Solutions: Real and Complex Analysis by Walter Rudin

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Chapter 1. Abstract Integration

Exercise 1

Does there exist an infinite σ-algebra which has only countably many members?

Solution: No. Impossible.

Exercise 2

Prove an analog of Theorem 1.8 for n functions.

Solution: We have that $u_1, u_2, ..., u_n$ are real measurable functions on a measurable space.

We let $f(x) = (u_1(x), u_2(x), ..., u_n(x))$. Since $h = \Phi \circ f$, Theorem 1.7 shows that it is enough to prove measurability of f.

We let $B = I_1 \times I_2 \times ... \times I_n$. We then have $f(B) = (u_1(I_1), u_2(I_2)), ..., u_n(I_n)$. We then have that $f^{-1}(B) = u_1^{-1}(I_1) \cap u_2^{-1}(I_2) \cap ... \cap u_n^{-1}(I_n)$, which is measurable by our measurability assumption on $u_1, u_2, ..., u_n$.

Every open set V in $I_1 \times I_2 \times ... \times I_n$ is a countable union of such B which we call B_i. Hence we have that $f^{-1}(V) = f^{-1}(\bigcup_{i=1}^{\infty} B_i) = \bigcup_{i=1}^{\infty} f^{-1}(B_i)$. Hence $f^{-1}(V)$ is measurable. $\square$

Exercise 3

Prove that if f is a real function on a measurable space X such that $\{x : f(x) \geq r\}$ is measurable for every rational r, then f is measurable.

Solution: We know that f is measurable if for every open set V in $\mathcal{O}_{std}$,

$f^{-1}(V)$ is measurable set. Here $\mathcal{O}_{std} : \{(a,b) : a < x < b \ \forall x \in \mathbb{R}\}$ is the standard topology on $\mathbb{R}$ and is just the collection of all open intervals (a,b). We know that $\{x \in X : f(x) \geq q\}$ is a measurable set $\forall q \in \mathbb{Q}$. Since we know that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, we can always get arbitrarily close to any $r \in \mathbb{R}$. We let $\forall r \in \mathbb{R}$, $(q_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathbb{Q}$ such that $\lim_{n\to\infty} q_n = r$. We then have that $\{x \in X : f(x) > r\} = \bigcup_{n=1}^{\infty}\{x \in X : f(x) > q_n\}$. By definition, the right hand side is measurable, hence every r is measurable. Hence, for every open interval in $I \in \mathcal{O}_{std}$, $f^{-1}(I)$ is a measurable set, hence f is measurable.

Exercise 4

Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty, \infty]$, and prove the following assertions:

(a)

$$\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$$

(b)

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

provided none of the sums is of the form $\infty - \infty$.

(c) If $a_n \leq b_n$ for all n, then

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$$

Show by an example that strict inequality can hold for (b).

Solution:

Exercise 5

Solution:

Exercise 6

Solution:

Exercise 7

Solution:

Exercise 8

Solution:

Exercise 9

Solution:

Exercise 10

Solution:

Exercise 11

Solution:

Exercise 12

Solution:

Exercise 13

Solution: