-selected in the state

$$\begin{aligned}|\Psi_{post}\rangle &= |V\pm\Delta\rangle\\ &= \frac{i}{\sqrt{2}}\left[|-\rangle\exp(\pm\Delta) - |+\rangle\exp(\mp\Delta)\right],\end{aligned} \tag{8}$$

here, Δ is the post-selected angle and denotes the overlap of the pre- and post-selections. A large weak value $\sigma_w = \mp i\cot\Delta$ can be obtained with $\Delta \ll 1$. According to Eqs. (4) and (5), we can obtain the amplified pointer shift of the SHEL[16]

$$\delta_{amp} = -\frac{2z\delta_{ori}|\sigma_w|}{k_0\left(\omega_0^2 + \delta_{ori}^2|\sigma_w|^2\right)}, \tag{9}$$

here, ω_0 and z denote the waist and the free propagation distance of the light beam, respectively.

The sensitivity of the amplified shift to the original shift is

$$\left|\frac{\partial\delta_{amp}}{\partial\delta_{ori}}\right| = \left|\frac{2z\sin\Delta\left(\delta_{ori}^3|\sigma_w|^2 - \omega_0^2|\sigma_w|\right)}{k_0\left(\delta_{ori}^2|\sigma_w|^2 + \omega_0^2\right)^2}\right|. \tag{10}$$

Using Eqs. (6) and (10), we can obtain the deviation σ_{ori} on the estimation of the original shift from the experimental results.

III. EXPERIMENTAL OBSERVATION

Our experimental setup is shown in Fig. 1. The intensity of the Gaussian light beam produced by a He-Ne laser is adjusted by a half-wave plate (HWP), the lenses (L1 and L2) are used for beam collimation ($w_0 \simeq 27\,\mu$m and $z = 250$ mm), and two Glan polarizers (P1 and P2) play the roles of pre- and

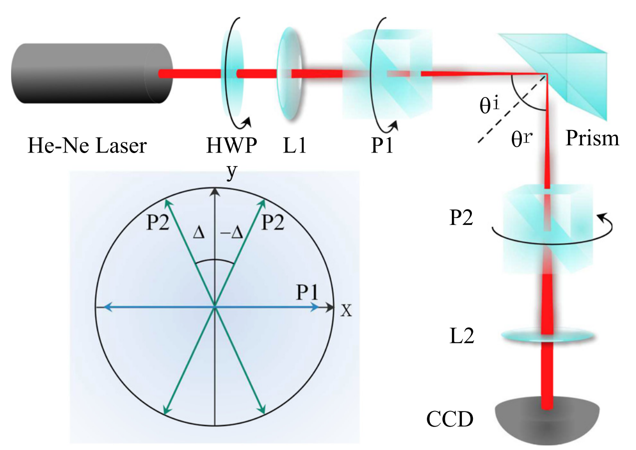

FIG. 1. Experiment setup: light source, He-Ne laser (Thorlabs HNL210L, 632.8 nm); HWP, half-wave plate; L1 and L2, lenses (f1 = 50 mm, f2 = 250 mm); P1 and P2, Glan polarizers (extinction ratio, 100 000:1); sample, BK7 prism; CCD, charge-coupled device (Thorlabs BC106N-VIS). Inset shows the pre- and post-selected states.

post-selections (see the inset). The incident angle at the prism surface was chosen as $\theta_i = 30°$ which corresponds to an original shift of $\delta_{ori} = 0.088\ \mu$m. In our experiment, the main technical noise sources are air-fluctuation, platform-vibration, and imperfections of experimental devices (such as unstability of laser, finite extinction ratio of polarizers, and electrical noise of CCD).

The experimental results of the amplified shift, along with the theoretical prediction, are shown in Fig. 2(a). For these measurements, the beam power collected by CCD is between 4.2 nW and 72 nW. We have performed the measurements with different integration time (2, 4, 10, and 20 ms). With the decrease of the post-selected angle (Δ, i.e., the overlap between the pre- and post-selections), the amplified shift increases and reaches its maximum value, then reduces to zero sharply. The measured values agree well with the theory, especially for the case with a longer integration time. The experimental standard deviation σ_{amp} of the amplified shift was obtained from the statistics of 100 measurements, see Fig. 2(b). σ_{amp} is inversely proportional to the post-selected angle Δ and the integration time. As we noted above, the post-selected probability is proportional to Δ, that is to say, a stronger signal which is more robust against the technical noise can be obtained with a larger Δ. Furthermore, by choosing a longer integration time, the influence of random noise (such as electrical, vibration, and fluctuation) can be suppressed. However, due to the intensity saturation of CCD, σ_{amp} is nearly constant with a large Δ and a long integration time.

Figure 2(c) shows the experimental values of the original shift obtained from Fig. 2(a). The experimental values show divergence from the theory when Δ is small. This is because the influence of the unwanted polarized light transmitted through polarizers is greater with respect to a weaker signal. From Eqs. (6) and (10), we can obtain the standard deviation σ_{ori} (measured precision) on the estimation of the original shift [see Fig. 2(d), this is the main result of this paper]. There is a complex relationship between σ_{ori} and Δ: σ_{ori} diverges when the pre- and post-selections become orthogonal to each other, i.e., $\Delta \rightarrow 0$. With Δ increasing, σ_{ori} reduces and then diverges again ($\Delta \simeq 0.192°$), finally, σ_{ori} reduces sharply and tends to be constant. For a fixed integration time, we found that a small σ_{ori} (i.e., a high measured precision of the original shift) can be obtained by choosing a large post-selected angle. In our experiment, most of the technical noise can be suppressed by the WVA technique with respect to the enhanced signal. The upper limit to the measured precision comes from the electrical noise of CCD which is always independent of the measurement itself. Note that, the measurement for a larger post-selected angle ($\Delta > 0.8°$) has not been performed due to the intensity saturation limit of CCD.

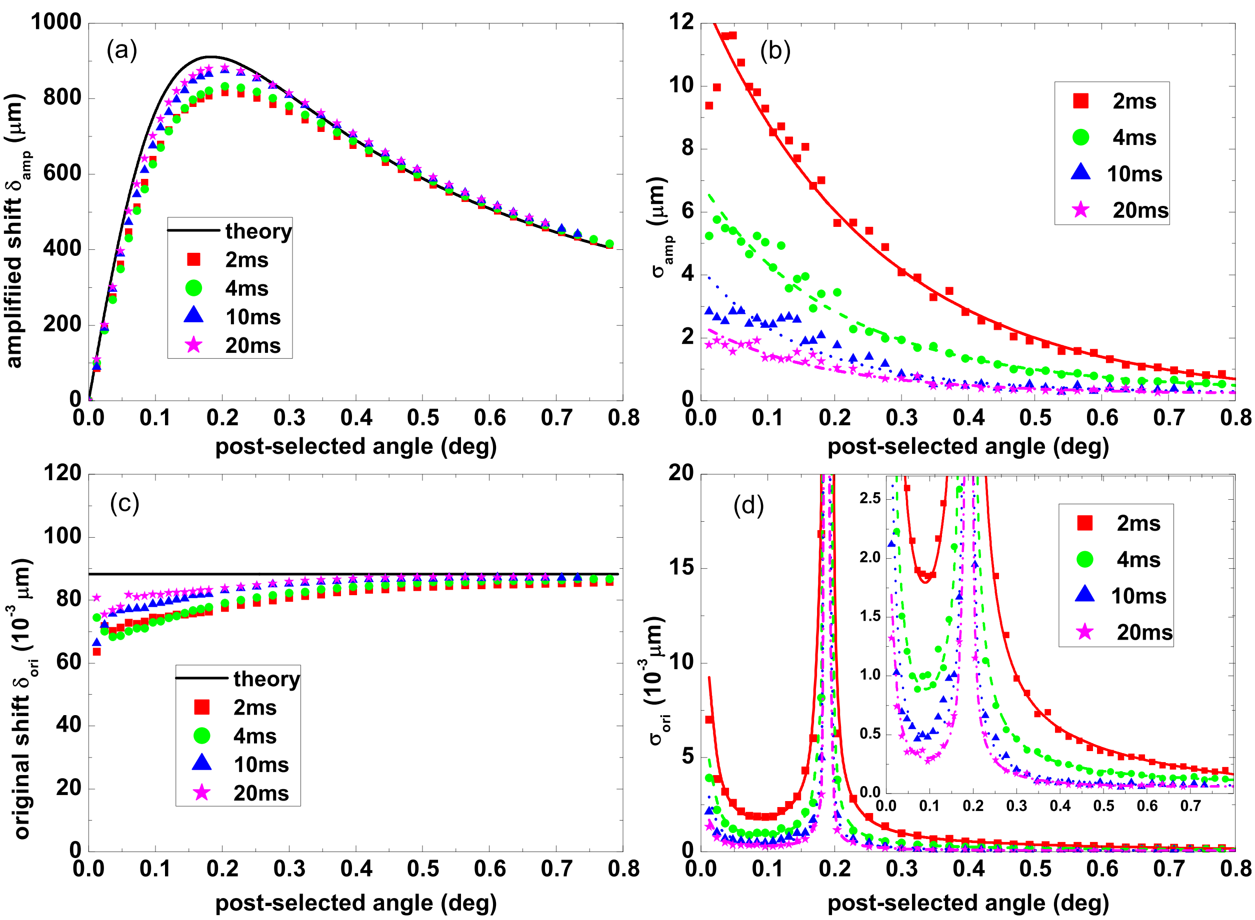

FIG. 2. Experimental results. (a) and (c) For the amplified shift δ_{amp} and the original shift δ_{ori}, respectively. The dots are experimental values and the curves denote the theoretical prediction. (b) and (d) For the standard deviation of the amplified shift and the original shift, respectively. The curves denote fit results. The integration time of CCD was chosen as 2, 4, 10, and 20 ms.

Compare Fig. 2(a) with Fig. 2(d), we can find that the largest amplified shift is not equivalent to the highest measured precision of the original shift of interest. From Eq. (6), the experimental standard deviation σ_{amp} of the amplified shift and the sensitivity to the original shift ($|\partial\delta_{amp}/\partial\delta_{ori}|$) codetermine the deviation σ_{ori} (i.e., the measured precision) of the original shift. In order to reveal the underlying mechanism, we have analyzed the relation between $|\partial\delta_{amp}/\partial\delta_{ori}|$ and the post-selected angle Δ. As shown in Fig. 3, $|\partial\delta_{amp}/\partial\delta_{ori}|$ (black solid line) reaches the maximum value at two different post-selected angles. The most interesting thing is that $|\partial\delta_{amp}/\partial\delta_{ori}|$ reduces to zero quickly, while the amplified shift (blue dashed line) reaches its maximum value ($\Delta \simeq 0.192^\circ$). This is responsible for the counterintuitive behaviour of the standard deviation of the original shift corresponding to the largest amplified shift. In addition, it is expected that $|\partial\delta_{amp}/\partial\delta_{ori}|$ will decrease for a larger post-selected angle ($\Delta > 0.8^\circ$); hence, the measured precision of the original shift cannot be improved by using the same experimental setup (the measured precision can be improved by incorporating lock-in amplifier or other metrology techniques; however, it is not the content of this work).

Furthermore, we have measured the amplified shift varying with the integration time. The measurements have been performed for three post-selections with $\Delta = 0.072^\circ$, 0.192°, and 0.42°. The standard deviation σ_{amp} of the measured results shown in Fig. 4(a) is inversely proportional to the integration time because of the suppression of random technical noise. Moreover, σ_{amp} is smaller for a larger post-selected angle Δ which leads to a more robust signal. From Eqs. (6) and (10), we obtained the standard deviation σ_{ori} of the original shift [see Fig. 4(b)]. In the cases of $\Delta = 0.072^\circ$ and 0.42°, the trend of the fit curves is similar to that of the amplified shift. However, in the case of $\Delta = 0.192^\circ$, σ_{ori} is significantly large due to the small value of $|\partial\delta_{amp}/\partial\delta_{ori}|$.

