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Short communication

Analytical approximations to the Lambert W function

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ABSTRACT

The Lambert W function is defined as the multivalued inverse of the function $w \mapsto we^w$. It has a wide range of applications. We propose a new method to construct a high-precision analytical approximation of the two branches of W. The method is based on Padé approximation and Schröder's iteration. This method can also be extended to solve other transcendental equations in science and engineering.

1. Introduction

Determining the roots of transcendental equations is a frequently encountered problem in science and engineering. However, it is difficult to obtain accurate analytical approximate solutions to the roots of such equations. Although there are a variety of root-finding algorithms that can be used to obtain solutions with the required precision, compared with purely numerical solutions, analytical approximate solutions provide explicit dependence of roots on the physical parameters of the problem, which is exactly what scientists and engineers expect.

This communication is concerned with the following transcendental equation

$$f(x) = xe^x - \beta = 0 \tag{1}$$

This equation appears in many scientific and engineering problems: for example, biochemical kinetics [1], ecological and evolutionary models [2], no-load induction machine speed calculation [3], the SIR epidemiological model [4], and underwater acoustic source positioning [5] and so on.

In mathematics, the Lambert W function is a set of functions, precisely the branches of the inverse function given below

$$x = W(\beta) \tag{2}$$

where W represents the solution of the Lambert Eq. (1). Note that graphical representation of the function $\beta = xe^x$ consists of two portions; a branch with $W \geq -1$ denoted as function W_0 and called the principal branch, and another with $W \leq -1$ denoted as function W_{-1}. The branch point is at $\beta = -1/e, W = -1$. The readers are referred to Fig. 1 for the details.

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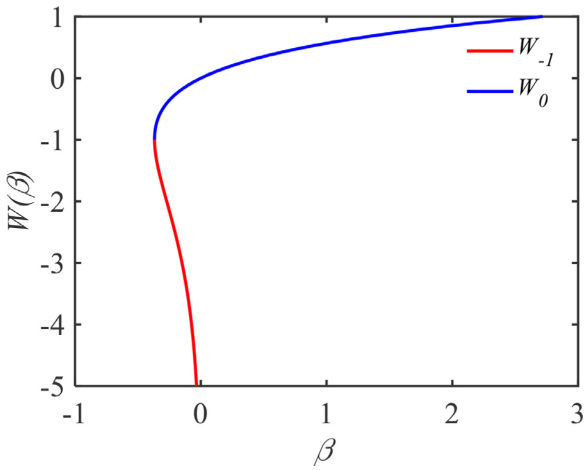

Fig. 1. Two branches of the Lambert W function.

Different techniques were proposed for solving Eq. (1) in the past. Fritsch et al. [6] suggested using a fourth-order method based on continued fraction expansion. They focused on the initial approximation in order to obtain an algorithm that provided six-digit precision by only one single iteration of the method. However, their initial approximations were less accurate. Corless et al. [7] presented comprehensive, analytical approximate methods such as Taylor series ($|\beta| < 1/e$), asymptotic expansions ($\beta > 1$ for W_0; $-1/e \leq \beta < 0$ for W_{-1}) and series expansion ($-1/e \leq \beta < 0$) about the branch point. They also described a numerical method based on Halley's iteration, in which the initial approximation was determined by the first two terms of the analytical approximations. Corless et al. [8] established a sequence of series expansions about various points for the Lambert function. The complex double-precision evaluation of the Wright ω function was further developed [9] by taking logarithms of both sides of Eq. (1) ($\beta > 0$). Chapeau-Blondeau and Monir [10] studied numerical evaluation of the branch W_{-1} of the Lambert W function with controlled accuracy. They applied the method to generate generalized Gaussian noise with exponent 1/2. In the evaluation, Halley's method was used and one-shot algorithm based on series or asymptotic expansions in [7] and rational fitting showed that the relative error was strictly lower than 10^{-4}. Veberič [11] used the iterations of Halley and Fritsch et. al [6] to solve Eq. (1), where the initial approximations were derived from the branch-point expansion [7], asymptotic series [7], rational fitting and continued-logarithm recursion. An analytical approximate solution based on Special Trans Function Theory (STFT) was also proposed [12]. Accuracy of the analytical approximate solutions discussed above is affected by the number of terms retained, and it is difficult to know how many terms should be retained such that the results attain the level of precision required.

