

Solutions: Galois Theory by Tom Leinster

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Chapter 1. Overview of Galois Theory

Exercise 1.1.3

Both proofs of ‘if’ contain little gaps: ‘It follows by induction’ in the first proof, and ‘it’s easy to see’ in the second. Fill them.

Solution: We show both both parts (i) and (ii) separately

(i) Follows from induction that for any polynomial p over \mathbb{R} , $\overline{p(w)} = p(\overline{w})$:

Let $p(w) = c_0 + c_1w^1 + c_2w^2 + \dots + c_nw^n$ where $w^n \in \mathbb{C}$ and $c_n \in \mathbb{C}$.

$$\begin{aligned}\overline{p(w)} &= \overline{c_0 + c_1w^1 + c_2w^2 + \dots + c_nw^n} \\ &= \overline{c_0} + \overline{c_1w^1} + \overline{c_2w^2} + \dots + \overline{c_nw^n} \\ &= c_0 + c_1\overline{w^1} + c_2\overline{w^2} + \dots + c_n\overline{w^n} \\ &= p(\overline{w})\end{aligned}$$

(ii) Checking that r is the zero polynomial:

Lemma. If $r(x) = a_0 + a_1x^1 + \dots + a_nx^n$ and $r(x) = 0, \forall x \neq 0$, then $a_0 = 0$. Since \mathbb{Q} is a field, and must contain a zero.

By this lemma we have that $\forall x, r(x) = 0$ and therefore $r(x) = 0 = x(a_1 + a_2x + \dots + a_nx^{n-1}) = a_1 + a_2x + \dots + a_nx^{n-1}$ and $\forall x \neq 0, a_1 = 0$. Hence, we repeat the Lemma and show that all $a_1, \dots, a_n = 0$. Therefore $r(x)$ is the zero polynomial.

Exercise 1.1.6

Let $z \in \mathbb{Q}$. Show that z is not conjugate to z' for any complex number $z' \neq z$.

Solution:

Exercise 1.1.10

Suppose that (z_1, \dots, z_k) and (z'_1, \dots, z'_k) are conjugate. Show that z_i and z'_i are conjugate, for each $i \in \{1, \dots, k\}$

Solution: By **Definition 1.1.9** have that:

$$p(z_1, \dots, z_k) = 0 \iff p(z'_1, \dots, z'_k) = 0$$

When $k = 1$ we have that:

$$p(z_1) = 0 \iff p(z'_1) = 0 \implies z_1 \text{ and } z'_1 \text{ are conjugate}$$

Similarly, for any k :

$$p(z_i) = 0 \iff p(z'_i) = 0 \implies z_i \text{ and } z'_i \text{ are conjugate}$$

Exercise 1.2.2

Show that $Gal(f)$ is a subgroup of S_k .

Chapter 2. Group actions, rings and fields

Exercise 2.1.3

Check that \bar{g} is a bijection for each $g \in G$. Also check that Σ is a homomorphism.

Solution: We show injectivity (i), surjectivity (ii) and homomorphism (iii) :

(i) Injectivity:

Let $x, y \in X$ and \bar{g} be our bijection

If we have $\bar{g}(x) = \bar{g}(y)$

$$\rightsquigarrow gx = gy \rightsquigarrow g^{-1}(gx) = g^{-1}(gy) \rightsquigarrow (g^{-1}g)x = (g^{-1}g)y \rightsquigarrow ex = ey \rightsquigarrow x = y \quad \square$$

(ii) Surjectivity:

We know that $f : X \rightarrow Y$ is surjective iff $\forall y \in Y \exists x \in X : f(x) = y$

We let $x \in X$ and e be the identity in G then,

$$x = ex = (gg^{-1})x = g(g^{-1}x) = gy = \bar{g}(y) \text{ where } y = g^{-1}x \in X \quad \square$$

Therefore \bar{g} is both injective and surjective, hence bijective.

(i) Σ is Homomorphism:

We have the map:

$$\begin{aligned}\Sigma : G &\rightarrow \text{Sym}(X) \\ g &\mapsto \bar{g}\end{aligned}$$

We know that \bar{g} is well defined

Then we take $g, h \in G, x \in X$, then by **Definition 2.1.1**.

$$\begin{aligned}\Sigma(gh)(x) &= gh(x) = g(hx) \\ &= \Sigma(g)(\Sigma(h)(x)) = \Sigma(g) \circ \Sigma(h)(x) \quad \square\end{aligned}$$

Exercise 2.1.10

Example **2.1.9(iii)** shows that the action of the isometry cube G of the cube on the set X of long diagonals is not faithful. By **Lemma 2.1.8**, there must be some non-identity isometry of the cube that fixes all four long diagonals. In fact, there is exactly one. What is it?

Solution: We show both both parts (i) and (ii) separately

Exercise 2.2.6

Prove that the only subring of a ring R that is also an ideal is R itself.

Solution: We know that I is an ideal of R if:

$$\begin{aligned}(I, +) &\leq (R, +) \quad [\text{I is additive subgroup of } R] \\ &\& \forall r \in R, x \in I : \\ (1) \quad &r \cdot x \in I \\ (2) \quad &x \cdot r \in I\end{aligned}$$

We know that a subring S of R is a subset $S \subseteq R$ containing 0 and 1.

Therefore if we take S to be an ideal as well then:

$$\begin{aligned}(S, +) &\leq (R, +) \\ &\& \forall r \in R, s \in S : \\ (1) \quad &r \cdot s \in S \\ (2) \quad &s \cdot r \in S\end{aligned}$$

But we know that $1 \in S$. Therefore, $\forall r \in R$, (1) $1 \cdot r = r \in S$ and (2) $r \cdot 1 = r \in S$
Therefore $\forall r \in R, r \in S \implies S = R \quad \square$

Exercise 2.2.8

The trivial ring or zero ring is the one-element set with its only possible ring structure. Show that the only ring in which $0 = 1$ is the trivial ring.

Solution: Let $(R, +, \cdot)$ be our commutative, unital ring. If $1 = 0$ in R , then $\forall r \in R$ we have $r = 1r = 0r = 0$ \square

Exercise 2.2.8

Fill in the details of Example 2.2.13.

Solution: We suppose that $I \subseteq \mathbb{Z}$ is an ideal and we take $n \in I$ to be the least positive integer in I . We have obviously that $\langle n \rangle \subseteq I$. Then we assume that $m \in I$, by the division algorithm we know that:

$$\begin{aligned} m &= qn + r & (0 \leq r < n) \\ r &= m - qn & \in I \end{aligned}$$

Therefore $r = 0 \rightsquigarrow m = qn$. Therefore $m \in \langle n \rangle$ and we have that $I \subseteq \langle n \rangle$. Hence we have equality, $I = \langle n \rangle$ \square

Exercise 2.2.15

Let r and s be elements of an integral domain. Show that $r|s|r \iff \langle r \rangle = \langle s \rangle \iff s = ur$ for some unit u .

Solution: If we have that $r|s|r$ then $\exists a \in R : s = ar$ and $\exists b \in R : r = bs$ then:

$$\begin{aligned} \frac{s}{a} &= bs \\ b &= \frac{1}{a} \rightsquigarrow ab = 1 \rightsquigarrow b = a^{-1} \end{aligned}$$

Then we have that $s = ar$, and we have just shown that a is a unit, hence $s = ur$. Therefore $r|s|r \implies s = ur$

If we have $\langle r \rangle = \langle s \rangle$, then $r = s$. Hence,

$$\begin{aligned} r &= 1s & \& \quad s &= 1r \\ r &= as & \& \quad s &= ar \quad (\text{where } a = 1) \\ \implies s &|r & \& \quad r &|s \end{aligned}$$

Therefore $\langle r \rangle = \langle s \rangle \implies r|s|r$

If we have that $s = ur$ for some unit u , then also we have that

$$\begin{aligned} u^{-1}s &= u^{-1}ur \rightsquigarrow r = u^{-1}s \\ \text{Therefore } s \in \langle r \rangle \text{ \& } r \in \langle s \rangle, \langle s \rangle &\subseteq \langle r \rangle \text{ \& } \langle r \rangle \subseteq \langle s \rangle \\ \implies \langle r \rangle &= \langle s \rangle \end{aligned}$$

Therefore $s = ur \implies \langle r \rangle = \langle s \rangle$

Exercise 2.3.1

Write down all the examples of fields that you know.

Solution: $\mathbb{C}, \mathbb{R}, \mathbb{Q}$

Exercise 2.3.5

Let $\phi : K \rightarrow L$ be a homomorphism of fields and let $0 \neq a \in K$. Prove that $\phi(a^{-1}) = \phi(a)^{-1}$. Why is $\phi(a)^{-1}$ defined?

Solution: Since K is a field, and the fact that $0 \neq a \in K$, we have that a is a unit, $aa^{-1} = 1$, and $a^{-1} \in K$. By **Lemma 2.3.3**, we have that $\phi : K \rightarrow L$ is injective. Hence, $\phi(a)\phi(a^{-1}) = \phi(a \circ a^{-1}) = \phi(1) = 1$, and $\phi(a^{-1})\phi(a) = \phi(a^{-1} \circ a) = \phi(1) = 1$. Therefore we have that $\phi(a^{-1})$ is both a left and right inverse of $\phi(a)$ and hence it is the only inverse of $\phi(a)$. Therefore, by injectivity $\phi(a^{-1}) = \phi(a)^{-1}$.

Exercise 2.3.13

This proof of Lemma 2.3.12 is quite abstract. Find a more concrete proof, taking equation (2.2) as your definition of characteristic. (You will still need the fact that ϕ is injective.)

Solution: By (2.2) we have:

$$\text{char} R = \begin{cases} \text{least } n > 0 : n * 1_R = 0_R & , \text{ if such an } n \text{ exists} \\ 0 & , \text{ otherwise} \end{cases}$$

We know that $\phi(1_K) = 1_L$ and $\phi(0_K) = 0_L$, since ϕ is injective, then also $\phi(n \cdot 1_K) = n \cdot 1_L \forall n \in \mathbb{N}$. We have two possible cases for the characteristic c of K ($\text{char} K$), $c = 0$ or $c > 0$.

If $c = 0$, then $\phi(0_K) = 0_L = 0$. Therefore $\text{char} L = c = \text{char} K$.

If $c > 0$, then $\phi(c \cdot 1_K) = c \cdot 1_L = 0$. Therefore $\text{char} L = c = \text{char} K$.

Exercise 2.3.15

What is the prime subfield of \mathbb{R} ? Of \mathbb{C} ?

Solution: For \mathbb{R} it is \mathbb{Q} . For \mathbb{C} it is also \mathbb{Q} . See **Lemma 2.3.16**.

Exercise 2.3.25

What are the irreducible elements of a field?

Solution: We know that for a ring R , r is irreducible if r is not 0 or a unit and if for $a, b \in R$, then $r = ab \implies a$ or b is a unit. However, we know that every element of a field K is either a unit or 0. Therefore, there are no irreducible elements in a field.

Chapter 3. Polynomials

Exercise 3.1.4

Show that whenever R is a finite nontrivial ring, it is possible to find distinct polynomials over R that induce the same function $R \rightarrow R$. (Hint: are there finitely or infinitely many polynomials over R ? Functions $R \rightarrow R$?)

Solution:

Exercise 3.1.8

What happens to everything in the previous paragraph if we substitute $t = u^2 + c$ instead?

Solution:

Exercise 3.1.13

Let p be a prime and consider the field $\mathbb{F}_p(t)$ of rational expressions over \mathbb{F}_p . Show that t has no p th root in $\mathbb{F}_p(t)$. (Hint: consider degrees of polynomials.)

Solution: A rational expression over K is $\frac{f(t)}{g(t)}$ where $f(t), g(t) \in K[t]$ with $g \neq 0$. For any $\frac{f(t)}{g(t)} \in \mathbb{F}_p(t)$ where $f(t), g(t) \in \mathbb{F}_p[t]$, suppose we have that $\left(\frac{f(t)}{g(t)}\right)^p = t$. We then have that $f^p = tg^p$. Then $\deg(f^p) = np$ where $n = \deg(f)$ and $\deg(tg^p) = \deg(t) + \deg(g^p) = 1 + mp$ where $m = \deg(g)$, hence we have $np = mp + 1 \rightsquigarrow p = \frac{1}{n-m}$. But this is impossible since p is prime, hence a contradiction, hence t has no p th root in $\mathbb{F}_p(t)$.

Exercise 3.2.4

Prove that the ideals in Warning 3.2.3 are indeed not principal.

Solution:

Exercise 3.3.5

If I gave you a quadratic over \mathbb{Q} , how would you decide whether it was reducible or irreducible?

Solution: By **Lemma 3.3.1 (ii)**, if the quadratic has a root in \mathbb{Q} , then it is reducible. By the same lemma **(iii)**, if the quadratic has no root in \mathbb{Q} , then it is irreducible.

Exercise 3.3.13

The last step in (3.9) was ' $\deg(\bar{h}) \leq \deg(h)'$ '. Why is that true? And when does equality hold?

Solution: $\bar{h} = h \bmod p$. Therefore if $p | a_{n_h}$ then $a_{n_{\bar{h}}} = 0$ and $\deg(\bar{h}) < \deg(h)$. If $p \nmid a_{n_h}$ then $a_{n_h} = a_{n_{\bar{h}}}$ and $\deg(\bar{h}) = \deg(h)$. Therefore $\deg(\bar{h}) \leq \deg(h)$. Equality holds on the preceding condition.

Exercise 3.3.15

Use Eisenstein's criterion to show that for every $n \geq 1$, there is an irreducible polynomial over \mathbb{Q} of degree n .

Solution: Let $f(t) = a_0 + \dots + a_n t^n \in \mathbb{Q}[t]$ with $n \geq 1$. For $n \geq 1$, we can always choose an $f \in \mathbb{Q}[t]$ such that $f(t) = a_n t^n + a_0$, and we can further always choose an a_n, a_0 and p such that $p \nmid a_n$, $p | a_0$, $p^2 \nmid a_0$. Hence, we have $f(t) = a_n t^n + a_0$ fulfilling the Eisenstein criterion, and hence $f(t)$ is irreducible over \mathbb{Q} . As an example, we can always choose $f(t) = t^n + 2$ and $p = 2$.

Chapter 4. Field extensions

Exercise 4.1.3

Find two examples of fields K such that $\mathbb{Q} \subsetneq K \subsetneq \mathbb{Q}(\sqrt{2}, i)$

Solution: $K = \mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ and $K = \mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}$

Exercise 4.1.5

Check the truth of all the statements in the previous paragraph.

Solution: Follow trivially from definitions of intersection and subfields. See **Lemma 2.2.3** for showing intersection of subfields still remains a subfield.

Exercise 4.1.7

What is the subfield of \mathbb{C} generated by $\{7/8\}$? By $\{2 + 3i\}$? By $\mathbb{R} \cup \{i\}$?

Solution: Since \mathbb{C} is of characteristic 0, by **Lemma 2.3.16** the prime subfield of \mathbb{C} is \mathbb{Q} . Since \mathbb{Q} contains $\{7/8\}$ and by definition of prime subfield, it is the intersection of all the subfields of \mathbb{C} containing $\{7/8\}$, hence \mathbb{Q} is generated by $\{7/8\}$.

Let L be the subfield of \mathbb{C} generated by $\{2 + 3i\}$. Then $L = \{2a + 3bi : a, b \in \mathbb{Q}\}$ by similar argument as **Example 4.1.6 (ii)**.

Similarly, let L be the subfield of \mathbb{C} generated by $\mathbb{R} \cup \{i\}$. Then $L = \mathbb{R} \cup \{a + bi : a, b \in \mathbb{Q}\} \stackrel{?}{=} \{a + bi : a \in \mathbb{R}, b \in \mathbb{Q}\}$.

Exercise 4.1.11

Let $M : K$ be a field extension. Show that $K(Y \cup Z) = (K(Y))(Z)$ whenever $Y, Z \subseteq M$.

Solution:

Exercise 4.2.2

Show that every element of K is algebraic over K .

Solution: Since K is a field, $\forall k \in K : \exists -k \in K : k + (-k) = (-k) + k = 0$. Therefore, $\forall k \in K$, we can choose $f(t) = t - k \in K[t]$. Hence we have that $f \neq 0$ and $f(k) = k - k = 0$. Therefore $\forall k \in K, k$ is algebraic over K .

Exercise 4.2.9

What is the minimal polynomial of an element of K ?

Solution: We can refer back to **Exercise 4.2.2**. If we let $m(t) = t - k$, then we see that it is indeed monic and unique $\forall k \in K$ satisfying condition (4.2).

Exercise 4.3.5

Let $M : K$ and $L : K$ be field extensions, and let $\phi : M \rightarrow L$ be a homomorphism over K . Show that if $\alpha \in M$ has minimal polynomial m over K then $\phi(\alpha) \in L$ also has minimal polynomial m over K .

Solution:

Exercise 4.3.9

Fill in the details of the last paragraph of that proof.

Solution: We show that there is at most one homomorphism $\phi : K(t) \rightarrow L$ over K such that $\phi(t) = \beta$. We let ϕ and ϕ' be two such homomorphisms. Then we have that $\phi(t) = \beta = \phi'(t)$. By **Lemma 4.3.1 (ii)** we have that t generates $K(t)$ over K , and hence by **Lemma 4.3.6** $\phi = \phi'$ \square

Exercise 4.3.15

Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

Solution: We know that $\sqrt{2} + \sqrt{3} \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and hence $\mathbb{Q}(\sqrt{2} + \sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Now we show the inclusion the other way. We use the hint and get that $(\sqrt{2} + \sqrt{3})^3 = 11\sqrt{2} + 9\sqrt{3} \in \mathbb{Q}(\sqrt{2} + \sqrt{3})$. Then we have that: $11\sqrt{2} + 9\sqrt{3} - 9(\sqrt{2} + \sqrt{3}) = 2\sqrt{2} \in \mathbb{Q}(\sqrt{2} + \sqrt{3})$, hence $\sqrt{2} \in \mathbb{Q}(\sqrt{2} + \sqrt{3})$. Similarly, we get that $\sqrt{3} \in \mathbb{Q}(\sqrt{2} + \sqrt{3})$. Therefore, $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2} + \sqrt{3})$ \square

Exercise 4.3.18

How many elements does the field $\mathbb{F}_3(\sqrt{2})$ have? What about $\mathbb{F}_2(\alpha)$, where α is a root of $1 + t + t^2$?

Solution: We know that $\mathbb{F}_3(\sqrt{2})$ can be constructed as $\mathbb{F}_3[t]/\langle t^2 - 2 \rangle$. Hence, any element of the field has the form $a_0 + a_1t + \langle t^2 - 2 \rangle$ with $a_i \in \mathbb{F}_3$. Hence, there are $3^2 = 9$ elements.

In a similar manner, we know that $\mathbb{F}_2(\alpha)$ can be constructed as $\mathbb{F}_2[t]/\langle t^2 + t + 1 \rangle$. Hence any element of the field has the form $a_0 + a_1t + \langle t^2 + t + 1 \rangle$ with $a_i \in \mathbb{F}_2$. Hence there are $2^2 = 4$ elements.

Chapter 5. Degree

Exercise 5.1.9

Write out the addition and multiplication tables of $\mathbb{F}_2(\alpha)$.

Solution: The tables are straightforward, using modulo arithmetic and the irreducible polynomial evaluated at α .

+	0	1	α	$1 + \alpha$
0	0	1	α	$1 + \alpha$
1	1	0	$1 + \alpha$	α
α	α	$1 + \alpha$	0	1
$1 + \alpha$	$1 + \alpha$	α	1	0

\times	0	1	α	$1 + \alpha$
0	0	0	0	0
1	0	1	α	$1 + \alpha$
α	0	α	$1 + \alpha$	1
$1 + \alpha$	0	$1 + \alpha$	1	α

Exercise 5.1.13

Give an example of to show that the inequality in Corollary 5.1.12 can be strict. Your example can be as trivial as you like.

Solution: We choose our fields and hence extensions to be $\mathbb{C} : \mathbb{R} : \mathbb{Q}$. We also choose $\beta = \sqrt{2} \in \mathbb{C}$. The minimal polynomial of $\sqrt{2}$ over \mathbb{Q} is $m = t^2 - 2$, then $\deg_{\mathbb{Q}}(\beta) = [\mathbb{Q}(\beta) : \mathbb{Q}] = 2$.

Similarly, the minimal polynomial of $\sqrt{2}$ over \mathbb{R} is $m = t - \sqrt{2}$, then $\deg_{\mathbb{R}}(\beta) = [\mathbb{R}(\beta) : \mathbb{R}] = 1$.

Hence we have that $[\mathbb{R}(\beta) : \mathbb{R}] < [\mathbb{Q}(\beta) : \mathbb{Q}] \quad \square$

Exercise 5.1.16

Let $M : K$ be a field extension and α a transcendental element of M . Can every element of $K(\alpha)$ be represented as a polynomial in α over K ?

Solution: We have that $K(\alpha) = \left\{ \frac{f(\alpha)}{g(\alpha)} : f, g \in F[t] \right\}$, which is just $K(t)$, the field rational expressions. Therefore it is not polynomial in α over K .

Exercise 5.1.20

Show that a field extension whose degree is a prime number must be simple.

Solution: Let $M : K(\alpha) : K$ be field extensions where M and K are arbitrary fields, $\alpha \in M$, and $[M : K] = p$, where p is prime. By **Theorem 5.1.17 (iii)** we have $[M : K] = [M : K(\alpha)][K(\alpha) : K]$. Hence, we must have that $[K(\alpha) : K] = 1$ or p , however, we also know that $K(\alpha) \neq K$, hence $[K(\alpha) : K] = p$, and therefore, $[M : K(\alpha)] = 1$, which by **Example 5.1.3** tells us $M = K(\alpha)$. Hence $M : K$ is a simple.

Exercise 5.1.23

Generalize Example 5.1.22. In other words, what general result does the argument of Example 5.1.22 prove, not involving the particular numbers chosen there?

Solution: Let $M : K$ be a field extension and $\alpha_1, \dots, \alpha_n \in M$. If $\gcd(\deg_K(\alpha_1), \dots, \deg_K(\alpha_n)) = 1$ (i.e., coprime), then we have that, $[K(\alpha_1, \dots, \alpha_n) : K] = [K(\alpha_1) : K] \dots [K(\alpha_n) : K]$

Exercise 5.2.5

Let $M : K$ be a field extension and $K \subseteq L \subseteq M$. In the proof of Proposition 5.2.4, I said that if L is a subfield of M then L is a K -linear subspace of M . Why is that true? And is the converse also true? Give proof or a counterexample.

Solution: We know that M acts as a vector space over K . If L is a subfield of M , then we can similarly conclude that L acts as a vector space over K . Since we have that L is a subset of M (a subfield) we can conclude that L is a linear (K -linear) subspace of M (by definition of a linear subspace).

The converse is not true.

Exercise 5.2.8

Let $M : K$ be a field extension and write L for the set of elements of M algebraic over K . By imitating the proof of Proposition 5.2.7, prove that L is a subfield of M .

Solution: We have that $L = \{\alpha \in M : [K(\alpha) : K] < \infty\}$.
Then $\forall \alpha, \beta \in L$, $[K(\alpha, \beta) : K] \leq [K(\alpha) : K][K(\beta) : K] < \infty$
Now $\alpha + \beta \in K(\alpha, \beta)$, so $K(\alpha + \beta) \subseteq K(\alpha, \beta)$, hence
 $[K(\alpha + \beta) : K] \leq [K(\alpha, \beta) : K] < \infty$, giving $\alpha + \beta \in L$. Similarly, $\alpha \cdot \beta \in L$.
Then $\forall \alpha \in L$, $[K(-\alpha) : K] = [K(\alpha) : K] < \infty$, giving $-\alpha \in L$. Similarly,
 $1/\alpha \in L$ (if $\alpha \neq 0$), and clearly $0, 1 \in L$ \square

Exercise 5.3.7

Find an example of Lemma 5.3.6 where $[LL' : L] = 2$, and another where $[LL' : L] = 1$.

Solution: If we let $L = \mathbb{Q}(\sqrt{2})$ and $L' = \mathbb{Q}(\sqrt{3})$, we then get $LL' = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Then $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{3})] = 2$.

If we let $L = \mathbb{Q}(\sqrt{4})$ and $L' = \mathbb{Q}(\sqrt{3})$, we then get $LL' = \mathbb{Q}(\sqrt{4}, \sqrt{3})$. Then $[\mathbb{Q}(\sqrt{4}, \sqrt{3}) : \mathbb{Q}(\sqrt{3})] = 1$.

Chapter 6. Splitting fields

Exercise 6.1.5

Show that if a ring homomorphism ψ is injective then so is ψ_* , and if ψ is an isomorphism then so is ψ_* .

Solution: We have that $\psi : R \rightarrow S$ and $\psi_* : R[t] \rightarrow S[t]$. Since ψ is injective we have that $\forall x, y \in R$ if $\psi(x) = \psi(y) \implies x = y$. Then we choose $f, f' \in R[t]$ and assume that $\psi_* f = \psi_* f'$. From **Definition 3.1.7** we then have that:

$$\begin{aligned} \psi_* f &= \psi_* f' \\ \psi_* \left(\sum_i a_i t^i \right) &= \psi_* \left(\sum_i b_i t^i \right) \\ \sum_i \psi(a_i) t^i &= \sum_i \psi(b_i) t^i \\ \psi(a_i) &= \psi(b_i) \\ \implies a_i &= b_i \end{aligned}$$

Hence we have that $f = \sum_i a_i t^i = f'$. Hence ψ_* is injective.

If ψ is an isomorphism then ψ is both surjective and injective. We have just shown that ψ_* is injective, so we show that it is also surjective to prove it is an isomorphism. We know that ϕ_* is surjective $\iff \forall s \in S[t] \exists r \in R[t] : \psi_* r = s$.

We choose $s \in S[t]$ and let e be the identity homomorphism. Then we have:

$$\begin{aligned}
s &= es \\
&= (\psi_* \psi_*^{-1})s \\
&= \psi_*(\psi_*^{-1}s) \\
&= \psi_* \left(\psi_*^{-1} \left(\sum_i a_i t^i \right) \right) \\
&= \psi_* \left(\sum_i \psi_*^{-1}(a_i) t^i \right) \\
&= \psi_* r
\end{aligned}$$

where $\psi_*^{-1}(a_i) \in R$ exists since ψ is an isomorphism, and $r = \sum_i \psi_*^{-1}(a_i) t^i \in R[t]$. Hence ψ_* is also surjective, hence it is an isomorphism.

Exercise 6.2.7

Show that (ii) can equivalently be replaced by: ‘if L is a subfield of M containing K , and f splits in L , then $L = M$ ’.

Solution: We first show (\implies). We have that $M = K(\alpha_1, \dots, \alpha_n)$ and so $M : K$ is well defined. We then take a basis $\alpha_1, \dots, \alpha_n$ of M over K . Then we have that every subfield L of M containing K is a K -linear subspace of M . So if $\alpha_1, \dots, \alpha_n \in L$, which would mean that f splits in L , then $L = M$.

We then show (\impliedby). We have that $K \subseteq L = M$, and $f(t) = \beta(t - \alpha_1) \cdots (t - \alpha_n)$ for some $n \geq 0$ and $\beta, \alpha_1, \dots, \alpha_n \in L = M$. Then the result follows trivially from **Proposition 5.2.4**.

