Solutions: Real and Complex Analysis by Walter Rudin

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Chapter 1. Abstract Integration

Exercise 1

Does there exist an infinite σ-algebra which has only countably many members?

Solution: No. Impossible.

Exercise 2

Prove an analog of Theorem 1.8 for n functions.

Solution: We have that $u_1, u_2, ..., u_n$ are real measurable functions on a measurable space.

We let $f(x) = (u_1(x), u_2(x), ..., u_n(x))$. Since $h = \Phi \circ f$, Theorem 1.7 shows that it is enough to prove measurability of f.

We let $B = I_1 \times I_2 \times ... \times I_n$. We then have $f(B) = (u_1(I_1), u_2(I_2)), ..., u_n(I_n)$. We then have that $f^{-1}(B) = u_1^{-1}(I_1) \cap u_2^{-1}(I_2) \cap ... \cap u_n^{-1}(I_n)$, which is measurable by our measurability assumption on $u_1, u_2, ..., u_n$.

Every open set V in $I_1 \times I_2 \times ... \times I_n$ is a countable union of such B which we call B_i. Hence we have that $f^{-1}(V) = f^{-1}(\bigcup\limits_{i=1}^{\infty} B_i) = \bigcup\limits_{i=1}^{\infty} f^{-1}(B_i)$. Hence $f^{-1}(V)$ is measurable. $\square$

Exercise 3

Prove that if f is a real function on a measurable space X such that $\{x : f(x) \geq r\}$ is measurable for every rational r, then f is measurable.

Solution: We know that f is measurable if for every open set V in $\mathcal{O}_{std}$,

$f^{-1}(V)$ is measurable set. Here $\mathcal{O}_{std} : \{(a,b) : a < x < b \ \forall x \in \mathbb{R}\}$ is the standard topology on $\mathbb{R}$ and is just the collection of all open intervals (a,b). We know that $\{x \in X : f(x) \geq q\}$ is a measurable set $\forall q \in \mathbb{Q}$. Since we know that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, we can always get arbitrarily close to any $r \in \mathbb{R}$. We let $\forall r \in \mathbb{R}$, $(q_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathbb{Q}$ such that $\lim_{n\to\infty} q_n = r$. We then have that $\{x \in X : f(x) > r\} = \bigcup_{n=1}^{\infty}\{x \in X : f(x) > q_n\}$. By definition, the right hand side is measurable, hence every r is measurable. Hence, for every open interval in $I \in \mathcal{O}_{std}$, $f^{-1}(I)$ is a measurable set, hence f is measurable.

Exercise 4

Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty, \infty]$, and prove the following assertions:

(a)
$$\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$$

(b)
$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

provided none of the sums is of the form $\infty - \infty$.

(c) If $a_n \leq b_n$ for all n, then
$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$$

Show by an example that strict inequality can hold for (b).

Solution:

a)
$$\begin{aligned}\limsup_{n\to\infty}(-a_n) &= \inf_{n\geq 0}\sup_{m\geq n}(-a_m)\\ &= \inf_{n\geq 0}(-\inf_{m\geq n} a_m)\\ &= -\sup_{n\geq 0}\inf_{m\geq n} a_m\\ &= -\liminf_{n\to\infty}(a_n)\end{aligned}$$

b) We let $a_n = \sup_{m\geq n} a_m$ and $b_n = \sup_{m\geq n} b_m$. It is trivial that $A \subseteq B \implies \sup A \leq \sup B$. We can then observe that $\forall m \geq n$, $a_n \geq a_m$ and $b_n \geq b_m$.

Hence we have that:

$$\begin{aligned} a_n + b_n &\geq a_m + b_m \\ \sup_{m \geq n}(a_n + b_n) = a_n + b_n &\geq \sup_{m \geq n}(a_m + b_m) \\ \lim_{n\to\infty}(a_n + b_n) &\geq \lim_{n\to\infty}\sup_{m \geq n}(a_m + b_m) \\ \lim_{n\to\infty}(\sup_{m \geq n} a_m + \sup_{m \geq n} b_m) &\geq \lim_{n\to\infty}\sup_{m \geq n}(a_m + b_m) \\ \lim_{n\to\infty}\sup_{m \geq n} a_m + \lim_{n\to\infty}\sup_{m \geq n} b_m &\geq \lim_{n\to\infty}\sup_{m \geq n}(a_m + b_m) \\ \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n &\geq \limsup_{n\to\infty}(a_n + b_n) \quad \square \end{aligned}$$

c) We have that $\forall n$:

$$\inf_{m \geq n} a_m \leq \inf_{m \geq n} b_m$$

Where for the sequences $(\inf_{m \geq n} a_m)_{n \in \mathbb{N}}$ and $(\inf_{m \geq n} b_m)_{n \in \mathbb{N}}$ we have:

$$\begin{aligned} \lim_{n\to\infty}\inf_{m \geq n} a_m &\leq \lim_{n\to\infty}\inf_{m \geq n} b_m \\ \liminf_{n\to\infty} a_n &\leq \liminf_{n\to\infty} b_n \quad \square \end{aligned}$$

Exercise 5

(a) Suppose $f : X \to [-\infty, \infty]$ and $g : X \to [-\infty, \infty]$ are measurable. Prove that the sets $\{x : f(x) < g(x)\}, \{x : f(x) = g(x)\}$ are measurable.

(b) Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

Solution:

a) Since f and g are measurable, from Excercise 3, we can deduce that $\forall q \in \mathbb{Q}$, $\{x : f(x) \geq q\}$ and $\{x : g(x) \geq q\}$ are measurable sets. Hence their complements and strict inequality conditioned sets $\{x : f(x) < q\}, \{x : g(x) < q\}, \{x : f(x) > q\}, \{x : g(x) > q\}, \{x : g(x) \leq q\}, \{x : g(x) \leq q\}$ are also measurable. We then have that $\{x : f(x) < g(x)\}, \{x : f(x) = g(x)\} \iff \{x : f(x) \leq g(x)\} = X$. Then we have:

$$\begin{aligned} X^C = \{x : f(x) > g(x)\} &= \bigcup_{q \in \mathbb{Q}} \{\{x : f(x) > q\} \cap \{x : g(x) < q\}\} \\ &= \bigcup_{q \in \mathbb{Q}} \{\{x : f(x) \leq q\} \cup \{x : g(x) \geq q\}\} \end{aligned}$$

which measurable since it is the union of countably many measurable sets. Hence we have that $(X^C)^C = X$ is measurable.

b) If we have a sequence of real-valued measurable functions $(f_n)_{n\in\mathbb{N}}$ which converge to a finite limit say a, we know that:

$$\lim_{n\to\infty} f_n = a \iff \liminf_{n\to\infty} f_n = \limsup_{n\to\infty} f_n = a$$

By **Theorem 1.14** we know that $h = \limsup_{n\to\infty} f_n$ and $g = \sup_{n\geq 1} f_n$ are measurable. From these it follows that $\inf_{n\geq 1} f_n$ and $\liminf_{n\to\infty} f_n$ are measurable. Then we have that $\{x : \liminf_{n\to\infty} f_n = \limsup_{n\to\infty}\} = X$ and:

$$X^C = \{x : \liminf_{n\to\infty} f_n > \limsup_{n\to\infty}\} \cup \{x : \liminf_{n\to\infty} f_n < \limsup_{n\to\infty}\}$$

which we know by (a) to be measurable. Hence X is measurable.

Exercise 6

Let X be an uncountable set, let $\mathfrak{M}$ be the collection of all sets $E \subset X$ such that such that either E or E^C is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. Prove that $\mathfrak{M}$ is a σ-algebra in X and that μ is a measure on $\mathfrak{M}$. Describe the corresponding measurable functions and their integrals.

Solution: By defintion $X, \emptyset \in \mathfrak{M}$. Additionally if $E \in \mathfrak{M}$, then $E^C \in \mathfrak{M}$. We then let $E = \cup_{i=1}^{\infty} E_i$ where $E_i \in \mathfrak{M}$. If all E_i are countable then we have that E is also countable, since a countable union of countable sets is countable, hence $E \in \mathfrak{M}$. If $\exists E_{i_u}$ such that $E_{i_u}^C$ is countable, then $E^C = \cap_{i=1}^{\infty} E_i^C \subseteq E_{i_u}^C$, and hence E^C is countable, hence $E \in \mathfrak{M}$. Hence $\mathfrak{M}$ is a σ-algebra in X.

We then look at μ. We let $E = \cup_{i=1}^{\infty} E_i$ be a countable collection of pairwise disjoint sets $E_i \in \mathfrak{M}$. Similar to before if all E_i are countable then E is countable and hence $\Sigma_{i=1}^{\infty} \mu(E_i) = \Sigma_{i=1}^{\infty} 0 = 0 = \mu(E)$. If $\exists E_{i_k}$ such that $E_{i_k}^C$ is countable, then E^C is countable. Hence $\Sigma_{i=1}^{\infty} \mu(E_i) = \Sigma_{i=1, i\neq i_k}^{\infty} \mu(E_i) + \mu(E_{i_k}) = \Sigma_{i=1}^{\infty} 0 + 1 = 1 = \mu(E)$. Hence we have that $\mu(\cup_{i=1}^{\infty} E_i) = \Sigma_{i=1}^{\infty} \mu(E_i)$. Hence μ is a measure on $\mathfrak{M}$.

Exercise 7

Solution:

Exercise 8

Solution:

Exercise 9

Solution:

Exercise 10

Solution:

Exercise 11

Solution:

Exercise 12

Solution:

Exercise 13

Solution: