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Atomic orbital search: A novel metaheuristic algorithm

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A R T I C L E I N F O

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A B S T R A C T

In this paper, the Atomic Orbital Search (AOS) is proposed as a novel metaheuristic algorithm for optimization purposes. The main concept of this algorithm is based on some principles of quantum mechanics and the quantum-based atomic model in which the general configuration of electrons around nucleus is in perspective. In order to evaluate the performance of this algorithm, a total number of 20 unconstrained mathematical test functions are utilized with different dimensions of 2–100 while a maximum number of 150,000 function evaluations is considered with 100 independent optimization runs for statistical purposes. A complete statistical analysis is also conducted by utilization of the Kolmogorov Smirnov, Wilcoxon and the Kruskal Wallis tests while 8 metaheuristics are also utilized as alternatives for comparative purposes. The latest Competitions on Evolutionary Computation (CEC) regarding the single objective real-parameter numerical optimization (CEC 2017) including 30 benchmark test functions is also considered in which the capability of the proposed algorithm is compared to the most state–of-the–art algorithms in the optimization field. In addition, a total number of 5 constrained engineering design problems are utilized as design examples including some of the constrained optimization problems of the recent Competitions on Evolutionary Computation (CEC 2020). The obtained results of the AOS algorithm in dealing with the constraint problems are compared to the results of different standard, improved and hybrid metaheuristic algorithms from the literature. The obtained results demonstrate that the proposed AOS algorithm provides very outstanding results in dealing with the mathematical and engineering design problems.

1. Introduction

Optimization is a numerical technique for solving real-life problems in different areas as diverse as engineering, management, accounting, computer science etc. In recent century, there has been an increasing desire in improving human activities in different fields. Optimization is considered as a proper tool for identifying better solutions for a design problem by utilization of some well-conceptualized optimization algorithms with well-established mathematical models. Based on the multiple applications of optimization in different fields, there is an increasing interest in proposing novel algorithms or even improving or hybridizing the existing ones in which some strong attachments to other scientific subjects including deep interactions with other aspects of mathematical computation can be considered.

In order to conduct an optimization process optimal configuration of a specific design problem, two important aspects should be clarified including a proper mathematical model for the design problem and a well-established algorithm for optimization purposes. The various parameters of the considered design problems should be determined through a fully-

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detailed mathematical model which includes the objective function alongside the required design constraints. By utilization of this model, an optimization algorithm should be conceptualized and formulated for solving the fully-defined design procedure in order to provide a feasible solution by consideration of design constraints. By repeating the mentioned process, some optimal solutions are obtained which can be utilized for construction purposes. Based on the multiple deficiencies of classical optimization algorithms including the gradient-based methods, some new optimization algorithms called "Meta-heuristics" have been proposed for optimization purposes. Some of the most successful metaheuristics algorithms are the Genetic Algorithm (GA) proposed by Holland [1] in which the natural selection and biological evolution are in perspective, Ant Colony Optimization (ACO) presented by Dorigo et al. [2] which is developed based on foraging behavior of ants, Particle Swarm Optimization (PSO) developed by Eberhart and Kennedy [3] which simulates social behavior of birds, Imperialistic Competitive Algorithm (ICA) presented by Atashpaz-Gargari and Lucas [4] which is developed based on human social evolution, Firefly Algorithm (FA) proposed by Yang [5] which is inspired based on the flashing behavior of fireflies, Whale Optimization Algorithm (WOA) developed by Mirjalili and Lewis [6] which mimics the hunting mechanism of humpback whales, Bees Algorithm (BA) proposed by Pham et al. [7] which is developed based on food foraging behavior of honey bee colonies, Charged System Search (CSS) developed by Kaveh and Talatahari [8] by using some principles of physics and mechanics, Jaya Algorithm (JA) presented by Rao [9] which is developed based on behavior of humans in achieving victory, Rain-fall Optimization Algorithm (ROA) proposed by Kaboli et al. [10] which mimics the nature of rain drops during rain falls, Flower Pollination Algorithm (FPA) presented by Yang [11] which is inspired by the pollination process of flowers, Earthworm Optimization Algorithm (EWA) proposed by Wang et al. [12] which is inspired based on reproduction process in earthworms and the Chaos Game Optimization (CGO) proposed by Talatahari and Azizi [13,14] which is developed based on chaos game theory and fractals. It also should be noted that some of the standard algorithms have been improved or hybridized for specific applications. The upgraded version of the WOA [15], quantum-behaved Developed Swarm Optimizer [16], Tribe−Charged System Search [17], Multiverse Optimizer [18], Tribe−Interior Search Algorithm [19] and hybrid Ant Lion Optimizer and Jaya algorithm [20] are some of the recently developed approaches.

In this paper, the Atomic Orbital Search (AOS) algorithm is proposed as a novel metaheuristic algorithm for optimization purposes. In this algorithm, the basic principles of quantum mechanics are utilized as inspirational concept while the methodology of quantum-based atomic model and the general configuration of electrons around nucleus are also in perspective. In recent years, the general concepts of atomic physics have been utilized as inspirations for novel metaheuristic algorithms. Zhao et al. [21] proposed Atom Search Optimization (ASO) algorithm for optimum design purposes in which the atomic motion model in nature is in perspective while the interacting forces between atoms are considered as key aspect of the mathematical model of this algorithm. Yildirim and Karci [22] developed Artificial Atom Algorithm (AAA) for optimization of discrete design problems which is inspired by chemical compounding process including the chemical ionic and covalent bonds. Biswas et al. [23] proposed an atomic model based optimization algorithm called Atom stabilization Algorithm (ASA) which mimics the excitation and dE−excitation mechanism of atoms in nature. Since the general principles of atomic physics have been utilized for developing novel algorithms, some other aspects of atomic physics form quantum point of view can also be utilized in this purpose. The basic principles of the electron density configuration and the absorption or emission of energy by atoms which are considered in the quantum-based atomic theory, are utilized as the main idea of the AOS optimization algorithm. This concept is utilized for the first time in developing a metaheuristic algorithm so the novelty of this paper can be considered from inspirational point of view while the complexity level of the utilized test functions is another aspect that has been considered for the first time in this paper.

A total number of 20 mathematical functions with different dimensions of 2–100 are utilized as test functions for performance evaluation of the proposed algorithm while a maximum number of 150,000 function evaluations is considered in the optimization process. For comparative purposes, a total number of 8 alternative metaheuristic algorithms are utilized while the statistical results for the alternative algorithms and the proposed AOS algorithm are provided by considering the best, mean, and the standard deviation of 100 independent optimization runs. Moreover, some of the well-known statistical analysis such as the Kolmogorov Smirnov (KS) test, Wilcoxon (W) sign rank test and the Kruskal Wallis (KW) test are also conducted for having a better judgment about the capability of the proposed algorithm.

One of the most important deficiencies of the recently proposed metaheuristic algorithms is the simplicity of the utilized test functions for evaluation purposes. In the last decade, multiple Competitions on Evolutionary Computation (CEC) have been held for different purposes in which different test functions with different levels of complexity have been proposed for detailed evaluation purposes. In this regard and in order to overcome this gap, the latest CEC regarding the single objective real-parameter numerical optimization as CEC 2017 [24] including 30 benchmark test functions is considered in which the capability of the proposed AOS algorithm is compared to the most state−of-the−art algorithms participated in this competition. In addition to the mentioned gap in recent attempts for optimization purposes, a fully detailed computational complexity analysis is also required for having a valid judgment in presenting a novel algorithm. In this purpose, a complete computational cost and complexity investigation is also conducted based on the computational scheme of the CEC 2017 and the "Big O notation". In order to evaluate the performance of the proposed algorithm in dealing with some complex problems, a total number of 5 constrained engineering design problems are considered which were presented in the latest CEC in constrained optimization problems as CEC 2020 [25]. In this regard, a total number of 25 independent optimization runs are considered in the optimization process for statistical purposes with a maximum number of 200,000 function evaluations. The obtained results of the AOS algorithm are compared to the results of different standard, improved and hybrid metaheuristic algorithms from the literature.

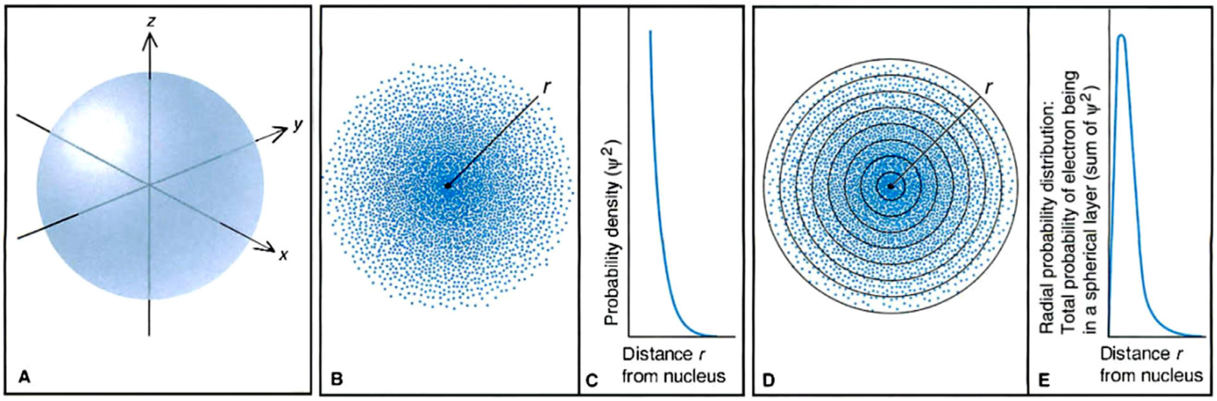

Fig. 1. Electron density configuration of an atom [26].

Based on the fact that the considered mathematical and constrained engineering design problems for testing the novel AOS algorithm are some of the well-known problems in the artificial intelligence field, the applicability of the proposed algorithm in dealing with real-world problems should be evaluated by considering some difficult and complex optimization problems such as structural size and topology optimization, optimum design of active vibration control systems, optimization of time−cost trade−off problem, optimization of fuzzy logic controllers and other optimization problems which can be considered for the future challenges. The rest of this paper is organized as follows:

In Section 2, the inspiration and mathematical model of the proposed AOS algorithm are presented. The numerical investigations including the mathematical test functions, CEC benchmark suit and the engineering design problems are all presented in Section 3. In Section 4, the key findings of this paper are presented as concluding remarks.

2. Atomic orbital search

2.1. Inspiration

Based on the classic atomic models, the atomic orbital refers to the physical space or region around the nucleus of an atom in which the electrons can be presented by a specific probability (Fig. 1.A). By taking a time−exposure photograph of an atom in which the electrons are changing their positions around nucleus by a wave−like motion, it is observed that the electrons appear as a cloud of charge which rapidly change their position over time (Fig. 1.B). The position of electrons around nucleus are not known in any moment but the probability of their existence around nucleus can be calculated by the electron probability density diagram (Fig. 1.C). In order to find the total probability of finding any electron in any distance from the nucleus, the volume around the nucleus is imaginarily divided into thin, spherical, concentric layers with radius of r (Fig. 1.D). The radial probability distribution plot is depicted in Fig. 1.E in which the volume of each layer increases faster than its probability density so the total probability of finding any electron in the second imaginary layer is higher than the first one.

Based on the fact that electrons create imaginary shells around nucleus, they are assumed to be in the ground state of energy. The quantum number n is determined and related to the radius of an electron orbit which demonstrates the energy level of the electrons. The electron in the layer with smaller n value is related to the orbital with smaller radius and lower energy level. The electrons around nucleus can be excited by the act of photons (lights), interaction with particles or magnetic fields which result in absorption or emission of energy. The amount of energy which is required to remove an electron from its shell is defined as binding energy. The quantum staircase analogy represents the movement of electrons between different orbitals by changing their energy level. If an electron absorbs an amount of energy less than its binding energy, it will undergo a transition to an exciting energy level in the outer orbital. Besides, if an electron absorbs an amount of energy more than its binding energy, it will be repositioned toward the lower energy level in inner orbital. The quantum staircase analogy for electrons around nucleus in atoms is described in Fig. 2.

2.2. Mathematical model

In this section, the AOS optimization algorithm which is derived of the previously presented atomic orbital principles is described in detail. The basic principles of the electron density configuration and the absorption or emission of energy by atoms which are considered in the quantum-based atomic theory, are utilized as the main idea of the AOS optimization algorithm. Based on the fact that most of the previously developed optimization algorithms utilize a population of solution candidates which are evolved by different random procedures, the proposed AOS algorithm considers a number of solution candidates (X) which represent the electrons around the nucleus in the quantum-based atomic model. The search space in

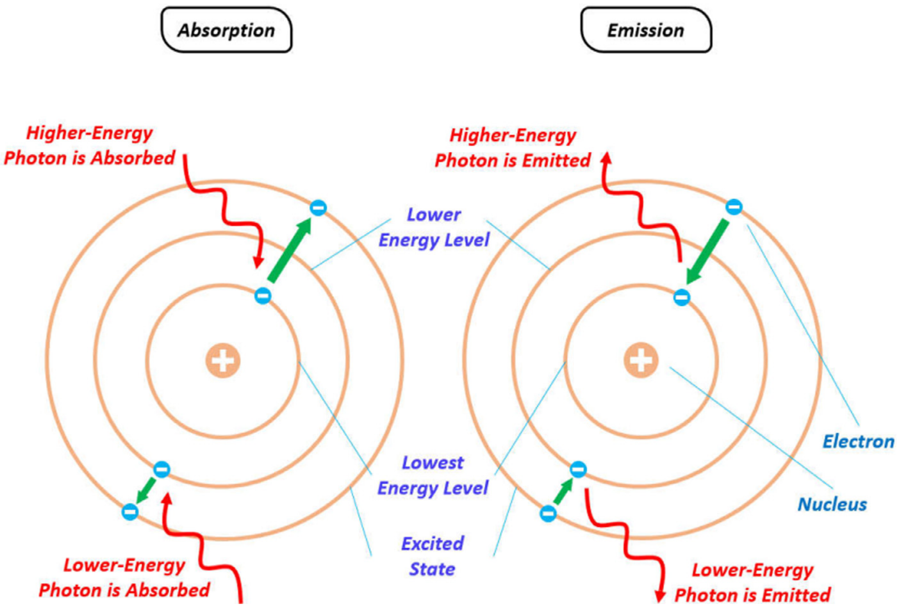

Fig. 2. Quantum staircase analogy for electrons around nucleus in atoms.

this algorithm is considered as the cloud of electrons around nucleus which is divided into thin, spherical, concentric layers. Each electron is represented by a solution candidate (X_i) in the search space while some decision variables ($x_{i,j}$) are also utilized for defining the position of solution candidates in the search space. The mathematical equations in this purpose are as follows:

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_i \\ \vdots \\ X_m \end{bmatrix} = \begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^j & \cdots & x_1^d \\ x_2^1 & x_2^2 & \cdots & x_2^j & \cdots & x_2^d \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ x_i^1 & x_i^2 & \cdots & x_i^j & \cdots & x_i^d \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ x_m^1 & x_m^2 & \cdots & x_m^j & \cdots & x_m^d \end{bmatrix}, \begin{cases} i = 1, 2, \ldots, m. \\ j = 1, 2, \ldots, d. \end{cases} \tag{1}$$

where m is the number of solution candidates (electrons) inside the search space (electron cloud), and d is the problem dimension which represent the position of candidates (electrons).

The initial positions of the electrons inside the electron cloud are randomly determined based on the following mathematical equation:

$$x_i^j(0) = x_{i,min}^j + rand.\left(x_{i,max}^j - x_{i,min}^j\right), \begin{cases} i = 1, 2, \ldots, m. \\ j = 1, 2, \ldots, d. \end{cases} \tag{2}$$

where $x_i^j(0)$ represent the initial position of the solution candidates; $x_{i,min}^j$ and $x_{i,max}^j$ are the minimum and maximum bounds of the jth decision variable for the ith solution candidate; $rand$ is a uniformly distributed random number in the range of [0,1].

Based on the provided details of the quantum-based atomic model, each electron has a state of energy which is considered in the mathematical model as the objective function value of a solution candidate. In this regard, the solution candidates with better objective function values represent the electrons with lower energy levels while the electrons with higher energy levels are considered in the mathematical model with solution candidates with worse objective function values. A vector equation is utilized for containing the objective function values (energy levels) of different solution candidates

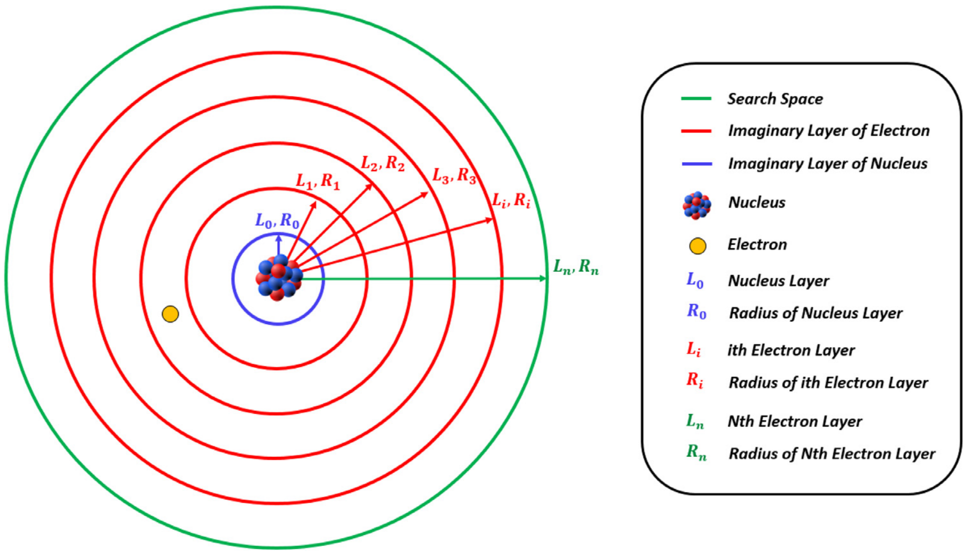

Fig. 3. Schematic presentation of imaginary layers around nucleus.

(electrons) as follows:

$$E = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_i \\ \vdots \\ E_m \end{bmatrix}, \quad i = 1, 2, \ldots, m. \tag{3}$$

where E is a vector containing the objective function values, E_i is the energy level of the ith solution candidate and m is the number of solution candidates (electrons) inside the search space (electron cloud).

In order to mathematically represent the atomic orbital model which have been developed based on the quantum mechanics, a random integer number (n) is generated for representing the number of imaginary spherical layers (L) around nucleus which mimics the quantum number in the quantum-based atomic model. These layers are utilized for dividing the entire search space into multiple sections in order to mathematically represent the wave−like behavior of electrons around nucleus. The radius of these layers demonstrate the way of distributing these layers around nucleus in which the layer with smaller radius is considered as the nucleus layer (L_0) and the layers with larger radiuses (L_i) are considered as the first to *nth* (L_n) spherical layers around nucleus. It should be noted that a zero index is utilized for the nucleus layer which denotes the fact that the nucleus of the atom is positioned in this layer and the electrons cannot be presented in this layer so the first (L_1) to *nth* (L_n) layers are considered for positioning the electrons. The schematic presentation of these aspects are depicted in Fig. 3.

The position of electrons around nucleus in quantum-based atomic model are determined by the electron probability density diagram which is considered in the mathematical model by a Probability Density Function (PDF). Based on the probability theory, a PDF of a variable is a function which represents the likelihood of this variable within a specific range. By considering the imaginarily created layers around nucleus, the PDF is utilized for determining the position of solution candidates in these layers. In this regard, the solution candidates are sorted in an ascending or descending order (based on the minimization or maximization optimization problems) in which the candidates with better objective function values are considered to have higher ranks. The solution candidates with better objective function values are considered with higher PDF values which represent the electrons with lower energy levels. Therefore, the solution candidates with higher PDF values are positioned in the inner imaginary electron layers while the ones with lower PDF values are positioned in the outer imaginary electron layers which mimics the electron configuration in the quantum-based atomic model.

The schematic presentation of determining the position of solution candidates (electrons) in the imaginary layers by utilization of the PDF based on a normal Gaussian distribution is depicted in Fig. 4. It should be noted that the total probability of finding any electron in the second imaginary layer is higher than the first one so the PDF for the second layer (between L_1 and L_2) has higher values than the first one (between L_0 and L_1).

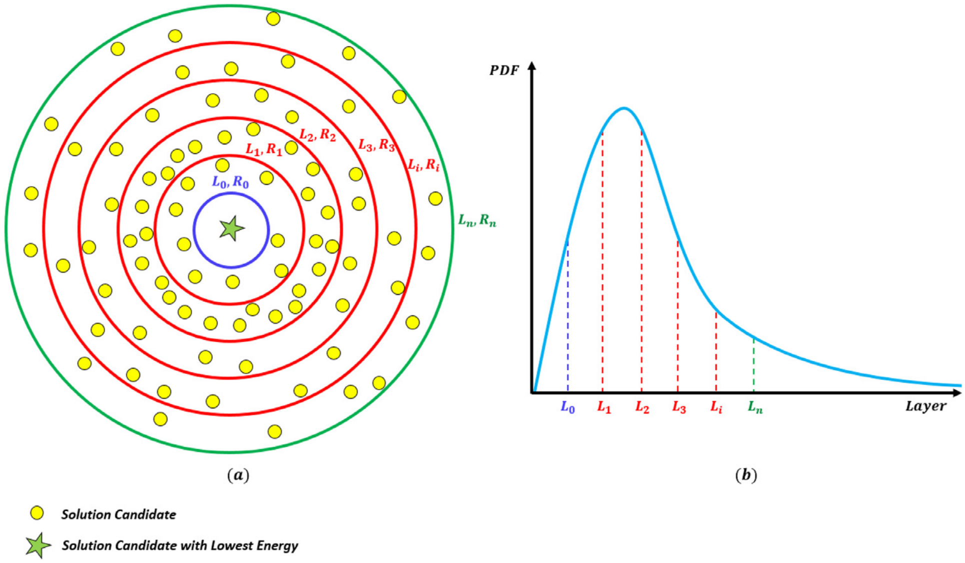

Fig. 4. Schematic presentation of determining the position of solution candidates by PDF.

Based on the provided details of determining the position of electrons by PDF, each of the imaginarily created layers contain some of the solution candidates. In this regard, the mathematical equations for the vectors for the positions (X^k) and the objective function values (E^k) of the solution candidates in the imaginary layers are presented as follows:

$$X^k = \begin{bmatrix} X_1^k \\ X_2^k \\ \vdots \\ X_i^k \\ \vdots \\ X_p^n \end{bmatrix} = \begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^j & \cdots & x_1^d \\ x_2^1 & x_2^2 & \cdots & x_2^j & \cdots & x_2^d \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ x_i^1 & x_i^2 & \cdots & x_i^j & \cdots & x_i^d \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ x_p^1 & x_p^2 & \cdots & x_p^j & \cdots & x_p^d \end{bmatrix}, \quad \begin{cases} i = 1, 2, \ldots, p. \\ j = 1, 2, \ldots, d. \\ k = 1, 2, \ldots, n. \end{cases} \tag{4}$$

$$E^k = \begin{bmatrix} E_1^k \\ E_2^k \\ \vdots \\ E_i^k \\ \vdots \\ E_p^n \end{bmatrix}, \quad \begin{cases} i = 1, 2, \ldots, p. \\ k = 1, 2, \ldots, n. \end{cases} \tag{5}$$

where X_i^k is the ith solution candidate in the kth imaginary layer, n is the maximum number of imaginarily created layers, p is the total number of solution candidates in the kth imaginary layer, d is the problem dimension, and E_i^k is the objective function value of the *ith* solution candidate in the kth imaginary layer.

The solution candidates with best objective function values in each of the imaginary layers are considered as the electrons with lowest energy levels in each of the layers (LE^k). In addition, the solution candidate with best objective function value between all of the candidates is considered as the electron with lowest energy level (LE) in the atom. As presented in Fig. 4, the nucleus layer is utilized for positioning the LE which has the best objective function value between all of the solution candidates.

Based on the quantum-based atomic model, electrons which are positioned around nucleus are assumed to be in the ground state of energy. In this regard, the solution candidates which are positioned in the imaginarily considered layers have not any information from other candidates of the same or other layers. The binding energy which represents the required amount of energy to remove an electron from its shell is determined in the mathematical model based on the position and objective function values of solution candidates in each layer. In this regard, the binding state and binding energy are determined for the solution candidates in each of the considered imaginary layers by considering the average of the positions and objective function values of all solution candidates in the considered layer. The mathematical equations in this purpose are as follows:

$$BS^k = \frac{\sum_{i=1}^{p} X_i^k}{p}, \begin{cases} i = 1, 2, \ldots, p. \\ k = 1, 2, \ldots, n. \end{cases} \tag{6}$$

$$BE^k = \frac{\sum_{i=1}^{p} E_i^k}{p}, \begin{cases} i = 1, 2, \ldots, p. \\ k = 1, 2, \ldots, n. \end{cases} \tag{7}$$

where BS^k and BE^k are the binding state and the binding energy of the kth layer; X_i^k and E_i^k are the position and objective function value of ith solution candidate in the kth layer; m is the total number of solutions candidates in the search space.

Based on the provided details, the binding state and the binding energy of an atom are also determined by considering the average of the positions and objective function values of all solution candidates in the search space as follows:

$$BS = \frac{\sum_{i=1}^{m} X_i}{m}, \ i = 1, 2, \ldots, m. \tag{8}$$

$$BE = \frac{\sum_{i=1}^{m} E_i}{m}, \ i = 1, 2, \ldots, m. \tag{9}$$

where BS and BE are the binding state and the binding energy of the atom; X_i and E_i are the position and objective function value of ith solution candidate in the atom.

Based on the provided details of the quantum-based atomic model, the electrons with different energy levels are capable of changing their position around nucleus between different layers with different energy states. In this regard, two position updating process for solution candidates are considered in the mathematical model in which the act of photons on electrons are considered as the main process while the other acts such as interaction with particles or magnetic fields are considered as the secondary process. For each of the solution candidates in the imaginarily create layers, these processes are conducted due to the characteristics of the candidate and its layer.

In order to mathematically represent the act of photons on electrons around nucleus, a uniformly distributed random number (ϕ) is generated in the range of (0, 1) for each electron. In addition, the Photon Rate (PR) is determined as a parameter which represent the probability of considering the act of photons on electrons. If the randomly generated number (ϕ) for each electron is larger than the PR ($\phi \geq PR$), the act of photons on the electron is probable so the movement of electrons between different layers around nucleus is considered based on the emission and absorption of photons. In this regard, the energy level (E_i^k) of each solution candidate (X_i^k) in each imaginary layer is compared to the binding energy of the layer (BE^k) in order to decide between the emission and absorption of photons for happening. If the energy level of a solution candidate in a specific layer is higher than the binding energy of the layer ($E_i^k \geq BE^k$), the emission of photon is considered. In this process, the solution candidates are tending to emit a photon with an amount of energy considered by β and γ in order to simultaneously reach to the binding state of the atom (BS) and to the state of the electron with lowest energy level (LE) in the atom. The mathematical equations for the position updating process of solution candidates in this process is as follows:

$$X_{i+1}^k = X_i^k + \frac{\alpha_i \times (\beta_i \times LE - \gamma_i \times BS)}{k}, \begin{cases} i = 1, 2, \ldots, p. \\ k = 1, 2, \ldots, n. \end{cases} \tag{10}$$

where X_i^k and X_{i+1}^k are the current and upcoming positions for the ith solution candidate of the kth layer; LE is the solution candidate with lowest energy level in the atom; BS is the binding state of the atom; α_i, β_i and γ_i are vectors containing randomly generated numbers which are distributed uniformly in the range of (0,1) for determining the amount of energy that is emitted.

If the energy level of a solution candidate in a specific layer is lower than the binding energy of the layer ($E_i^k < BE^k$), the absorption of photon is considered. In this process, the solution candidates are tending to absorb a photon with an amount of energy considered by β and γ in order to simultaneously reach to the binding state of the layer (BS^k) and to the state of the electron with lowest energy level (LE^k) inside the considered layer. The mathematical equation for the position updating process of solution candidates in this process is as follows:

$$X_{i+1}^k = X_i^k + \alpha_i \times \left(\beta_i \times LE^k - \gamma_i \times BS^k\right), \begin{cases} i = 1, 2, \ldots, p. \\ k = 1, 2, \ldots, n. \end{cases} \tag{11}$$

where X_i^k and X_{i+1}^k are the current and upcoming positions for the ith solution candidate of the kth layer; LE^k is the solution candidate with lowest energy level of the kth layer; BS^k is the binding state of the kth layer; α_i, β_i and γ_i are vectors containing randomly generated numbers which are distributed uniformly in the range of (0,1) for determining the amount of energy that is absorbed.

If the randomly generated number (ϕ) for each electron is smaller than the PR ($\phi < PR$), the act of photons on the electron is not probable so the movement of electrons between different layers around nucleus is considered based on

```
Procedure Atomic Orbital Search (AOS) Algorithm
    Determine initial positions of solution candidates (X_i) in the search space with m candidates
    Evaluate fitness values (E_i) for initial solution candidates
    Determine binding state (BS) and binding energy (BE) of atom
    Determine candidate with lowest energy level in atom (LE)
    while Iteration < Maximum number of iterations
        Generate n as the number of imaginary layers
        Create imaginary layers
        Sort solution candidates in an ascending or descending order
        Distribute solution candidates in the imaginary layers by PDF
        for k=1: n
            Determine binding state (BS^k) and binding energy (BE^k) of the kth layer
            Determine the candidate with lowest energy level in the kth layer (LE^k)
            for i=1:p
                Generate φ, α, β, γ
                Determine PR
                if φ ≥ PR
                    if E_i^k ≥ BE^k
                        X_{i+1}^k = X_i^k + α_i×(β_i×LE−γ_i×BS) / k
                    else if E_i^k < BE^k
                        X_{i+1}^k = X_i^k + α_i × (β_i × LE^k − γ_i × BS^k)
                    end
                else if φ < PR
                    X_{i+1}^k = X_i^k + r_i
                end
            end
        end
        Update binding state (BS) and binding energy (BE) of atom
        Update candidate with lowest energy level in atom (LE)
    end while
end Procedure
```

Fig. 5. Pseudo-code of the AOS algorithm.

some other acts such as interaction with particles or magnetic fields which also results in absorption or emission of energy. In this regard, the position updating process of solution candidates based on these effects is considered as follows:

$$X_{i+1}^k = X_i^k + r_i, \quad \begin{cases} i = 1, 2, \ldots, p. \\ k = 1, 2, \ldots, n. \end{cases} \tag{12}$$

where X_i^k and X_{i+1}^k are the current and upcoming positions for the ith solution candidate of the kth layer; r_i is a vector containing randomly generated numbers which are distributed uniformly in the range of (0,1).

In order to consider the general aspects of an optimization algorithm in the mathematical model, the violation of boundary condition by solution variables alongside the termination criterion should be dealt with properly. In this purpose, a mathematical flag is determined in which the flag considers a boundary change for the violating variables. In addition, a maximum number of iterations or a maximum number of objective function evaluations can be determined in which the optimization process is terminated after a fixed number of iterations or objective function evaluations. The pseudo-code of the AOS algorithm is presented in Fig. 5 while the flowchart of this algorithm is also provided in Fig. 6.

3. Numerical tests of the algorithm

In this section, the results of the numerical investigations are presented in which the capability of the proposed AOS algorithm is verified through some mathematical test functions, benchmark suits and some of the well-known engineering design problems.

3.1. Tests of benchmark mathematical functions

In this section, a total number of 20 mathematical test functions are collected with different dimensions of 2–100 which are some of the widely known unconstrained test functions in the global optimization field. In this collection, 7 functions

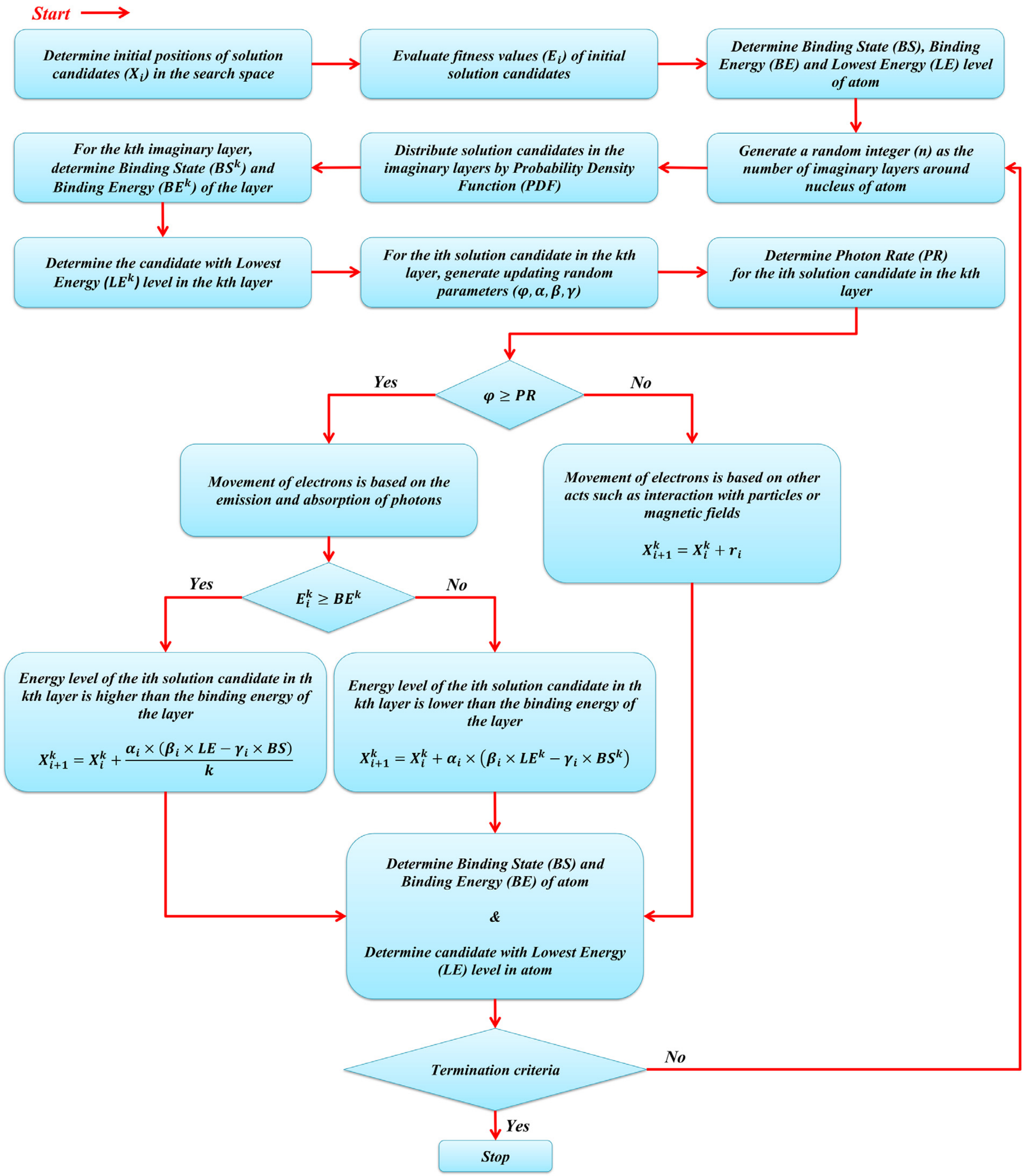

Fig. 6. Flowchart of the AOS algorithm.

are considered with 2 dimensions (F_1 to F_7), 7 functions are determined with 50 dimensions (F_8 to F_{17}) and 6 functions (F_{15} to F_{20}) are collected with 100 dimensions. The basic properties of the mentioned functions are presented in Table 1 in the appendix while the complete mathematical formulation of these functions have been presented by Jamil and Yang [27] and Jamil et al. [28]. In addition, the 3D plots for these functions are provided in Fig. 7 in the appendix.

In order to have a better judgement about the capability of the proposed AOS algorithm in dealing with the considered mathematical test functions, some of the recently proposed metaheuristic algorithms should also be determined as alternatives for comparative purposes. In this regard, 12 different metaheuristic algorithms are considered as the Genetic Algorithm

Table 3
The best results of different metaheuristics in dealing with mathematical functions.

No.	Alternative Metaheuristic Algorithms								
	GA	MVO	SCA	BIA	BBBC	CSA	ASO	WDO	AOS
F_1	0	2.20E−11	2.00E−06	0	5.03E−11	0	5.97E−09	3.77E−10	0
F_2	−24.15682	−24.15682	−24.1568	−24.15682	−24.15682	−24.1568	−24.1568	−24.15682	−24.15682
F_3	1.57E−05	5.38E−10	2.74E−08	5.04E−12	2.34E−10	0	1.01E−05	4.23E−10	0
F_4	0	8.64E−10	3.35E−05	5.86E−12	1.86E−10	0	6.39E−08	4.28E−08	0
F_5	−176.1375	−176.1376	−176.1372	−176.1375	−176.1375	−176.138	−176.137	−176.1375	−176.1375
F_6	−2.5	−2.499998	−2.5	−2.5	−2.499997	−2.5	−2.5	−2.5	−2.5
F_7	−1.5	−1.5	−1.5	−1.5	−1.5	−1.5	−1.5	−1.5	−1.5
F_8	0.2263613	0.0690873	5.93E−05	13.617307	0.1214692	3.424855	1.61E−08	0	0
F_9	0.0070754	1.9555316	2.43E−10	3.1044347	0.605799	9.844303	0.000154	0	0
F_{10}	−0.999995	−0.999843	−0.560407	−1	−0.999937	−0.88069	−0.65809	−0.967569	−0.874969
F_{11}	−1	−0.999996	−1	−0.999987	−0.999991	−1	−1	−1	−1
F_{12}	−33.0243	−13.6182	−49	−19.08807	−7.764056	−15.0676	−49	−33.82127	−33.89885
F_{13}	2.2582399	2.0790961	0	464.99531	2.646938	0.344853	7.00E−−07	0	0
F_{14}	10.145915	11.452348	8.6790533	8.6647214	11.288089	18.97202	1.915405	0.2628051	0
F_{15}	−0.999675	−0.998391	−0.481324	−0.998701	−0.999267	−0.75885	−0.82113	−0.852041	−0.790344
F_{16}	19.750851	56.678358	10.698172	16.832774	75.307851	0.445065	6.367419	0.0404401	2.9418597
F_{17}	3.05E−12	1.08E−10	0.2190689	0	5.18E−11	0	4.52E−07	0	0
F_{18}	2.3396616	1.4998733	1.5001277	18.299873	1.3005743	4.199935	0.199873	0.0998733	0.0998733
F_{19}	17.789144	8.9483061	66.48675	34.480032	7.0175024	11.87716	0.067599	0	0
F_{20}	−575	−512	−294	−575	−465	−479	−263	−575	−346

Table 4
The mean values of different metaheuristics in dealing with mathematical functions.

No.	Alternative Metaheuristic Algorithms								
	GA	MVO	SCA	BIA	BBBC	CSA	ASO	WDO	AOS
F_1	2.00E−10	4.11E−09	0.000206	3.20E−11	1.95E−09	0	1.26E−06	2.29E−07	1.38E−05
F_2	−24.15682	−24.15682	−24.15025	−24.15682	−24.15682	−24.1568	−24.1525	−20.11458	−24.15515
F_3	0.1362871	1.76E−07	0.0001338	0.4130361	3.61E−08	0	0.003908	1.91E−06	6.00E−08
F_4	4.48E−08	3.91E−08	0.0016476	9.01E−10	1.35E−08	0	4.19E−05	8.09E−05	4.38E−08
F_5	−173.2744	−166.5378	−176.1028	−122.5159	−163.2873	−176.138	−175.987	−158.3966	−176.1375
F_6	−2.5	−2.499985	−2.5	−2.5	−2.476151	−2.5	−2.49999	−2.5	−2.499998
F_7	−1.5	−1.499991	−1.5	−1.5	−1.475423	−1.5	−1.49999	−1.5	−1.499997
F_8	1.1116862	3.2039079	19.972762	16.069339	12.028634	9.174743	4.39E−08	0	0
F_9	0.050886	7.6556393	0.2700079	11.628547	0.8813084	20.88773	0.073201	1.4654054	0
F_{10}	−0.991149	−0.991312	−0.481712	−1	−0.99973	−0.8526	−0.58269	−0.891054	−0.800144
F_{11}	−0.999998	−0.999992	−1	−0.999982	−0.999986	−1	−1	−1	−1
F_{12}	−27.43647	−8.734473	−46.52209	−6.197204	−4.134486	−11.5189	−43.3022	−18.66162	−16.48442
F_{13}	22.462287	4.1223507	1.53E−−08	739.43264	3.0166386	0.60657	5.02E−−06	0	0
F_{14}	13.871364	17.181914	40.769231	12.89748	14.21646	27.05712	2.580487	6.7067959	0.7579332
F_{15}	−0.944491	−0.974328	−0.442668	−0.940168	−0.999091	−0.729	−0.61985	−0.738516	−0.720921
F_{16}	41.594667	105.89162	60.563083	38.594141	112.5574	1.806629	7.158375	1.6335048	4.3906572
F_{17}	4.03E−11	2.09E−10	0.3879549	2.29E−08	7.88E−11	1.76E−05	7.99E−07	0	1.33E−08
F_{18}	2.9888975	1.9168734	4.9814234	24.626873	1.6006534	5.187077	0.200873	0.2488734	0.1098733
F_{19}	31.274789	22.177555	76.826252	49.813328	18.112028	16.01277	0.076152	0	0
F_{20}	−575	−477.28	−264.51	−575	−266.94	−440.51	−190.74	−544.85	−266.33

(GA) [29], Multi verse Algorithm (MVO) [30], Sine Cosine Algorithm (SCA) [31], Bat Inspired Algorithm (BIA) [32], Big-Bang Big-Crunch (BBBC) algorithm [33], Cuckoo Search Optimization Algorithm (CSA) [34], Wind Driven Optimization (WDO) algorithm [35] alongside one of the recently proposed atom-based algorithms (ASO) [21] and the novel algorithm which is proposed in this paper (AOS). Based on the fact that some of the mentioned metaheuristic algorithms require a number of parameters to be identified through the optimization process, a complete parameter presentation for these algorithms are provided in Table 2 in the appendix.

The results of the optimization process for the proposed AOS algorithm alongside the alternative approaches in dealing with the considered mathematical test functions are presented based on a total number of 150,000 function evaluations with a tolerance of 1×10^{-12} as a stopping criteria. For statistical purposes, a total number of 100 optimization runs are conducted for each of the novel and alternative metaheuristics in dealing with the considered functions which are utilized for obtaining the mean and standard deviation of the results. In addition, the initialization process for all of the algorithms in dealing with the mentioned mathematical functions are determined by means of common (fixed) random state so the algorithms are evaluated under equal conditions.

In Tables 3–5, the best, mean and standard deviation of the 100 optimization runs for the considered mathematical functions (F_1 to F_{20}) are presented. The results demonstrate that the AOS algorithms is capable of outranking the alternative

Table 5
The standard deviation of results for different metaheuristics in dealing with mathematical functions.

No.	Alternative Metaheuristic Algorithms								
	GA	MVO	SCA	BIA	BBBC	CSA	ASO	WDO	AOS
F_1	1.11E−09	3.95E−09	0.000218	3.13E−11	1.63E−09	0	1.44E−06	3.23E−07	4.82E−05
F_2	5.50E−09	1.06E−07	0.0067011	3.57E−15	3.86E−08	3.57E−15	0.006086	7.4435735	0.0051339
F_3	0.4497599	2.96E−07	0.000126	1.5404609	3.47E−08	0	0.006909	2.84E−06	3.03E−07
F_4	1.32E−07	3.86E−08	0.0019004	9.55E−10	1.32E−08	0	0.000104	0.0001068	2.23E−07
F_5	9.1500149	16.571701	0.0369596	35.921281	17.128004	3.43E−13	0.201693	24.453517	3.43E−13
F_6	5.58E−07	8.00E−06	0	1.47E−06	0.0472141	0	6.67E−05	0	7.17E−06
F_7	5.68E−07	5.47E−06	0	0	0.05839	0	3.78E−05	0	1.14E−05
F_8	0.2602001	6.1339822	2.8680349	0.9993233	9.0419373	3.58599	1.76E−08	0	0
F_9	0.04356	3.0451564	1.4543991	5.6490211	0.156042	4.369272	0.121216	3.2755118	0
F_{10}	0.0124335	0.0130797	0.0275599	0	0.0017527	0.010674	0.028471	0.044816	0.0381749
F_{11}	1.14E−06	1.65E−06	4.76E−07	1.65E−06	1.24E−06	1.08E−06	1.34E−06	0	0
F_{12}	2.286376	1.5087051	6.55176	3.0149446	1.0489446	1.058497	12.81357	5.9571004	5.8525845
F_{13}	14.874601	1.3959398	6.60E−08	132.96207	0.1448848	0.133581	4.24E−06	0	0
F_{14}	1.7273302	2.2541866	8.3368637	2.4342556	1.459643	5.298037	0.377378	6.4580204	3.4543559
F_{15}	0.0216846	0.015657	0.014864	0.0303496	5.67E−05	0.011503	0.131255	0.0473529	0.0342164
F_{16}	13.15083	21.297517	34.846783	10.782323	15.487651	1.120121	0.262966	6.7591497	0.5142734
F_{17}	3.68E−11	6.10E−11	0.1000701	2.29E−07	1.36E−11	1.32E−05	1.30E−07	0	1.57E−08
F_{18}	0.2588053	0.1563827	2.1200129	2.8559297	0.1214815	0.47323	0.01	0.0784895	0.0301511
F_{19}	4.6114359	5.2335965	3.7849378	6.0415648	4.1654099	2.209228	0.003111	0	0
F_{20}	0	15.04544	11.40485	0	157.22221	12.50414	20.19482	81.581635	39.941385

Table 7
The KS test results (p-values) for different metaheuristic algorithms.

Main Algorithm	Data Type	Alternative Metaheuristic Algorithms							
		GA	MVO	SCA	BIA	BBBC	CSA	ASO	WDO
AOS	Min.	1.35E−01	7.25E−04	8.16E−03	2.75E−01	7.25E−04	1.35E−01	1.83E−04	9.65E−01
	Mean	4.97E−01	1.35E−01	5.91E−02	1.35E−01	1.35E−01	4.97E−01	5.91E−02	7.71E−01
	Std.	2.75E−01	2.75E−01	5.91E−02	1.35E−01	5.91E−02	4.97E−01	1.35E−01	2.75E−01
	Fun. Evl.	1.35E−01	1.83E−04	7.25E−04	1.35E−01	7.25E−04	2.75E−01	1.83E−04	4.97E−01

metaheuristics in all of the cases; however, a complete statistical investigation is also conducted in the following in order to have a better judgment about the performance of the proposed algorithm. In Table 6 (see appendix), the mean values of maximum function evaluations in each optimization run for the considered alternative metaheuristics alongside the proposed AOS algorithm are presented. It can be concluded that the AOS is capable of converging to better results with lower required function evaluations which demonstrates the capability of the proposed method in dealing with computational complexity issues.

Based on the presented results, a complete statistical analysis is also conducted in which the capability of the proposed AOS algorithm is evaluated from different perspectives. In this regard three of the well-known statistical tests are considered as the Kolmogorov Smirnov (KS) test for normality issues, Wilcoxon (W) signed ranks test for comparing the summation and mean of the metaheuristics' ranks in a two by two comparing procedure, and the Kruskal Wallis (KW) test for comparing the mean of the metaheuristics' ranks which leads to evaluate the overall rankings of different metaheuristic algorithms.

The KS test demonstrates that the distribution of obtained data is either normal or non-normal which results in the utilization of either parametric or non-parametric statistical tests. If the p-value of this test is less than 0.05, the obtained data are not distributed normally and the non-parametric statistical tests should be utilized for further investigations. In Table 7, the p-values of different metaheuristics in dealing with the mathematical functions are presented. Based on the results that are less than 0.05, the non-parametric statistical tests such as the W and the KW should be considered.

The W test is one of the well-known non-parametric statistical tests for comparing the mean ranks of different data samples in a two by two manner. In this test, the smaller mean of ranks between two data samples represents the superiority of this sample from statistical point of view. In Table 8, the results of W test including the mean of ranks for different metaheuristics in dealing with the mathematical functions are presented in a two by two manner. In this table, the bolded values represent the metaheuristics with lower mean of ranks which are superior to the other metaheuristics. It can be concluded that the proposed AOS algorithm has lower mean of ranks in most of the cases which represents its superiority to other alternatives. In Table 9 (see appendix), the p-values of this statistical test are also provided for further investigations.

In the KW statistical test, multiple independent data sets can be compared with different or equal sample sizes. This test extends the previously conducted W test which is implemented in a two by two manner while this test is proposed for comparing multiple data sets in the same time. In the KW test, the mean of ranks for different data sets are determined while the data sets with lower mean values have better statistical behavior. In Table 10, the KW results for different metaheuristics including the mean of ranks in dealing with the mathematical functions are presented. In this table, the bolded

Table 8
The W test results (mean of ranks) for different metaheuristic algorithms.

Main Algorithm	Data Type	Alternative Metaheuristic Algorithms							
		GA	MVO	SCA	BIA	BBBC	CSA	CSS	WDO
AOS	Min.	80	134	122	83	134	59	160	31
		25	37	14	22	37	19	30	24
	Mean	131	153	186	153	163	116	184	70
		79	57	24	57	47	74	26	66
	Std.	119	117	165	125	148	107	124	123
		91	93	45	85	62	83	86	13
	Fun.	75	105	86	80	105	70	97	60
	Evl.	45	0	19	56	0	35	8	55

Table 10
The KW test results including mean of the ranks considering mathematical test functions.

Rankings	Data Type							
	Min.		Mean		Std.		Fun. Evl.	
	Algorithms	Mean of Ranks	Algorithms	Mean of Ranks	Algorithms	Mean of Ranks	Algorithms	Mean of Ranks
1	AOS	61.85	AOS	68.1	AOS	69.35	AOS	59.1
2	WDO	63.8	WDO	77.3	CSA	80.45	WDO	67.85
3	GA	89.25	CSA	86.15	WDO	85.55	BIA	79.4
4	BIA	92	GA	90.4	ASO	90.45	CSA	83.15
5	CSA	96	ASO	91.1	GA	92.95	GA	84.35
6	ASO	98.85	BIA	95.6	BIA	93.35	SCA	101.75
7	SCA	99.35	BBBC	101.55	BBBC	100.05	BBBC	111.95
8	BBBC	106.6	SCA	102.05	SCA	100.95	MVO	112.45
9	MVO	106.8	MVO	102.25	MVO	101.4	ASO	114.5
Chi-sq.	17.04741		8.240445		6.69786		36.33312	
Prob>Chi-sq.	0.029619		0.410342		0.569554		1.53E−05	

values represent the metaheuristics with lower mean of ranks which are considered as the algorithms with better behavior. It can be concluded that the proposed AOS algorithm has higher ranks than other approaches in most of the cases which proves its capability in dealing with these kinds of problems.

3.2. Comparative investigation based on CEC 2017

In most of the comparative investigations for performance evaluation of the novel algorithms, there should be some difficult challenges in which the proposed algorithm can be tested through some complex optimization problems. In this regard, the CEC 2017 are considered as the latest completion on unconstrained mathematical optimization. The problem summary of this competition is presented in Table 11 in the appendix while the complete mathematical equations of these functions has been provided in detail by Awad et al. [24].

In Table 12, the statistical results of AOS algorithm alongside the other competitors in CEC 2017 such as the Kumar et al. [36], Awad et al. [37] and Sallam et al. [38] are presented for comparative purposes. In this comparing investigation, a total number of 51 optimization runs are considered while the error values of these runs are utilized in statistical representation. By comparing the results of the AOS algorithm with the other approaches, it is concluded that the proposed algorithm is capable of providing very competitive results in dealing with these complex problems.

3.3. Computational cost and complexity

In this section, a complete computational cost and complexity analysis is conducted for the proposed AOS algorithm in order to have a better perspective about the mathematical formulation of the algorithm. The computational cost analysis can be determined by considering the CEC 2017 protocol in this purpose in which four specific computational times are determined as T_0 (run time of a specific mathematical procedure), T_1 (computational time for 200,000 function evaluations of G_{18} function), T_2 (computational time of the proposed AOS algorithm considering 200000 function evaluations in dealing with G_{18} function) and $\hat{\mathbf{T}}_2$ (mean values of considering T_2 for five times) all in seconds. In Table 13, the computational complexity results for the AOS and the other approaches are presented in which the superiority of the proposed algorithm is demonstrated.

In order to conduct the computational complexity analysis for a novel metaheuristic algorithm, the “Big O notation” can be utilized as a well-known mathematical notation in computer science in which the required run time and memory usage of the algorithms are determined for comparative purposes. Considering the NP as the number of initial population

Table 12
The statistical results of the AOS and other approaches in dealing with the CEC 2017 functions.

Fun.	Metaheuristics																			
	EBO with CMAR [36]					LSHADE−cnEpSin [37]					MM_OED [38]					AOS (Present Study)				
	Best	Worst	Median	Mean	Std	Best	Worst	Median	Mean	Std	Best	Worst	Median	Mean	Std	Best	Worst	Median	Mean	Std
G_{1}	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
G_{2}	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
G_{3}	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	1.56E−07	3.36E−04	1.20E−05	4.22E−05	7.36E−05
G_{4}	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	1.38E−06	1.35E−07	2.10E−07	2.49E−07
G_{5}	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	2.99E+00	1.99E+00	1.69E+00	7.53E−01	0.00E+00	2.99E+00	9.95E−01	1.11E+00	7.35E−01	8.19E−07	9.95E−01	1.57E−05	7.81E−02	2.70E−01
G_{6}	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	2.71E−04	3.25E−02	9.57E−03	1.18E−02	7.94E−03
G_{7}	1.04E+01	1.10E+01	1.05E+01	1.06E+01	1.75E−01	1.06E+01	1.29E+01	1.20E+01	1.20E+01	4.80E−01	1.04E+01	1.31E+01	1.15E+01	1.15E+01	6.71E−01	3.71E−05	2.63E+00	2.02E+00	1.55E+00	9.13E−01
G_{8}	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	1.70E−03	2.99E+00	1.99E+00	1.80E+00	7.71E−01	0.00E+00	2.99E+00	9.95E−01	1.11E+00	9.68E−01	0.00E+00	9.95E−01	3.12E−05	3.21E−01	4.65E−01
G_{9}	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	3.61E−09	7.76E−06	3.60E−07	9.73E−07	1.42E−06
G_{10}	1.25E−01	2.17E+02	1.36E+01	3.72E+01	5.39E+01	3.71E−01	1.55E+02	1.51E+01	4.30E+01	5.57E+01	2.50E−01	1.42E+02	6.83E+00	1.79E+01	3.64E+01	4.24E−05	1.71E+01	3.13E−01	3.60E+00	6.62E+00
G_{11}	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
G_{12}	0.00E+00	2.37E+02	1.18E+02	9.02E+01	7.44E+01	2.08E−01	2.38E+02	1.19E+02	1.01E+02	7.30E+01	0.00E+00	2.64E+02	1.19E+02	1.02E+02	5.96E+01	1.99E+02	4.94E+04	7.57E+02	2.56E+03	7.05E+03
G_{13}	0.00E+00	7.95E+00	3.13E−02	2.17E+00	2.53E+00	0.00E+00	8.32E+00	4.84E+00	3.66E+00	2.66E+00	6.82E−05	9.63E+00	5.05E+00	4.19E+00	2.66E+00	3.63E+00	2.44E+01	1.43E+01	1.48E+01	5.57E+00
G_{14}	0.00E+00	9.95E−01	0.00E+00	6.05E−02	2.36E−01	0.00E+00	9.95E−01	0.00E+00	7.80E−02	2.70E−01	0.00E+00	1.03E+00	0.00E+00	8.80E−02	2.74E−01	3.18E−02	2.18E+01	3.29E+00	4.32E+00	3.51E+00
G_{15}	3.81E−05	5.00E−01	3.07E−02	1.09E−01	1.74E−01	7.03E−06	5.00E−01	4.91E−01	3.24E−01	2.16E−01	2.00E−05	5.00E−01	2.13E−02	6.71E−02	1.19E−01	7.26E−01	8.03E+00	3.21E+00	3.37E+00	1.47E+00
G_{16}	2.62E−02	9.35E−01	4.43E−01	4.17E−01	1.98E−01	3.92E−03	1.08E+00	4.98E−01	5.37E−01	2.93E−01	2.11E−02	8.80E−01	2.16E−01	2.53E−01	2.01E−01	5.86E−01	2.87E+02	1.22E+01	5.91E+01	8.06E+01
G_{17}	1.00E−02	1.01E+00	4.94E−02	1.47E−01	2.03E−01	2.66E−03	2.63E+00	3.23E−01	3.07E−01	3.81E−01	0.00E+00	3.76E−01	1.97E−02	5.64E−02	1.13E−01	6.44E−01	6.15E+01	1.79E+01	1.78E+01	1.55E+01
G_{18}	3.92E−04	2.00E+01	4.09E−01	7.00E−01	2.77E+00	2.22E−04	2.05E+01	4.62E−01	3.86E+00	7.63E+00	1.11E−04	2.00E+01	9.00E−02	9.69E−01	3.89E+00	6.51E+00	2.41E+01	1.81E+01	1.69E+01	5.64E+00
G_{19}	0.00E+00	1.22E−01	1.79E−02	1.50E−02	1.88E−02	0.00E+00	1.50E+00	1.97E−02	4.47E−02	2.09E−01	0.00E+00	1.94E−02	0.00E+00	3.80E−03	7.49E−03	4.15E−01	3.59E+00	1.43E+00	1.59E+00	8.21E−01
G_{20}	0.00E+00	3.12E−01	0.00E+00	1.47E−01	1.57E−01	0.00E+00	6.24E−01	3.12E−01	2.57E−01	2.31E−01	0.00E+00	6.24E−01	0.00E+00	6.73E−02	1.57E−01	0.00E+00	2.25E+01	1.31E+00	3.78E+00	6.04E+00
G_{21}	1.00E+02	2.02E+02	1.00E+02	1.14E+02	3.52E+01	1.00E+02	2.04E+02	1.00E+02	1.46E+02	5.17E+01	1.00E+02	2.03E+02	1.00E+02	1.04E+02	1.99E+01	1.00E+02	2.17E+02	1.00E+02	1.19E+02	4.06E+01
G_{22}	2.17E+01	1.00E+02	1.00E+02	9.85E+01	1.10E+01	1.00E+02	1.00E+02	1.00E+02	1.00E+02	6.80E−02	1.00E+02	1.00E+02	1.00E+02	1.00E+02	6.88E−02	1.00E+02	1.06E+02	1.01E+02	1.01E+02	1.01E+00
G_{23}	3.00E+02	3.03E+02	3.00E+02	3.00E+02	7.07E−01	3.00E+02	3.05E+02	3.03E+02	3.02E+02	1.64E+00	1.00E+02	3.08E+02	3.03E+02	2.98E+02	2.84E+01	1.99E+02	4.94E+04	7.57E+02	2.56E+03	7.05E+03
G_{24}	1.00E+02	3.30E+02	1.00E+02	1.66E+02	9.97E+01	1.00E+02	3.32E+02	3.30E+02	3.16E+02	5.45E+01	1.00E+02	2.01E+02	1.00E+02	1.04E+02	1.97E+01	3.23E+00	1.16E+02	1.04E+02	1.03E+02	1.45E+01
G_{25}	3.98E+02	4.43E+02	3.98E+02	4.12E+02	2.12E+01	3.98E+02	4.43E+02	4.43E+02	4.26E+02	2.24E+01	3.98E+02	4.43E+02	3.98E+02	4.14E+02	2.19E+01	2.79E+01	1.05E+02	1.03E+02	9.83E+01	1.85E+01
G_{26}	2.00E+02	3.00E+02	3.00E+02	2.65E+02	4.74E+01	3.00E+02	3.00E+02	3.00E+02	3.00E+02	0.00E+00	2.00E+02	3.00E+02	3.00E+02	2.94E+02	2.38E+01	1.90E+02	3.54E+02	3.24E+02	3.21E+02	2.18E+01
G_{27}	3.90E+02	3.95E+02	3.90E+02	3.92E+02	2.40E+00	3.84E+02	3.95E+02	3.89E+02	3.89E+02	1.96E+00	3.89E+02	3.90E+02	3.90E+02	3.90E+02	1.22E−01	6.73E+00	3.71E+02	3.47E+02	2.84E+02	1.13E+02
G_{28}	0.00E+00	5.84E+02	3.00E+02	3.07E+02	7.18E+01	3.00E+02	6.11E+02	3.00E+02	3.85E+02	1.19E+02	3.00E+02	6.47E+02	3.00E+02	3.37E+02	1.02E+02	1.04E+02	5.24E+02	4.44E+02	4.24E+02	5.48E+01
G_{29}	2.27E+02	2.45E+02	2.30E+02	2.31E+02	3.77E+00	2.26E+02	2.33E+02	2.28E+02	2.28E+02	1.72E+00	2.30E+02	2.48E+02	2.34E+02	2.36E+02	4.19E+00	2.00E+00	6.83E+02	4.55E+02	4.13E+02	1.48E+02
G_{30}	3.95E+02	4.43E+02	3.95E+02	4.07E+02	1.78E+01	3.39E+02	4.65E+05	4.07E+02	1.76E+04	8.61E+04	3.95E+02	1.25E+06	3.95E+02	5.69E+04	2.34E+05	3.89E+02	4.87E+02	4.00E+02	4.07E+02	1.92E+01

EBO with CMAR: Effective Butterfly Optimizer with Covariance Matrix Adapted Retreat
LSHADE−cnEpSin: Ensemble Sinusoidal Differential Covariance Matrix Adaptation with Euclidean Neighborhood
MM_OED: Multi-Method Based Orthogonal Experimental Design

Table 13
Computational complexity results of the AOS algorithm compared to other approaches.

Metaheuristics	Properties	Results (sec)
EBO with CMAR [36]	T0	0.0413
	T1	0.8218
	$\hat{\mathbf{T}}_2$	7.5794
	$(\hat{\mathbf{T}}_2 - \mathbf{T}1)/\mathbf{T}0$	163.6223
LSHADE−cnEpSin [37]	T0	0.1093
	T1	0.8391
	$\hat{\mathbf{T}}_2$	2.1835
	$(\hat{\mathbf{T}}_2 - \mathbf{T}1)/\mathbf{T}0$	12.30009
MM_OED [38]	T0	2.157784
	T1	0.146416
	$\hat{\mathbf{T}}_2$	6.704923
	$(\hat{\mathbf{T}}_2 - \mathbf{T}1)/\mathbf{T}0$	3.039417
AOS (Present Study)	T0	0.031683
	T1	0.138048
	$\hat{\mathbf{T}}_2$	6.540569
	$(\hat{\mathbf{T}}_2 - \mathbf{T}1)/\mathbf{T}0$	202.0806

and D as the problem dimension, the computational complexity of the proposed AOS algorithm in the initialization phase is calculated as O(NP × D). Besides, the objective function evaluation in the initialization phase has a computational complexity of O(NP) × O(F(x)) while the F(x) represents the objective function evaluation. In the main loop of the algorithm in which the optimization process is handled, each line has a computational complexity equal to maximum number of iterations (MxItr). In addition, a position updating process is also conducted for all of the initially generated solution candidates so the computational complexity of this process is as O(MxItr × NP × D) considering D as the problem dimension and NP as the number of population. Moreover, the objective function evaluation in the main loop has a computational complexity of O(MaxIter × NP × D) × O(F(x)).

It is obvious that the capacity of a metaheuristic algorithms depends on the balance between exploration and exploitation while the convergence speed is also an important factor in evaluation of novel algorithms. In order to visualize the balance between exploration and exploitations and investigate convergence speed as critical properties of metaheuristic algorithms, the diversity graphs for the proposed AOS algorithm are depicted through Figs. 8–10 for F1, F4 and F5 functions. It is proved that the population in the optimization process by AOS tend to localize the search for achieving better results.

3.4. Engineering design problems

In this section, a total number of 5 constrained engineering design problems are collected and considered as design examples for performance evaluation of the proposed AOS algorithm including the constrained optimization problems of the recent Competitions on Evolutionary Computation named "CEC 2020" presented by Kumar et al. [25]. The basic characteristics of these design examples are presented in Table 14 in the appendix while the complete mathematical equations of these problems are provided in the literature.

The numerical results for optimum design of different engineering problems by utilization of the proposed AOS algorithm are presented in which a total number of 25 independent optimization runs are considered in the optimization process with a maximum number of 200,000 function evaluations. For comparative purposes, the results of other metaheuristics from the literature are also presented for having a valid judgment about the performance of the proposed AOS algorithm.

3.4.1. Speed reducer

In this design example, optimum design configuration of the speed reducers which are implemented in the aircraft engines is considered. There are seven design variables in this engineering design problem as the teeth module (m), face width (b), length of the first shaft between bearings (l_1), diameter of the first shaft (d_1), number of teeth on pinion (z), length of the second shaft between bearings (l_2), and the diameter of the second shaft (d_2) with only eleven inequality design constraints. The schematic view of a speed reducer is depicted in Fig. 11 (see appendix) while the complete mathematical formulation of this problem is presented by Yildiz et al. [39].

The best results of multiple optimization runs for the AOS and other algorithms in dealing with the speed reducer problem are presented in Table 15 while the statistical results including the mean, worst and standard deviation are also provided in Table 16. It is obvious that the proposed AOS algorithm is capable of providing better results than the other approaches. In addition, the statistical results of the AOS are also superior to the results of other metaheuristics.

3.4.2. Pressure vessel

The optimum design of a pressure vessel is considered in this example in which the overall cost optimization is determined as the objective function. This design example has four design variables including the head thickness (T_h), shell

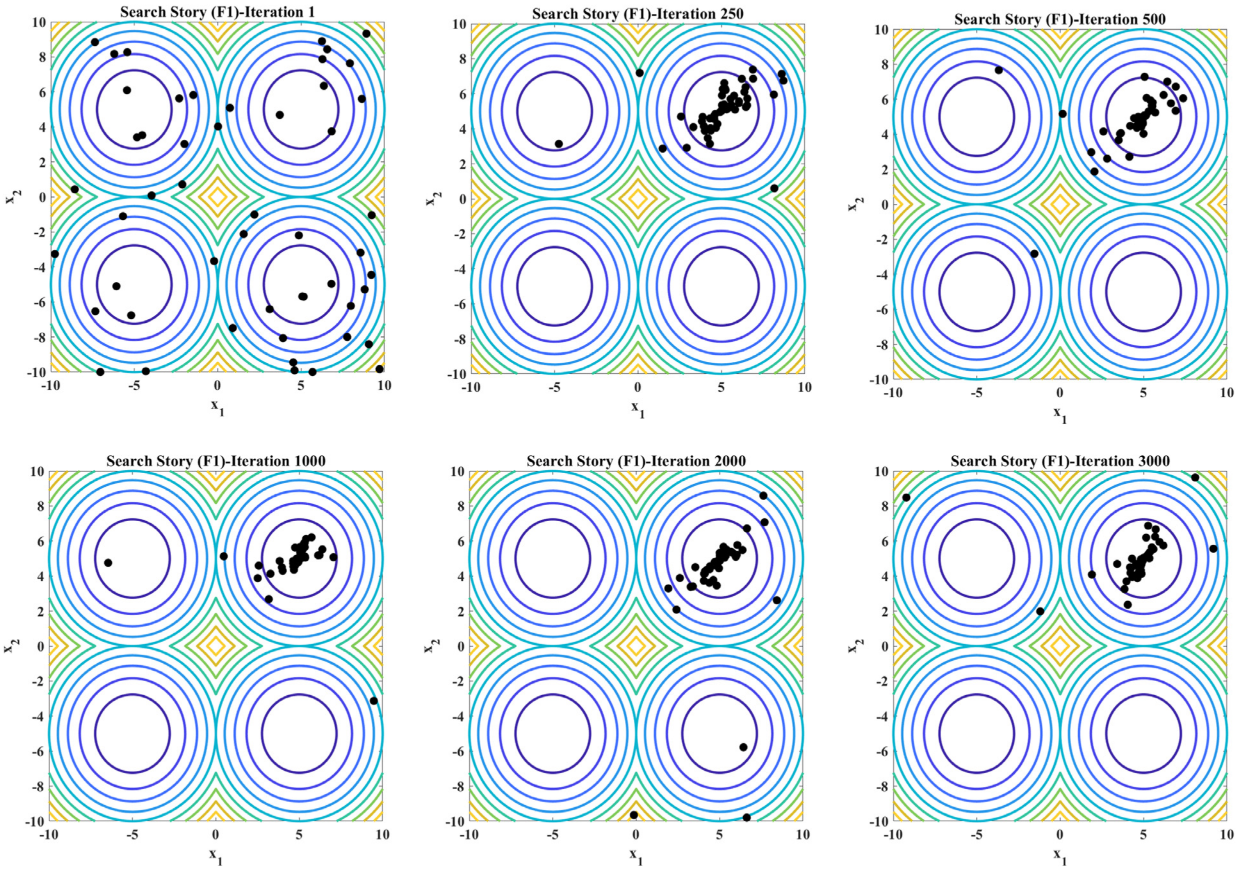

Fig. 8. Diversity plots of F1 function considering AOS.

Table 15
Best results of different approaches for the speed reducer problem.

	Montes et al. [40]	Akhtar et al. [41]	Gandomi et al. [42]	Zhang et al. [43]	Present Study (AOS)
Best	3025.005000000	3008.080000000	3000.981000000	2994.471066000	2994.445819000
b	3.506163000	3.506122000	3.501500000	3.500000000	3.500011428
m	0.700831000	0.700006000	0.700000000	0.700000000	0.700000000
z	17.000000000	17.000000000	17.000000000	17.000000000	17.000030500
l_1	7.460181000	7.549126000	7.605000000	7.300000000	7.300000000
l_2	7.962143000	7.859330000	7.818100000	7.715319912	7.715359846
d_1	3.362900000	3.365576000	3.352000000	3.350214666	3.350543190
d_2	5.309000000	5.289773000	5.287500000	5.286654465	5.286670439
$g_1(x)$	−0.077700000	−0.075500000	−0.074300000	−0.073915200	−2.155147507
$g_2(x)$	−0.201300000	−0.199400000	−0.198300000	−0.197998500	−98.138396990
$g_3(x)$	−0.474100000	−0.456200000	−0.434900000	−0.999996700	−1.925139008
$g_4(x)$	−0.897100000	−0.899400000	−0.900800000	−0.999999500	−18.309889350
$g_5(x)$	−0.011000000	−0.013200000	−0.001100000	−0.666852600	−0.002230861
$g_6(x)$	−0.012500000	−0.001700000	−0.000400000	−0.000000000	−0.007700733
$g_7(x)$	−0.702200000	−0.702500000	−0.702500000	−0.702500000	−28.099978650
$g_8(x)$	−0.000600000	−0.001700000	−0.000400000	−0.000000000	−0.000016300
$g_9(x)$	−0.583100000	−0.582600000	−0.583200000	−0.583333300	−6.999983675
$g_{10}(x)$	−0.069100000	−0.079600000	−0.089000000	−0.051325700	−0.374185215
$g_{11}(x)$	−0.027900000	−0.017900000	−0.013000000	−0.000000000	−0.000022400

thickness (T_s), length (L) and the inner radius (R) of the vessel with only four inequality design constraints. A schematic view for this engineering design problem is depicted in Fig. 12 (see appendix) while the mathematical formulation of this problem is presented by Gandomi et al. [44].

The comparative best results of different optimization approaches for the pressure vessel design problem alongside the proposed AOS algorithm are presented in Table 17 while the optimum decision variables and the constraints are also provided. Besides, the statistical results of different optimization runs are provided in Table 18 for comparative purposes. By

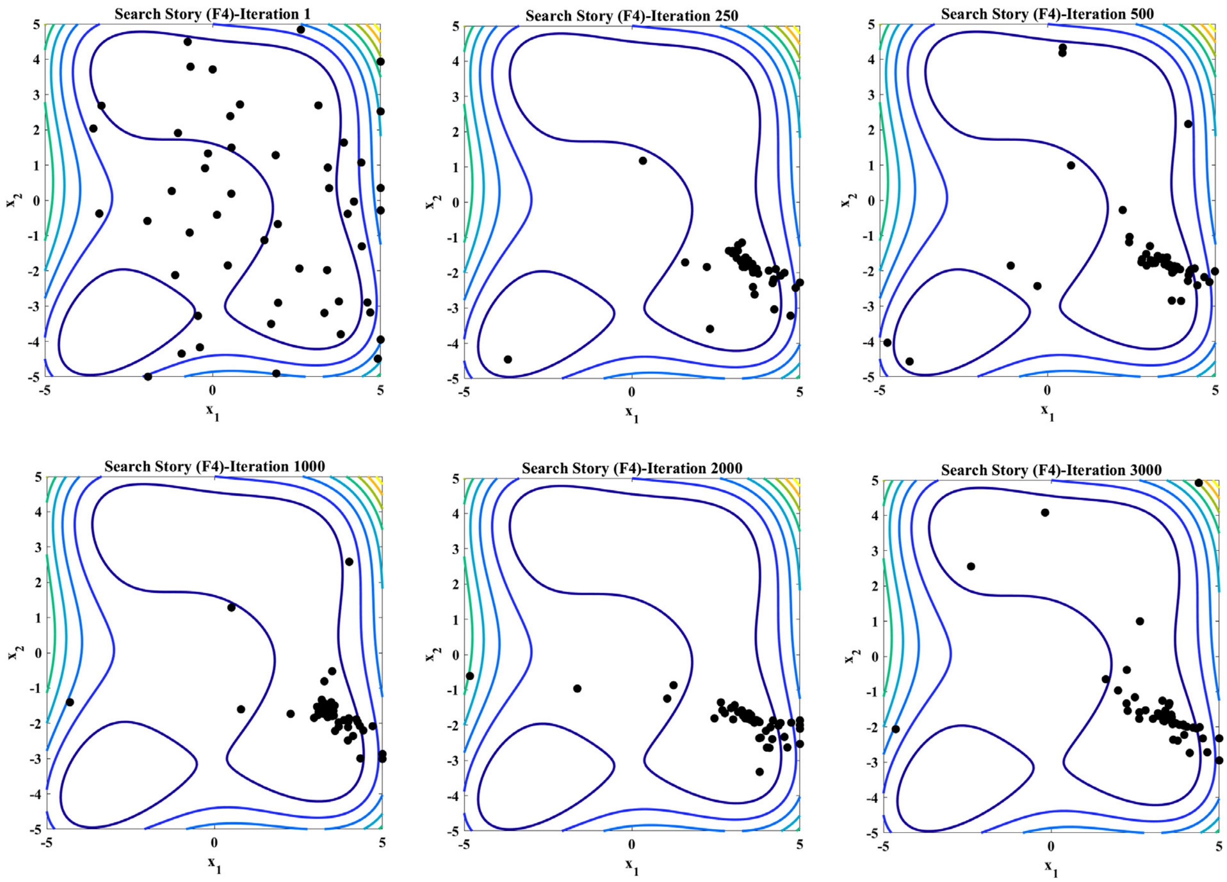

Fig. 9. Diversity plots of F4 function considering AOS.

Table 16
Statistical results for the speed reducer problem considering different approaches.

Approaches	Best	Mean	Worst	Std−Dev
[40]	3025.005000000	3088.777800000	3078.591800000	NA
[41]	3008.080000000	3012.120000000	3028.280000000	NA
[42]	3000.981000000	3007.199700000	3.009000000	4.963400000
[43]	2994.471066000	2994.471066000	2994.471066000	0.000000000
Present Study (AOS)	2994.445819000	2994.452861000	2994.463279000	2.456907000

comparing the results of the AOS to the results of other approaches, it can be concluded that the AOS is capable of obtaining very outstanding results than the other metaheuristics which demonstrates its capability in dealing with these kinds of difficult problems. It also should be concluded that the AOS provides very better statistical results than the other approaches.

3.4.3. Welded beam

The cost optimization of a welded beam including the optimum topology configuration of the beam and the welding are considered in this problem. This design example has four design varies including the width of the welded joint (h), width of the beam (t) length of the welded joint (l), and the thickness of the beam (b) with only seven inequality design constraints. A schematic view for this design problem is depicted in Fig. 13 (see appendix) while the mathematical formulation of this problem is presented by Gandomi et al. [44].

The best results of different optimization runs for the welded beam design problem considering the AOS and other metaheuristics are presented in Table 19 while the statistical results are provided in Table 20. By comparing the best results of the proposed AOS algorithm with the considered alternatives from the literature, it is demonstrated that the novel algorithm is capable of providing very competitive results; however, the statistical results of this algorithm are completely better than the results of other alternatives.

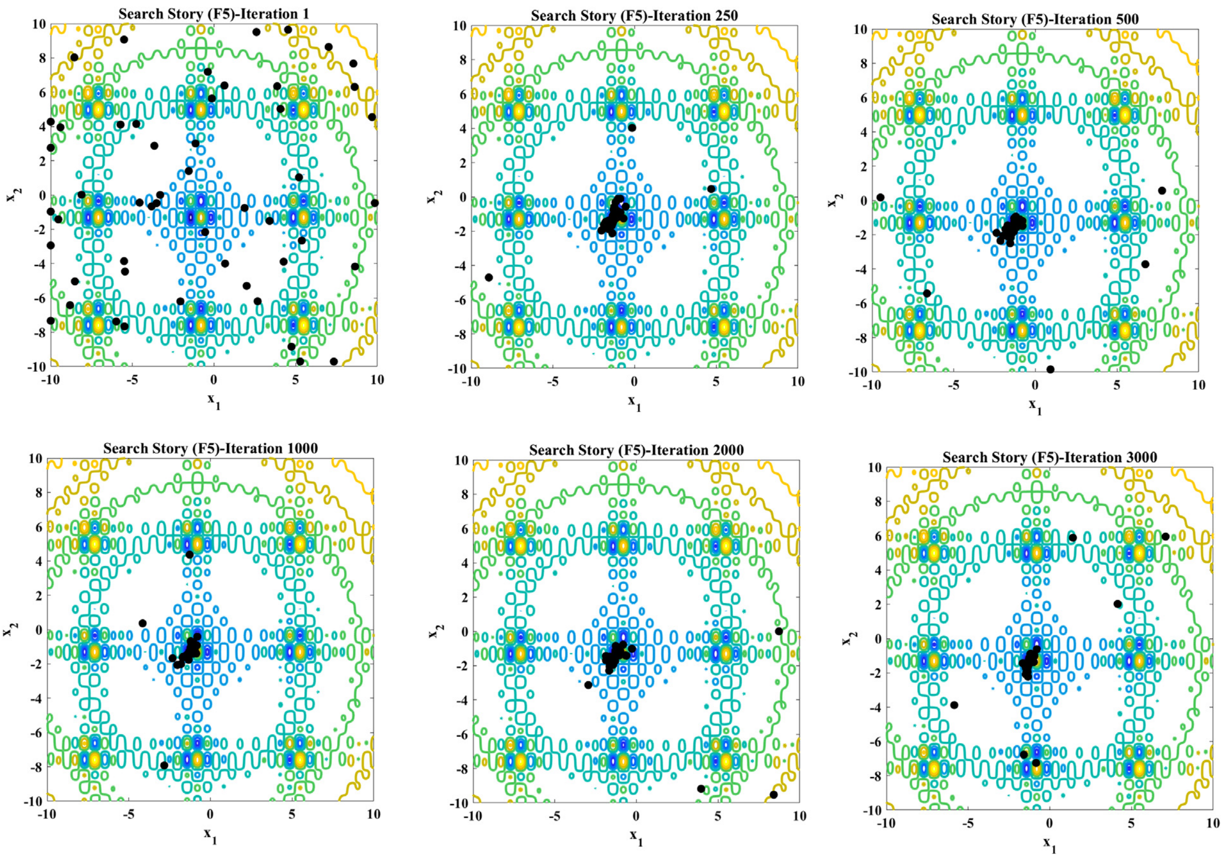

Fig. 10. Diversity plots of F5 function considering AOS.

Table 17
Best results of different approaches for the pressure vessel problem.

	He & Wang [45]	Coelho [46]	Zahara & Kao [47]	Sadollah et al. [48]	Present Study (AOS)
Best	6061.077700000	6059.720800000	5930.313700000	5889.321600000	5888.457948000
T_s	0.812500000	0.812500000	0.803600000	0.780200000	0.778674389
T_h	0.437500000	0.437500000	0.397200000	0.385600000	0.385321793
R	42.091300000	42.098400000	41.639200000	40.429200000	40.340890640
L	176.746500000	176.637200000	182.412000000	198.496400000	199.721517800
$g_1(x)$	−0.000001370	−0.000000879	0.000036500	0.000000000	−0.000095200
$g_2(x)$	−0.000359000	−0.035800000	0.000037900	0.000000000	−0.000469696
$g_3(x)$	−118.768700000	−0.217900000	−1.591400000	−86.364500000	−89.094885520
$g_4(x)$	−63.253500000	−63.362800000	−57.587900000	−41.503500000	−40.278482220

Table 18
Statistical results for the pressure vessel problem considering different approaches.

Approaches	Best	Mean	Worst	Std-Dev
[45]	6061.077700000	6147.133200000	6363.804100000	86.450000000
[46]	6059.720800000	6440.378600000	7544.492500000	448.471100000
[47]	5930.313700000	5946.790100000	5960.055700000	9.161000000
[48]	5889.321600000	6200.647700000	6392.506200000	160.340000000
Present Study (AOS)	5888.457948000	5888.480501000	5894.840682000	2.199072000

3.4.4. Tension or compression spring

In this engineering problem, the optimum weight of a tension or compression spring is considered in which there are three design variables including the mean diameter of the coil (D), diameter of the wire (d), and the number of active coils (N) with only four inequality design constraints. A schematic view for this design example is depicted in Fig. 14 (see appendix) while the formulation of this problem is presented by Gandomi et al. [44].

Table 19
Best results of different approaches for the welded beam problem.

	Huang et al. [49]	Eskandar et al. [50]	Guedria [51]	Mirjalili et al. [52]	Present Study (AOS)
Best	1.733461000	1.724856000	1.724852000	1.726240000	1.724852309
h	0.203137000	0.205728000	0.205730000	0.205676000	0.205729640
l	3.542998000	3.470522000	3.470489000	3.478377000	3.470488666
t	9.033498000	9.036620000	9.036624000	9.036810000	9.036623910
b	0.206179000	0.205729000	0.205730000	0.205778000	0.205729640
$g_1(x)$	−44.578560000	−0.034128000	0.000000000	−21.233804060	0.000000000
$g_2(x)$	−44.663530000	−0.000034900	−0.000000001	−8.285588702	0.000000000
$g_3(x)$	−0.003042000	−0.000001190	0.000000000	−0.000102000	0.000000000
$g_4(x)$	−3.423726000	−3.432980000	−3.432984000	−3.431880875	−3.432983785
$g_5(x)$	−0.078137000	−0.080728000	−0.080730000	−0.080676000	−0.080729640
$g_6(x)$	−0.235557000	−0.235540000	−0.235540000	−0.235544614	−0.235540323
$g_7(x)$	−38.028260000	−0.013503000	−0.000000001	−4.313487255	0.000000000

Table 20
Statistical results for the welded beam problem considering different approaches.

Approaches	Best	Mean	Worst	Std-Dev
[49]	1.733461000	1.768158000	1.824105000	0.022194000
[50]	1.724856000	1.726427000	1.744697000	0.004290000
[51]	1.724852000	1.724853000	1.724862000	0.000002020
[52]	1.731206500	1.878656000	2.345579300	0.267798900
Present Study (AOS)	1.724852309	1.725673538	1.732516334	0.024786984

Table 21
Best results of different approaches for the tension or compression spring.

	Coello [53]	Ray & Liew [54]	Han et al. [55]	Gandomi et al. [44]	Present Study (AOS)
Best	0.012704780	0.012669200	0.012665340	0.012665220	0.012665233
d	0.051480000	0.052160000	0.051680000	0.051690000	0.051689535
D	0.351661000	0.368159000	0.356500100	0.356730000	0.356729145
N	11.632201000	10.648442000	11.301833500	11.288500000	11.288297130
$g_1(x)$	−0.003337000	−0.000000007	−0.000006218	0.000000000	0.000000000
$g_2(x)$	−0.000110000	−0.000000004	−0.000001691	0.000000000	0.000000000
$g_3(x)$	−4.026318000	−4.075805000	−4.053315000	−4.053800000	−4.053808155
$g_4(x)$	−0.731239000	−0.719787000	−0.727879900	−0.727700000	−0.727720880

Table 22
Statistical results for the tension or compression spring considering different approaches.

