UNIT 4

SOLVING INQUALITIES

Unit Outcomes

By the end of this unit, you will be able to:

- **Solve linear inequalities in one variable.**
- **Solve inequalities involving absolute value.**
- **Solve system of linear inequalities.**
- **Solve quadratic inequalities.**
- **Apply inequalities in real life situations.**

Unit Contents

KEY Words

- linear equation
- Quadratic inequality
- linear inequality
- sign chart
- Absolute value
- open interval
- solution set
- closed intervals
- Product property
- Quadratic equation
- discriminant
- complete listing

INTRODUCTION

An algebraic inequality is a mathematical sentence connecting an expression to a value, variable, or another expression with an inequality sign ($<$, $\leq$, $>$, or $\geq$). Solving inequalities is similar to solving equations.

4.1 Revision on Linear Inequalities in one Variable

Activity 4.1

a. If you add or subtract the same number from both sides of an inequality, will the inequality sign remain the same or have to be reversed? For instance, if $2 < 5$, then can you conclude that $2 + c < 5 + c$? Why? $2 - c < 5 - c$? Why?

b. If you multiply or divide by a negative number, will the inequality sign change? Why?

Review on some properties of inequalities.

a. If $a > b$, then $a + c > b + c$

b. If $a > b$, then $a - c > b - c$

c. If $a > b$ and $m > 0$, then $ma > mb$ and $\frac{a}{m} > \frac{b}{m}$

d. If $a > b$ and $m < 0$, then $ma < mb$ and $\frac{a}{m} < \frac{b}{m}$

Example 1

Solve the following inequalities.

a. $x - 5 > 2$ **b.** $2x > 4$ **c.** $\frac{1}{3}x < 1$

d. $-3x > 6$ **e.** $2x - 3 \geq 1$

Solution:

a. $x - 5 > 2$

$x > 2 + 5$

$x > 7$

b. $2x > 4$

$x > \frac{4}{2}$

$x > 2$

c. $\frac{1}{3}x < 1$

$x < 3$

d. $-3x > 6$

$x < -2$

e. $2x - 3 \geq 1$

$2x \geq 4, \ x \geq 2$

Exercise 4.1.1

Solve the following inequalities.

a. $x - 1 > 3$ b. $4x < 12$

c. $-\frac{1}{5}x \leq 2$ d. $3x + 1 \geq 7$

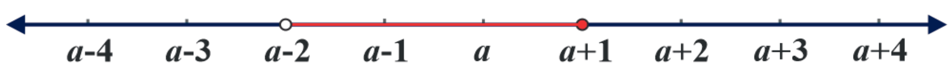
Example 2

Express the following using the notation of intervals.

a

b

c



Solution:

a. $[1, 6]$ **b.** $[-3, 2)$ **c.** $(a - 2, a + 1]$

Note

The symbol $[a, b]$ is closed interval with end-points a and b. It shows the set of all real numbers x such that $a \leq x \leq b$.

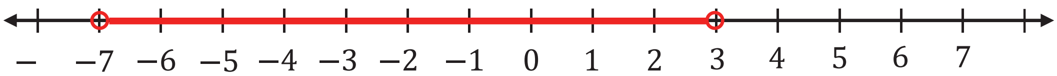
The symbol (a, b) is open interval with end-points a and b. It shows the set of all real numbers x such that $a < x < b$.

The symbol $[a, b)$ and $(a, b]$ is half-open interval. They show the set of all real numbers x such that $a \leq x < b$ and $a < x \leq b$, respectively.

Example 3

Write down $(-7,\ 3)$ on the number line

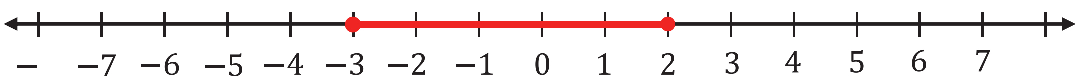
Solution:



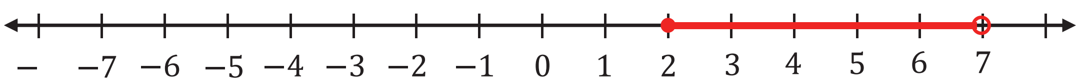
Exercise 4.1.2

1. Express the following using the notation of intervals.

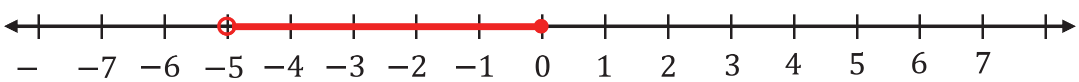
a.



b.



c.



2. Express the following intervals on a number line.

   a. $(-3,1)$ b. $[-2,4)$ c. $[-5,1]$

Example 1

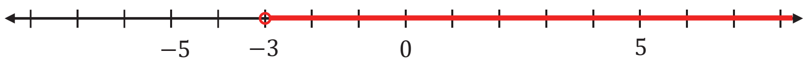
Solve the inequality below and express it on a number line.

$$x + 1 > -2$$

Solution:

$$x + 1 > -2$$

$$x > -3$$



Using interval notation, the solution is $(-3, \infty)$. To represent the solution on a number line, we place the solution (boundary point) on the number line. Since $x = -3$ is not a solution to the inequality $(<)$ we use an open dot as shown. Using the set notation, the solution is $\{x \mid x > -3\}$

Example 2

Solve $3x - (x + 2) \geq 0$, and express it on a number line

Solution:

$$3x - x - 2 \geq 0$$

$$2x \geq 2$$

$$x \geq 1 \text{ (dividing both sides by 2)}$$

Now, we place the solution (boundary point) on the number line. Since, $x = 1$ is also a solution to the inequality $(\geq)$ we use a filled-in dot as follows:

The solution is $x \geq 1$. Using interval notation, the solution is $[1, \infty)$.

Example 3

Solve $x + 2 < -x + 6$, and express it on a number line

Solution:

$$x + x < 6 - 2$$
$$2x < 4$$
$$x < 2$$

Using interval notation, the solution is $(-\infty, 2)$. The solution is represented on a number line as follows:

Example 4

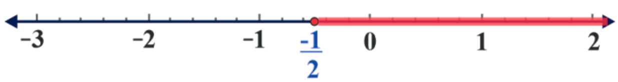
Solve the inequality below and graph the solution set and write it in an interval notation.

$$-\frac{2}{3}x + \frac{1}{2} \leq \frac{5}{6}$$

Solution:

$$-\frac{2}{3}x + \frac{1}{2} \leq \frac{5}{6}$$
$$6\left(-\frac{2}{3}x + \frac{1}{2}\right) \leq 6\left(\frac{5}{6}\right)$$
$$-4x + 3 \leq 5$$
$$-4x \leq 2$$
$$x \geq -\frac{1}{2} \text{ (dividing both sides by } -4 \text{)}$$

The solution in interval notation is $\left[-\frac{1}{2}, \infty\right)$



Exercise 4.2

1. Solve the inequality below, express it on a number line, and write it in an interval notation.

   a. $x - 2 > 3$ b. $x + 1 \leq 5$

2. Solve the following linear inequalities.

a. $x + 3 < 5$

b. $x + 6 > 2$

c. $x - 10 \leq -7$

d. $3x - (2x + 2) \leq 7$

e. $\frac{1}{3}x \geq 4$

f. $\frac{2}{3}x \leq 6$

g. $-5x > -15$

h. $4x - (2x + 8) \geq 0$

i. $5x - 3(x - 1) < 5$

j. $8x - 5 > 2x + 4$

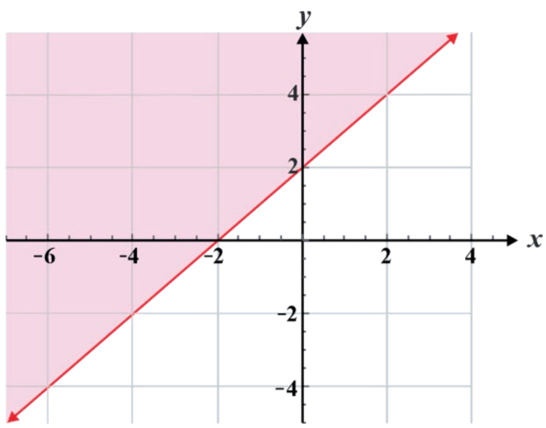
k. $\frac{x+2}{5} \geq \frac{2x}{3}$

l. $\frac{1}{4}x + 6 > 5$

4.2 Systems of Linear Inequalities in Two variables

Activity 4.2

1. How do you draw the graph of a linear equation given the y-intercept and the slope?
2. What mathematical statement describe all the points above the line $y = x + 3$, below the line $y = x + 3$, on the line $y = x + 3$?
3. What linear inequality is represented by the graph below?



a. $x - y \geq 2$

b. $x - y \leq 2$

c. $-x + y \geq 2$

d. $-x + y \leq 2$

Definition 4.1

Linear inequalities in two variables represent the unequal relation between two algebraic expressions that includes two distinct variables.

Important procedures in solving inequalities in two variables graphically

To solve an inequality in two variables graphically, first sketch the corresponding equations on the same xy-plane. This graph is the boundary line (or curve); since all points that make the inequality true lie on one side or the other of the line. Before graphing the equations, decide whether the line or curve is part of the solution or not, that is, whether it is solid or dashed.

If the inequality symbol is either $\leq$ or $\geq$, then the points on the boundary line are solutions to the inequality and the line must be solid. If the symbol is either $<$ or $>$, then the points on the boundary line are not solutions to the inequality and the line is dashed.

Next, decide which side of the boundary line must be shaded to show the part of the graph that represents all (x, y) coordinate pairs that make the inequality true. To do this, choose a point not on the boundary line. Substitute the x- and y-values of this point into the original inequalities. If the inequality is true for the test point, then shade the region on the side of the boundary line that contains the test point. If the inequality is false for the test point, then shade the opposite region. The shaded portion represents all of the (x, y) coordinate pairs that are solutions to the original inequality.

Example

Solve the inequality $y > -x$ using graphical method.

Solution:

Sketch the corresponding equation $y = -x$ on the xy-plane.

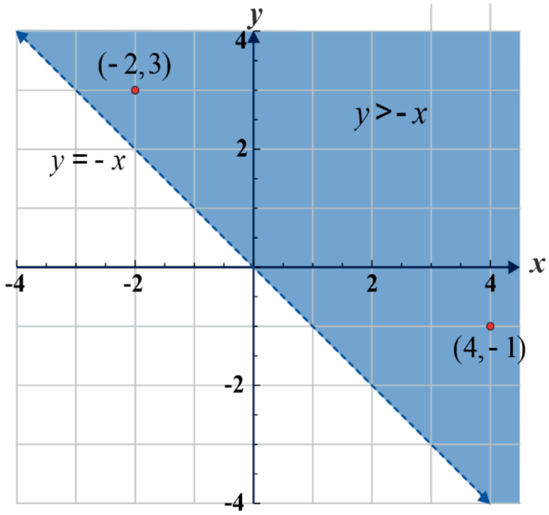


Figure 4.1

You can check a couple of points to determine which side of the boundary line to shade. Checking points $(-2,3)$ and $(4,-1)$ yield true statements. So, we shade the area above the line. The line is dashed as points on it are not part of the solution.

Exercise 4.3

Solve the following linear inequalities in two variables using graphical method.

a. $y \leq -x$

b. $y > x$

c. $y < x + 2$

d. $y \geq -x + 1$

e. $x > -y$

Solving systems of linear inequalities in two variables using graphical method

Activity 4.3

By drawing the paired inequalities on the same xy-plane, identify the common region

a. $y > x$ and $y > -x$

b. $y < x + 2$ and $y > 2x$

The solution to the system of inequalities is the overlap of the shading from the individual inequalities. When two boundary lines are graphed, there are often four regions. The region containing the coordinate pairs that make both of the inequalities true is the solution region.

Example 1

Solve the system of linear inequalities $\begin{cases} y < 2x + 5 \\ y > -x \end{cases}$ graphically

Solution:

First, draw the graph of the linear equation, $y = 2x + 5$. As shown in Fig 4.2, the dashed line is $y = 2x + 5$. Every ordered pair in the shaded area below the line is a solution to $y < 2x + 5$, as all the points below the line will make the inequality true. To confirm this, you may substitute x and y coordinate of points such as $(3, 1)$, $(-1, -1)$, etc. into the inequalities. So, the shaded area (Fig 4.2) shows all the solutions for this inequality. The boundary line divides the coordinate plane in half. In this case, it is shown as a dashed line as the points on the line do not satisfy the inequality.

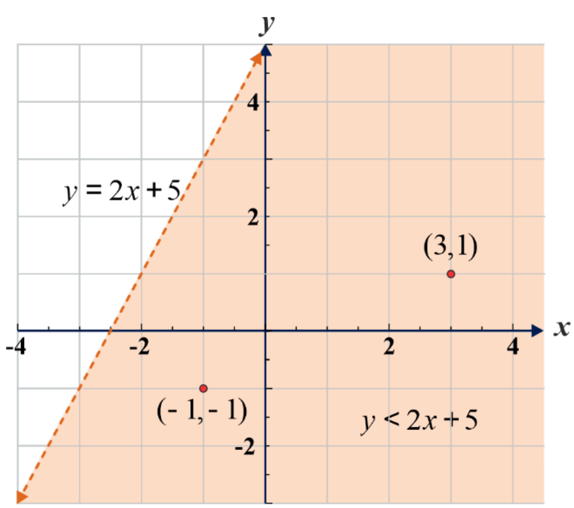


Figure 4.2

If the inequality had been $y \leq 2x + 5$, then the boundary line would have been solid. Next, draw the graph of $y > -x$ in the same plane. As shown in Fig 4.3, the area shared by light blue shows the region $y > -x$. The points on the line $y = -x$ do not satisfy $y > -x$, the line was dashed.

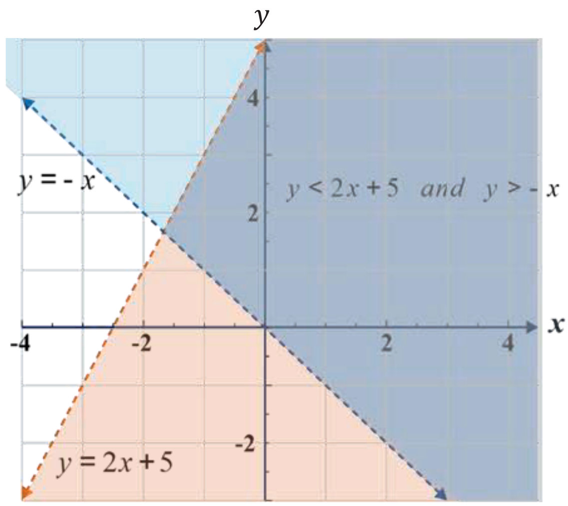


Figure 4.3

The dark blue area (Fig 4.3) shows where, the solutions of the two inequalities overlap. This area is the solution to the ***system of inequalities***. Any point within this blue region will be true for both $y < 2x + 5$ and $y > -x$.

Exercise 4.4

Solve the following linear inequalities in two variables using graphical method.

a. $\begin{cases} y < 2x + 1 \\ y > -x \end{cases}$ b. $\begin{cases} y > 2x + 1 \\ y < -x \end{cases}$ c. $\begin{cases} y < x + 1 \\ y > -3x \end{cases}$

d. $\begin{cases} y > x + 1 \\ y > -3x \end{cases}$ e. $\begin{cases} y < -x + 2 \\ y < x \end{cases}$

Example 2

Solve the system of linear inequalities $y \leq x - 2$ and $y > -3x + 5$ using graphical method.

Solution:

First, graph the inequality $y \leq x - 2$. The related equation is $y = x - 2$ (Fig. 4.4). Since the inequality is $\leq$ (less than or equal to), any points on the line satisfy the inequality. Thus, you use the solid line for the border line.

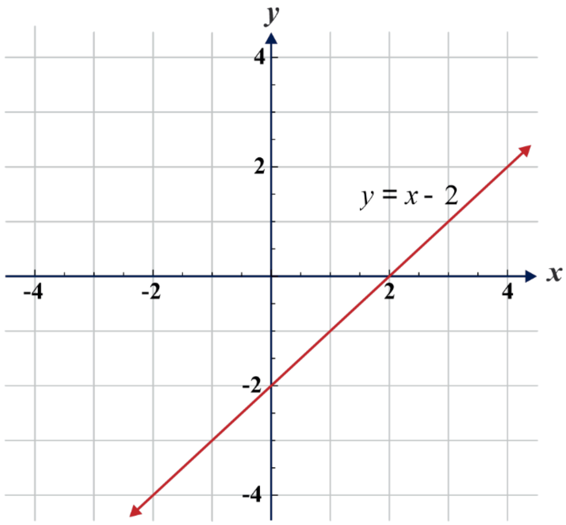


Figure 4.4

Consider a point that is not on the line, for instance,$(0, 0)$, and substitute in the inequality.

$$y \leq x - 2$$

$$(\mathrm{L.H.S}) = 0,\ (\mathrm{R.H.S}) = 0 - 2 = -2$$

Thus, $(\mathrm{L.H.S}) \geq (\mathrm{R.H.S})$. This is not true. So, the solution does not contain the point$(0, 0)$. Hence, shade the lower half of the line $y = x - 2$ (Fig 4.5).

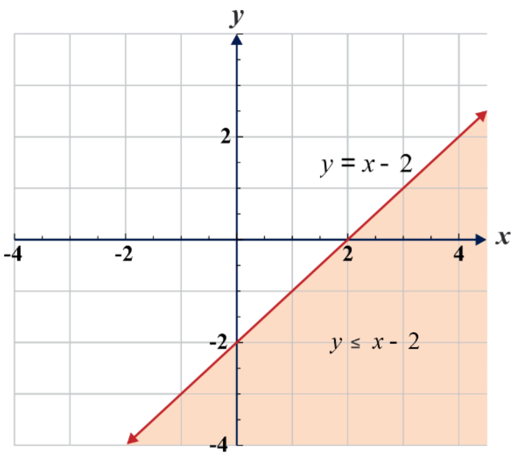


Figure 4.5

Similarly, draw a dashed line for the related equation $y > -3x + 5$ of the second inequality which has a strict inequality (Fig 4.6). The point $(0,0)$ does not satisfy the inequality, so shade the half that does not contain the point $(0,0)$.

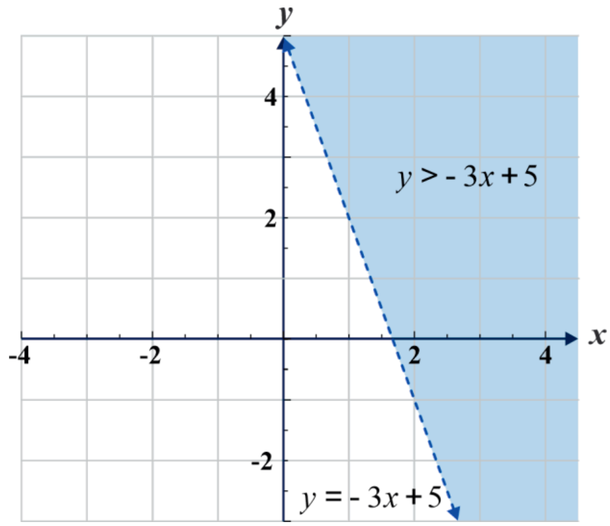


Figure 4.6

The solution of the system of inequalities is the intersection region of the solutions of the two inequalities (Fig 4.7).

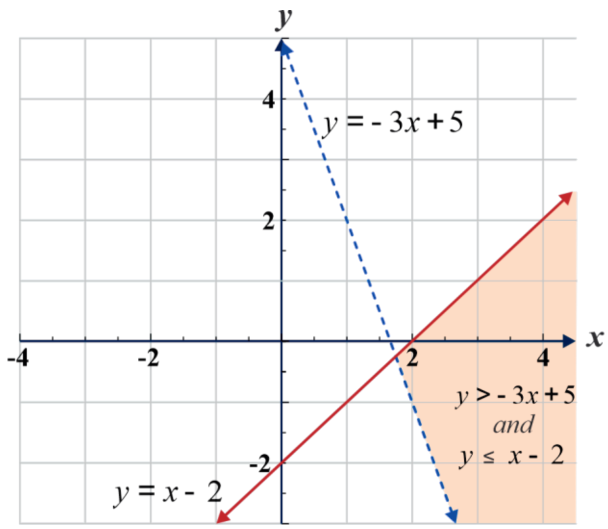


Figure 4.7

Example 3

Solve the following $\begin{cases} 2x + 3y \geq 12 \\ 8x - 4y > 1 \\ x < 4 \end{cases}$

Solution:

Rewrite the first two inequalities with y alone on one side.

$$3y \geq -2x + 12$$

This leads to $y \geq -\frac{2}{3}x + 4$.

Similarly, the second inequality,

$8x - 4y > 1$, $4y < 8x - 1$, then, $y < 2x - \frac{1}{4}$. Now, graph the inequality $y \geq -\frac{2}{3}x + 4$. The related equation is $y = -\frac{2}{3}x + 4$. Since the inequality is $\geq$, the border line is solid. Shade the upper half of the line (Fig 4.8), as $y \geq -\frac{2}{3}x + 4$.

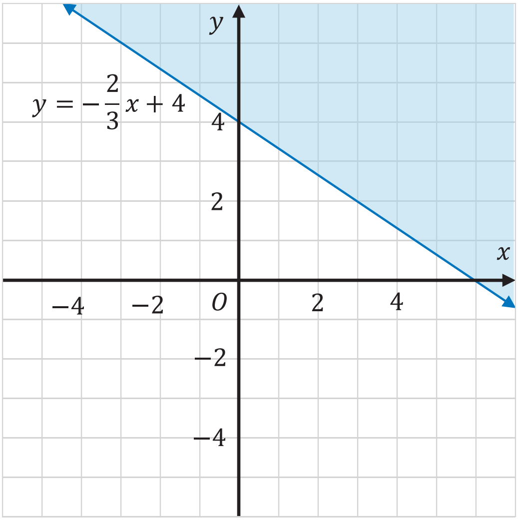


Figure 4.8

Similarly, draw a dashed line of related equation $y = 2x - \frac{1}{4}$ of the second inequality $y < 2x - \frac{1}{4}$ which has a strict inequality. Shade the lower half of the line(Fig 4.9)

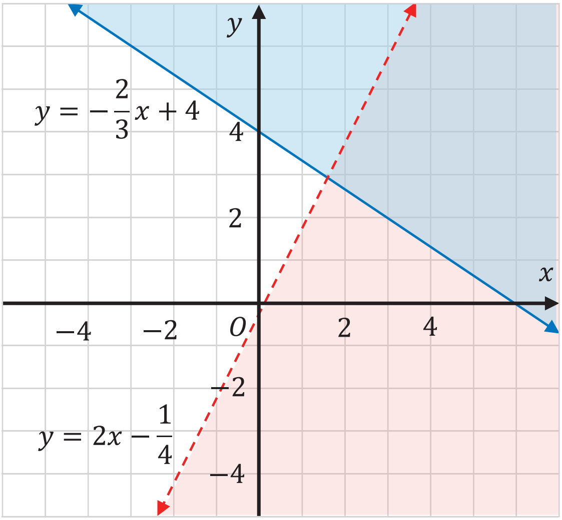


Figure 4.9

Draw a dashed vertical line $x = 4$, which is the related equation of the third inequality. Since $x < 4$, shade the left half of the line (Fig 4.10).

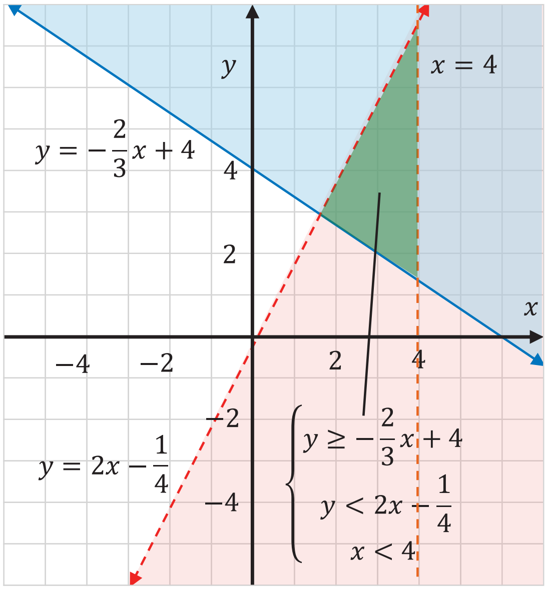


Figure 4.10

The solution of the system of inequalities is the intersection region of the solutions of the three inequalities.

Exercise 4.5

Solve the following systems of linear inequalities by using graphical method.

a. $\begin{cases} y > 3x - 4 \\ y \leq -2x \end{cases}$

b. $\begin{cases} y \geq -3x - 6 \\ y > 4x - 4 \end{cases}$

c. $\begin{cases} y \leq \frac{1}{2}x + 2 \\ y \leq -\frac{2}{3}x + 1 \end{cases}$

d. $\begin{cases} y < -\frac{3}{5}x + 4 \\ y \leq \frac{1}{3}x + 3 \end{cases}$

e. $\begin{cases} y < -\frac{3}{7}x - 1 \\ y > \frac{4}{5}x + 1 \end{cases}$

f. $\begin{cases} 2x + y < 4 \\ x - y \leq 4 \\ x > 2 \end{cases}$

4.3 Inequalities Involving Absolute Value

Graphing absolute value inequalities

Activity 4.4

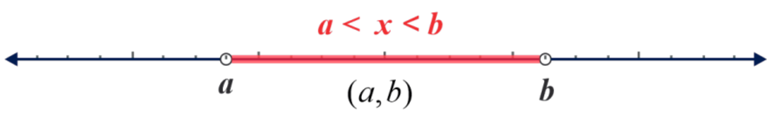
Which of the absolute value inequalities represent the number line indicated by red color below?

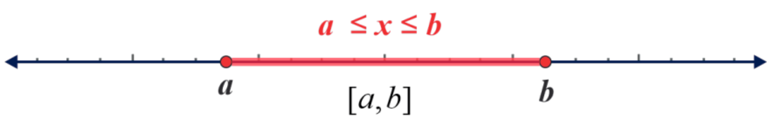
a. $|x+4| > 3$ b. $|x+4| \geq 3$ c. $|x+4| < 3$ d. $|x+4| \leq 3$

While graphing absolute value inequalities on a number line, you have to keep the following things in mind.

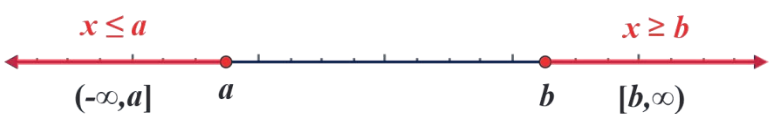
- Use open dots at the endpoints of the open interval (a, b).
- Use closed dots at the endpoints of the closed interval $[a, b]$.