According to the above analysis, the optimal post-selected angle for the estimation of the original shift should be relatively large, rather than the angle corresponding to the largest amplified shift. To vividly clarify this point, we have theoretically investigated that the amplified shift varies with the original shift for different post-selected angles ($\Delta = 0.072^\circ$, 0.192°, 0.42°, and 0.6°, see Fig. 5). In all cases,

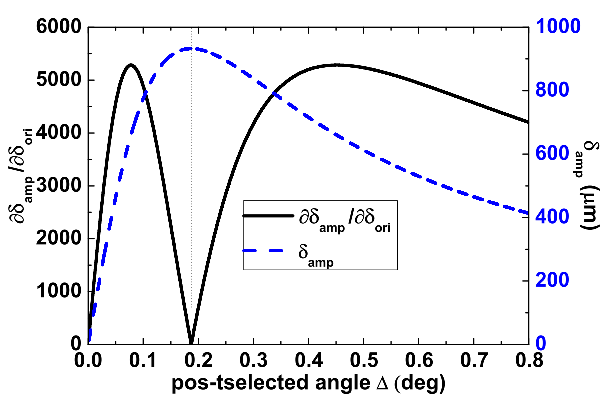

FIG. 3. The blue dashed line denotes the amplified shift. The black solid line shows the sensitivity of the amplified shift to the original shift [Eq. (10)] as a function of the post-selected angle.

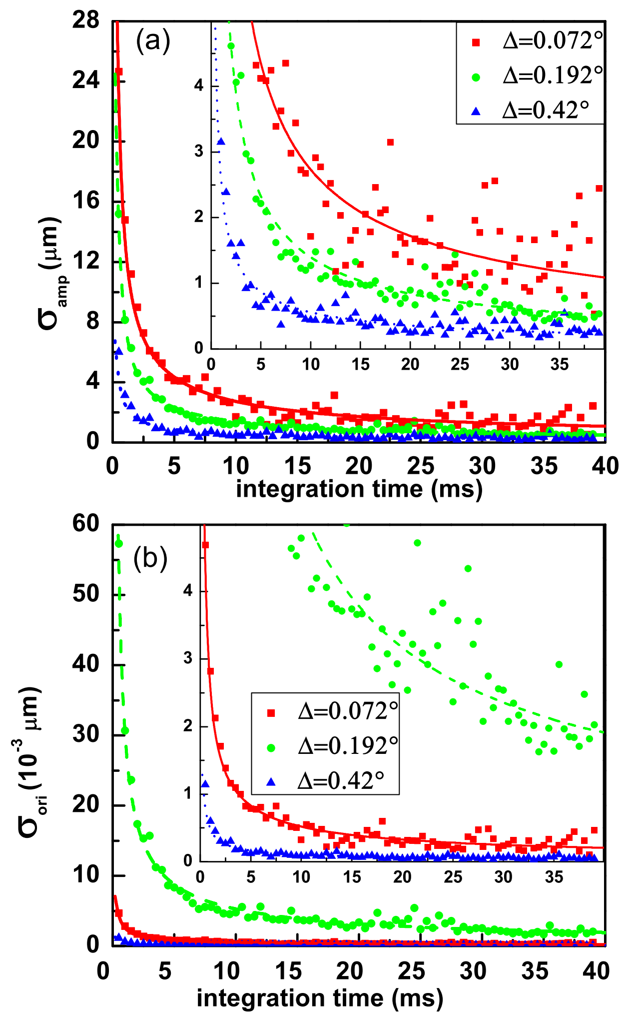

FIG. 4. The standard deviations versus the integration time of CCD choosing different post-selected angles. (a) and (b) For the amplified and original shifts, respectively. The curves denote fit results.

the original shift changes from 0.086 to 0.090 μm. In the cases of $\Delta = 0.072^\circ$, 0.42°, and 0.6° [Figs. 5(a), 5(c), and 5(d)], the variation of the amplified shift is $\sim 20\mu$m which is measurable in our experiment. In the case of $\Delta = 0.192^\circ$ [Fig. 5(b)], the amplified shift is nearly largest. However, its variation is even smaller than the experimental standard deviation of the measurements, i.e., we cannot measure the change of the original shift. In addition, we should choose a relatively large post-selected angle to obtain a robust signal which has small standard deviation. In the presence of the intensity saturation of CCD, we found that the optimal post-selection can be obtained in the nonlinear intermediate regime with $|\sigma_w|\delta_{ori}/\omega_0 \sim 1$.

