Solutions: Real and Complex Analysis by Walter Rudin

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Chapter 1. Abstract Integration

Exercise 1

Does there exist an infinite σ-algebra which has only countably many members?

Solution: No. Impossible.

Exercise 2

Prove an analog of Theorem 1.8 for n functions.

Solution: We have that $u_1, u_2, ..., u_n$ are real measurable functions on a measurable space.

We let $f(x) = (u_1(x), u_2(x), ..., u_n(x))$. Since $h = \Phi \circ f$, Theorem 1.7 shows that it is enough to prove measurability of f.

We let $B = I_1 \times I_2 \times ... \times I_n$. We then have $f(B) = (u_1(I_1), u_2(I_2)), ..., u_n(I_n)$. We then have that $f^{-1}(B) = u_1^{-1}(I_1) \cap u_2^{-1}(I_2) \cap ... \cap u_n^{-1}(I_n)$, which is measurable by our measurability assumption on $u_1, u_2, ..., u_n$.

Every open set V in $I_1 \times I_2 \times ... \times I_n$ is a countable union of such B which we call B_i. Hence we have that $f^{-1}(V) = f^{-1}(\bigcup\limits_{i=1}^{\infty} B_i) = \bigcup\limits_{i=1}^{\infty} f^{-1}(B_i)$. Hence $f^{-1}(V)$ is measurable. $\square$

Exercise 3

Prove that if f is a real function on a measurable space X such that $\{x : f(x) \geq r\}$ is measurable for every rational r, then f is measurable.

Solution: We know that f is measurable if for every open set V in $\mathcal{O}_{std}$,

$f^{-1}(V)$ is measurable set. Here $\mathcal{O}_{std} : \{(a,b) : a < x < b \ \forall x \in \mathbb{R}\}$ is the standard topology on $\mathbb{R}$ and is just the collection of all open intervals (a,b). We know that $\{x \in X : f(x) \geq q\}$ is a measurable set $\forall q \in \mathbb{Q}$. Since we know that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, we can always get arbitrarily close to any $r \in \mathbb{R}$. We let $\forall r \in \mathbb{R}$, $(q_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathbb{Q}$ such that $\lim_{n\to\infty} q_n = r$. We then have that $\{x \in X : f(x) > r\} = \bigcup_{n=1}^{\infty}\{x \in X : f(x) > q_n\}$. By definition, the right hand side is measurable, hence every r is measurable. Hence, for every open interval in $I \in \mathcal{O}_{std}$, $f^{-1}(I)$ is a measurable set, hence f is measurable.

Exercise 4

Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty, \infty]$, and prove the following assertions:

(a)
$$\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$$

(b)
$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

provided none of the sums is of the form $\infty - \infty$.

(c) If $a_n \leq b_n$ for all n, then
$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$$

Show by an example that strict inequality can hold for (b).

Solution:

a)
$$\begin{aligned}\limsup_{n\to\infty}(-a_n) &= \inf_{n\geq 0}\sup_{m\geq n}(-a_m)\\ &= \inf_{n\geq 0}(-\inf_{m\geq n} a_m)\\ &= -\sup_{n\geq 0}\inf_{m\geq n} a_m\\ &= -\liminf_{n\to\infty}(a_n)\end{aligned}$$

b) We let $a_n = \sup_{m\geq n} a_m$ and $b_n = \sup_{m\geq n} b_m$. It is trivial that $A \subseteq B \implies \sup A \leq \sup B$. We can then observe that $\forall m \geq n$, $a_n \geq a_m$ and $b_n \geq b_m$.

Hence we have that:

$$\begin{aligned}
a_n + b_n &\geq a_m + b_m \\
\sup_{m \geq n} (a_n + b_n) = a_n + b_n &\geq \sup_{m \geq n} (a_m + b_m) \\
\lim_{n\to\infty} (a_n + b_n) &\geq \lim_{n\to\infty} \sup_{m \geq n} (a_m + b_m) \\
\lim_{n\to\infty} (\sup_{m \geq n} a_m + \sup_{m \geq n} b_m) &\geq \lim_{n\to\infty} \sup_{m \geq n} (a_m + b_m) \\
\lim_{n\to\infty} \sup_{m \geq n} a_m + \lim_{n\to\infty} \sup_{m \geq n} b_m &\geq \lim_{n\to\infty} \sup_{m \geq n} (a_m + b_m) \\
\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n &\geq \limsup_{n\to\infty} (a_n + b_n) \quad \square
\end{aligned}$$

c) We have that $\forall n$:

$$\inf_{m \geq n} a_m \leq \inf_{m \geq n} b_m$$

Where for the sequences $(\inf_{m \geq n} a_m)_{n \in \mathbb{N}}$ and $(\inf_{m \geq n} b_m)_{n \in \mathbb{N}}$ we have:

$$\begin{aligned}
\lim_{n\to\infty} \inf_{m \geq n} a_m &\leq \lim_{n\to\infty} \inf_{m \geq n} b_m \\
\liminf_{n\to\infty} a_n &\leq \liminf_{n\to\infty} b_n \quad \square
\end{aligned}$$

Exercise 5

(a) Suppose $f : X \to [-\infty, \infty]$ and $g : X \to [-\infty, \infty]$ are measurable. Prove that the sets $\{x : f(x) < g(x)\}, \{x : f(x) = g(x)\}$ are measurable.

(b) Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

Solution:

a) Since f and g are measurable, from Excercise 3, we can deduce that $\forall q \in \mathbb{Q}$, $\{x : f(x) \geq q\}$ and $\{x : g(x) \geq q\}$ are measurable sets. Hence their complements and strict inequality conditioned sets $\{x : f(x) < q\}, \{x : g(x) < q\}, \{x : f(x) > q\}, \{x : g(x) > q\}, \{x : g(x) \leq q\}, \{x : g(x) \leq q\}$ are also measurable. We then have that $\{x : f(x) < g(x)\}, \{x : f(x) = g(x)\} \iff \{x : f(x) \leq g(x)\} = X$. Then we have:

$$\begin{aligned}
X^C = \{x : f(x) > g(x)\} &= \bigcup_{q \in \mathbb{Q}} \{\{x : f(x) > q\} \cap \{x : g(x) < q\}\} \\
&= \bigcup_{q \in \mathbb{Q}} \{\{x : f(x) \leq q\} \cup \{x : g(x) \geq q\}\}
\end{aligned}$$

which measurable since it is the union of countably many measurable sets. Hence we have that $(X^C)^C = X$ is measurable.

b) If we have a sequence of real-valued measurable functions $(f_n)_{n\in\mathbb{N}}$ which converge to a finite limit say a, we know that:

$$\lim_{n\to\infty} f_n = a \iff \liminf_{n\to\infty} f_n = \limsup_{n\to\infty} f_n = a$$

By **Theorem 1.14** we know that $h = \limsup_{n\to\infty} f_n$ and $g = \sup_{n\geq 1} f_n$ are measurable. From these it follows that $\inf_{n\geq 1} f_n$ and $\liminf_{n\to\infty} f_n$ are measurable. Then we have that $\{x : \liminf_{n\to\infty} f_n = \limsup_{n\to\infty}\} = X$ and:

$$X^C = \{x : \liminf_{n\to\infty} f_n > \limsup_{n\to\infty}\} \cup \{x : \liminf_{n\to\infty} f_n < \limsup_{n\to\infty}\}$$

which we know by (a) to be measurable. Hence X is measurable.

Exercise 6

Let X be an uncountable set, let $\mathfrak{M}$ be the collection of all sets $E \subset X$ such that such that either E or E^C is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. Prove that $\mathfrak{M}$ is a σ-algebra in X and that μ is a measure on $\mathfrak{M}$. Describe the corresponding measurable functions and their integrals.

Solution: By defintion $X, \emptyset \in \mathfrak{M}$. Additionally if $E \in \mathfrak{M}$, then $E^C \in \mathfrak{M}$. We then let $E = \cup_{i=1}^{\infty} E_i$ where $E_i \in \mathfrak{M}$. If all E_i are countable then we have that E is also countable, since a countable union of countable sets is countable, hence $E \in \mathfrak{M}$. If $\exists E_{i_u}$ such that $E_{i_u}^C$ is countable, then $E^C = \cap_{i=1}^{\infty} E_i^C \subseteq E_{i_u}^C$, and hence E^C is countable, hence $E \in \mathfrak{M}$. Hence $\mathfrak{M}$ is a σ-algebra in X.