In view of the shortcoming and algorithm inadequacy as discussed above, the purpose of this communication is to construct high-precision analytical approximations for the two branches of Eq. (2). We first use the Padé approximation [13, 14] to construct a high-precision initial approximation, then the Schröder's iteration [15] is applied to further improve precision of the initial approximation.

2. Initial approximations

In this section, we will construct the high-precision initial approximation for the two branches of the Lambert W function.

2.1. Padé approximation of exponential function

A Padé approximant is the "best" approximation of a function by a rational function of given order. By using this technique, the approximant's power series agrees with the power series of the function it targets to approximate. The Padé approximant often gives better approximation of the function than that of truncating its Taylor series, and it may still work where the Taylor series does not converge. Thus, there are many applications using this approach in science and engineering.

Given a function $f(x)$ and two integers $m \geq 0$ and $n \geq 1$, the Padé approximant of order $[m/n]$ is the rational function [13]

$$R(x) = \frac{\sum_{j=0}^{m} a_j x^j}{1 + \sum_{k=1}^{n} b_k x^k} \tag{3}$$

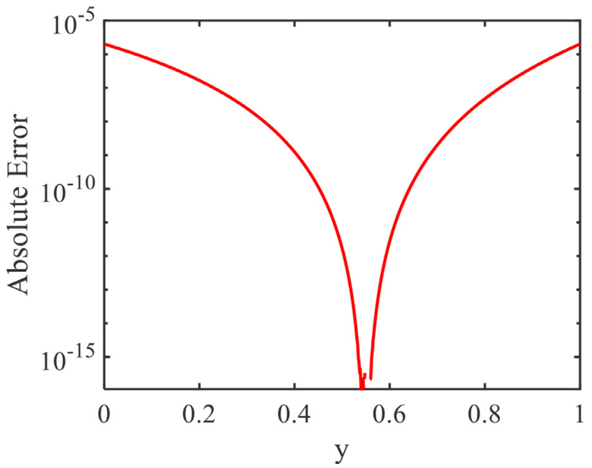

Fig. 2. Absolute error between e^{y-a} and its Padé approximation.

which satisfies $f(x) - R(x) = O(x^{m+n+1})$. The Padé approximant is unique for given m and n, that is, the coefficients $a_j (j = 0, 1, \cdots, m)$ and $b_k (k = 1, \cdots, n)$ can be uniquely determined. It is known that in many cases a higher-precision of approximation is achieved for small integers m and n. Thus the degrees of both numerator and denominator are set to be small hereafter so that analytical approximations to roots of a transcendental equation can be obtained.

The Padé approximation of order [2/3] for function e^{y-a} is

$$e^{y-a} \approx \frac{1 + \frac{2}{5}(y-a) + \frac{1}{20}(y-a)^2}{1 - \frac{3}{5}(y-a) + \frac{3}{20}(y-a)^2 - \frac{1}{60}(y-a)^3} \equiv f(y), 0 \le y \le 1 \tag{4}$$

where the constant $a = 0.548459$ is determined by ensuring that the absolute errors are equal between e^{y-a} and its Padé approximation $f(y)$ at $y = 0$ and $y = 1$. Figure 2 shows the curve of the absolute error $|e^{y-a} - f(y)|$ for $y \in [0, 1]$, from which we can observe that the absolute error is smaller than 2.02×10^{-6} for $y \in [0, 1]$. The blank gap in the figure refers to the cases where the absolute error of Matlab is indistinguishable from the rounding error.

2.2. Initial analytical approximations to the Lambert W *function*

Using Eq. (1), we have

$$\frac{d\beta}{dx} = (x+1)e^x \tag{5}$$

Based on Eq. (5), for $x > -1$, β can be derived as a monotonically increasing function of x for the branch W_0; while for $x < -1$, β is a monotonically decreasing function of x for the other branch W_{-1}.