IV. CONCLUSIONS

In conclusion, we have investigated the optimal pre- and post-selections of weak measurements for precision parameter estimation. The spin splitting of the SHEL plays the role of original shift and has been detected with weak measurements. We have found that, in the presence of technical noise, the optimal pre- and post-selections can be obtained

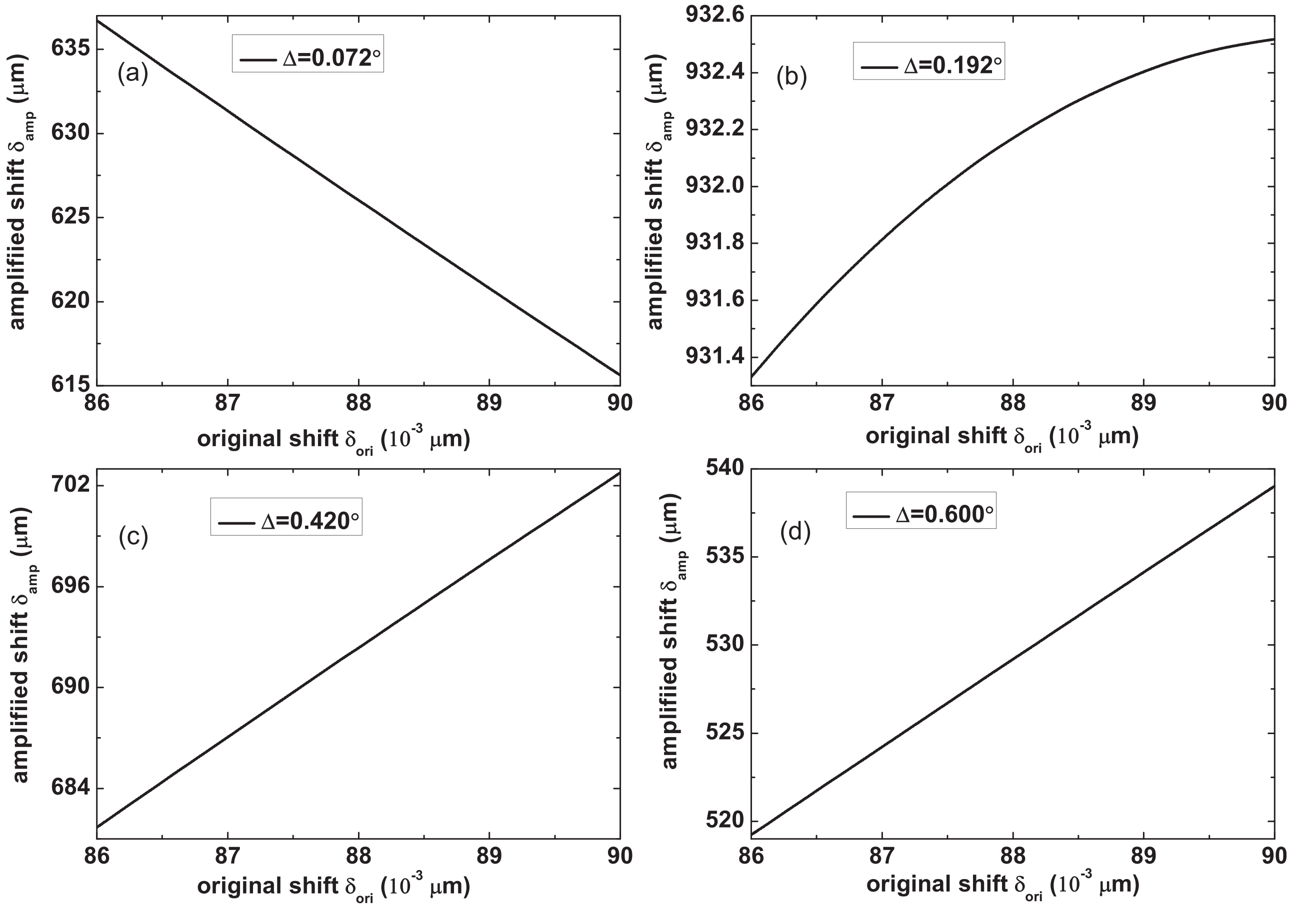

FIG. 5. The amplified shifts δ_{amp} changing with the original shift δ_{ori} for different post-selected angles.

by choosing a relatively large post-selected angle in the nonlinear intermediate regime. In addition, the largest WVA is incapable of estimating the original shift, since the amplified shift is nearly independent of the original shift with the corresponding pre- and post-selections. These results could help by acquiring the highest parameter estimation precision of weak measurements and by promoting its practical applications in the field of precision metrology.

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