UNIT 6

REGULAR POLYGONS

Unit Outcomes

By the end of this unit, you will be able to:

- **Identify regular polygons.**
- **Solve problems on area and perimeter of regular polygons.**
- **Find the measure of each interior or exterior angle of a regular polygon.**
- **Explain properties of regular polygons.**

Unit Contents

KEY Words

- concave polygon
- area of regular polygon
- convex polygon
- perimeter of regular polygon
- in circle
- interior angles
- line of symmetry
- circumcircle
- exterior angles
- regular polygon
- apothem
- central angle

INTRODUCTION

In the previous grades, you studied different plane figures like triangles, quadrilaterals (rectangles, squares, and rhombus) and circles. You discussed how to get area and perimeter of such plane figures. In this unit, you will revise some of the basic concepts on polygons and study about regular polygon and their related properties.

6.1 Sum of Interior Angles of a Convex Polygon

Concave and convex polygons

Activity 6.1

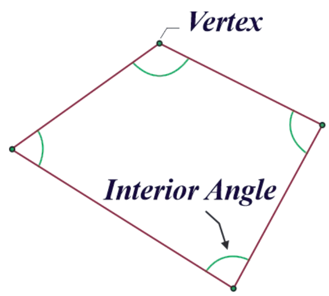
Answer each of the following.

1. How many sides and vertices (corners) does a triangle have?
2. How many sides and vertices (corners) does a quadrilateral have?
3. Can you draw a plane figure that has 5 sides and corners?

In the above activity, you discussed some plane figures. Now let us define a polygon.

Definition 6.1

A polygon is a simple closed plane figure formed by three or more line segments joined end to end, no two of which in succession are collinear. The line segments forming the polygon are called ***sides***. The common end point of any two sides is called ***vertex*** of the polygon. The angles formed inside the polygon are called ***interior angles***.



For instance:- Triangle (three sides) is the simplest polygon. Other polygons are quadrilaterals (four sides), and pentagons (five sides), and so on.

Common naming of polygon

In general, if a polygon has n sides, it is named as "n-gon", for instance a polygon with 16 sides is called 16-gon. The following table and figure provide the names of some polygons.

Table 6.1 Name of Polygons and Their Number of Sides and Vertices

Name of polygon	Number of sides	Number of vertices
Triangle	3	3
Quadrilateral	4	4
Pentagon	5	5
Hexagon	6	6
Heptagon	7	7
Octagon	8	8
Nonagon	9	9
Decagon	10	10

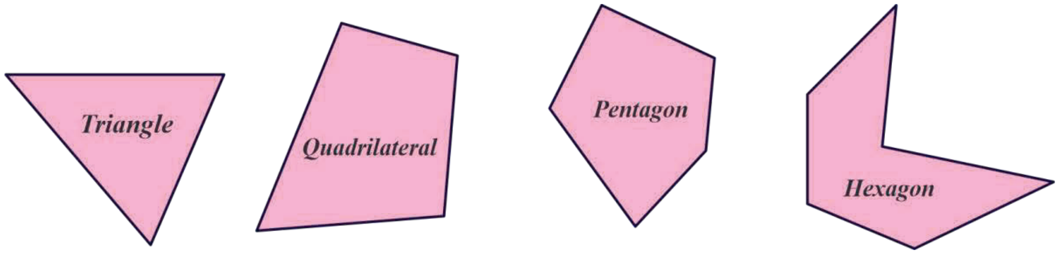


Figure 6.1 Polygons

Example

How many sides and vertices does a hexagon have?

Solution:

A hexagon is a polygon with 6 sides and 6 vertices.

Exercise 6.1

1. How many sides and vertices does a nonagon have?
2. What is the name of a polygon with ten sides?
3. Draw an octagon using a ruler.
4. When a triangle is cut off from a quadrilateral by a straight line which passes through vertices, write down the names of the possible polygons to be formed.

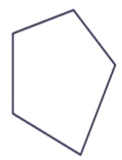
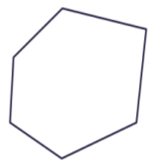
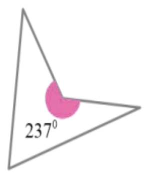
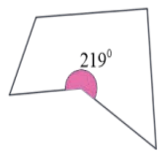
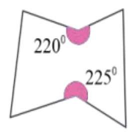
Definition 6.2

i. A polygon is said to be a convex polygon when the measure of each interior angle is less than 180 degrees. The vertices of a convex polygon are always outwards.

ii. A polygon is said to be a concave polygon when there is at least one interior angle whose measure is more than 180 degrees. The vertices of a concave polygon present inwards and also outwards.

Definition 6.2 could be easily observed by the following table 6.2.

Table 6.2

	Quadrilateral	Pentagon	Hexagon
Convex			
Concave			

The following could be another description for the above definitions.

Note

If none of the side extensions intersects the polygon, then the polygon is convex; otherwise it is concave.

Example

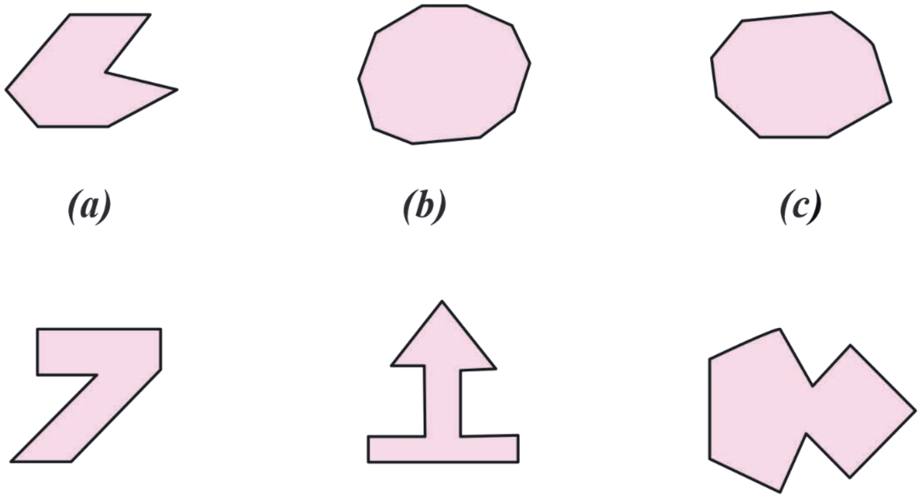
Is the quadrilateral in figure 6.1 convex or concave?

Solution:

The quadrilateral in figure 6.1 is convex since the measure of each interior angle is less than 180°. You can check this by measuring each angle using protractor.

Exercise 6.2

1. Is the hexagon in figure 6.1 convex or concave?
2. Name the following polygons and which of them are convex?



(a) *(b)* *(c)*

(d) *(e)* *(f)*

Sum of interior angles of a convex polygon

Activity 6.2

1. What is the sum of interior angles of a triangle?
2. What is the sum of interior angles of a quadrilateral?
3. Does the sum of interior angles of a polygon depend on the number of sides of the polygon?
4. How can you find the sum of interior angles of n-gon?

From your discussion in the activity and your previous grade geometry lessons, you learned that the sum of the measure of interior angles of any triangle is 180°.

For any triangle ABC, note that

$a + b + c = 180°$.

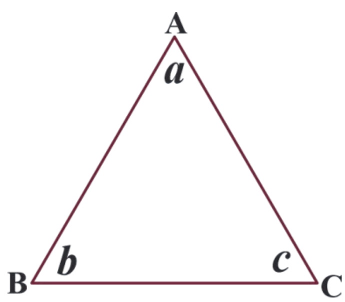


Figure 6.2

Example 1

Determine the measure of interior angles of a quadrilateral.

Solution:

Consider a quadrilateral $ABCD$ as shown below in figure 6.3. Draw a line segment from one vertex, say A, to its opposite vertex C as shown below. Here, the line segment AC is one of the diagonals of the given quadrilateral.

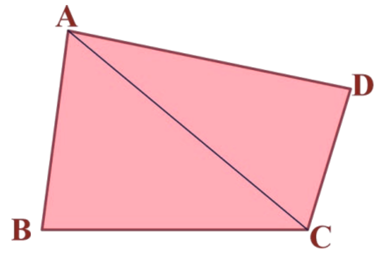


Figure 6.3

As you see from the figure 6.3 above, the quadrilateral is divided into two triangles so that the sum of interior angles of a quadrilateral is two times the sum of interior angles of a triangle. Hence, the sum of the measure of interior angles of a quadrilateral is $360°$. Note that a quadrilateral can be divided into two triangles. This is $\times 180° = 2 \times 180°$, where $n = 2$, the number of triangles forming the quadrilateral.

Example 2

a. In how many triangles can we divide the pentagon?

b. What is the sum of the measures of interior angles of a pentagon?

Solution:

a. Consider a pentagon (a five-sided polygon). Draw a line segment (diagonal) from A to another vertex in the polygon (You do not have to draw lines to the adjacent vertices since they are already connected by a side) as shown in figure 6.4.

You can easily observe that the pentagon is divided into three triangles. So that the sum of interior angles of a pentagon is three times the sum of interior angles of a triangle.

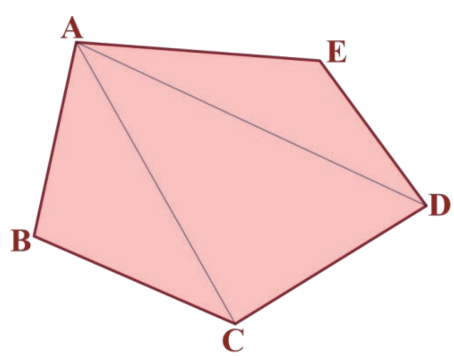


Figure 6.4

Note also that a pentagon can be divided into three triangles.

b. Sum of the measure of interior angles of a pentagon will be

$n \times 180° = 3 \times 180° = 540°$, where $n = 3$ is the number of triangles forming the pentagon.

Exercise 6.3

1. How many triangles can be made for a heptagon?
2. What is the sum of the measure of interior angles of a heptagon?

Deriving sum of the interior angles of a polygon

Can you derive a formula to determine the sum of the interior angles of a polygon?

Can you fill in the blank space of the following table?

Number of sides of a polygon	Name of the Polygon	Number of triangles	Sum of measure of interior angles
3	Triangle	1	$1 \times 180°$
4	Quadrilateral	2	$2 \times 180°$
5	Pentagon	3	$3 \times 180°$
6	Hexagon		
	.		
	.		
	.		
n	n-gon	$(n-2)$	$\_\_ \times 180°$

This indicates as the number of sides of a polygon increases, the number of triangles also increases. Furthermore, the sum of measure of interior angles of the polygon increases.

To summarize the above procedure, if n is the number of sides of a polygon, then $n - 2$ non-overlapping triangles are formed. Hence, we conclude the following statement:

Sum of the measure of interior angles of n-sided polygon is equal to $(n-2)\times 180°$.

Example 1

Find the sum of the measure of interior angles of hexagon.

Solution:

Hexagon is a 6-sided polygon. We use $n = 6$ in the above formula. Hence, the sum of measure of interior angles of a hexagon is equal to

$$(n-2)\times 180° = (6-2)\times 180° = 4\times 180° = 720°.$$

Example 2

The sum of interior angle of a polygon is 1,080°. How many sides does the polygon have?

Solution:

We need the value of n which satisfies $(n-2)\times 180° = 1{,}080°$.

$n-2 = \frac{1{,}080°}{180°} = 6$. So, $n = 8$. The polygon is octagon.

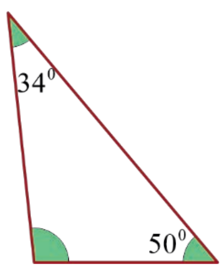
Exercise 6.4

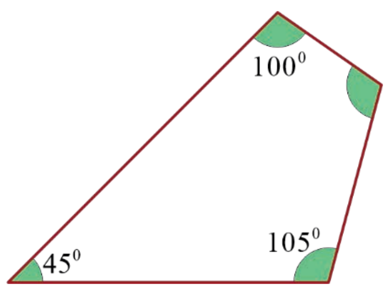
1. Find the sum of the measure of interior angles of:

   a. Nonagon   b. a 12-sided polygon

2. The sum of interior angles of a polygon is 1,440° . How many sides does the polygon have?

3. Find the incomplete measure of interior angle of the given polygons.





4. An architect design to construct recreation area of a school with a heptagonal shape with each interior angle are put in increasing order, each differs from the next by 25°. Find the measure of the smallest interior angle of the given heptagon to the nearest tenth of degree.

6.2 Sum of Exterior Angles of a Convex Polygon

In the previous section, you have studied about convex polygons and how to determine the sum of the measure of interior angles of a polygon. In this section, you will learn about exterior angles of convex polygons.

Activity 6.3

Given triangle ABC as shown in the figure 6.5. Construct exterior angles of the triangle at each vertex.

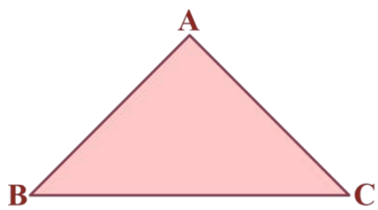
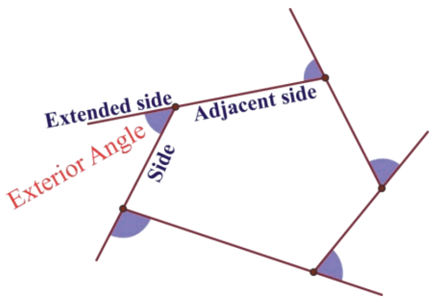


Figure 6.5

Definition 6.3

An exterior angle of a convex polygon is an angle outside the polygon formed by one of its sides and the extension of an adjacent side.



Example

Which ones from a to e are the exterior angles of the hexagon as shown in figure 6.6?

Solution:

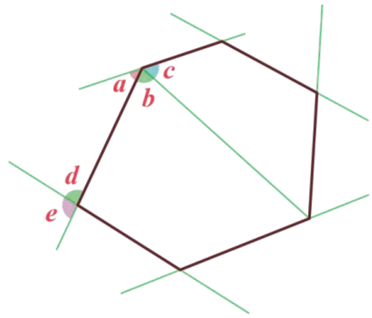


Figure 6.6

By the definition, an exterior angle is an angle formed outside of the polygon which is formed by extending one of the sides with the next side as defined in definition 6.3. Hence, a and d are exterior angles, whereas e is not an exterior angle since it does not satisfy the definition.

Exercise 6.5

1. Using a ruler, construct a heptagon and show its exterior angles.
2. Write **true** if the statement is correct and **false** otherwise.
    a. For a triangle, there are 6 exterior angles.
    b. For convex polygon, each exterior angle is less than $180°$.
    c. Each exterior angle of a rectangle is $90°$.
    d. If the measure of each interior angle of a regular polygon is $x°$, then the measure of each exterior angle of the given regular polygon is $(360 - x)°$.

Sum of the exterior angles of a polygon

What can you say about the sum of exterior angles of a polygon?

Does the sum depend on the number of sides of the polygon?

Let us see how to find sum of exterior angles of a polygon.

Example 1

Determine the sum of the exterior angles of a triangle.

Solution:

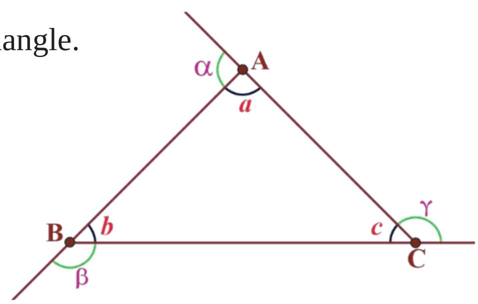


Figure 6.7

Consider a triangle ABC as figure 6.7. You will observe that angles α, β and γ are exterior angles, and angles a, b and c are interior angles of ΔABC.

The sum of the interior angles of a triangle is 180°. $a + b + c = 180°$. Also, we note that $(a + b + c) + (\alpha + \beta + \gamma) = 3 \times 180°$ (measure of three straight lines). This leads to $\alpha + \beta + \gamma = 360°$.

Hence, the sum of the measure of exterior angles of a triangle is 360°.

Example 2

Find the sum of the measure of exterior angles of a quadrilateral.

Solution:

Consider a quadrilateral $ABCD$ as shown in the figure 6.8 below with exterior angles α, β, γ and δ. Its interior angles are a, b, c and d. The sum of interior angles of a quadrilateral is $(n - 2) \times 180° = 2 \times 180° = 360°$ where n is the number of sides of the polygon. For our case, we know that $n = 4$.

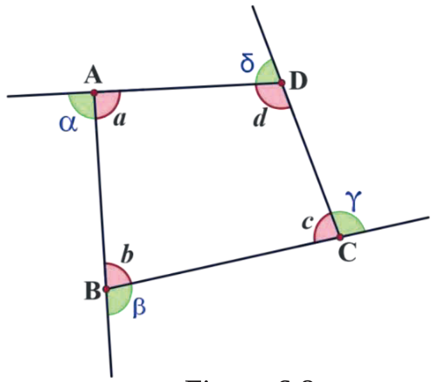


Figure 6.8

Hence, $a + b + c + d = 360°$. From the figure we conclude that $(a + b + c + d) + (\alpha + \beta + \gamma + \delta) = 4 \times 180° = 720°$ and $\alpha + \beta + \gamma + \delta = 360°$.

Hence, the measure of the exterior angles of a quadrilateral is 360°.

Exercise 6.6

1. Find the sum of the measure of exterior angles of a hexagon.
2. The exteriors angles of a pentagon are $(n + 5)°$, $(2n + 3)°$, $(3n + 2)°$, $(4n + 1)°$ and $(5n + 4)°$ respectively. Find the measure of each angle.

Sum of the exterior angles of n-sided polygon

Similarly, if we take an n-sided polygon, the sum of the interior angles of a polygon is $(n-2) \times 180°$.

For such a polygon, we also have n straight lines. Each one is the sum of interior and exterior angle. Therefore, we have:

> The sum of interior and exterior angles of n-sided polygon is $n \times 180^0$.

Using this concept, we can derive the sum of exterior angles of a polygon as follows.

For an n-sided polygon,

Sum of interior angles of a polygon + sum of exterior angles of a polygon

$$= n \times 180° \qquad \text{(i)}$$

Using $(n-2) \times 180°$ in place of sum of interior angles of a polygon, equation (i) will be

$(n-2) \times 180°$ + sum of exterior angles of a polygon = $n \times 180°$. (ii)

Using distribution on equation (ii) we get

$n \times 180° - 360°$+ sum of exterior angles of a polygon = $n \times 180°$. (iii)

Collecting like terms and rearranging equation (iii) gives us

Sum of the measure of exterior angles of a polygon = $360°$. (iv)

Hence, we can conclude that:

> For any n-sided polygon, the sum of the measure of exterior angles of the polygon is $360°$.

Note

At each vertex of a polygon, the sum of interior and exterior angles is $180°$.

Example

Consider a triangle whose two interior angles are 60° and 40°. Then, find

a. the remaining interior angle of the triangle.

b. the measure of exterior angle at each vertex.

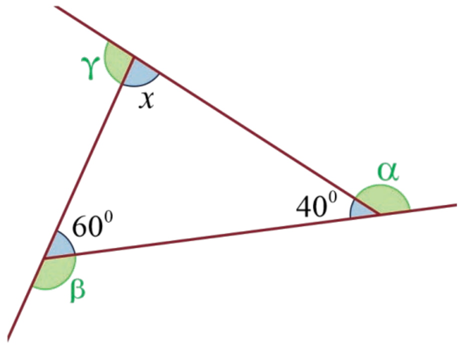


Figure 6.9

Solution:

a. Consider a triangle with two of its interior angles are 60° and 40° as shown in figure 6.9 . We know that the sum of the measure of interior angles of a triangle is 180°. Assume the degree measure of the remaining interior angle of the triangle be x. Since the sum of the measure of interior angles of a triangle is 180°, then $x + 60° + 40° = 180°$. $x = 180° - (60° + 40°) = 80°$.

b. We also know that a straight line measures 180°, then the exterior angle

$\alpha = 140°$, $\beta = 120°$ and $\gamma = 100°$.

Is the sum of these exterior angles in Example 1, 360°? Yes,

$100° + 120° + 140° = 360°$.

Exercise 6.7

1. Given a pentagon with four of its interior angles are 120° , 95°, 100° and 70°. Find the measure of exterior angle at each vertex.

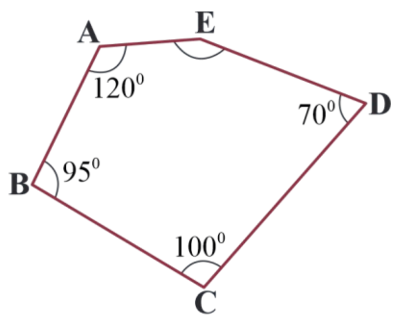


Figure 6.10

2. Find the measure of angle x and y from the following figure 6.11. The dashes on each line are to mean the line segments are equal in length.

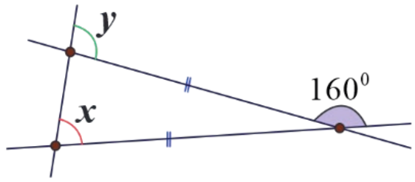


Figure 6.11

6.3 Measures of Each Interior Angle and Exterior Angle of a Regular Polygon

In the previous two subsections you learned how to determine the sum of the interior and exterior angles of a polygon. In this section you will see the definition of a regular polygon, and how to determine each interior and exterior angle of such polygon.

A summary of Greek mathematics was given by Euclid in about 300 BCE in his work the *Elements*. Euclid taught in Alexandria, Egypt, and wrote a number of mathematical treatises. His *Elements* consists of thirteen books and includes topics on number theory and irrationals as well as geometry.

Euclid gave constructions and proofs for an equilateral triangle (Proposition 1 in Book I), the square and pentagon (Book IV). He also shows that using the triangle and pentagon a 15-sided polygon can be constructed; furthermore, by bisecting angles further polygons can be constructed from all these, e.g. 6-,12-,24-,48-sided n-gons .

Euclid (325 BCE-265 BCE)

Activity 6.4

What common property do you observe for the following in figures 6.12?

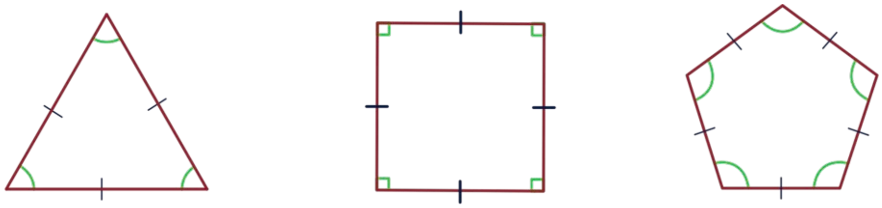


Figure 6.12

For a polygon of the above types, we have the following definition:

Definition 6.4

A polygon is said to be regular if it is equiangular (having equal angles) and equilateral (having equal sides).

For example: -Equilateral triangles and squares are regular polygons. Students, do you know honeycomb? We also have such polygons in our day-to-day life. For instance, figure 6.13 shows a honeycomb which is formed by honeybees as regular hexagons.

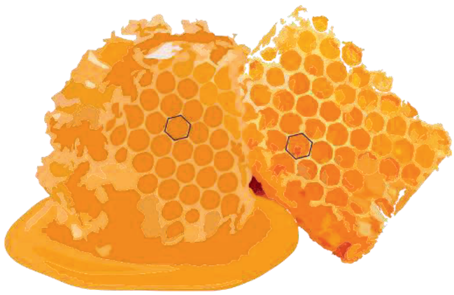


Figure 6.13

What is the measure of each interior and exterior angle of n-sided regular polygon?

For n-sided polygon, recall that

i. The sum of the measure of interior angles is $(n-2) \times 180°$.

ii. The sum of the measure of exterior angles is $360°$.

Using this and the definition of regular polygon, the measure of each interior angle of n-sided regular polygon is determined by

$$\frac{(n-2)\times 180°}{n}$$ (Since a regular polygon is equiangular)

and the measure of each exterior angle is $\frac{360°}{n}$ (since there are n exterior angles with same measure).

Example

Find the measure of each interior and each exterior angle of a regular pentagon.

Solution:

Given a regular pentagon which is five-sided ($n = 5$). So each interior angle degree measure is

$$\frac{(n-2)\times 180°}{n} = \frac{(5-2)\times 180°}{5} = \frac{3\times 180°}{5} = 108°$$

The measure of each exterior angle is $\frac{360°}{5} = 72°$.

Exercise 6.8

1. Find the measure of each interior and exterior angle of

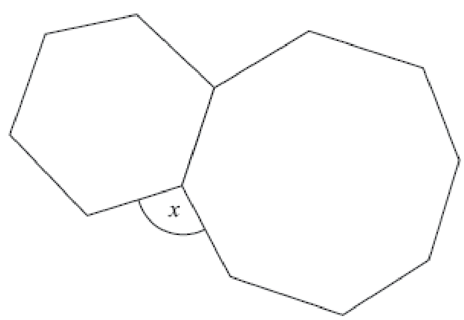
   i) a regular hexagon ii) a regular octagon

2. What is the sum of each interior and its corresponding exterior angle of the above two regular polygons?

3. A Mathematics teacher gave a group work for students to construct a convex regular polygon with one of the interior angle measures 135°. What is the name of this polygon?

4. For n-sided regular polygon, the measure of each interior angle is 5 times the measure of each exterior angle. What is the number of sides of this polygon?

5. The diagram shows a regular hexagon and a regular octagon. Calculate the size of the angle marked x.



6.4 Properties of Regular Polygons: Pentagon, Hexagon, Octagon and Decagon

In section 6.3, you studied about regular polygon, interior and exterior angles of such polygons. In this section, you will discuss the basic properties of regular polygons.

Line of symmetry for regular polygons

You have learned about symmetry of simple polygons in lower grades. You will apply the concept of symmetry to do the following activity.

Activity 6.5

On a sheet of paper draw an ***equilateral triangle*** and ***isosceles triangle***. Then cutout the above polygonal region and practice the following (you may need to use ruler, protractor and pencil.)

1. For each of the above polygonal region, how many lines of symmetry can you get? (Fold it in different ways till you obtain the folding line which serves as a mirror or line of symmetry)
2. Is there a relation between the number of lines of symmetry and the sides of a polygon?

Now, let us discuss symmetry of regular polygons.

- If two parts of a figure are identical after folding or flipping, then it is said to be ***symmetric***. To be symmetrical, one half must be the mirror image of the other. If the figure is not symmetrical then it is said to be ***asymmetrical***.
- ***Line of symmetry*** is a line through which the two halves of the figure match exactly. It is also called ***mirror line***. A figure can have more than one line of symmetry. An asymmetrical figure has no line of symmetry.

Note

All regular polygons are symmetrical shapes so that we have lines of symmetry for each.

The following figure shows the first four regular polygons with lines of symmetry.

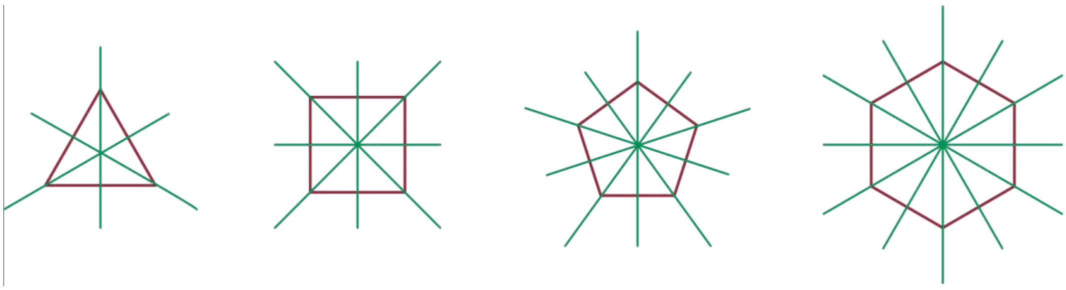


Figure 6.14: Some regular polygons with lines of symmetry

From figure 6.14 above, you can observe that

- An equilateral triangle has 3 lines of symmetry.
- A square has 4 lines of symmetry (two horizontal and vertical lines of symmetry, as well as two diagonal lines of symmetry).
- A regular pentagon has 5 lines of symmetry.
- A regular hexagon has 6 lines of symmetry.
- The lines of symmetry meet at a point inside the polygon called center of the polygon.
- If n is the number of sides of a regular polygon:
  - If n is odd, the line of symmetry connect vertex to side.
  - If n is even, the line of symmetry connect vertex to vertex and side to side.

From the above discussion, we can conclude the following: -

The n -sided regular polygon has n lines of symmetry.

Example

How many lines of symmetry does a regular heptagon have?

Solution:

A regular heptagon has 7 sides so that it has 7 lines of symmetry. It can be observed from the following figure 6.15.

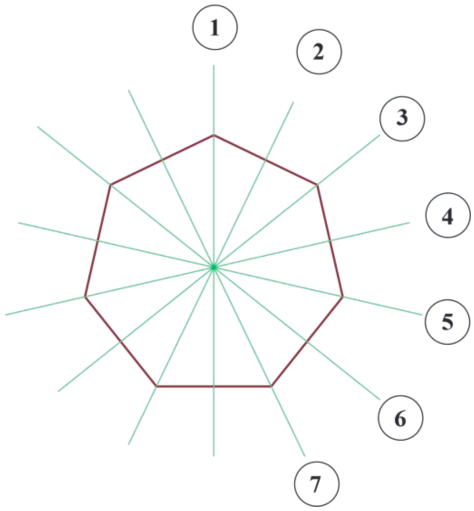


Figure 6.15

Exercise 6.9

1. How many lines of symmetry does a regular octagon have?
2. For each of the following statements write 'true' if the statement is correct and 'false' otherwise. If your answer is false give justification why it is false.

   a) If a polygon is not regular, then it is asymmetric.

   b) A right-angled triangle has at most one line of symmetry.

   c) A rhombus has 4 lines of symmetry.

   d) For n-sided regular polygon, the lines of symmetry divide the polygon into $2n$ triangles.

Inscribed and circumscribed polygon

Activity 6.6

1. Construct the following plane figures.
   a. A right-angled triangle with sides 6 cm and 8 cm that form the right angle.
   b. A square of side 10 cm.
   c. An equilateral triangle of side length 7 cm.
2. Using each of the above polygons,
   a. Try to draw a circle which passes through all vertices of the polygon.
   b. Try to draw a circle in the polygon where it touches all its sides.
3. Is it possible to draw such circles for the above polygons? (Identify the type of polygon for which such types of circles can be made).

In the above activity you observe facts about circles which could be made in and out of a polygon. Based on the discussion you had, we can study facts based on regular polygons. Let us consider n-sided regular polygon where O be the center of a circle and a polygon. In the figure below $\overline{AB}$, $\overline{BC}$, $\overline{CD}$,...are sides of n-sided regular polygon, E is the mid-point of $\overline{BC}$ which is one of the n sides of the polygon. Then we have:

- The n-sided polygon is inscribed in the bigger circle whose radius is $\overline{OB}$. Such a circle is called **a circumcircle**, and it connects all vertices (corner points) of the polygon.
- The line segment from the center perpendicular to the sides of a regular

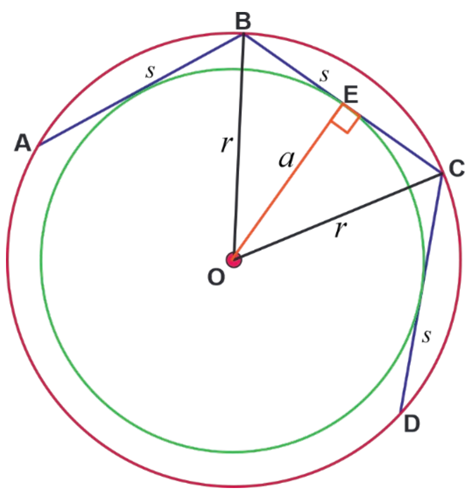


Figure 6.16

polygon is called an **apothem** of a polygon (for instance $a = \overline{OE}$ is an apothem which is perpendicular bisector of $\overline{BC}$).

- The inside circle (the smaller circle) is called an **in circle** and it just touches each side of the polygon at its midpoint. The radius of the incircle is the **apothem** of the polygon. (Not all polygons have these properties, but triangles and regular polygons do).
- A circle is inscribed in and circumscribed about any regular polygon.

Measure of central angles

The central angle for a regular polygon is an angle formed at the center of a circle by two consecutive radii from the vertex of a polygon (see figure 6.16, $\angle BOC$ is a central angle). For n-sided regular polygon, we can obtain n isosceles triangles (see also figure 6.16, ΔBOC is one of such isosceles triangles). Recall that the sum of measures of angles at a point is $360°$. Therefore, the sum of the central angles is $360°$. Hence, the measure of each central angle of n-sided regular polygon is $\frac{360°}{n}$ (each interior angle of a regular polygon has the same measure).

Example

Find the measure of central angle of a regular polygon inscribed in a circle which has the given number of sides:

a. 5 **b.** 8

Solution:

a. The measure of the central angle depends on the number of sides of a regular polygon. For $= 5$, the measure of each central angle is $\frac{360°}{5} = 72°$.

b. For $n = 8$, the measure of each central angle is $\frac{360°}{8} = 45°$.

Exercise 6.10

1 Determine the measure of each central angle of a regular polygon which is

a. 10 sided b. 15 sided

2. Find the number of sides of a regular polygon whose measure of central angle is $12°$.

3. Which of the following measure of central angles yields a regular polygon?

a. $6°$ b. $14°$ c. $80°$ d. $40°$

Perimeter, area and apothem of regular polygons inscribed in a circle

There is a close relationship between the perimeter, area, side length and apothem of an n-sided regular polygon which is inscribed in a circle. We discuss a particular case as an illustrative example and we develop the formula through the process for the general n-sided regular polygon.

Example

In figure 6.17, the regular hexagon is inscribed in a circle with center O and radius r. Express the side length s, perimeter P, apothem a, and area A of the regular hexagon using r.

Solution:

Given that ABCDEF is a regular hexagon. It has 6 sides. Triangle AOB is one of equilateral triangle as shown in figure 6.17.

(since $\angle AOB = 360° \div 6 = 60°$,

$\angle OAB = \angle OBA = (180° - 60°) \div 2 = 60°$)

There are 6 congruent equilateral triangles inside this regular hexagon. So that $\underline{s = r}$, $\underline{P = 6s = 6r}$.

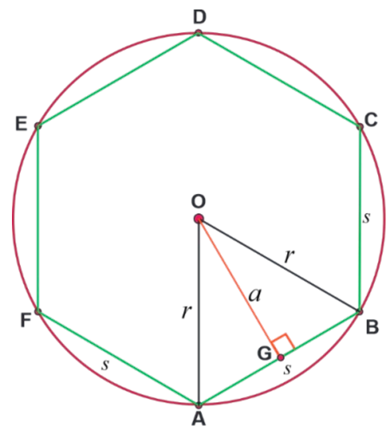


Figure 6.17

Draw a perpendicular bisector from O to $\overline{AB}$ = s. It meets $\overline{AB}$ at G. $\overline{OG}$ is the apothem a of the hexagon.

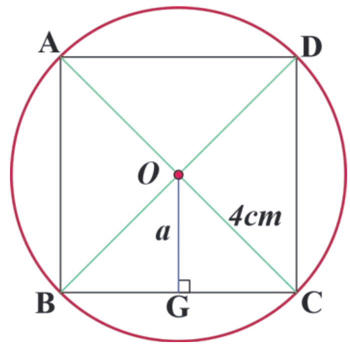
Here $\Delta AOG \equiv \Delta BOG$ (SSS). So that, m (∠AOG) = m (∠BOG) = 60° ÷ 2 = 30°.

Then, $\underline{a = r\ \cos 30° = \frac{\sqrt{3}}{2}r}$. The area of $\Delta AOB = \frac{1}{2}\ \overline{AB} \times \overline{OG} = \frac{1}{2}\ ar = \frac{\sqrt{3}}{4}r^2$.

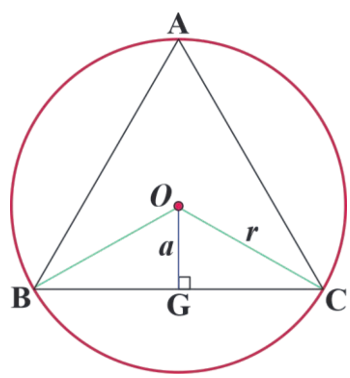
The area of the pentagon ABCDEF $= A = 6 \times \Delta AOB = 6 \times \frac{\sqrt{3}}{4}r^2 = \underline{\frac{3\sqrt{3}}{2}r^2}$.

Exercise 6.11

1. The figure below shows that a square is inscribed in a circle with the radius of 4 cm. Find the length of the side, the apothem and the perimeter, and the area of the square.



2. The figure below shows that an equilateral triangle is inscribed in a circle with the radius of r. Show the length of the side, the apothem, the perimeter, and the area of the triangle in terms of the radius r.



Example 1

Suppose a regular polygon (n-gon) is inscribed in a circle with center O and radius r. Find the side length s, perimeter P, apothem a, and area A of the regular n-gon. (In this case, n-gon is not necessarily a hexagon, but Fig 6.17 may help our understanding).

A regular n-gon is given in the figure. It has n sides. Triangle AOB is one of such isosceles triangles as shown in figure 6.16. Moreover, there are n congruent isosceles triangles inside this regular n-gon. Triangle AOB is one of such isosceles triangles as shown in figure 6.17.

Draw a perpendicular bisector from O to $\overline{AB}$ at G.

$\overline{OG}$ is the apothem a of the n-gon. Here, $\Delta AOG \equiv \Delta BOG$ (SSS).

And $m(\angle AOB) = 360° \div n = \frac{360°}{n}$

So that, $m(\angle AOG) = m(\angle AOG) = \frac{360°}{n} \div 2 = \frac{180°}{n}$

Then, $\frac{s}{2} = r\sin(\frac{180°}{n})$ which implies $\underline{s = 2r\sin(\frac{180°}{n})}$.

$\underline{P = n \times s = 2nr\sin(\frac{180°}{n})}$ and $\underline{a = r\cos\left(\frac{180°}{n}\right)}$.

The area of $\Delta AOG = \frac{1}{2}\ \overline{AB} \times \overline{OG} = \frac{1}{2}sa$. So, the area of the n-gon is

$A = n \times$ area of isosceles triangle. This is $A = \frac{1}{2}nas$, but $P = ns$, so that $\underline{A = \frac{1}{2}aP}$.

The above example 1 leads us to state the following theorem.

Theorem 6.1

For n-sided regular polygon inscribed in a circle of radius r, length of side s, apothem a, perimeter P and area A are determined by

$s = 2r\sin\left(\frac{180°}{n}\right)$ $\qquad$ $a = r\cos\left(\frac{180°}{n}\right)$

$P = 2nr\sin\left(\frac{180°}{n}\right)$ $\qquad$ $A = \frac{1}{2}aP$

Example 2

a. Find the length of side of a square inscribed in a circle of radius 5 cm.

b. Find the apothem of a regular pentagon inscribed in a circle of radius 6 cm.

c. Find the area of a regular nonagon inscribed in a circle whose radius is 10 cm.

(Use a calculator or a trigonometric table whenever necessary)

Solution:

a. By the formula in the above theorem 6.1, $s = 2r\sin\left(\frac{180°}{n}\right)$. Using $r = 5$ cm and $n = 4$, the length of the side is

$$s = 2r\sin\left(\frac{180°}{n}\right) = 2(5\text{ cm})\sin\left(\frac{180°}{4}\right)$$
$$= 10\text{ cm} \times \sin(45°)$$
$$= 10\text{ cm} \times \frac{1}{\sqrt{2}} = 5\sqrt{2}\text{ cm} \quad \text{(rationalizing the denominator)}$$

b. Given $r = 6$ cm and $n = 5$, we can determine the apothem (a) by using the formula $a = r\cos\left(\frac{180°}{n}\right)$, $a = 6\text{ cm} \times \cos\left(\frac{180°}{5}\right) = 6\text{cm} \times \cos(36°)$. Using scientific calculator or trigonometric table $\cos(36°) \approx 0.8090$.

Hence, $a \approx 4.854$ cm

c. The area of a regular polygon is determined by

$$\boldsymbol{A} = \frac{1}{2}aP = \frac{1}{2}\left(r\cos\left(\frac{180°}{n}\right)\right) \times 2nr\sin\left(\frac{180°}{n}\right)$$

Using $n = 9$ and $r = 10$ cm,

$$\boldsymbol{A} = nr^2\cos\left(\frac{180°}{n}\right)\sin\left(\frac{180°}{n}\right) = 9(100\text{ cm}^2)\cos(20°)\sin(20°)$$

(Using trigonometric table/Scientific calculator)

$\cos(20°) \approx 0.9397$ and $\sin(20°) \approx 0.3420$, hence, $A = 289.240\text{ cm}^2$.

Exercise 6.12

1. Find the perimeter, area and apothem of a decagon which is inscribed in a circle of radius $r = 8$ cm.
2. Suppose central angle of a regular polygon is $60°$. Then find the radius of a circle circumscribing the given polygon if its side is of length 9 cm .
3. One student from your class states that 'the radius of a regular polygon is never less than its apothem'. Do you agree? If so provide justification. If not, provide a counter example.

Summary

1. A polygon is a simple closed plane figure formed by three or more line segments joined end to end, no two of which in succession are collinear. The line segments forming the polygon are called ***sides***. The common end point of any two sides is called ***vertex*** of the polygon. The angles formed inside the polygon are called ***interior angles.***
2. i) A polygon is said to be a convex polygon when the measure of each interior angle is less than 180°.

   ii) A polygon is said to be a concave polygon when there is at least one interior angle whose measure is more than 180 degrees.
3. The sum of the measure of interior angles of n-sided polygon is determined by $(n-2)\times 180°$.
4. For any n-sided polygon, the sum of the measure of exterior angles of the polygon is 360°.
5. At each vertex of a polygon, the external and internal angle add up to 180° .
6. A regular polygon is a convex polygon with all sides equal and all interior angles equal.
7. i) The measure of each interior angle of a regular n-sided polygon is $\frac{(n-2)\times 180°}{n}$.

   ii) The measure of each exterior angle of a regular n- sided polygon is $\frac{360°}{n}$.

   iii) The measure of each central angle of a regular n-sided polygon is $\frac{360°}{n}$.
8. If two parts of a figure are identical after folding through the line of symmetry, then it is said to be symmetric. A symmetrical figure has at least one line of symmetry. An asymmetrical figure has no line of symmetry.
9. An n-sided regular polygon has n lines of symmetry.

10. An inscribed polygon is a polygon in which all vertices lie on a circle. The polygon is **inscribed** in the circle and the circle is **circumscribed** about the polygon.

11. A circumscribed polygon is a polygon in which each side touches a circle. The circle is **inscribed** in the polygon and the polygon is **circumscribed** about the circle.

12. A circle can always be inscribed in or circumscribed about any regular polygon.

13. An apothem of a regular polygon is a perpendicular segment from a midpoint of a side of a regular polygon to the center of the circle circumscribed about the polygon.

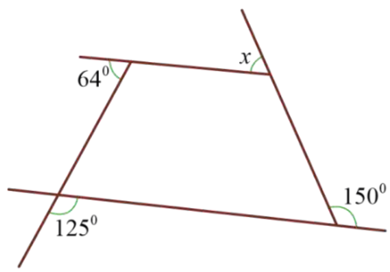
14. Formula for the length of side s, apothem a, perimeter P and area A of the regular polygon with n-sides and radius r are given by

$$s = 2r\sin\left(\frac{180°}{n}\right) \qquad a = r\cos\left(\frac{180°}{n}\right)$$

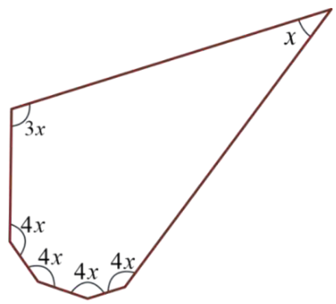
$$P = 2nr\sin\left(\frac{180°}{n}\right) \qquad A = \frac{1}{2}aP$$

Review Exercise

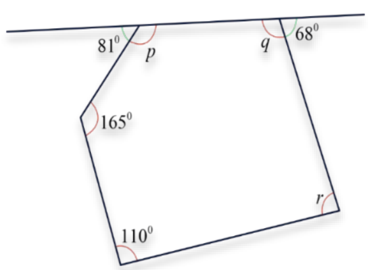
1. Write true if the statement is correct and false otherwise.
   - a. If a polygon is equiangular, then it is equilateral.
   - b. Equiangular hexagons are regular.
   - c. All symmetric plane figures are polygons.
   - d. If a polygon is inscribed in a circle, then the polygon is regular.
   - e. Fixing the radius of a polygon inscribed in a circle, increasing the number of sides of a regular polygon increases the length of the apothem.
   - f. If a regular polygon of n sides has every line of symmetry passing through a vertex, then n is even.
2. Find the sizes of the angles marked by letters.



(a)



(b)



(c)

3. Find the number of sides of a regular polygon if each exterior angle is equal to

   i) its adjacent interior angle.

   ii) twice its adjacent interior angle.

4. Is it possible to construct a polygon whose sum of interior angles is 26 right angles? If yes, find the number of sides of the polygon.

5. One of the interior angles of a polygon is $100°$ and each of the other angles is $110°$. Find the number of sides of the polygon.

6. The interior angles of a polygon are in the ratio 2:3:5:9:11. Find the measure of each angle. Is the polygon convex or concave? Why?

7. Each side of a regular hexagon is 10 units long. What is the area of this hexagon?

   A. 27 unit^2 **B.** 75 unit^2 **C.** $150\sqrt{3}$ unit^2 **D.** $54\sqrt{3}$ unit^2

8. The area of a regular hexagon is $96\sqrt{3}$ cm^2. What is its perimeter?

   A. 48 cm **B.** 54 cm **C.** 42 cm **D.** 60 cm

9. Which one of the following statements is correct about a regular octagon?

   A. From one vertex we can draw 4 diagonals only.

   B. The sum of the measures of all its interior angles is $1080°$.

   C. The sum of the measures of all its central angles is $180°$.

   D. The measure of each exterior angle is $135°$.

10. Which one of the following is not correct about n-sided regular polygon?

   A. It has n lines of symmetry.

   B. The measure of each interior angle is $\frac{(n-2)\times 180°}{n}$.

   C. The measure of each central angle and each exterior angle is $\frac{360°}{n}$.

   D. The sum of the measure of exterior angles is $n \times 360°$.

11. If the measure of each exterior angle of a regular polygon is $40°$, then how many sides does this polygon have?

A. 7 **B.** 9 **C.** 8 **D.** 10

12. The perimeter of regular pentagon inscribed in a circle with radius 10 cm is equal to :

(use $\sin 36^\circ = 0.59$, $\cos 36^\circ = 0.81$, $\sin 72^\circ = 0.95$, $\cos 72^\circ = 0.31$)

A. 81 cm **B.** 50 cm **C.** 91 cm **D.** 59 cm

13. Is it possible to have a regular polygon with each of whose exterior angle is 50° ? Give reason to support your answer.

14. What is the perimeter of a square with a diagonal of $5\sqrt{2}$cm?

15. How many sides does a regular polygon have if the measure of an interior angle is 165°?

16. There are two regular polygons with number of sides equal to $(n - 1)$ and $(n + 2)$. Their exterior angles differ by 6°. Find the number of sides of the two polygons.

17. The figure (fig 6. 18) shows two regular pentagons and an equilateral triangle(ΔEDF). Determine the measure of $\angle AEI$.

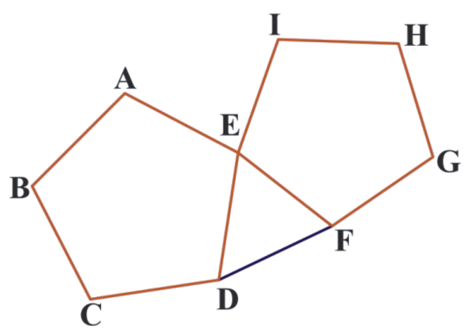


Figure 6.18

18. A regular polygon has a perimeter 143 unit and the sides are 11 unit long. How many sides does the polygon have?

19. Suppose an equilateral triangle is inscribed in a circle of radius 8 cm. Find the area of the region which is in a circle and outside the triangle.

(use $\pi \approx 3.14$, $\sqrt{3} \approx 1.73$)

20. Find the possible measure of minimum interior angles and maximum exterior angles in a regular polygon. Give a reason to support your answer.

21. The school pedagogical center is making a design of a regular octagonal stop sign (as shown in fig. 6.19) which will be placed at the main gate of the school to prevent car accidents. Each side of the sign is 40cm long. What is the area of the sign? (to the nearest integer)

Figure 6.19

22. A gardener has enough material for 660 meters of fence. She knows that a circle maximizes the area to perimeterration, but circular fence is difficult to construct. So, she wants to simulate a circle by constructing a fence in a shape of a regular polygon with as many sides as possible but with more than 5 and not more than 15 sides. If she wishes to make each side a length of an even meter, and she wishes to have no more than 2 meter left over, which regular polygon would you suggest she build, and how long would each side be?

A. Octagon; 83 m **B.** 13-gon; 51 m **C.** 15-gon; 44 m **D.** 11-gon; 60 m