Inequalities with $<$ or $\leq$





Inequalities with $<$, $>$ or $\leq$, $\geq$



Example

Graph the following absolute value inequality on a number line.

a. $|x| \leq 1$ **b.** $|x| > 1$

Solution:

a. $|x| \leq 1$ are the numbers that satisfy $-1 < x < 1$.

b. $|x| > 1$ are the numbers that satisfy $x < -1$ or $x > 1$.

Exercise 4.6

Put the following absolute value inequalities on a number line.

a. $|x| < 2$ b. $|x| \leq 3$ c. $|x| \leq 5$

d. $|x| > 4$ e. $|x| > \frac{3}{2}$

Inequalities involving absolute value

Assume k is an algebraic expression and c is a positive number.

The solutions of $|k| < c$ are the numbers that satisfy $-c < k < c$

The solutions of $|k| > c$ are the numbers that satisfy $k > c$ or $k < -c$. These rules are valid if $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$.

Example 1

Solve the absolute value inequality $|x + 2| < 4$.

Solution:

The solutions of $|k| < c$ are the numbers that satisfy $-c < k < c$.

Hence, $|x + 2| < 4$

$-4 < x + 2 < 4$

$-4 < x + 2$ **and** $x + 2 < 4$

$-4 < x + 2$

$-6 < x$

$x > -6$

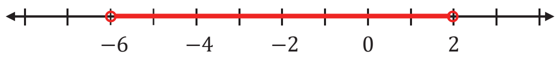
On the other hand, $x + 2 < 4$

$x < 2.$

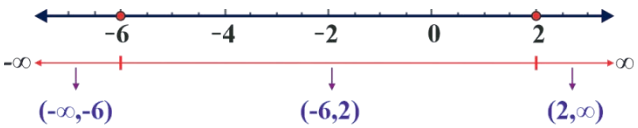
Therefore, the solution is $x > -6$ and $x < 2$ or equivalently, $-6 < x < 2$

The solution using interval notation is $(-6, 2)$

The solution using a number line is shown as follows:

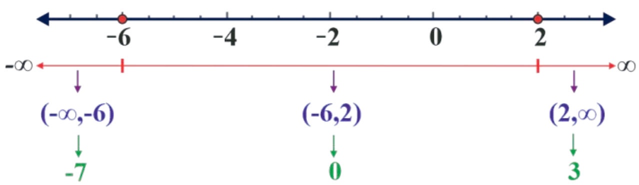


Note: Here, we can see that the number line is divided into 3 parts.



Take a random number from each of these intervals and substitute it in the given inequality.

Identify which of these numbers satisfy the given inequality. For instance, $x = 0$ does satisfy the inequality, while neither $x = -7$ nor $x = 3$ satisfy it.



Therefore, the solution of the given inequality is, $(-6, 2)$ or $-6 < x < 2$

Alternatively, one can solve $|x + 2| < 4$ using $|k| < c$ if and only if $-c < k < c$

$-4 < x + 2 < 4$

$-4 - 2 < x < 4 - 2$

$-6 < x < 2$

We also note the following:

i. If the problem was $|x+2| \leq 4$, then the solution would have been $[-6, 2]$ or $-6 \leq x \leq 2$

If $|x+2| > 4$, then the solution would have been $(-\infty, -6) \cup (2, \infty)$ or $x < -6$ or $x > 2$.

ii. Also If $|x+2| \geq 4$, then the solution would have been $(-\infty, -6] \cup [2, \infty)$ or $x \leq -6$ or $x \geq 2$.

Example 2

Solve the absolute value inequality $|3x-2| \geq 7$.

Solution:

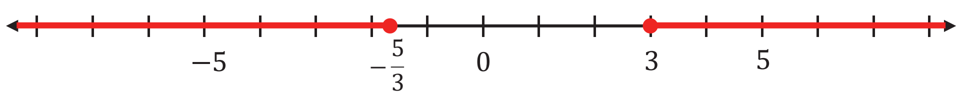
The solutions of $|k| \geq c$ are the numbers that satisfy $k \geq c$ **or** $k \leq -c$.

Hence, $|3x-2| \geq 7$

$3x - 2 \geq 7$ **or** $3x - 2 \leq -7$

$3x \geq 9$ **or** $3x \leq -5$

$x \geq 3$ **or** $x \leq -\frac{5}{3}$. (Represent on a number line)



Steps for Solving Linear Absolute Value Inequalities:

1. Identify what the absolute value inequality is set "equal" to…

a. If the absolute value is less than zero, there is no solution.

b. If the absolute value is less than or equal to zero, there is one solution. Just set the argument equal to zero and solve.

c. If the absolute value is greater than or equal to zero, the solution is all real numbers.

d. If the absolute value is greater than zero, the solution is all real numbers except for the value which makes it equal to zero. This will be written as a union.

2. Graph the answer on a number line and write the answer in interval notation.

Example 3

Solve the absolute value inequality $|-x+4| < 0$.

Solution:

As said above in step (a), there is no solution as the absolute value is less than 0. We can prove it as follows. Using $|k| < c$, if and only if $-c < k < c$, the inequality becomes,

$$-0 < -x+4 < 0$$

$$0 < -x+4 < 0$$

$$0 < -x+4 \quad \text{and} \quad -x+4 < 0$$

This leads to $x < 4$ and $x > 4$. Hence, there is no real number satisfies x. Thus, this inequality has no solution.

Example 4

Solve the absolute value inequality $|-5x-7| \leq 0$.

Solution:

As said above in steps (b.), there is one solution when the given inequality is equal to zero. Thus, the solution of $|-5x-7| = 0$ is the solution of $|-5x-7| \leq 0$.

$$|-5x-7| = 0$$

$$-5x-7 = 0$$

$$x = -\frac{7}{5}$$

Example 5

Solve the absolute value inequality $|6x+8| > -2$.

Solution:

As absolute value of a number is always greater than a negative number, the value of x is all real numbers.

Exercise 4.7

Solve the following absolute value inequalities.

a. $|x+3| < 7$ b. $|x-5| \leq 2$ c. $|x-7| > 4$

d. $|2x - 3| \geq 5$ e. $|x + 8| < 0$ f. $|3 - x| \leq 0$

g. $|4x - 5| \geq -2$

4.4 Quadratic Inequalities

In unit 3 section 3.3.3, you have learned how to solve quadratic equations using factorization, completing the square method or quadratic formula.

Definition 4.2

The general forms of the quadratic inequalities are: $ax^2 + bx + c < 0$, $ax^2 + bx + c \leq 0$, $ax^2 + bx + c > 0$ and $ax^2 + bx + c \geq 0$ where $\neq a$ and a, b, c $\in \mathbb{R}$

For instance, inequalities such as $x^2 - 6x - 16 \geq 0$, $2x^2 - 11x + 12 > 0$, $x^2 + 4 > 0$, $x^2 - 3x + 2 \leq 0$ are quadratic inequalities.

4.4.1 Solving quadratic inequalities using product properties

Activity 4.5

1. a) If the product of two real numbers is greater than zero, what can you say about the numbers?

   b) If the product of two real numbers is less than zero, what can you say about the numbers?

   c) If the product of two real numbers is equal to zero, what can you say about the numbers?

2. If possible, factorize each of the following expressions.

   a. $x^2 - 8x + 16$ b. $x^2 - x - 6$

   c. $x^2 + 4x + 7$ d. $3x^2 - 17x + 10$

Product properties:

1. $mn > 0$, if and only if

i. $m > 0$ **and** $n > 0$ or *ii.* $m < 0$ **and** $n < 0$

2. $m \cdot n < 0$, if and only if

i. $m > 0$ **and** $n < 0$ or *ii.* $m < 0$ **and** $n > 0$

Example 1

Solve $(x-3)(x-1) < 0$

Solution:

Using the product property 2, the first case is

$$x - 3 > 0 \text{ and } x - 1 < 0$$

This leads to $x > 3$ and $x < 1$, which is not possible.

The second case is $x - 3 < 0$ and $x - 1 > 0$

This leads to $x < 3$ and $x > 1$, which is $1 < x < 3$.

Therefore, the solution is $1 < x < 3$, or $(1, 3)$ in interval notation.

Example 2

Solve the inequality $(x-4)(x-2) > 0$

Solution:

Using the product property,

Case i: $x - 4 > 0$ and $x - 2 > 0$

$x > 4$ and $x > 2$

$x > 4$ or

Case ii: $x - 4 < 0$ and $x - 2 < 0$

$x < 4$ and $x < 2$

Thus, $x < 2$

Hence, the solution is $x < 2$ or $x > 4$. This is expressed as $(-\infty, 2) \cup (4, \infty)$ in interval notation.

Exercise 4.8

Solve the following quadratic inequalities.

a. $(x-5)(x-4)<0$

b. $(x+1)(x-5)<0$

c. $(x+3)(x-6)\geq 0$

d. $(x-2)(x-7)>0$

e. $(2x-1)(x-3)\leq 0$

f. $(3x+5)(3x+1)>0$

Rearrange the given formula, factorize it, and solve the quadratic inequality.

Example 3

Solve the inequality $x^2-x\geq 12$.

Solution:

$x^2-x\geq 12$ implies $x^2-x-12\geq 0$.

Factorize the quadratic inequality to get $(x-4)(x+3)\geq 0$.

Using the product rule, the product will be greater than or equal to zero if

i. $x-4\geq 0$ and $x+3\geq 0$. Hence, $x\geq 4$ and $x\geq -3$. Thus, $x\geq 4$.

Or

ii. $x-4\leq 0$ and $x+3\leq 0$. Hence, $x\leq 4$ and $x\leq -3$. Thus, $x\leq -3$.

Therefore, the solution of this quadratic inequality is $x\leq -3$ or $x\geq 4$. Alternatively, the solution is $(-\infty,\ -3]\cup[4,\ \infty)$.

Example 4

Solve an inequality $-x^2+x+12\leq 0$

Solution:

$-x^2+x+12\leq 0$ implies $x^2-x-12\geq 0$

The rest of the solution is the same as Example 3 above.

Therefore, the solution of this quadratic inequality is $x\leq -3$ or $x\geq 4$.

Alternatively, the solution is $(-\infty,\ -3]\cup[4,\ \infty)$.

Exercise 4.9

Solve the following quadratic inequalities using product rule.

a. $x^2 - 6x + \quad < 0$

b. $x^2 + 6x + \quad \geq 0$

c. $x^2 - 2x - \quad > 0$

d. $x^2 - 7x \leq -6$

e. $-x^2 + 4 < 0$

f. $2x^2 + x - 15 \leq 0$

g. $-2x^2 + 5x + 12 \geq 0$

4.4.2 Solving quadratic inequalities using sign chart

Activity 4.6

Fill in the table as +++, − − − or 0 as indicated in the table.

Factors	$x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$x > 2$
$x + 1$	$- - -$				
$x - 2$		$-$	$- - -$	0	
$f(x) = (x + 1)(x - 2)$					$+ + +$

Sign chart is a table or number line used to solve inequalities which can be factorized into linear binomials. For example, $(ax + b)(cx + d) > 0$. It could also be less than or equal or greater than or equal.

Steps in solving quadratic inequalities by sign chart:

1. Re-write the inequality to get a 0 on the right-hand side.
2. Factor (if possible) the left-hand side.
3. "Graph" each factor on a sign chart using a display of signs. A root of any factor is a key point.
4. Each key point or interval is displayed using a column.

Example 1

Solve $(x-4)(x+3)<0$

Solution:

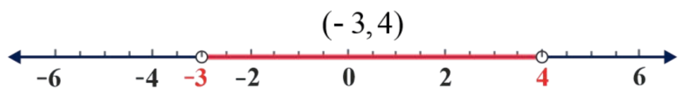
The factors of the left-hand side is $x=-3$ and $x=4$. Using the factors, we prepare the sign chart as follows:

Factors	$x<-3$	$x=-3$	$-3<x<4$	$x=4$	$x>4$
$x-4$	$---$	$-$	$---$	0	$+++$
$x+3$	$---$	0	$+++$	$+$	$+++$
$(x-4)(x+3)$	$+++$	0	$---$	0	$+++$

From the table above, we see that $(x-4)(x+3)<0$ if $-3<x<4$.

Alternatively, $x \in (-3,4)$

We represent this on a number line as follows:



Example 2

Solve $(x+3)(x+2)\geq 0$

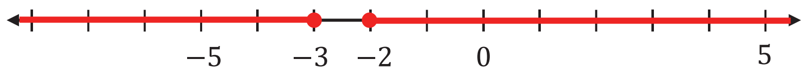
Solution:

Factors	$x<-3$	$x=-3$	$-3<x<-2$	$x=-2$	$x>-2$
$x+3$	$---$	0	$+++$	$+$	$+++$
$x+2$	$---$	$-$	$---$	0	$+++$
$(x+3)(x+2)$	$+++$	0	$---$	0	$+++$

From the table above, we see that $(x+3)(x+2)\geq 0$ when $x\leq -3$ *or* $x\geq -2$.

The solution can be written using interval notation as $(-\infty,-3]\cup[-2,\ \infty)$

We represent this on a number line as follows:



Exercise 4.10

Use sign chart to solve the following quadratic inequalities. Check the solution with the product property method.

a. $(x-3)(x+1) < 0$

b. $(x-3)(x+2) > 0$

c. $(2x-1)(3x+4) > 0$

d. $x^2 - x - 12 \leq 0$

e. $x^2 + 2x - 15 > 0$

f. $5 - 4x - x^2 > 0$

g. $1 - x - 2x^2 < 0$

h. $10x^2 - 19x + 6 \leq 0$

4.5 Some Applications of Inequalities

Activity 4.8

Write each statement as linear inequality in two variables.

a. The sum of the costs of 10 books and 4 pens is greater than Birr 180.

b. Given that Ariat is taller than Medina. The difference between the height of Medina and Ariat is at least 10cm.

c. Five times the length of a ruler decreased by 5cm is less than the height of Musa.

d. In a month, the total amount the family spends for food and educational expenses and recreation is at most Birr 5,000.

Activity 4.9

The total amount Meskerem paid for 5 kilos of rice and 2 kilos of coffee is less than Birr 600.

a. What mathematical statement represents the total amount Meskerem paid? Define the variables used.

b. Suppose a kilo of rice costs Birr 35. What could be the greatest cost of a kilo of coffee to the nearest birr?

c. Suppose Meskerem paid more than Birr 600 and each kilo of rice costs Birr 34. What could be the least amount she will pay for 2 kilos of coffee? Of coffee to the nearest birr?

Example 1

Kebede has started a saving account for buying a tractor. He saved Birr 6,000 up to last month and plans to add Birr 1,000 each month until he saves more than Birr 80,000. Write and solve the inequality. Then, interpret the answer. How many months does he need to save more than Birr 80,000?

Solution:

More than Birr 80,000 means greater than Birr 80,000. *Let* "m" be the number of months he saves.

$$6{,}000 + 1{,}000m > 80{,}000$$

$$1{,}000m > 74{,}000$$

$$m > 74$$

Since the number of months is greater than 74, it will take him more than 6 years 2 months to save enough money to buy a tractor.

Example 2

Chaltu likes to text messages to her friends using her cell phone. She is charged Birr 0.20 each time she types a message. Her telephone has Birr 50.00 in her account. She is allowed to have a bill that is at most Birr 80.00. Write and solve the inequality. Then, interpret the answer. How many messages can she text message at most?

Solution:

At most means less than or equal to and then we have

$$0.2m + 50 \leq 80$$

$$0.2m \leq 30$$

$$m \leq 150$$

Since the number of messages is less than or equal to 150, she can send a maximum of 150 text messages.

Example 3

The length of a normal human pregnancy is from 37 to 41 weeks, inclusive.

a. Write an inequality representing the normal length of a pregnancy.

b. Write a compound inequality representing an abnormal length for a pregnancy.

Solution:

Let w be the number of weeks that a human pregnancy takes, then

a. The inequality representing the normal length of a pregnancy will be $37 \leq w \leq 41$

b. A compound inequality representing an abnormal length for a pregnancy will be $w < 37$ or $w > 41$

Example 4

Dendir works two part time jobs in order to earn enough money to meet his obligations of at least Birr 240 a week. His job in food service pays Birr 10 per hour and his tutoring job on campus pays Birr 15 per hour. How many hours does Dendir need to work at each job to earn at least Birr 240?

a. Let x be the number of hours Dendir works at the job in food service and let y be the number of hours he works tutoring. Write an inequality that would model this situation.

b. Graph the inequality.

c. Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Dendir.

Solution:

a. Dendir earns Birr 10 per hour at the job in food service and Birr 15 per hour tutoring. At each job, the number of hours multiplied by the hourly wage will give the amount earned at that job.

Amount earned at the food service job plus the amount earned tutoring is at least Birr 240.

$10x + 15y \geq 240.$

b. To graph the inequality (Fig 4.11), we put in slope-intercept form.

$10x + 15y \geq 240$

$15y \geq -10x + 240$

$y \geq -\frac{2}{3}x + 16$

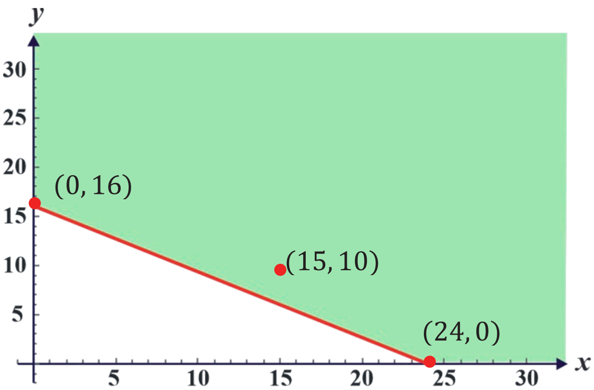


Figure 4.11

c. From the graph, we see that the ordered pairs represent three of the infinitely many solutions.

Check the values in the inequality, $10x + 15y \geq 240$.

$(15, 10)$	$(0, 16)$	$(24, 0)$
$(LHS) = 10(15) + 15(10)$ $= 300$ $(LHS) > (RHS)$, true.	$(LHS) = 10(0) + 15(16)$ $= 240$ $(LHS) = (RHS)$, true.	$(LHS) = 10(24) + 15(0)$ $= 240$ $(LHS) = (RHS)$, true.
This means that Dendir works for food service for 15 hours and for tutoring for 10 hours per week respectively. Then, he earns Birr 300 per week.	This means that Dendir works for tutoring for 16 hours per week. Then, he earns Birr 240 per week.	This means that Dendir works for food service for 24 hours per week. Then he would earns Birr 240 per week.

Exercise 4.11

1. Answer the following questions.
    a. The difference of a number and 6 is between 0 and 8. Find all such numbers.
    b. Twice a number is between 2 and 12. Find all such numbers.
    c. One plus twice a number is greater than 1and less than 5. Find all such numbers.
    d. One-third of a number is either less than 2 or greater than 5. Find all the natural numbers.
2. On Addis Ababa's interstate highway, the speed limit is 55 km/h. The minimum speed limit is 45 km/h. Write a compound inequality that represents the allowable speeds.
3. The normal number of white blood cells for human blood is between 4800 and 10,800 cells per cubic millimeter, inclusive.
    a. Write an inequality representing the normal range of white blood cells per cubic millimeter.
    b. Write a compound inequality representing abnormal levels of white blood cells per cubic millimeter.
4. The ideal diameter of a piston for one type of car is 88 mm. The actual diameter can vary from the ideal by at most 0.007 mm. Find the range of acceptable diameter for the piston.
5. Assume that you are allowed to go within 9 km/h of the speed limit of 65 km/h without getting a ticket. Write an absolute value inequality that models this situation. Write an absolute inequality that shows the speed you are allowed to go without getting a ticket.
6. One leg of a right triangle is 3 cm longer than the other. How long should the shorter leg be to ensure the area is at least 14 cm^2?

7. Fatima, who lives in Jimma zone, needs to make a rectangular coffee plot which has an area of at most 18 m^2. The length should be 3 m longer than the width. What are the possible dimensions of the plot?
8. Almaz is selling sambusa at a school. She sells two sizes, small (which has 1 scoop of beans inside) and large (which has 2 scoops of beans). She knows that she can get a maximum of 70 scoops of beans out of her supply. She charges Birr 3 for a small sambusa and Birr 5 for a large one. Almaz wants to earn at least Birr 120 to give back to the school. Let x be the number of small sambusa, and y be the number of large one? Write and graph a system of inequalities that models this situation.

Summary

1. The open interval (a, b) with end-points a and b is the set of all real numbers x such that $a < x < b$.
2. The closed interval $[a, b]$ with end-points a and b is the set of all real numbers x such that $a \leq x \leq b$.
3. The half-open interval or half-closed interval $(a, b]$ with end points a and b is the set of all real numbers x such that $a < x \leq b$.
4. The half-open interval or half-closed interval $[a, b)$ with end points a and b is the set of all real numbers x such that $a \leq x < b$.
5. For any positive real number a, the solution set of:
   - **i.** the inequality $|x| < a$ is $-a < x < a$ and
   - **ii.** the inequality $|x| > a$ is $x < -a$ or $x > a$.
6. Linear inequalities in two variables represent the inequality between two algebraic expressions involving the two variables.
7. An inequality that can be reduced to either $ax^2 + bx + c \leq 0$, $ax^2 + bx + c < 0, ax^2 + bx + c \geq 0$ or $ax^2 + bx + c > 0$ where a, b and c are constants and $a \neq 0$ is called a **quadratic inequality.**
8. A quadratic inequality can be solved by using product properties, sign chart or graphically.
9. Property of product rule.
   - **a.** If $m \cdot n > 0$, then $m > 0$ and $n > 0$ or $m < 0$ and $n < 0$.
   - **b.** If $m \cdot n < 0$, then $m > 0$ and $n < 0$ or $m < 0$ and $n > 0$.

Review Exercise

1. If $x > y$, what can you say about the inequality sign if we multiply both side of the inequality by a negative number? By a positive number? By zero?

2. If x is a natural number, then the solution set for the inequality $3x + 7 < 16$ is

   a. $\{1,2\}$ **b.** $\{2,3\}$ **c.** $\{\ldots..0, 1, 2, 3\}$ **d.** $\{4, 5, 6, \ldots..\}$

3. Set up an algebraic inequality for

   a. The sum of 7 and three times a number is less than or equal to 1.

   b. If three is subtracted from two times a number, then the result is greater than or equal to nine.

4. Solve the following inequalities.

   a. $\frac{2}{3}\left[x - \left(1 - \frac{x-2}{3}\right) + 1\right] \leq x$ **b.** $2 - [-2(x+1)] - \frac{x-3}{2} \leq \frac{2}{3}x - \frac{5x-3}{12} + 3x$

   c. $\dfrac{3x+1}{7} - \dfrac{2-4x}{3} \geq \dfrac{-5x-4}{14} + \dfrac{7}{6}x$

5. Solve the following simultaneous inequalities.

   a. $\begin{cases} x + 7 > 4 \\ x + 2 < 5 \end{cases}$ **b.** $\begin{cases} 3x + 2 > 7 \\ x + 1 < 5 \end{cases}$ **c.** $\begin{cases} 2x + 1 \geq 9 \\ 3x + 4 < 25 \end{cases}$

6. Solve the following simultaneous inequalities:

   a. $\begin{cases} -2x + y > -4 \\ 3x - 6y \geq 6 \end{cases}$ **b.** $\begin{cases} -3x + 2y > 6 \\ 6x - 4y > 8 \end{cases}$ **c.** $\begin{cases} y \geq -4 \\ y < x + 3 \\ y \leq -3x + 3 \end{cases}$

7. Solve the following inequalities:

   a. $|2x - 1| - 7 \leq -5$ **b.** $|x - 1| \geq -3$

   c. $|x - 1| \leq -3$ **d.** $\left|\frac{2x-9}{3}\right| < 4$

   e. $|6 - 2x| \leq 2$

8. Solve the following quadratic inequalities using product rule, sign chart and graphically:

   a. $(x-1)(x-2) > 0$ **b.** $(x+3)(x-5) \leq 0$

   c. $x^2 + 2x - 3 \geq 0$ **d.** $x^2 + 6x > -2x$

   e. $4x - x^2 > 12$ **f.** $x^2 + 8x + 16 \geq 0$

g. $x^2 - 4x - 12 \leq 0$ **h.** $4x^2 > 12x$

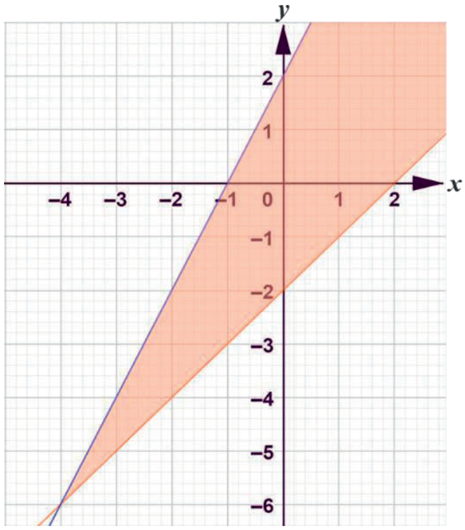
9. Which of the following is the solution for the inequality $|3x + 7| \leq 5$?

A. $\frac{2}{3} \leq x \leq 4$ B. $x \leq -\frac{2}{3}$ C. $x \leq -4$ D. $-4 \leq x \leq -\frac{2}{3}$

10. Which of the following point is a solution to the inequality $\begin{cases} x - 2y > -1 \\ 3x - y < -3 \end{cases}$

A. $(2, 1)$ B. $(-2, -1)$ C. $(2, -1)$ D. $(-2, 1)$

11. Which of the following system of inequalities represent the following graph?



A. $\begin{cases} y > x - 2 \\ y \leq 2x + 2 \end{cases}$ B. $\begin{cases} y \geq x - 2 \\ y \leq 2x + 2 \end{cases}$ C. $\begin{cases} y > x - 2 \\ y > 2x + 2 \end{cases}$ D. $\begin{cases} y \geq x - 2 \\ y < 2x + 2 \end{cases}$

12. The solution for the quadratic inequality $x^2 - x > 12$ is:

A. $x > -3$ or $x > 4$ B. $x < -3$ or $x > 4$

C. $-3 < x < -4$ D. $x > -3$ and $x < 4$

13. A technician measures an electric current which 0.036A with a possible error of± 0.002 A. Write this current, i, as an inequality with absolute values.

14. A shop owner has determined that the relationship between monthly rent charged for store space r (in hundred birr per square meter) and monthly profit $P(r)$ (in thousands of birr) can be approximated by the function $P(r) = -9r^2 + 8r + 1$. Solve each quadratic equation or inequality. Explain what each answer tells about the relationship between monthly rent and profit for the shop owner?