For the branch W_0, the interval $[-1/e, +\infty)$ can be split into a union of some intervals $[ne^n, (n+1)e^{n+1})$ for $n = -1, 0, 1, \cdots$. For a given β, there is an integer $n \ge -1$ such that $ne^n - \beta \le 0$ and $(n+1)e^{n+1} > \beta$, i.e., $\beta \in [ne^n, (n+1)e^{n+1})$. For the branch W_{-1}, the interval $[-1/e, 0)$ can also be split into a union of intervals $[(n+1)e^{n+1}, ne^n)$ for $n = -2, -3, -4 \cdots$. For a given $\beta \in [-1/e, 0)$, there is also an integer $n \le -2$ such that $(n+1)e^{n+1} \le \beta$ and $ne^n > \beta$, i.e., $\beta \in [(n+1)e^{n+1}, ne^n)$. Let

$$x = n + y \tag{6}$$

then there exists $y \in [0, 1)$ for W_0 or $y \in (0, 1]$ for W_{-1}, so that $(n+y)e^{n+y} = \beta$. Substituting Eq. (6) into Eq. (1) yields

$$(n+y)e^{n+a}e^{y-a} = \beta \tag{7}$$

Replacing e^{y-a} with its Padé approximant in Eq. (4) and combining with Eq. (7) lead to

$$(n+y)e^{n+a}f(y) = \beta \tag{8}$$

Eq. (8) can be expressed as cubic equation in $u = y - a$, as

$$\left(\frac{e^{n+a}}{20\beta} + \frac{1}{60}\right)u^3 + \left(\frac{n+a}{20\beta}e^{n+a} + \frac{2}{5\beta}e^{n+a} - \frac{3}{20}\right)u^2 + \left[\frac{2(n+a)+5}{5\beta}e^{n+a} + \frac{3}{5}\right]u + \frac{(n+a)e^{n+a}}{\beta} - 1 = 0 \tag{9}$$

Solving Eq. (9) and taking the real root y within the interval $[0, 1)$ for W_0 or $(0, 1]$ for W_{-1} may produce the initial approximate solution to Eq. (1) as

$$x_0 = n + y \tag{10}$$

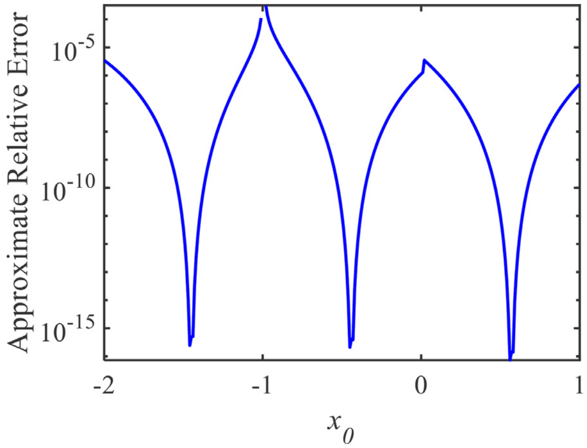

Fig. 3. Approximate relative error for the initial approximate solutions x_0.

Next, we investigate the error between the exact solution $x_e = n + y_e$ to Eq. (1) and the approximate solution $n + y$ to Eq. (10). Note that y_e and y satisfy the following two equations

$$(n+y_e)e^{n+a}e^{y_e-a} = \beta \tag{11}$$

and

$$(n+y)e^{n+a}f(y) = \beta \tag{12}$$

respectively. Combining Eqs. (11) and (12) results in

$$(n+y_e)e^{y_e-a} = (n+y)f(y) \tag{13}$$

Expanding the left-hand side of Eq. (13) into a Taylor series at y, and keeping only the linear term in $y_e - y$ yield

$$y_e - y \approx \frac{(n+y)}{(n+y+1)}\frac{(f(y)-e^{y-a})}{e^{y-a}} \tag{14}$$

Defining the approximate relative error $e_{ar} = |x_e - x_0|/|x_0|$, and using Eqs. (10) and (14) yield

$$e_{ar} \approx \frac{|e^{y-a} - f(y)|}{|n+y+1|e^{y-a}} \tag{15}$$

Therefore the approximate relative error can be estimated by using Eq. (15). The approximate relative errors of the initial approximate solution in Eq. (10) for $-2 \le x_0 \le 1$ is presented in Fig. 3. From this figure, it can be observed that, except for a small neighborhood close to the branch point at $x_0 = -1$, the approximate relative error is very small. In the other intervals not shown, the approximate relative error is similar to the corresponding part in the interval [0, 1] of Fig. 3, and it gradually decrease as the distance from the branch point increases.

