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A simple approximation for the current-voltage characteristics of high-power, relativistic diodes

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A simple approximation for the current-voltage characteristics of a relativistic electron diode is presented. The approximation is accurate from non-relativistic through relativistic electron energies. Although it is empirically developed, it has many of the fundamental properties of the exact diode solutions. The approximation is simple enough to be remembered and worked on almost any pocket calculator, so it has proven to be quite useful on the laboratory floor. [http://dx.doi.org/10.1063/1.4953654]

I. INTRODUCTION

With the advent of modern high-speed, multiple-core desktop computers, predicting the space-charge limited current for any voltage applied to a high-power diode is simply a matter of running a particle in cell (PIC) or ray-trace simulation using the actual diode geometry. Examples of commercially available codes suitable for this purpose are LSP (PIC)[1] and Trak (ray-trace).[2] On the other hand, it is often useful to have a simple formula to interpolate sparse experimental or simulation data. This note describes such a formula that the author has found valuable over many years of work with several different diodes in both the relativistic and non-relativistic regimes.[3–5]

II. UN-MAGNETIZED DIODES

Following Langmuir,[6–9] in the non-relativistic regime, the current/voltage characteristic can be universally expressed as

$$I = K_G V^{3/2}, \tag{1}$$

where K_G is only a function of the diode geometry and is independent of voltage.[9] This universal law can be expressed in convenient non-dimensional form as

$$4\pi j d^2/I_0 = \eta (V/V_0)^{3/2}, \tag{2}$$

where j is the current density at the emission surface, and V is the voltage of the anode with respect to the cathode.

The constants in this equation are $I_0 = 4\pi m_e c^3/e = 17.050$ kA and $V_0 = 1.36 m_e c^2/e = 695$ kV. In this formulation, the geometric constant η is unity for a planar diode of infinite extent, and a function of the anode-cathode (AK) gap d (cm) and the emission area A (cm^2) otherwise. It follows that the current for a given voltage remains the same if all dimensions are scaled by the same factor. That is, η remains unchanged for a proportional scaling of dimensions.[9] Generations of vacuum tube engineers provided ample experimental data validating the universality of this law for widely varied diode geometries and sizes.

The relativistically correct solution for space charge limited flow is arrived at by solving Poisson's equation in one dimension with conservation of charge and relativistic dynamics. This approach reduces to solving the equation

$$\frac{d^2\gamma}{dx^2} = \frac{ej}{m_e c^3 \varepsilon_0} \frac{\gamma}{\sqrt{\gamma^2 - 1}}, \tag{3}$$

where γ is the Lorentz relativistic mass factor.[10] The resulting relativistically correct law has the form

$$4\pi j d^2/I_0 = \eta f\,(V/V_1)\,, \tag{4}$$

where $f\,(V/V_1)$ is the exact planar diode solution, and V_1 is a constant. This solution can be expressed as an infinite series[11] or as elliptic integrals.[10] Eq. (4) converges to Eq. (2) in the non-relativistic regime ($eV \ll m_e c^2$) and to $4\pi j d^2/I_0 \propto V/V_1$ in the relativistic regime ($eV \gg m_e c^2$). Here, $V_1 = m_e c^2/2e = 256$ kV. The arguments of Langmuir and Compton in Ref. 9 hold equally well in the relativistic regime, so it follows that η in Eq. (4) is also independent of voltage, and only a function of geometry.

The exact solutions are unwieldly, so simpler approximations have been sought.[10–13] However, although elegant, some approximations are themselves somewhat cumbersome, or fail in either the relativistic or non-relativistic limits ($I \propto V$ or $I \propto V^{3/2}$, respectively). For many years, I have used a simple approximation that does follow the proper scaling in these limits. This is

$$4\pi j d^2/I_0 = \eta\,(V/V_2)\left[1 - \exp\left(-\sqrt{V/V_3}\right)\right], \tag{5}$$

where the constants V_2 and V_3 are chosen to give the best fit to the exact planar solution over the voltage range of interest with η set to unity. The geometric constant η is then chosen by fitting to experimental data or to simulations of the actual geometry. This fit to theory is accurate over a wide range of energies, only requires remembering two or three constants, and is simpler than previous fits,[12,13] so it is perhaps more useful in applicable conditions.

For example, over the energy range 0 < Vak < 15 MV, Eq. (5) fits the exact Jory-Trivelpiece planar solution with $V_2 = 0.293$ MV and $V_3 = 2.83$ MV with an average rms error of less than 0.003%. (I do not plot a comparison here,

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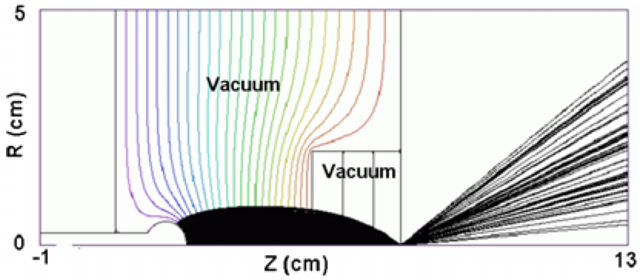

FIG. 1. 14-MV grid-focused diode showing 45-kA beam and applied equipotential contours. The AK gap spacing is 3 cm and the emission area is 0.785 cm^2.

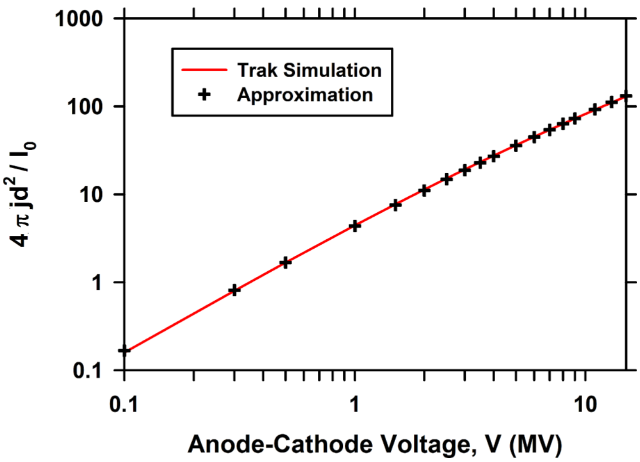

FIG. 2. Current-voltage characteristic for the grid-focused diode. The solid red curve is the Trak simulation, and the crosses are from the approximation to a planar diode ($V_2 = 0.293$ MV and $V_3 = 2.83$ MV) multiplied by the geometrical factor $\eta = 2.86$.

because the two curves are indistinguishable.) An example of relativistic space charge limited flow in a distinctly non-planar geometry is afforded by a pulse-power driven 14-MeV diode with focusing grids proposed for radiography.[14] Fig. 1 shows the geometry and focused beam produced by this diode, and Fig. 2 shows the current-voltage characteristic as calculated by the Trak e-gun design code.[2] Also shown in Fig. 2 is the approximation Eq. (5) with V_2 and V_3 set as above for the relativistic planar diode, and $\eta = 2.86$, which gave a fit with an average rms error of less than 0.03%. This is an excellent example of the universality of the relativistic solution, Eq. (4).

III. MAGNETIZED DIODES

So far we have only discussed diodes with no externally applied magnetic fields, so conservation of energy and continuity enables one to relate beam current density to AK voltage through solutions to Poisson's equation (e.g., Eq. (3)). Adding an external magnetic field adds the additional constraint of conservation of canonical angular momentum, and η is no longer independent of AK gap voltage.[15] Nevertheless, the scaling of approximation (Eq. (5)) is still quite accurate, albeit with different values of V_2, V_3, and η.

As an example of applying Eq. (5) to a diode with an external magnetic field, we consider the injector diode for the long-pulse linear induction accelerator (LIA) at Los

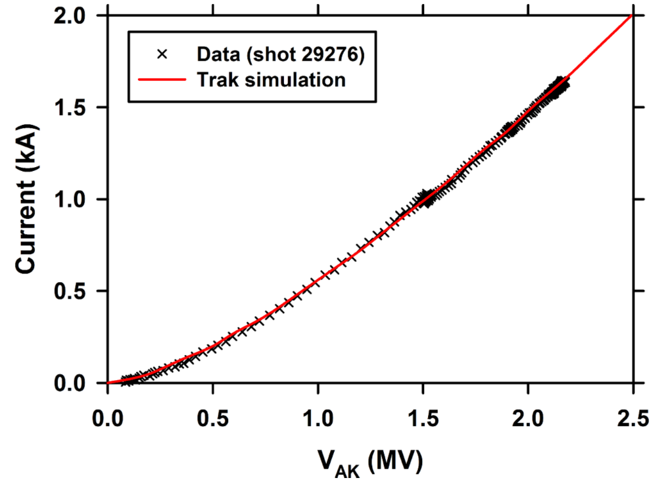

FIG. 3. Diode current-voltage data for a single shot of Los Alamos long-pulse LIA compared with Trak simulation (solid red curve). There are 445 measured data on this plot ranging from the non-relativistic regime through fully relativistic.

Alamos National Laboratory.[5,16] This shielded diode (zero total magnetic flux linking the cathode) has a voltage pulse with a slow rise time (~500 ns) and a long (>1 μs) flattop, so it is possible to trace the I-V characteristic from the non-relativistic regime into the relativistic regime on each and every shot. For example, shown in Fig. 3 is the current/voltage characteristic for a recent single shot with data measured every 1 ns. The solid red curve shows the prediction of this characteristic by Trak simulations that used the actual diode geometry and applied axial magnetic field. Evidently, the data and simulation agree throughout the range, from non-relativistic to fully relativistic.

In order to directly compare with data and Trak simulations, we set the left hand side of the approximation Eq. (5) equal to the diode current (in kA). Fig. 4 shows the Trak simulation compared to the so modified approximation with V_2 = 0.531 MV/kA, V_3 = 8.091 MV, and η = 1.

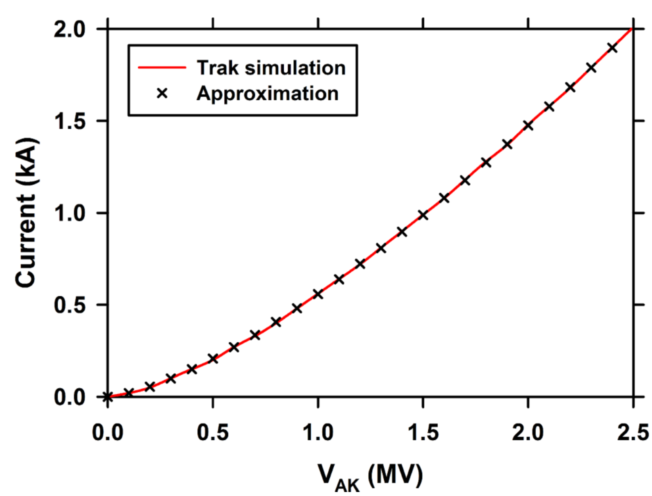

FIG. 4. The approximation $I = (V/V_2)[1 - \exp(-\sqrt{V/V_3})]$ with V_2 = 0.531 MV/kA and V_3 = 8.091 MV compared to Trak simulation of the Los Alamos long-pulse LIA diode. This comparison is over the same range as the data shown in Fig. 3.

IV. CONCLUSION

In summary, the approximation Eq. (5) is accurate over relativistic and non-relativistic regimes for diodes with, or without, applied magnetic fields. For un-magnetized diodes, it obeys the geometric universality derived by Langmuir.[9] For specific experiments, only two constants and the simple formula need to be remembered, making it useful for calculations "on the fly."

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