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A sequentially rejective test procedure based on a modified Bonferroni inequality

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SUMMARY

A sharper Bonferroni procedure for multiple tests of significance is derived. This procedure is an improvement of Hochberg's (1988) procedure which contrasts the individual P-values with corresponding critical points. It is shown that Hochberg's original procedure is conservative, and can be made more powerful by enlarging the rejection region so that the type-one error is exactly at the nominal level. It is also shown that the modified procedure retains all the desired properties of the original procedure.

Some key words: Consonant test; Multiple tests of significance; Strong control of family-wise error rate.

1. Introduction

Multiple tests of significance based on individual P-values are considered in several recent papers. Two sequential procedures suggested by Simes (1986) and Hochberg (1988) are based on a modified Bonferroni inequality. Consider a set of ordered P-values $p_{(1)}, \ldots, p_{(n)}$ corresponding to the hypotheses $H_{(1)}, \ldots, H_{(n)}$. Let $H = \{H_{(1)}, \ldots, H_{(n)}\}$ and let H_0 be the intersection of the hypotheses in H. In what follows, H_0 is considered as the global null hypothesis. Both Simes and Hochberg reject the global null hypothesis when $p_{(i)} \leqslant c_i$ for at least one i $(1 \leqslant i \leqslant n)$, the procedures differing in the way they determine c_i. For Simes's procedure, $c_i = i\alpha/n$, whereas for Hochberg's procedure, $c_i = \alpha/(n+1-i)$. Simes proved that his test procedure has a type-one error equal to α when the P-values are independent, and suggested by simulation that his procedure controls the type-one error for a large variety of distributions, when the test statistics are correlated.

Hochberg (1988) gave a simple sequential way of making inferences on individual hypotheses. The simplicity of his procedure stems from the fact that it is able to reject at least one individual hypothesis when the global null hypothesis is rejected. This property of consonance (Gabriel, 1969) makes Hochberg's procedure easy to interpret.

Hommel (1988) extended Simes's procedure to make inferences on individual hypotheses, and subsequently (1989), showed that his procedure, although more complicated, is more powerful than Hochberg's procedure. However, as shown in the present paper, the superiority of Hommel's over Hochberg's procedure is due to the conservatism of the latter test, i.e. its size is strictly less than α for $n > 2$. This undesired property can be corrected by modifying the critical points of Hochberg's procedure. It is also shown that the modified procedure, as the original, strongly controls the family-wise error rate (Hochberg & Tamhane, 1987).

2. Modification of Hochberg's procedure

We start by considering Hochberg's procedure with the critical points α/n, $\alpha/(n-1), \ldots, \alpha$, replaced by $c_{1n}, \ldots, c_{n_n}$, respectively. To control the type-one error for testing the global null hypothesis H_0, we need

$$A_n(\alpha) = \mathrm{pr}\,(p_{(1)} \geqslant c_{1n}, \ldots, p_{(n)} \geqslant c_{n_n}) = 1 - \alpha. \tag{1}$$

Under the global null hypothesis, $p_{(1)}, \ldots, p_{(n)}$ are the order statistics of n independent uniform (0, 1) random variables with joint density $n!$ $(0 \le p_1 \le \ldots \le p_n \le 1)$; 0, otherwise. Thus (1) can be written as

$$A_n(\alpha) = \int \int \ldots \int n!\, dp_1 \ldots dp_n = 1 - \alpha,$$

where the integrals are over the ranges $(c_{n_n}, 1)$, $(c_{(n-1)_n}, p_n)$ and (c_{1_n}, p_2), respectively.

Clearly, $c_{1_1} = \alpha$. As in the original procedure, let $c_{i_j} = c_{(i+1)_{j+1}}$ $(1 \le i \le j)$. On integrating and simplifying we have

$$A_n(\alpha) = \sum_{i=0}^{n-1} c_{n_n}^i A_1(\alpha) - \sum_{i=1}^{n-1} \binom{n}{i} c_{(n-i)_n}^{n-i} A_i(\alpha) = 1 - \alpha.$$

Letting $A_i(\alpha) = 1 - \alpha$ $(i = 1, \ldots, n)$, we obtain the recurrence relationship

$$\sum_{i=1}^{n-1} c_{n_n}^i - \binom{n}{i} c_{(n-i)_n}^{n-i} = 0. \quad (2)$$

Iteratively substituting $n = 2, 3, \ldots$ in (2), we obtain the modified critical points in Table 1.

The change in the critical points is more pronounced for $\alpha = 0{\cdot}05$. Since the modified critical points are greater than the original critical points, except for $n \le 2$, the modified procedure always rejects the global null hypothesis whenever the original one does. The closure principle of Marcus, Peritz & Gabriel (1976), ensures that the modified procedure strongly controls the family-wise error rate at the designated significance level α.

Table 1. *Critical points for the modified,* MH, *and original,* H, *procedures of Hochberg*

	$\alpha = 0{\cdot}05$		$\alpha = 0{\cdot}01$	
i	MH	H	MH	H
1	$5{\cdot}00 \times 10^{-2}$	$5{\cdot}00 \times 10^{-2}$	$1{\cdot}00 \times 10^{-2}$	$1{\cdot}00 \times 10^{-2}$
2	$2{\cdot}50 \times 10^{-2}$	$2{\cdot}50 \times 10^{-2}$	$5{\cdot}00 \times 10^{-3}$	$5{\cdot}00 \times 10^{-3}$
3	$1{\cdot}69 \times 10^{-2}$	$1{\cdot}67 \times 10^{-2}$	$3{\cdot}34 \times 10^{-3}$	$3{\cdot}33 \times 10^{-3}$
4	$1{\cdot}27 \times 10^{-2}$	$1{\cdot}25 \times 10^{-2}$	$2{\cdot}51 \times 10^{-3}$	$2{\cdot}50 \times 10^{-3}$
5	$1{\cdot}02 \times 10^{-2}$	$1{\cdot}00 \times 10^{-2}$	$2{\cdot}01 \times 10^{-3}$	$2{\cdot}00 \times 10^{-3}$
6	$8{\cdot}51 \times 10^{-3}$	$8{\cdot}33 \times 10^{-3}$	$1{\cdot}67 \times 10^{-3}$	$1{\cdot}67 \times 10^{-3}$
7	$7{\cdot}30 \times 10^{-3}$	$7{\cdot}14 \times 10^{-3}$	$1{\cdot}43 \times 10^{-3}$	$1{\cdot}43 \times 10^{-3}$
8	$6{\cdot}39 \times 10^{-3}$	$6{\cdot}25 \times 10^{-3}$	$1{\cdot}26 \times 10^{-3}$	$1{\cdot}25 \times 10^{-3}$
9	$5{\cdot}68 \times 10^{-3}$	$5{\cdot}56 \times 10^{-3}$	$1{\cdot}12 \times 10^{-3}$	$1{\cdot}11 \times 10^{-3}$
10	$5{\cdot}11 \times 10^{-3}$	$5{\cdot}00 \times 10^{-3}$	$1{\cdot}01 \times 10^{-3}$	$1{\cdot}00 \times 10^{-3}$

Since rejecting the global null hypothesis is equivalent to rejecting at least one individual hypothesis, the modified procedure, as the original, is consonant.

As in Hochberg's original procedure, the inferences on the individual hypotheses can be done in the following simple sequential way: if $p_{(n)} \le c_n$ then all $H_{(i)}$'s are rejected; otherwise, $H_{(n)}$ cannot be rejected and one goes on to compare $p_{(n-1)}$ with c_{n-1}. If smaller, then all the remaining hypotheses are rejected; otherwise, $H_{(n-1)}$ cannot be rejected, and one goes on to compare $p_{(n-2)}$ with c_{n-2}, etc.

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