We then look at μ. We let $E = \cup_{i=1}^{\infty} E_i$ be a countable collection of pairwise disjoint sets $E_i \in \mathfrak{M}$. Similar to before if all E_i are countable then E is countable and hence $\Sigma_{i=1}^{\infty} \mu(E_i) = \Sigma_{i=1}^{\infty} 0 = 0 = \mu(E)$. If $\exists E_{i_k}$ such that $E_{i_k}^C$ is countable, then E^C is countable. Hence $\Sigma_{i=1}^{\infty} \mu(E_i) = \Sigma_{i=1, i\neq i_k}^{\infty} \mu(E_i) + \mu(E_{i_k}) = \Sigma_{i=1}^{\infty} 0 + 1 = 1 = \mu(E)$. Hence we have that $\mu(\cup_{i=1}^{\infty} E_i) = \Sigma_{i=1}^{\infty} \mu(E_i)$. Hence μ is a measure on $\mathfrak{M}$.

Let $f : X \to R$, where R is an arbitrary range. We know that if f is measurable then either $f^{-1}(x)$ or $(f^{-1}(x))^C$ is measurable and hence countable. If we have that $(f^{-1}(x))^C$ is countable then by the measure μ that we have $f(x)$ is almost constant.

Exercise 7

Suppose $f_n : X \to [0, \infty]$ is measurable for $n = 1, 2, 3, ..., f_1 \geq f_2 \geq f_3 \geq ... \geq 0$, $f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$, and $f_1 \in L^1(\mu)$. Prove that then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

and show that this conclusion does *not* follow if the condition "$f_1 \in L^1(\mu)$" is omitted.

Solution: If $\forall x \in X,\ f_1(x) < \infty$. We have that $\lim_{n\to\infty} f_n(x) = f(x)$ where each f_n is measurable on X. We also have that $f_1(x) \geq f_n(x) \geq 0$ hence $|f_n(x)| \leq f_1(x) \in L^1(\mu)$. By Lebesgue's Dominated Convergence Theorem (LDCT), the conclusion then follows.

Else we can let $E = \{x \in X : f_1(x) = \infty\}$. If we assume that $\mu(E) > 0$, then we also have that $\int_X |f_1| d\mu = \infty$, but since $f_1 \in L^1(\mu)$, this is a contradiction, hence $\mu(E) = 0$. Therefore we have that $\int_X |f_1| d\mu = \int_{X\setminus E} |f_1| d\mu + \int_E |f_1| d\mu = \int_{X\setminus E} |f_1| d\mu < \infty$. Hence the conclusion again follows by above.

As seen above if "$f_1 \in L^1(\mu)$" is omitted, the conclusion does not follow.

Exercise 8

Put $f_n = \chi_E$ if n is odd, $f_n = 1 - \chi_E$ if n is even. What is the relavance of this example to Fatou's Lemma ?

Solution:

Exercise 9

Suppose μ is a positive measure on $X, f : X \to [0, \infty]$ is measurable, $\int_X f d\mu = c$, where $0 < c < \inf$, and α is a constant. Prove that

$$\lim_{n\to\infty} \int_X n \log[1 + (f/n)^\alpha]\ d\mu = \begin{cases} \infty & \text{if } 0 < \alpha < 1 \\ c & \text{if } \alpha = 1 \\ 0 & \text{if } 1 < \alpha < \infty \end{cases}$$

Solution:

Exercise 10

Suppose $\mu(X) < \infty$, $\{f_n\}$ is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Prove that

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f_n\, d\mu$$

and show that the hypothesis "$\mu(X) < \infty$" cannot be omitted.

Solution: Since we have a sequence of bounded complex measurable functions, we have that $|f_n(x)| \leq g(x)$, where $g(x)$ is some bound. Since $\mu(X) < \infty$, then $\int_X |g(x)| < \infty$, hence $g(x) \in L^1(\mu)$. By uniform convergence, we also have that $f(x) = \lim_{n\to\infty} f_n(x)$. The conclusion then follows by LDCT.

If "$\mu(X) < \infty$" is omitted, then we can have $\int_X |g(x)| = \infty$, and hence $g(x) \notin L^1(\mu)$ and hence LDCT does not apply, and the conclusion does not follow.

Exercise 11

Show that

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

in Theorem 1.41, and hence prove the theorem without any reference to integration.

Solution: If $x \in \bigcup_{k\in K} E_k$ then x is in at least one of the E_k. If $x \in \bigcap_{k\in K} E_k$ then x is in all of the E_k. Hence we can conclude that $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$ are all the elements in either $E_n, E_{n+1}, E_{n+2}, ...$, without regard for the the size of n. Hence this must be the same as being part of infinitely many of the E_k.

More formally, if x is in infinitely many E_k, then for all $n = 1, 2, 3, ..., \infty$ $\exists\, k > n : x \in E_k$. Hence $x \in \bigcup_{k=n}^{\infty} E_k$, however it is so for all n, hence it is in the interesection of all such unions, hence $x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$. For the other direction, it is enough to show that if x is not in infinitely many E_k then $A \neq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$. Hence, if x is not in infinitely many E_k, then there is some largest k and hence E_k in which x lies. Therefore, $\exists\, n > k : x \notin \bigcup_{k=n}^{\infty} E_k$. Hence $x \notin \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$ $\square$

We let $F_n = \cup_{k=n}^{\infty} E_k$. From this it is obvious that $F_{n+1} \subset F_n$. We then have that

$$\begin{aligned}
\mu(A) &= \mu(\bigcap_{n=1}^{\infty} F_n) \\
&= \lim_{n\to\infty} \mu(F_n) \quad \text{(Theorem 1.19 (e))} \\
&= \lim_{n\to\infty} \mu(\bigcup_{k=n}^{\infty} E_k) \\
&\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k) = 0
\end{aligned}$$

since we know that $\sum_{k=1}^{\infty} \mu(E_k) < \infty$, as well as the fact that the tail must go to 0 (Theorem 3.23, Principles of Mathematical Analysis; Rudin).

Exercise 12

Suppose $f \in L^1(\mu)$. Prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f|\, d\mu < \epsilon$ whenever $\mu(E) < \delta$.

Solution: Since $f \in L^1(\mu)$ we know that $\int_E |f|\, d\mu$ is finite. By definition $\int_E |f|\, d\mu = \sup \sum_{i=1}^{\infty} \alpha_i \mu(A_i \cap E)$. In addition we know that $\mu(E) < \delta$. Therefore for each ϵ we can choose $\delta = \frac{\epsilon}{\sum_{i=1}^{n} \alpha_i}$. hence we have:

$$\begin{aligned}
\int_E |f|\, d\mu &= \sup \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E) \\
&\leq \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E) \\
&\leq \sum_{i=1}^{n} \alpha_i \mu(E) = \mu(E) \sum_{i=1}^{n} \alpha_i = \delta \sum_{i=1}^{n} \alpha_i = \epsilon
\end{aligned}$$

Exercise 13

Show that proposition 1.24(c) is also true when $c = \infty$.

Solution: If $f = 0$, then we have that $\int_E cf\, d\mu = \int_E \infty \cdot 0\, d\mu = \int_E 0\, d\mu = 0 = \infty \cdot 0 = \infty \cdot \int_E 0\, d\mu = c \int_E f\, d\mu$.

If $f > 0$, we know that $\infty \cdot f = \infty\ \forall f > 1$. Then we have that if $\mu(E) = 0$ then $\int_E cf\, d\mu = 0 = c \int_E f\, d\mu$. If $\mu(E) \neq 0$, then $\int_E cf\, d\mu = \int_E \infty \cdot f\, d\mu = \int_E \infty\, d\mu = \infty = \infty \cdot \int_E f\, d\mu = c \int_E f\, d\mu$. test