2.3. Initial approximations of other methods

For comparison, we list the initial approximations of some other methods. For various expansion series below, only the first five terms are listed.

The Taylor expansion about $\beta = 0$ for a small β [7] can be expressed as

$$x_1 = \beta - \beta^2 + \frac{3}{2}\beta^3 - \frac{8}{3}\beta^4 + \frac{125}{24}\beta^5 \tag{16}$$

The branch point expansion [7] is

$$x_2 = -1 + p - \frac{1}{3}p^2 + \frac{11}{72}p^3 - \frac{43}{540}p^4 + \cdots,\ p = \pm\sqrt{2(e\beta+1)},\ \beta \ge -1/e \tag{17}$$

where sign $\pm$ correspond to W_0 and W_{-1}, respectively. The asymptotic series [7] is

$$x_3 = L_1 - L_2 + \frac{L_2}{L_1} + \frac{(-2+L_2)L_2}{2{L_1}^2} + \frac{(6-9L_2+2{L_2}^2)L_2}{6{L_1}^3} + \cdots \tag{18}$$

where $L_1 = \ln(-\beta)$, $L_2 = \ln(-\ln(-\beta))$ for $-1/e \le \beta < 0$ and $L_1 = \ln\beta$, $L_2 = \ln(\ln\beta)$ for $\beta > 1$. For $\beta > 1$, the series expansion is (Eq. (72) in [8])

$$x_4 = v + \frac{v}{1+v}p + \frac{v}{2(1+v)^3}p^2 - \frac{v(-1+2v)}{6(1+v)^5}p^3 + \frac{v(6v^2-8v+1)}{24(1+v)^7}p^4,\ v = \ln\beta,\ p = -\ln v \tag{19}$$

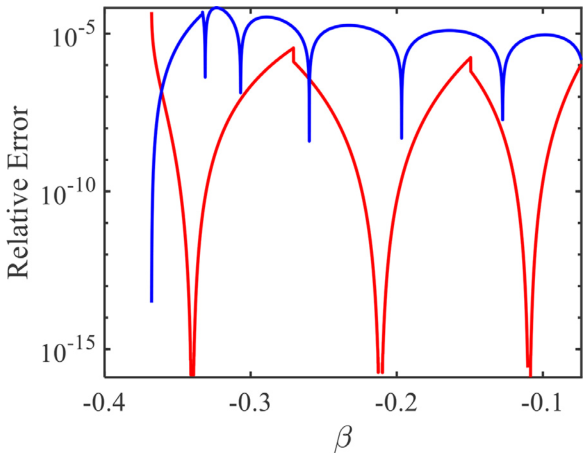

Fig. 4. Relative error of initial approximation of the Lambert function $W_{-1}(\beta)$ for $-1/e \le \beta \le -1/(5e)$. The result in [10] is shown in blue and the proposed expression in Eq. (10) in red.

A series of polynomials in $\ln(\beta/e)$ is (Eq. (78) in [8])

$$x_5 = 1 + \frac{1}{2}\ln\left(\frac{\beta}{e}\right) + \frac{1}{16}\ln^2\left(\frac{\beta}{e}\right) - \frac{1}{192}\ln^3\left(\frac{\beta}{e}\right) - \frac{1}{3072}\ln^4\left(\frac{\beta}{e}\right), \beta > 0 \tag{20}$$

For small and moderate β, the series containing only terms rational in β is (Eq. (79) in [8])

$$\begin{aligned} x_6 = {} & \frac{\beta}{e} - \frac{(1-e/\beta)\beta^2}{e^2(1+\beta/e)} + \frac{(1-e/\beta)^2\beta^3}{2e^3(1+\beta/e)^3} - \frac{(1-e/\beta)^3\beta^3(-2\beta^2/e^2+\beta/e)}{6e^3(1+\beta/e)^5} \\ & + \frac{\left[6(\beta/e)^3 - 8(\beta/e)^2 + \beta/e\right](1-\beta/e)^4}{24(1+\beta/e)^7} \end{aligned} \tag{21}$$

The STFT solution is (Eq. (17) Eq. (17) in [3])

$$x_7 = \frac{\beta\left(1+4\beta+9\beta^2/2+4\beta^3/3+\beta^4/24\right)}{1+5\beta+8\beta^2+9\beta^3/2+2\beta^4/3+\beta^5/120}(M=5) \tag{22}$$

2.4. Comparison of various initial approximations

The relative error e_r of an approximation solution x to Eq. (1) is defined as $e_r = |x - x_e|/|x_e|$ where x_e represents the exact solution (via a numerical approach). A comparison study for the relative errors of various initial approximations is presented.

The relative error of the initial approximation proposed in this paper and the initial approximation in [10] for W_{-1} is presented in Figs. 4 and 5. Note that the interval $[-1/e, 0)$ is divided into two parts to show an enlarged region close to $\beta = 0$. From these two figures, the proposed initial approximation has higher accuracy than those in [10] except a few intervals.

For W_0 in the interval $[-1/e, 0)$, Fig. 6 displays the relative error of the initial approximation proposed in this paper and the initial approximation in [3,7]. It is observed in Fig. 6 that the proposed approximation has higher accuracy except for two small intervals containing $-1/e$ and 0.

For W_0 in the interval (0, 100], the relative error of the initial approximation proposed in this paper and the initial approximation in [3,6-8] are divided into three parts as shown in Figs. 7-9. It is illustrated in these three figures that in general the proposed initial approximation has higher accuracy than other initial approximations. Furthermore, the approximate relative error in Eq. (15) indicates that, the proposed initial approximation in this paper has higher accuracy for $\beta > 100$.

3. Improving numerical accuracy by iteration

Many different iterative methods can be applied to determine the zeros of function $f(x)$. Let $\hat{f}(x) = 1/f(x)$ and $\hat{f}^{(k)}(x)$ the k-fold derivative of $\hat{f}(x)$. Schröder's iteration of the second kind of order m (S2-m formula) [15] is based on the iteration function defined by

$$K_m(f, x) = x + (m-1)\frac{\hat{f}^{(m-2)}(x)}{\hat{f}^{(m-1)}(x)} \tag{23}$$

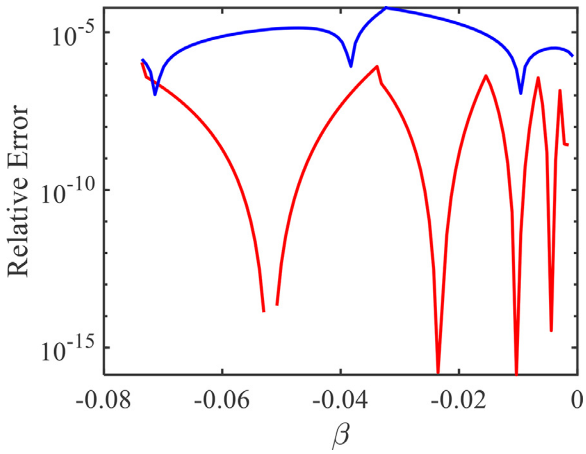

Fig. 5. Relative error of initial approximation of the Lambert function $W_{-1}(\beta)$ for $-1/(5e) < \beta < 0$. The result in [10] is shown in blue and the proposed expression in Eq. (10) in red.

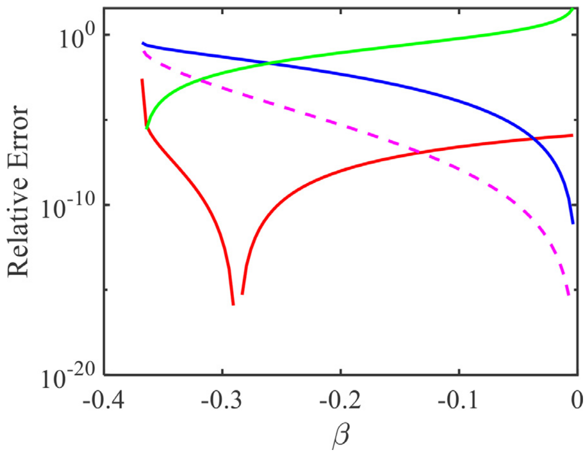

Fig. 6. Relative error of initial approximation of the Lambert function $W_0(\beta)$ for $-1/e \le \beta < 0$. The Taylor series expansion in Eq. (16) is shown in blue, the branch point expansion in Eq. (17) in green, the STFT solution in Eq. (22) in magenta dashed line and the proposed expression in Eq. (10) in red.

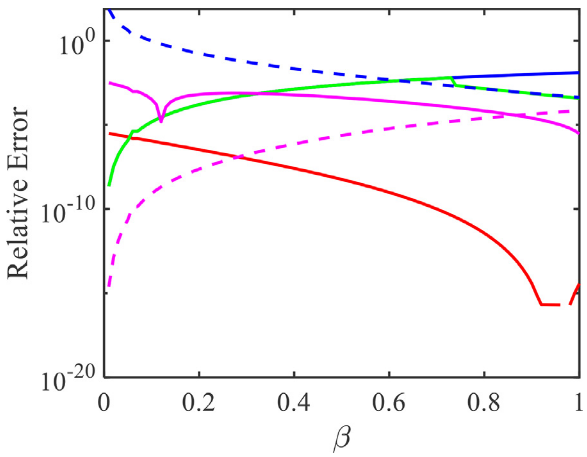

Fig. 7. Relative error of initial approximation of the Lambert function $W_0(\beta)$ for $0 < \beta \le 1$. Version A in [6] is shown in blue, Version B in [6] in green, the series expansion in Eq. (20) in blue dashed line, the series expansion in Eq. (21) in magenta, the STFT solution in Eq. (22) in magenta dashed line and the proposed expression in Eq. (10) in red.

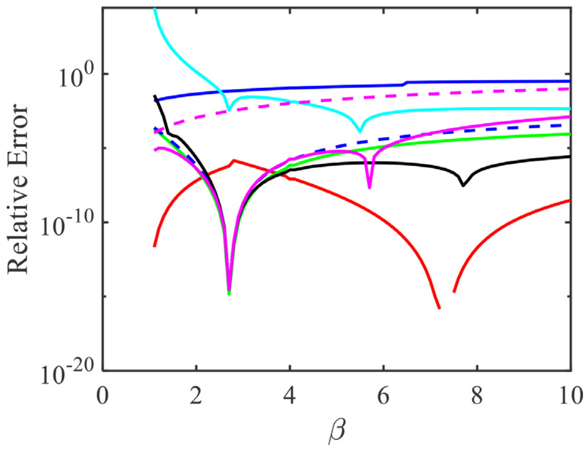

Fig. 8. Relative error of initial approximation of the Lambert function $W_0(\beta)$ for $1 < \beta \le 10$. Version A in [6] is shown in blue, Version B in [6] in green, the series expansion in Eq. (18) in cyan, the series expansion in Eq. (19) in black, the series expansion in Eq. (20) in blue dashed line, the series expansion in Eq. (21) in magenta, the STFT solution in Eq. (22) in magenta dashed line and the proposed expression in Eq. (10) in red.

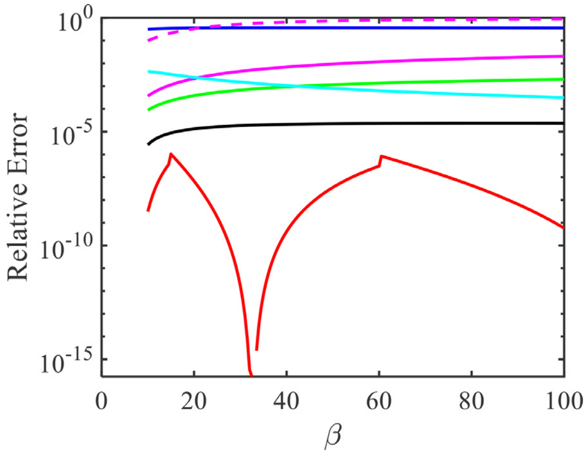

Fig. 9. Relative error of initial approximation of the Lambert function $W_0(\beta)$ for $10 < \beta \le 100$. Version A in [6] is shown in blue, Version B in [6] in green, the series expansion in Eq. (18) in cyan, the series expansion in Eq. (19) in black, the series expansion in Eq. (20) in magenta, the STFT solution in Eq. (22) in magenta dashed line and the proposed expression in Eq. (10) in red.

With an initial value x_0, the iteration formula is as follows

$$x_{k+1} = K_m(f, x_k)(k = 0, 1, \cdots) \tag{24}$$

Note that, Eqs. (23) and (24) for m=2 and m=3 correspond to Newton's and Halley's iterations [15], respectively. In a sufficiently close neighborhood of a simple zero of $f(x)$, the order of convergence for Schröder's iteration in Eqs. (23) and (24) is m.

Using x_0 in Eq. (10) as an initial approximation to solution of Eq. (1), applying Schröder's iteration in Eqs. (23) and (24) once yield three approximations to the solution of Eq. (1) as

$$x_0^{(2)} = x_0 - \frac{x_0 - \beta e^{-x_0}}{(x_0 + 1)}(m = 2) \tag{25}$$

$$x_0^{(3)} = x_0 - \frac{2(x_0 + 1)(x_0 - \beta e^{-x_0})}{2(x_0 + 1)^2 - (x_0 + 2)(x_0 - \beta e^{-x_0})}(m = 3) \tag{26}$$

$$x_0^{(4)} = x_0 - \frac{6(x_0 + 1)^2(x_0 - \beta e^{-x_0}) - 3(x_0 + 2)(x_0 - \beta e^{-x_0})^2}{6(x_0 + 1)^3 - 6(x_0 + 1)(x_0 + 2)(x_0 - \beta e^{-x_0}) + (x_0 + 3)(x_0 - \beta e^{-x_0})^2}(m = 4) \tag{27}$$

The convergence order of approximation in Eqs. (25)-(27) is m, which means that x_0= W(β)+ O(ε) will result in $x_0^{(m)}$= W(β)+O(ε^m) for m = 2, 3 and 4. The relative error of x_0, $x_0^{(2)}$ and $x_0^{(3)}$ for $0 < \beta \le 100$ is shown in Fig. 10. It should be highlighted that the numerical accuracy is remarkably improved with only one Schröder's iteration step for m = 2 and

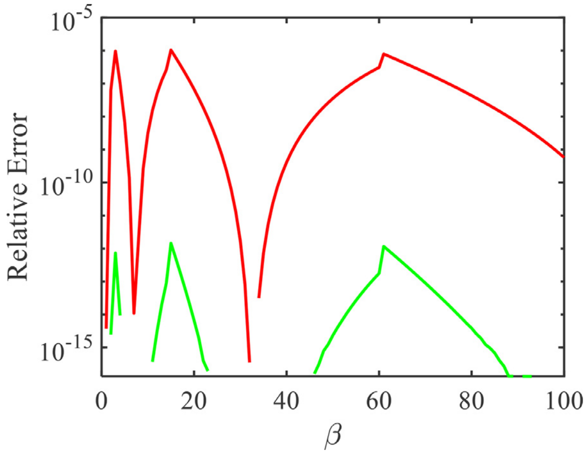

Fig. 10. Relative error of initial approximation in Eq. (10) in red, and the result for one Schröder's iteration step for $m = 2$ in Eq. (25) is in green.

3. In particular for $m = 3$ (Halley's iteration), the relative error by Matlab is indistinguishable from the rounding error. Therefore, the result for $m = 3$ is not shown in Fig. 10.

4. Conclusions

In this communication, the Padé approximation has been further developed to establish a high-precision initial analytical approximation approach for the Lambert W function. The initial approximation is valid for small as well as large value of β except for a small interval containing the branch point. With only one Schroeder iteration, a further higher-precision approximation can be achieved. The proposed method can be extended to construct explicit and accurate approximate solutions for the roots of many other transcendental equations in general.

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