Exercise 6.2.9

In Example 6.2.8(iii), I said that $\mathbb{Q}(\xi, \omega\xi, \omega^2\xi) = \mathbb{Q}(\xi, \omega)$. Why is that true?

Solution: $\omega = e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2}$, and hence $\omega^2 = \frac{-1-i\sqrt{3}}{2}$. and hence we see that ω^2 is a rational multiple of ω , hence $\mathbb{Q}(\xi, \omega\xi, \omega^2\xi) = \mathbb{Q}(\xi, \omega)$.

As an aside, we know that $\mathbb{Q}(\xi) = \{a + b\xi + c\xi^2 : a, b, c \in \mathbb{Q}\}$, and hence $\{1, \xi, \xi^2\}$ forms a basis for $\mathbb{Q}(\xi) : \mathbb{Q}$. Similarly, $\{1, \omega\}$ forms a basis for $\mathbb{Q}(\xi, \omega) : \mathbb{Q}(\xi)$. By **Theorem 5.1.17 (Tower Law)(i)** we then have that $\{1, \xi, \xi^2, \omega, \xi\omega, \xi^2\omega\}$ forms a basis for $\mathbb{Q}(\xi, \omega) : \mathbb{Q}$.

Exercise 6.2.12

Why does the proof of **Proposition 6.2.11** not show that there are *exactly* $[M : K]$ isomorphisms ϕ extending ψ ? How could you strengthen the hypothe-

sis in order to obtain that conclusion?

Solution: It can be strengthened by ...

Exercise 6.3.2

Check that this really does define a group.

Solution: Firstly we have that $Gal(M : K) = Aut(M : K)$. By definition we have $Aut(M : K) = \{f : M : K \rightarrow M : K \mid f \text{ is an isomorphism of } M : K\}$ and $\circ : M : K \times M : K \rightarrow M : K$. We show that the pair $(Aut(M : K), \circ)$ is a group.

Firstly, for $f, g \in Aut(M : K)$ and $\forall a, b \in M : K$, we have that:

$$\begin{aligned} (g \circ f)(ab) &= g(f(ab)) \\ &= g(f(a)f(b)) \\ &= g(f(a))g(f(b)) \\ &= (g \circ f)(a)(g \circ f)(b) \end{aligned}$$

Since f, g are bijective by definition, $g \circ f$ is bijective, and by above is a homomorphism, hence it is an automorphism.

We now show associativity. $\forall f, g, h \in Aut(M : K)$ and $a \in M : K$ we have:

$$\begin{aligned} ((h \circ g) \circ f)(a) &= h(g(f(a))) \\ &= h(g \circ f(a)) \\ &= h \circ (g \circ f(a)) \\ &= (h \circ (g \circ f))(a) \quad \square \end{aligned}$$

Next, we check for an identity. $\forall f \in Aut(M : K)$ and $e_{M:K} : (M : K) \rightarrow (M : K) : a \mapsto a$ we have, $f \circ e_{M:K} = e_{M:K} \circ f = f$. Hence $e_{M:K}$ is the identity element.

Finally, we check for the inverse. $\forall f \in Aut(M : K)$, we have that $f \circ f^{-1} = e_{M:K} = f^{-1} \circ f$. This follows by definition since f is an isomorphism.

Exercise 6.3.4

Prove that $Gal(\mathbb{Q}(e^{2\pi i/3}) : \mathbb{Q}) = \{id, \kappa\}$, where $\kappa(z) = \bar{z}$.

Solution: We know that the identity is an automorphism of $\mathbb{Q}(e^{2\pi i/3})$ over \mathbb{Q} . By **Lemma 1.1.2**, since $\mathbb{Q} \subset \mathbb{R}$ we have that κ is also an automorphism of $\mathbb{Q}(e^{2\pi i/3})$ over \mathbb{Q} .

Hence we have that $\{id, \kappa\} \subseteq Gal(\mathbb{Q}(e^{2\pi i/3}) : \mathbb{Q})$. We also know that $\mathbb{Q}(e^{2\pi i/3}) : \mathbb{Q} = \mathbb{Q}(\frac{-1+i\sqrt{3}}{2}) : \mathbb{Q} = \mathbb{Q}(i\sqrt{3}) : \mathbb{Q}$

We let $\theta \in Gal(\mathbb{Q}(e^{2\pi i/3}) : \mathbb{Q})$. Since θ is a homomorphism we have that:

$$\begin{aligned}(\theta(i\sqrt{3}))^2 &= \theta((i\sqrt{3})^2) \\ &= \theta(-3) \\ &= -\theta(3) \\ &= -3\end{aligned}$$

Hence $\theta(i\sqrt{3}) = \pm i\sqrt{3}$. If $\theta(i\sqrt{3}) = i\sqrt{3}$ then $\theta = id$, and if $\theta(i\sqrt{3}) = -i\sqrt{3}$ then $\theta = \kappa$. Hence $Gal(\mathbb{Q}(e^{2\pi i/3}) : \mathbb{Q}) = \{id, \kappa\}$.

Exercise 6.3.11

I skipped two small bits in that proof: ‘ θ is surjective because σ is a permutation’ (why?), and ‘You can check that θ is a homomorphism of fields’. Fill in the gaps.

Solution: This follows by definition of permutation. A permutation is a bijective map from a set to itself, hence it is surjective. θ is a homomorphism of fields.

Secondly, by **Definition 6.3.5** $Gal_K(f)$ is $Gal(SF_K(f) : K)$. Then by **Definition 6.3.1** we know that an element of $Gal(M : K)$ is an isomorphism $\theta : M \rightarrow M$, hence by definition we have that $\theta \in Gal_K(f)$ is a homomorphism of fields.

Chapter 7. Preparation for the fundamental theorem

Exercise 7.1.4

What happens if you drop the word ‘irreducible’ from Lemma 7.1.2? Is it still true?

Solution:

Exercise 7.1.4

What happens if you drop the word ‘irreducible’ from Lemma 7.1.2? Is it still true?

Solution:

Exercise 7.2.1

Try to find an example of an irreducible polynomial of degree d with fewer than d distinct roots in its splitting field.

Solution: An irreducible polynomial over a field of characteristic 0 has distinct roots in its splitting field. Therefore we must consider field of characteristic $p > 0$, where p is prime. Hence if we have the field extension $\mathbb{F}_p(t) : \mathbb{F}_p(t^p)$, and we consider $t \in \mathbb{F}_p(t)$ its minimal polynomial over $\mathbb{F}_p(t^p)$ is $X^p - t^p = (X - t)^p$. We get to the last step from the Frobenius automorphism.

Exercise 7.2.8

Check one or two of the properties in Lemma 7.2.7.

Solution: We check the additive property.

We let $f(t) = \sum_{i=0}^n a_i t^i \in K[t]$ and $g(t) = \sum_{i=0}^n b_i t^i \in K[t]$. Then we have that $f(t) + g(t) = \sum_{i=0}^n (a_i + b_i) t^i = \sum_{i=0}^n c_i t^i \in K[t]$, where $c_i = (a_i + b_i)$. Then by **Definition 7.2.6** we have that $D(f + g)(t) = \sum_{i=1}^n i c_i t^{i-1} = \sum_{i=1}^n i(a_i + b_i) t^{i-1} = \sum_{i=1}^n i a_i t^{i-1} + \sum_{i=1}^n i b_i t^{i-1} = Df + Dg \in K[t]$

Exercise 7.2.15

Let $M : L : K$ be field extensions. Show that if $M : K$ is algebraic then so are $M : L$ and $L : K$.

Solution: By definition of an algebraic extension, we have that if $M : K$ is algebraic then $\forall \alpha \in M \exists f \neq 0 \in K[t] : f(\alpha) = 0$. Since we have that M is a field extension of L it must contain all of L , therefore we have that $\forall \alpha \in L \exists f \neq 0 \in K[t] : f(\alpha) = 0$, hence $L : K$ is algebraic. Similarly, since L extends K any $f \neq 0 \in K[t]$ must exist in $L[t]$, hence we that $\forall \alpha \in M \exists f \neq 0 \in L[t] : f(\alpha) = 0$, hence $M : L$ is algebraic.

Exercise 7.3.2

Using Lemma 7.3.1, show that every automorphism of a field is an automorphism over its prime subfield. In other words, $\text{Aut}(M) = \text{Gal}(M : K)$ whenever M is a field with prime subfield K .

Solution: We know by definition, that $\text{Gal}(M : K) = \text{Aut}(M)$

Exercise 7.3.5

Find another example of Theorem 7.3.3.

Solution: We follow **Example 7.3.4**. If we have $\kappa : \mathbb{Q}(\sqrt{2}) \rightarrow \mathbb{Q}(\sqrt{2})$ representing complex conjugation, then $H = \{id, \kappa\}$ is a subgroup of $Aut(\mathbb{Q}(\sqrt{2}))$. By **Theorem 7.3.3**, we have that $[\mathbb{Q}(\sqrt{2}) : \text{Fix}(H)] \leq |H| = 2$. Since $\text{Fix}(H) = \mathbb{Q}$, and we know that $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, the inequality holds.