

Solutions: Real and Complex Analysis by Walter Rudin

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Chapter 1. Abstract Integration

Exercise 1

Does there exist an infinite σ -algebra which has only countably many members?

Solution: No. Impossible.

Exercise 2

Prove an analog of Theorem 1.8 for n functions.

Solution: We have that u_1, u_2, \dots, u_n are real measurable functions on a measurable space.

We let $f(x) = (u_1(x), u_2(x), \dots, u_n(x))$. Since $h = \Phi \circ f$, Theorem 1.7 shows that it is enough to prove measurability of f .

We let $B = I_1 \times I_2 \times \dots \times I_n$. We then have $f(B) = (u_1(I_1), u_2(I_2), \dots, u_n(I_n))$. We then have that $f^{-1}(B) = u_1^{-1}(I_1) \cap u_2^{-1}(I_2) \cap \dots \cap u_n^{-1}(I_n)$, which is measurable by our measurability assumption on u_1, u_2, \dots, u_n .

Every open set V in $I_1 \times I_2 \times \dots \times I_n$ is a countable union of such B which we call B_i . Hence we have that $f^{-1}(V) = f^{-1}(\bigcup_{i=1}^{\infty} B_i) = \bigcup_{i=1}^{\infty} f^{-1}(B_i)$. Hence $f^{-1}(V)$ is measurable. \square

Exercise 3

Prove that if f is a real function on a measurable space X such that $\{x : f(x) \geq r\}$ is measurable for every rational r , then f is measurable.

Solution: We know that f is measurable if for every open set V in \mathcal{O}_{std} ,

$f^{-1}(V)$ is measurable set. Here $\mathcal{O}_{std} : \{(a, b) : a < x < b \ \forall x \in \mathbb{R}\}$ is the standard topology on \mathbb{R} and is just the collection of all open intervals (a, b) . We know that $\{x \in X : f(x) \geq q\}$ is a measurable set $\forall q \in \mathbb{Q}$. Since we know that \mathbb{Q} is a dense subset of \mathbb{R} , we can always get arbitrarily close to any $r \in \mathbb{R}$. We let $\forall r \in \mathbb{R}$, $(q_n)_{n \in \mathbb{N}}$ be a decreasing sequence in \mathbb{Q} such that $\lim_{n \rightarrow \infty} q_n = r$. We then have that $\{x \in X : f(x) > r\} = \bigcup_{n=1}^{\infty} \{x \in X : f(x) > q_n\}$. By definition, the right hand side is measurable, hence every r is measurable. Hence, for every open interval in $I \in \mathcal{O}_{std}$, $f^{-1}(I)$ is a measurable set, hence f is measurable.

Exercise 4

Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty, \infty]$, and prove the following assertions:

(a)

$$\limsup_{n \rightarrow \infty} (-a_n) = -\liminf_{n \rightarrow \infty} a_n$$

(b)

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$$

provided none of the sums is of the form $\infty - \infty$.

(c) If $a_n \leq b_n$ for all n , then

$$\liminf_{n \rightarrow \infty} a_n \leq \liminf_{n \rightarrow \infty} b_n$$

Show by an example that strict inequality can hold for (b).

Solution:

a)

$$\begin{aligned} \limsup_{n \rightarrow \infty} (-a_n) &= \inf_{n \geq 0} \sup_{m \geq n} (-a_m) \\ &= \inf_{n \geq 0} (-\inf_{m \geq n} a_m) \\ &= -\sup_{n \geq 0} \inf_{m \geq n} a_m \\ &= -\liminf_{n \rightarrow \infty} (a_n) \end{aligned}$$

b) We let $a_n = \sup_{m \geq n} a_m$ and $b_n = \sup_{m \geq n} b_m$. It is trivial that $A \subseteq B \implies \sup A \leq \sup B$. We can then observe that $\forall m \geq n$, $a_n \geq a_m$ and $b_n \geq b_m$.

Hence we have that:

$$\begin{aligned}
a_n + b_n &\geq a_m + b_m \\
\sup_{m \geq n} (a_n + b_n) &= a_n + b_n \geq \sup_{m \geq n} (a_m + b_m) \\
\lim_{n \rightarrow \infty} (a_n + b_n) &\geq \lim_{n \rightarrow \infty} \sup_{m \geq n} (a_m + b_m) \\
\lim_{n \rightarrow \infty} (\sup_{m \geq n} a_m + \sup_{m \geq n} b_m) &\geq \lim_{n \rightarrow \infty} \sup_{m \geq n} (a_m + b_m) \\
\lim_{n \rightarrow \infty} \sup_{m \geq n} a_m + \lim_{n \rightarrow \infty} \sup_{m \geq n} b_m &\geq \lim_{n \rightarrow \infty} \sup_{m \geq n} (a_m + b_m) \\
\limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n &\geq \limsup_{n \rightarrow \infty} (a_n + b_n) \quad \square
\end{aligned}$$

c) We have that $\forall n$:

$$\inf_{m \geq n} a_m \leq \inf_{m \geq n} b_m$$

Where for the sequences $(\inf_{m \geq n} a_m)_{n \in \mathbb{N}}$ and $(\inf_{m \geq n} b_m)_{n \in \mathbb{N}}$ we have:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \inf_{m \geq n} a_m &\leq \lim_{n \rightarrow \infty} \inf_{m \geq n} b_m \\
\liminf_{n \rightarrow \infty} a_n &\leq \liminf_{n \rightarrow \infty} b_n \quad \square
\end{aligned}$$

Exercise 5

(a) Suppose $f : X \rightarrow [-\infty, \infty]$ and $g : X \rightarrow [-\infty, \infty]$ are measurable. Prove that the sets $\{x : f(x) < g(x)\}$, $\{x : f(x) = g(x)\}$ are measurable.

(b) Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

Solution:

a) Since f and g are measurable, from Exercise 3, we can deduce that $\forall q \in \mathbb{Q}$, $\{x : f(x) \geq q\}$ and $\{x : g(x) \geq q\}$ are measurable sets. Hence their complements and strict inequality conditioned sets $\{x : f(x) < q\}$, $\{x : g(x) < q\}$, $\{x : f(x) > q\}$, $\{x : g(x) > q\}$, $\{x : g(x) \leq q\}$, $\{x : f(x) \leq q\}$ are also measurable. We then have that $\{x : f(x) < g(x)\}, \{x : f(x) = g(x)\} \iff \{x : f(x) \leq g(x)\} = X$. Then we have:

$$\begin{aligned}
X^C &= \{x : f(x) > g(x)\} = \bigcup_{q \in \mathbb{Q}} \{\{x : f(x) > q\} \cap \{x : g(x) < q\}\} \\
&= \bigcup_{q \in \mathbb{Q}} \{\{x : f(x) \leq q\} \cup \{x : g(x) \geq q\}\}
\end{aligned}$$

which is measurable since it is the union of countably many measurable sets. Hence we have that $(X^C)^C = X$ is measurable.

b) If we have a sequence of real-valued measurable functions $(f_n)_{n \in \mathbb{N}}$ which converge to a finite limit say a , we know that:

$$\lim_{n \rightarrow \infty} f_n = a \iff \liminf_{n \rightarrow \infty} f_n = \limsup_{n \rightarrow \infty} f_n = a$$

By **Theorem 1.14** we know that $h = \limsup_{n \rightarrow \infty} f_n$ and $g = \sup_{n \geq 1} f_n$ are measurable. From these it follows that $\inf_{n \geq 1} f_n$ and $\liminf_{n \rightarrow \infty} f_n$ are measurable. Then we have that $\{x : \liminf_{n \rightarrow \infty} f_n = \limsup_{n \rightarrow \infty} f_n\} = X$ and:

$$X^C = \{x : \liminf_{n \rightarrow \infty} f_n > \limsup_{n \rightarrow \infty} f_n\} \cup \{x : \liminf_{n \rightarrow \infty} f_n < \limsup_{n \rightarrow \infty} f_n\}$$

which we know by (a) to be measurable. Hence X is measurable.

Exercise 6

Let X be an uncountable set, let \mathfrak{M} be the collection of all sets $E \subset X$ such that either E or E^C is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. Prove that \mathfrak{M} is a σ -algebra in X and that μ is a measure on \mathfrak{M} . Describe the corresponding measurable functions and their integrals.

Solution: By definition $X, \emptyset \in \mathfrak{M}$. Additionally if $E \in \mathfrak{M}$, then $E^C \in \mathfrak{M}$. We then let $E = \cup_{i=1}^{\infty} E_i$ where $E_i \in \mathfrak{M}$. If all E_i are countable then we have that E is also countable, since a countable union of countable sets is countable, hence $E \in \mathfrak{M}$. If $\exists E_{i_u}$ such that $E_{i_u}^C$ is countable, then $E^C = \cap_{i=1}^{\infty} E_i^C \subseteq E_{i_u}^C$, and hence E^C is countable, hence $E \in \mathfrak{M}$. Hence \mathfrak{M} is a σ -algebra in X .

We then look at μ . We let $E = \cup_{i=1}^{\infty} E_i$ be a countable collection of pairwise disjoint sets $E_i \in \mathfrak{M}$. Similar to before if all E_i are countable then E is countable and hence $\sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1}^{\infty} 0 = 0 = \mu(E)$. If $\exists E_{i_k}$ such that $E_{i_k}^C$ is countable, then E^C is countable. Hence $\sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1, i \neq i_k}^{\infty} \mu(E_i) + \mu(E_{i_k}) = \sum_{i=1}^{\infty} 0 + 1 = 1 = \mu(E)$. Hence we have that $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$. Hence μ is a measure on \mathfrak{M} .

Let $f : X \rightarrow R$, where R is an arbitrary range. We know that if f is measurable then either $f^{-1}(x)$ or $(f^{-1}(x))^C$ is measurable and hence countable. If we have that $(f^{-1}(x))^C$ is countable then by the measure μ that we have $f(x)$ is almost constant.

Exercise 7

Suppose $f_n : X \rightarrow [0, \infty]$ is measurable for $n = 1, 2, 3, \dots$, $f_1 \geq f_2 \geq f_3 \geq \dots \geq 0$, $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$, for every $x \in X$, and $f_1 \in L^1(\mu)$. Prove that then

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu$$

and show that this conclusion does *not* follow if the condition “ $f_1 \in L^1(\mu)$ ” is omitted.

Solution: If $\forall x \in X, f_1(x) < \infty$. We have that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ where each f_n is measurable on X . We also have that $f_1(x) \geq f_n(x) \geq 0$ hence $|f_n(x)| \leq f_1(x) \in L^1(\mu)$. By Lebesgue’s Dominated Convergence Theorem (LDCT), the conclusion then follows.

Else we can let $E = \{x \in X : f_1(x) = \infty\}$. If we assume that $\mu(E) > 0$, then we also have that $\int_X |f_1| d\mu = \infty$, but since $f_1 \in L^1(\mu)$, this is a contradiction, hence $\mu(E) = 0$. Therefore we have that $\int_X |f_1| d\mu = \int_{X \setminus E} |f_1| d\mu + \int_E |f_1| d\mu = \int_{X \setminus E} |f_1| d\mu < \infty$. Hence the conclusion again follows by above.

As seen above if “ $f_1 \in L^1(\mu)$ ” is omitted, the conclusion does not follow.

Exercise 8

Put $f_n = \chi_E$ if n is odd, $f_n = 1 - \chi_E$ if n is even. What is the relevance of this example to Fatou’s Lemma ?

Solution:

Exercise 9

Suppose μ is a positive measure on X , $f : X \rightarrow [0, \infty]$ is measurable, $\int_X f d\mu = c$, where $0 < c < \infty$, and α is a constant. Prove that

$$\lim_{n \rightarrow \infty} \int_X n \log[1 + (f/n)^\alpha] d\mu = \begin{cases} \infty & \text{if } 0 < \alpha < 1 \\ c & \text{if } \alpha = 1 \\ 0 & \text{if } 1 < \alpha < \infty \end{cases}$$

Solution:

Exercise 10

Suppose $\mu(X) < \infty$, $\{f_n\}$ is a sequence of bounded complex measurable functions on X , and $f_n \rightarrow f$ uniformly on X . Prove that

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu$$

and show that the hypothesis “ $\mu(X) < \infty$ ” cannot be omitted.

Solution: Since we have a sequence of bounded complex measurable functions, we have that $|f_n(x)| \leq g(x)$, where $g(x)$ is some bound. Since $\mu(X) < \infty$,

then $\int_X |g(x)| < \infty$, hence $g(x) \in L^1(\mu)$. By uniform convergence, we also have that $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. The conclusion then follows by LDCT.

If “ $\mu(X) < \infty$ ” is omitted, then we can have $\int_X |g(x)| = \infty$, and hence $g(x) \notin L^1(\mu)$ and hence LDCT does not apply, and the conclusion does not follow.

Exercise 11

Show that

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

in Theorem 1.41, and hence prove the theorem without any reference to integration.

Solution: If $x \in \bigcup_{k \in K} E_k$ then x is in at least one of the E_k . If $x \in \bigcap_{k \in K} E_k$ then x is in all of the E_k . Hence we can conclude that $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$ are all the elements in either $E_n, E_{n+1}, E_{n+2}, \dots$, without regard for the size of n . Hence this must be the same as being part of infinitely many of the E_k .

More formally, if x is in infinitely many E_k , then for all $n = 1, 2, 3, \dots, \infty \exists k > n : x \in E_k$. Hence $x \in \bigcup_{k=n}^{\infty} E_k$, however it is so for all n , hence it is in the intersection of all such unions, hence $x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$. For the other direction, it is enough to show that if x is not in infinitely many E_k then $A \neq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$. Hence, if x is not in infinitely many E_k , then there is some largest k and hence E_k in which x lies. Therefore, $\exists n > k : x \notin \bigcup_{k=n}^{\infty} E_k$. Hence $x \notin \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$ \square

We let $F_n = \bigcup_{k=n}^{\infty} E_k$. From this it is obvious that $F_{n+1} \subset F_n$. We then have that

$$\begin{aligned} \mu(A) &= \mu\left(\bigcap_{n=1}^{\infty} F_n\right) \\ &= \lim_{n \rightarrow \infty} \mu(F_n) \quad (\text{Theorem 1.19 (e)}) \\ &= \lim_{n \rightarrow \infty} \mu\left(\bigcup_{k=n}^{\infty} E_k\right) \\ &\leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \mu(E_k) = 0 \end{aligned}$$

since we know that $\sum_{k=1}^{\infty} \mu(E_k) < \infty$, as well as the fact that the tail must go to 0 (Theorem 3.23, Principles of Mathematical Analysis; Rudin).

Exercise 12

Suppose $f \in L^1(\mu)$. Prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| d\mu < \epsilon$ whenever $\mu(E) < \delta$.

Solution: Since $f \in L^1(\mu)$ we know that $\int_E |f| d\mu$ is finite. By definition $\int_E |f| d\mu = \sup \sum_{i=1}^{\infty} \alpha_i \mu(A_i \cap E)$. In addition we know that $\mu(E) < \delta$. Therefore for each ϵ we can choose $\delta = \frac{\epsilon}{\sum_{i=1}^n \alpha_i}$. hence we have:

$$\begin{aligned} \int_E |f| d\mu &= \sup \sum_{i=1}^n \alpha_i \mu(A_i \cap E) \\ &\leq \sum_{i=1}^n \alpha_i \mu(A_i \cap E) \\ &\leq \sum_{i=1}^n \alpha_i \mu(E) = \mu(E) \sum_{i=1}^n \alpha_i = \delta \sum_{i=1}^n \alpha_i = \epsilon \end{aligned}$$

Exercise 13

Show that proposition 1.24(c) is also true when $c = \infty$.

Solution: If $f = 0$, then we have that $\int_E cf d\mu = \int_E \infty \cdot 0 d\mu = \int_E 0 d\mu = 0 = \infty \cdot 0 = \infty \cdot \int_E 0 d\mu = c \int_E f d\mu$.

If $f > 0$, we know that $\infty \cdot f = \infty \forall f > 1$. Then we have that if $\mu(E) = 0$ then $\int_E cf d\mu = 0 = c \int_E f d\mu$. If $\mu(E) \neq 0$, then $\int_E cf d\mu = \int_E \infty \cdot f d\mu = \int_E \infty d\mu = \infty = \infty \cdot \int_E f d\mu = c \int_E f d\mu$. test

Chapter 2. Positive Borel Measures

Exercise 1

Let $\{f_n\}$ be a sequence of real nonnegative functions on \mathbb{R}^1 , and consider the following four statements:

- (a) If f_1 and f_2 are upper semicontinuous, then $f_1 + f_2$ is upper semicontinuous.
- (b) If f_1 and f_2 are lower semicontinuous, then $f_1 + f_2$ is lower semicontinuous.
- (c) If each f_n is upper semicontinuous, then $\sum_1^{\infty} f_n$ is upper semicontinuous.
- (d) If each f_n is lower semicontinuous, then $\sum_1^{\infty} f_n$ is lower semicontinuous.

Show that three of these are true and that one is false. What happens when the word “nonnegative” is omitted? Is the truth of the statements affected? Is the truth of the statements affected if \mathbb{R}^1 is replaced by a general topological

space?

Solution: If $f_1(x)$ and $f_2(x)$ are upper semicontinuous, we work with the following set $\{x : f_1(x) + f_2(x) < \alpha\}$, then:

$$\begin{aligned}\{x : f_1(x) + f_2(x) < \alpha\} &= \{x : f_1(x) < \alpha - f_2(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} (\{x : f_1(x) < q < \alpha - f_2(x)\}) \\ &= \bigcup_{q \in \mathbb{Q}} (\{x : f_1(x) < q\} \cap \{x : f_2(x) < \alpha - q\})\end{aligned}$$

In the last equality the two sets on the right are open, given their upper continuity. Then, the intersection of open sets are open, as well as their union. Hence we that $\{x : f_1(x) + f_2(x) < \alpha\}$ is open, hence $f_1(x) + f_2(x)$ is upper semicontinuous.

By similar argument, if $f_1(x)$ and $f_2(x)$ are lower semicontinuous, then:

$$\begin{aligned}\{x : f_1(x) + f_2(x) > \alpha\} &= \{x : f_1(x) > \alpha - f_2(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} (\{x : f_1(x) > q > \alpha - f_2(x)\}) \\ &= \bigcup_{q \in \mathbb{Q}} (\{x : f_1(x) > q\} \cap \{x : f_2(x) > \alpha - q\})\end{aligned}$$

is open, and hence $f_1(x) + f_2(x)$ is lower semicontinuous.

If each $f_n(x)$ is lower semicontinuous, then:

$$\begin{aligned}\{x : \sum_1^\infty f_n(x) > \alpha\} &= \{x : \lim_{N \rightarrow \infty} \sum_{n=1}^N f_n(x) > \alpha\} \\ &= \{x : \sup_{N \geq 1} \sum_{n=1}^N f_n(x) > \alpha\}\end{aligned}$$

Previously, we have shown that a finite sum of lower semicontinuous functions is lower semicontinuous. By **Definition 2.8 (c)**, we have that supremum of any collection of lower semicontinuous functions is lower semicontinuous. Hence the we get that the above is lower semicontinuous.

If each $f_n(x)$ is upper semicontinuous, then firstly let:

$$f_n = \begin{cases} 1 & \text{if } x \geq \frac{1}{n} \\ 0 & \text{if } x < \frac{1}{n} \end{cases}$$

Then we have that each f_n is upper semicontinuous. However we have:

$$\sup_n f_n = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

which is not upper semicontinuous. Then by a similar argument as before we have:

$$\begin{aligned}\{x : \sum_1^\infty f_n(x) < \alpha\} &= \{x : \lim_{N \rightarrow \infty} \sum_{n=1}^N f_n(x) < \alpha\} \\ &= \{x : \sup_{N \geq 1} \sum_{n=1}^N f_n(x) < \alpha\}\end{aligned}$$

Again we know that a finite sum of upper semicontinuous functions is upper semicontinuous. However we have just shown that for our f_n the supremum is not necessarily upper semicontinuous. Hence the implication does not follow.

Exercise 2

Let f be an arbitrary complex function on \mathbb{R}^1 , and define

$$\begin{aligned}\phi(x, \delta) &= \sup\{|f(s) - f(t)| : s, t \in (x - \delta, x + \delta)\}, \\ \phi(x) &= \inf\{\phi(x, \delta) : \delta > 0\}\end{aligned}$$

Prove that ϕ is upper semicontinuous, that f is continuous at a point x if and only if $\phi(x) = 0$, and hence that the set of points of continuity of an arbitrary complex function is a G_δ . Formulate and prove an analogous statement for general topological spaces in place of \mathbb{R}^1 .

Solution: We choose arbitrary $x_a \in \mathbb{R}$. By the second equation we then let $y > \phi(x_a)$. Hence there must be a $\delta_a > 0 : \phi(x_a, \delta_a) < y$. We then let $x \in (x_a - \frac{\delta_a}{2}, x_a + \frac{\delta_a}{2})$ and $s, t \in (x - \frac{\delta_a}{2}, x + \frac{\delta_a}{2})$. Hence we must have that $(x - \frac{\delta_a}{2}, x + \frac{\delta_a}{2}) \subseteq (x_a - \delta_a, x_a + \delta_a)$, since $x_a - \delta_a < x - \frac{\delta_a}{2} < x + \frac{\delta_a}{2} < x_a + \delta_a$. Hence $\phi(x, \frac{\delta_a}{2}) \leq \phi(x_a, \delta_a) < y$ (due to supremums). Hence for any δ , we have that $\inf\{\phi(x, \delta) : \delta > 0\} \leq \phi(x_a, \delta_a) < y$. Hence we have that $\forall x \in (x_a - \frac{\delta_a}{2}, x_a + \frac{\delta_a}{2}), \phi(x) < y$. Notice that x_a and y were arbitrary. Hence ϕ is upper semicontinuous.

Exercise 3

Let X be a metric space, with metric ρ . For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf\{\rho(x, y) : y \in E\}$$

Show that ρ_E is a uniformly continuous function on X . If A and B are disjoint nonempty closed subsets of X , examine the relevance of the function

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

Solution: By definition, ρ_E is uniformly continuous on X if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\rho_E(f(p), f(q)) < \epsilon$ for all $p, q \in X$ for which $\rho_X(p, q) < \delta$.