Approaches	Best	Mean	Worst	Std-Dev
[53]	0.012704780	0.012769200	0.012822080	0.000039390
[54]	0.012669200	0.012922700	0.016717200	0.000051985
[55]	0.012665340	0.012685920	0.012729680	0.000021672
[44]	0.012665220	0.013500520	0.016895400	0.001420272
Present Study (AOS)	0.012665233	0.012737649	0.013596859	0.000121146

The best results of different optimization runs for the tension or compression spring design problem are presented in Table 21 while the optimum design variables and constraints for different metaheuristic approaches are also presented. It is demonstrated that the proposed AOS algorithm is capable of providing very competitive results in this case. In addition, the statistical results of different metaheuristic approached alongside the AOS algorithm are presented in Table 22 in which the superiority of the proposed algorithm in providing better mean, worst and standard deviation values are proved.

3.4.5. Multiple disk clutch brake

In this engineering design problem, the total mass optimization of a multiple disk clutch brake is considered in which there are five design variables including the inner radius (r_1), thickness of the disk (t), outer radius (r_0), number of friction surfaces (Z) and the actuating force (F) with only eight inequality design constraints. A schematic view for this design problem is depicted in Fig. 15 (see appendix) while the mathematical formulation of this problem is presented by Ferreira et al. [56].

The best results of different optimization runs for the AOS and other metaheuristics in dealing with the multiple disk clutch brake problem are presented in Table 23 while the design variables and constraints are also provided. By comparing the best results of the AOS to other approaches, it is concluded that the AOS is capable of converging to outstanding results

Table 23
Best results of different approaches for the multiple disk clutch brake problem.

	Deb & Srinivasan [57]	Eskandar et al. [50]	Rao et al. [58]	Ferreira et al. [56]	Present Study (AOS)
Best	0.470400000	0.313656000	0.313656611	0.313656000	0.235242480
r_1	70.000000000	70.000000000	70.000000000	70.000000000	69.999997820
r_0	90.000000000	90.000000000	90.000000000	90.000000000	90.000000000
t	1.500000000	1.000000000	1.000000000	1.000000000	1.000000000
F	1000.000000000	910.000000000	810.000000000	830.000000000	349.399656100
Z	3.000000000	3.000000000	3.000000000	3.000000000	2.000000000
$g_1(x)$	0.000000000	0.000000000	0.000000000	0.000000000	−0.000002180
$g_2(x)$	−22.000000000	−24.000000000	−24.000000000	−24.000000000	−25.500000000
$g_3(x)$	−0.900500000	−0.909480000	−0.919427810	−0.917438000	−0.965244577
$g_4(x)$	−9.790600000	−9.809429000	−9830.371094000	−9.826183000	−9.994100622
$g_5(x)$	−7.894700000	−7.894696000	−7894.696590000	−7.894697000	−9.830260219
$g_6(x)$	−3.352700000	−2.231421000	−0.702013203	−0.173855000	−14.957298520
$g_7(x)$	−60.625000000	−49.768749000	−37706.250000000	−40.118750000	−33657.066400000
$g_8(x)$	−11.647300000	−12.768578000	−14.297986800	−14.826145000	−0.042701477

Table 24
Statistical results for the multiple disk clutch brake problem considering different approaches.

Approaches	Best	Mean	Worst	Std-Dev
[50]	3.14E−01	3.14E−01	3.14E−01	1.69E−16
[58]	3.14E−01	3.27E−01	3.92E−01	6.70E−01
[56]	3.14E−01	3.14E−01	3.14E−01	1.13E−16
Present Study (AOS)	2.35E−01	2.35E−01	2.35E−01	6.45E−25

than the other algorithms. By comparing the statistical results of different approaches in Table 24, it is also obvious that the proposed AOS algorithm even provides very better mean, worst and standard deviation values.

4. Conclusions

The Atomic Orbital Search (AOS) algorithm is proposed in this paper as a novel metaheuristic algorithm for optimization purposes. For numerical purposes, 20 mathematical functions and 5 engineering design problems are considered alongside of two of the most important CEC as CEC 2017 and CEC 2020. A complete computational cost and complexity analysis with "Big O notation" is also conducted. The results and core findings of this paper are summarized in the following:

- AOS algorithm outranks the other alternative metaheuristics in converging to the global best in dealing with different mathematical test functions.
- AOS algorithm is capable of converging to better results with lower required function evaluations which represents the effectiveness of the proposed algorithm in dealing with the computational complexity issues.
- The AOS has lower mean of the ranks in most of the cases in a two by two comparing manner considering the W statistical test which demonstrates its effectiveness among other approaches.
- The results of KW test prove that the AOS has the first ranking considering the best, mean, standard deviation and function evaluation values of different 2D, 50D and 100D functions.
- Regarding the convergence speed of the AOS, this algorithm is capable of localizing the searching process for achieving better results.
- Considering the CEC 2017 benchmark suite, the AOS is capable of providing very competitive results in dealing with complex mathematical problems comparing to the state−of-the−art-metaheuristics.
- The proposed algorithm is capable of outranking the other metaheuristic algorithms in all of the engineering design problems in which the best and statistical results of the proposed AOS algorithm is outstanding.
- The maximum difference between the results of the AOS and other approaches in dealing with the engineering design problems are about 40% in some cases.
- Regarding pressure vessel problem, AOS is capable of calculating 5888.45 which is the best among other challenges in the literature.
- The proposed AOS is capable of obtaining 0.2352 for multiple disk clutch brake problem as one of the most important constrained optimization problems.
- Considering ten-bar truss problem as one of the important design examples in the structural optimization field, AOS is capable of achieving 525.67 which is way better than the previously reported results in the literature.
- Based on the fact that most of the engineering design problems in the constraint optimization filed are considered in this paper, the capability of the proposed AOS algorithm is proved due to the provided results.

Considering the future challenges, the proposed algorithm can be utilized in different applications regarding the fact that the capability of the AOS should be considered in dealing with some real-size complex optimization problems. It also should be noted that the limitation of the proposed AOS algorithm in dealing with these complex problems is another challenge that should be considered properly. One of the key factors in considering these limitations is the initial population size and photon rate that should be considered wisely in the future challenges.

Appendix

Table 1
Basic properties of the collected mathematical test functions.

No.	Name	Type	R	D	Formulation	Min.
F_1	Becker-Lago function	S	[-10, 10]	2	[28]	0
F_2	Carrom table function	NS	[-10, 10]	2	[28]	−24.1568
F_3	Cube Function	C, D, NS, NSc, U	[-10, 10]	2	[27]	0
F_4	Himmelblau Function	C, D, NS, NSc, M	[-5, 5]	2	[27]	0
F_5	Levy 5 function	NS	[-10, 10]	2	[28]	−176.138
F_6	Ursem 3 Function	NS	[-2, 2] & [-1.5, 1.5]	2	[27]	−2.5
F_7	Ursem 4 Function	NS	[-2, 2]	2	[27]	−1.5
F_8	Ackley 1 Function	C, D, NS, Sc,M	[-35, 35]	50	[27]	0
F_9	Alpine 1 Function	C, ND, S, NSc,U	[-10, 10]	50	[27]	0
F_{10}	Deb 3 Function	C, D, S, Sc, M	[0, 1]	50	[27]	−1
F_{11}	Exponential Function	C, D, NS, Sc, M	[-1, 1]	50	[27]	−1
F_{12}	Inverted cosine wave function	NS	[-10, 10]	50	[28]	−49
F_{13}	Schwefel 2.20 Function	C, ND, S, Sc, U	[-100, 100]	50	[27]	0
F_{14}	Streched V Sine Wave Function	C, D, NS, Sc, U	[-10, 10]	50	[27]	0
F_{15}	Deb 1 Function	C, D, S, Sc, M	[-1, 1]	100	[27]	−1
F_{16}	Levy 8 function	NS	[-10, 10]	100	[28]	0
F_{17}	Mishra 11 Function	C, D, NS, NSc, M	[-10, 10]	100	[27]	0
F_{18}	Salomon Function	C, D, NS, Sc, M	[-100, 100]	100	[27]	0
F_{19}	Schwefel 2.21 Function	C, ND, S, Sc, U	[-100, 100]	100	[27]	0
F_{20}	Stepint Function	DC, ND, S, Sc, U	[-5.12, 5.12]	100	[27]	−575

C: Continuous, NC: Non-Continuous, D: Differentiable, ND: Non-Differentiable, S: Separable, NS: Non-Separable, Sc: Scalable, NSc: Non-Scalable, U: Unimodal and M: Multi-modal, R: Variables Range, D: Variables Dimension, Min: Global Minimum Of The Functions.

Table 2
Parameter presentation of the alternative metaheuristic algorithms.

Metaheuristic	Parameter	Description	Value
GA	$\mathbf{N_{pop}}$	Number of Population	50
	$\mathbf{p_c}$	Crossover Percentage	0.8
	$\mathbf{p_m}$	Mutation Percentage	0.3
	$\boldsymbol{\mu}$	Mutation Rate	0.02
	$\boldsymbol{\beta}$	Roulette wheel selection pressure	1
BIA	$\mathbf{N_{pop}}$	Number of Bats	50
	$\mathbf{A}$	Loudness	0.5
	$\mathbf{r}$	Pulse Rate	0.5
	$\mathbf{Q_{Min}}$	Frequency Minimum	0
	$\mathbf{Q_{Max}}$	Frequency Maximum	2
BBBC	$\mathbf{N_{pop}}$	Number of Population	50
	$\boldsymbol{\beta}$	Parameter for the center of mass	0.2
	$\boldsymbol{\alpha}$	Parameter for size of the initial search space	1
CSA	$\mathbf{N_{pop}}$	Number of Butterflies	50
	$\mathbf{P_a}$	Discovery rate of alien eggs	0.25
WDO	$\mathbf{N_{pop}}$	Population Size	50
	$\mathbf{RT}$	RT Coefficient	3
	$\mathbf{g}$	Gravitational Constant	0.2
	$\boldsymbol{\alpha}$	Constant for Updating Process	0.4
	$\mathbf{c}$	Coriolis Effect	0.4
	$\mathbf{V}$	Maximum Allowed Speed	0.3

Table 6
The mean of function evaluations for different metaheuristics in dealing with mathematical functions.

No.	Alternative Metaheuristic Algorithms								
	GA	MVO	SCA	BIA	BBBC	CSA	ASO	WDO	AOS
F_1	28015.98	150000	150000	149294.5	150000	25485	150000	150000	100300
F_2	6602.346	149945.5	150000	50	127557	13697	150000	149692.5	74723.5
F_3	150000	150000	150000	150000	150000	69619	150000	150000	67212.5
F_4	49061.88	150000	150000	150000	150000	39899	150000	150000	86238
F_5	16872.83	144346.5	150000	129175.5	146371	17983	150000	139934.5	5715.5
F_6	23272.19	150000	9655	4110	150000	47159	150000	6914.5	110816
F_7	38261.99	150000	10527	159.5	150000	49692	36327.5	6666.5	137049
F_8	150000	150000	150000	150000	150000	150000	150000	22132.5	28728.31
F_9	150000	150000	150000	150000	150000	150000	150000	51420.5	26830.3
F_{10}	150000	150000	150000	50	150000	150000	150000	150000	150000
F_{11}	150000	150000	147229	150000	150000	150000	150000	10024	13052.29
F_{12}	150000	150000	150000	150000	150000	150000	150000	150000	150000
F_{13}	150000	150000	147668	150000	150000	150000	150000	25202.5	30748.24
F_{14}	150000	150000	150000	150000	150000	150000	150000	150000	146463.2
F_{15}	150000	150000	150000	150000	150000	150000	150000	150000	150000
F_{16}	150000	150000	150000	150000	150000	150000	150000	150000	150000
F_{17}	150000	150000	150000	108006	150000	122241	150000	12935	80233.31
F_{18}	150000	150000	150000	150000	150000	150000	150000	150000	150000
F_{19}	150000	150000	150000	150000	150000	150000	150000	25965.5	32859.24
F_{20}	6276.947	150000	150000	51	150000	150000	150000	74166	150000

Table 9
The W test results (p-values) for different metaheuristic algorithms.

Main Algorithm	Data Type	Alternative Metaheuristic Algorithms							
		GA	MVO	SCA	BIA	BBBC	CSA	CSS	WDO
AOS	Min.	9.06E−02	3.47E−02	5.23E−03	5.80E−02	3.47E−02	1.29E−01	8.90E−03	7.70E−01
		3.32E−01	7.31E−02	2.50E−03	7.31E−02	3.04E−02	3.98E−01	3.19E−03	9.18E−01
	Mean	6.01E−01	6.54E−01	2.51E−02	4.55E−01	1.08E−01	6.29E−01	4.78E−01	4.46E−03
		4.21E−01	1.22E−04	3.53E−02	5.35E−01	1.22E−04	2.96E−01	3.05E−03	1.00E+0
	Std.	4.49E−01	8.03E−03	2.81E−06	1.03E−04	1.73E−02	1.73E−03	6.56E−05	5.48E−01
		1.21E−01	1.48E−03	1.63E−07	7.12E−05	8.65E−03	2.50E−03	5.82E−04	2.51E−01
	Fun.	2.34E−02	1.26E−03	7.06E−05	6.05E−05	2.82E−02	1.59E−01	1.37E−02	9.98E−03
	Evl.	5.03E−06	2.17E−07	5.86E−07	5.38E−04	5.85E−07	2.95E−07	2.70E−07	8.97E−02

Table 11
Problem summary of the CEC 2017 mathematical test functions [24].

Function Type	Fun. No.	Function Detail	Fun. Min.
Unimodal Functions (Shifted and Rotated)	G_1	Bent Cigar Function	100
	G_2	Sum of Different Power Function	200
	G_3	Zakharov Function	300
Simple Multimodal Functions (Shifted and Rotated)	G_4	Rosenbrock Function	400
	G_5	Rastrigin Function	500
	G_6	Expanded Scaffer F6 Function	600
	G_7	Lunacek Bi_Rastrigin Function	700
	G_8	Non-Continuous Rastrigin Function	800
	G_9	Levy Function	900
	G_{10}	Schwefel Function	1000
Hybrid Functions	G_{11}	Zakharov, Rosenbrock, Rastrigin Functions	1100
	G_{12}	High Conditioned Elliptic, Modified Schwefel, Bent Cigar Functions	1200
	G_{13}	Bent Cigar, Rosenbrock, Lunache Bi-Rastrigin Functions	1300
	G_{14}	High Conditioned Elliptic, Ackley, Schaffer, Rastrigin Functions	1400
	G_{15}	Bent Cigar, HGBat, Rastrigin, Rosenbrock Functions	1500
	G_{16}	Expanded Schaffer F6, HGBat, Rosenbrock, Modified Schwefel Functions	1600
	G_{17}	Katsuura, Ackley, Expanded Griewank plus Rosenbrock, Modified Schwefel, Rastrigin Functions	1700
	G_{18}	High Conditioned Elliptic, Ackley, Rastrigin, HGBat, Discu Functions	1800
	G_{19}	Bent Cigar, Rastrigin, Expanded Griewank plus Rosenbrock, Weierstrass, Expanded Schaffer Functions	1900
	G_{20}	Happycat, Katsuura, Ackley, Rastrigin, Modified Schwefel, Schaffer Functions	2000
Composition Functions	G_{21}	Rosenbrock, High Conditioned Elliptic, Rastrigin Functions	2100
	G_{22}	Rastrigin, Griewank, Modifed Schwefel Functions	2200
	G_{23}	Rosenbrock, Ackley,Modified Schwefel, Rastrigin Functions	2300
	G_{24}	Ackley, High Conditioned Elliptic, Girewank, Rastrigin Functions	2400
	G_{25}	Rastrigin, Happycat, Ackley, Discus, Rosenbrock Functions	2500
	G_{26}	Expanded Scaffer, Modified Schwefel, Griewank, Rosenbrock, Rastrigin Functions	2600
	G_{27}	HGBat, Rastrigin, Modified Schwefel, Bent-Cigar, High Conditioned Elliptic, Expanded Scaffer's F6 Functions	2700
	G_{28}	Ackley, Griewank, Discus, Rosenbrock, HappyCat, Expanded Scaffer's F6 Functions	2800
	G_{29}	G15, G16, G17 Functions	2900
	G_{30}	G15, G18, G19 Functions	3000

Search Range: [-100,100]
Dimensions: 10

Table 14
Basic characteristics of the considered engineering design problems.

No.	Name	D	g	h
C_1	Speed Reducer	7	11	0
C_2	Pressure Vessel	4	4	0
C_3	Welded Beam	4	7	0
C_4	Tension/Compression Spring	3	4	0
C_5	Multiple Disk Clutch Brake	5	8	0

D: Dimensions
g: Number of inequality constraints
h: Number of equality constraints

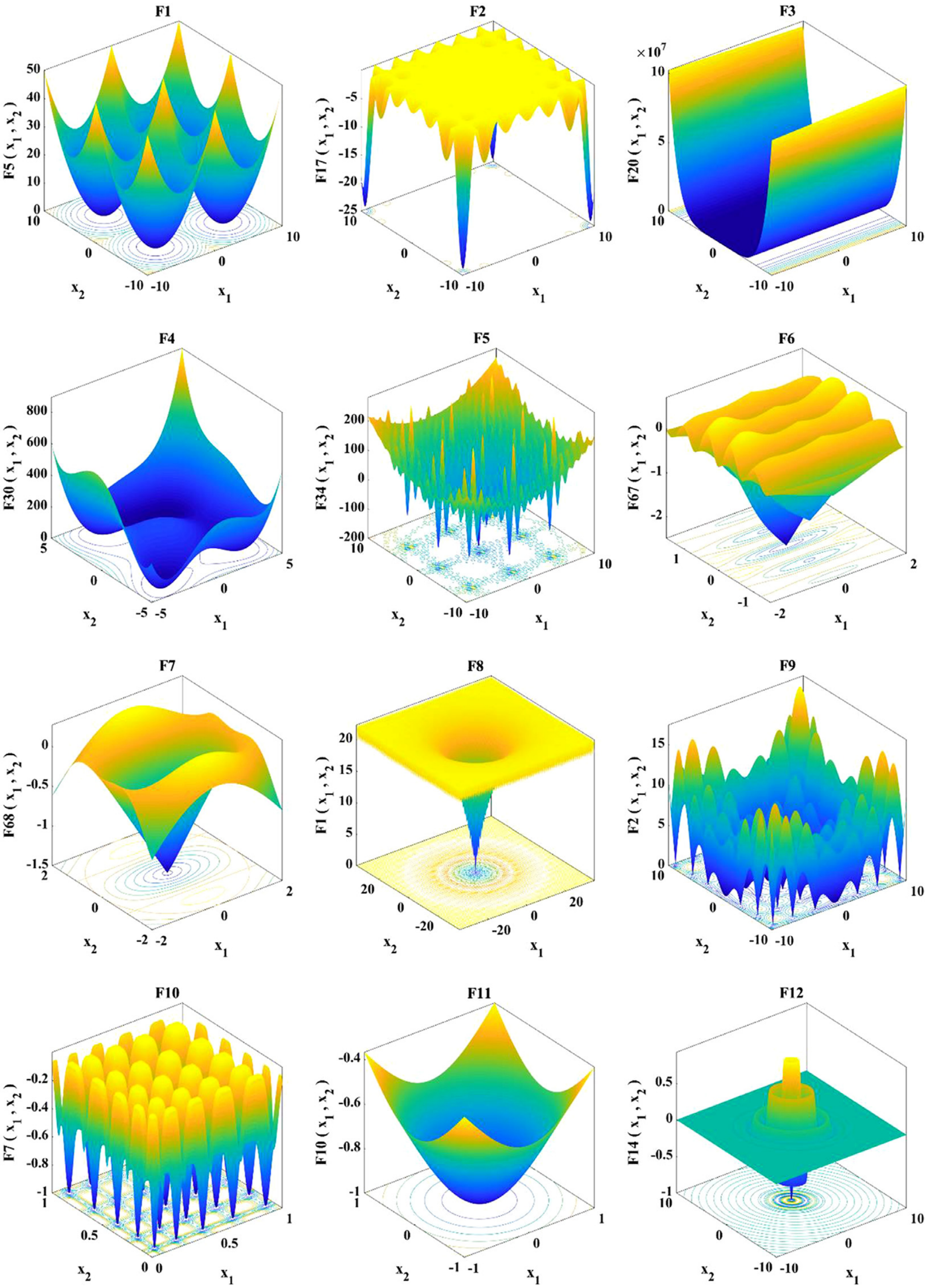

Fig. 7. 3D plots of the mathematical functions.

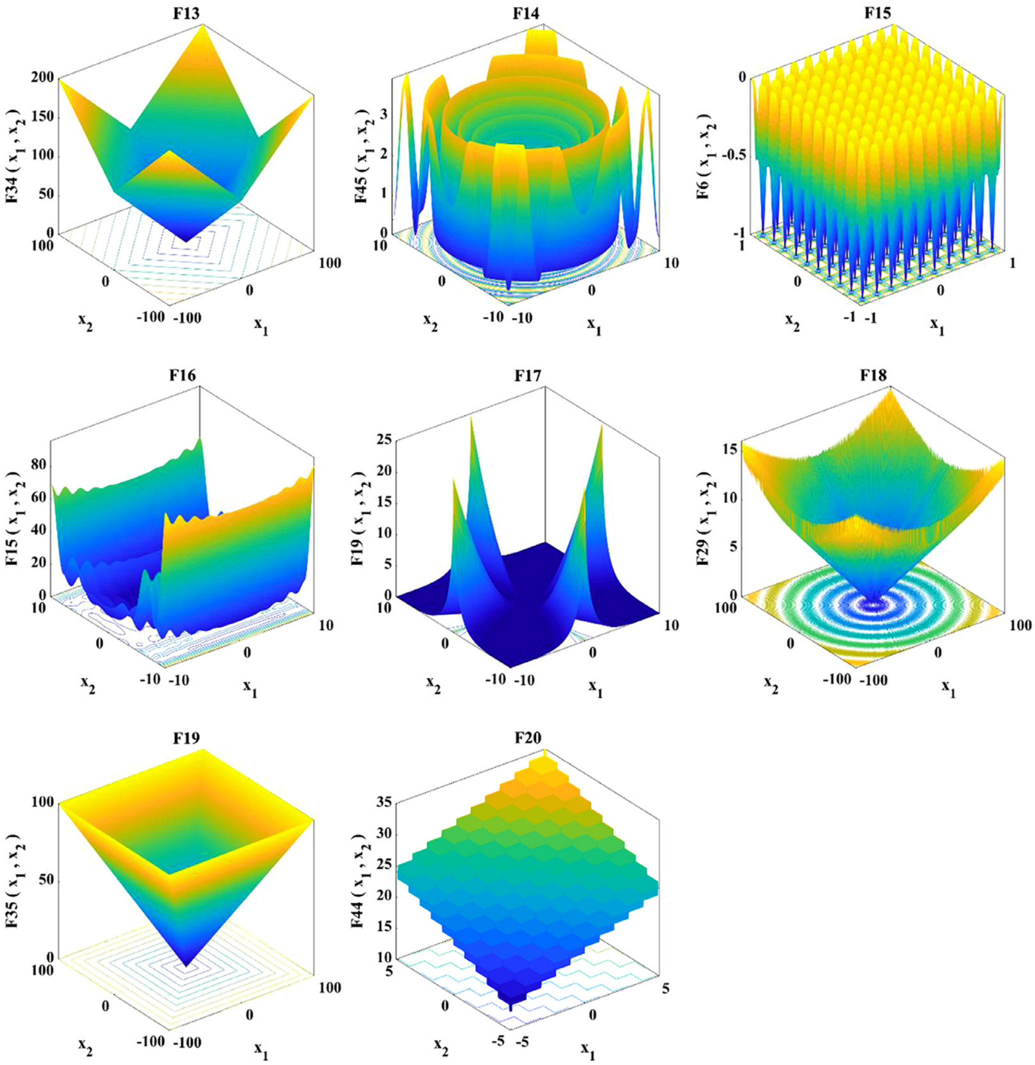

Fig. 7. Continued

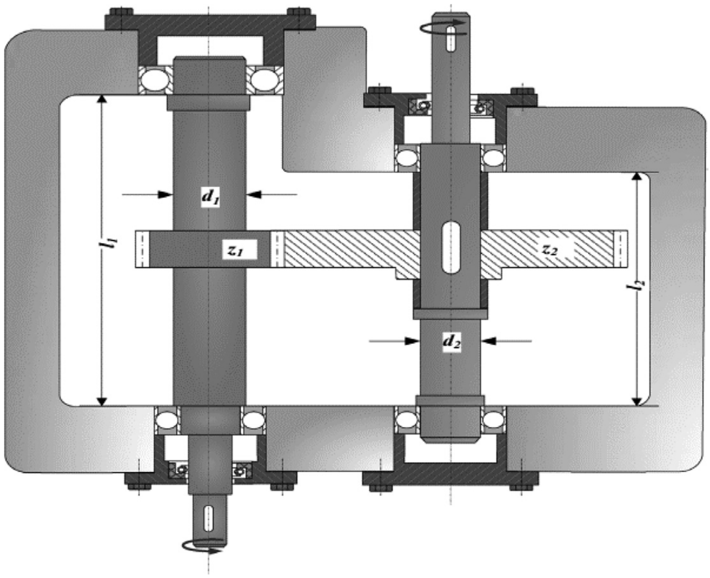

Fig. 11. Schematic view of the speed reducer [39].

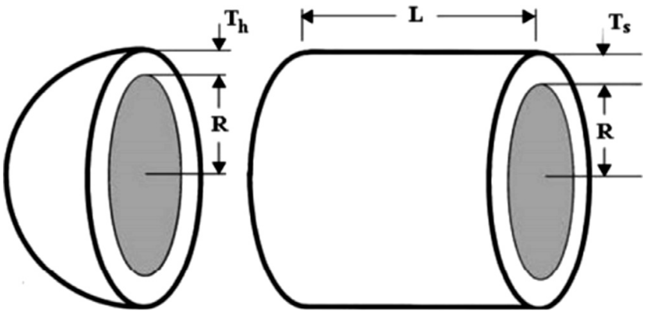

Fig. 12. Schematic view of the pressure vessel problem [44].

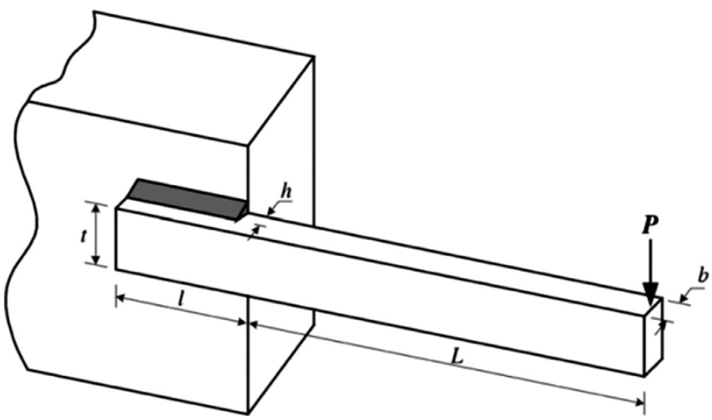

Fig. 13. Schematic view of the welded beam problem [44].

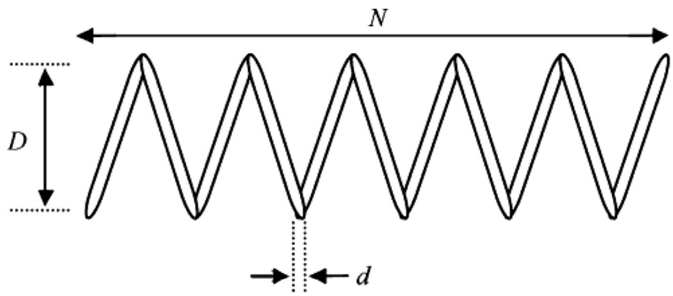

Fig. 14. Schematic view of the tension or compression spring [44].

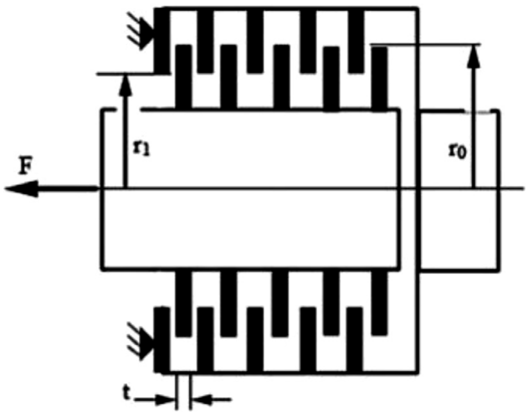

Fig. 15. Schematic view of the multiple disk clutch brake problem [56].

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