UNIT 8

VECTORS IN TWO DIMENSIONS

Unit Outcomes

By the end of this unit, you will be able to:

- **Conceptualize vectors in the sense of direction and magnitude symbolically.**
- **Perform operations on vectors.**

Unit Contents

KEY Words

- direction of a vector
- equality of vectors
- parallel vectors
- addition of vectors
- triangle law of vector addition
- parallelogram law of vector addition
- scalar quantities
- subtraction of vectors
- vector quantities
- magnitude of a vector
- position vector

INTRODUCTION

In your day to day activities, you have an experience of measurements of different physical quantities. In the process of measuring physical quantities, you may express what you measure by using real numbers with appropriate unit. For example, the dimension of your class room may be 6 meter by 8 meter, the average speed of a car is 45 km/hr, the area of a football field is 5,000 m^2, the average annual temperature of Dallol is 34.6° C, the amount of water a bottle contains 1.5 Liter, etc. In each of these measurements of quantities, direction is not involved.

On the other hand, there are physical quantities which need the direction with its magnitude. For example: A boy is riding a bike with a velocity of 30 km/hr in the north-east direction. Here if we want to define the velocity, we need two things, that is, the magnitude of the velocity (30 km/hr) and its direction (north-east).

In this unit, you will learn scalar and vector quantities, representation of vectors, addition and subtraction of vectors, scar multiplication of vectors and some application of vectors in two dimensions.

Activity 8.1

a. List at least 6 physical quantities.

b. What common property and difference you observed among them?

8.1 Vector and Scalar Quantities

Definition 8.1

Scalars are quantities that are fully described by a magnitude (or numerical value) alone.

Example 1

The mass of a stone is 10 kg. Here, the mass is represented by a single number (10) with appropriate measuring unit(kg) . No direction is needed to represent mass of an object. Hence, mass is a scalar quantity.

Example 2

'The density of pure water is 1 g/cm^3 '. Density is represented by a single number so that it is a scalar quantity.

Example 3

' Temprature of a room is $27°C$ '. Temprature is a scalar quantity.

Definition 8.2

Vectors are quantities that are fully described by both a magnitude and a direction.

There are many engineering applications where vectors are important. Force, acceleration, velocity, electric and magnetic fields are vector quantities and all are described by vectors.

If a projectile is projected up in the atmosphere, its position is described by a vector. Furthermore, computer software is used to control the position of a robot, and in this case the position is also described by a vector.

Example 4

The force action causes an object to move or speed up. When the object shown in figure 8.1 is moved by applying a force, we achieve different effects depending on the direction of the force.

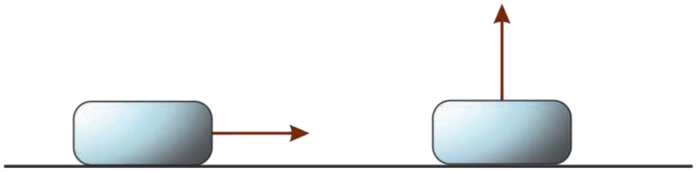


Figure 8.1

In order to specify the force completely, we must state not only its magnitude (its 'strength') but also the direction in which the force acts. For example, we might state that 'a force of 10 Newton's is applied vertically upward (Figure 8.1 of the second part). Clearly this force would have a different effect from one applied horizontally (Figure 8.1 of the first part). The direction in which the force acts is crucial. This indicates that force is a vector quantity.

Example 5

Displacement and *distance* are related quantities which people use interchangeably. However, in more precise language, they are not the same. Distance is a *scalar* whereas displacement is 'directed distance', that is, distance together with a specified direction so that it is a *vector.*

Figure 8.2

So, referring to figure 8.2. Suppose $AB = 20$ meters. We can state that the distance moved is 20 meters, but the displacement is 20 meters in the direction from A to B.

Example 6

Explain the vectors in the following figures

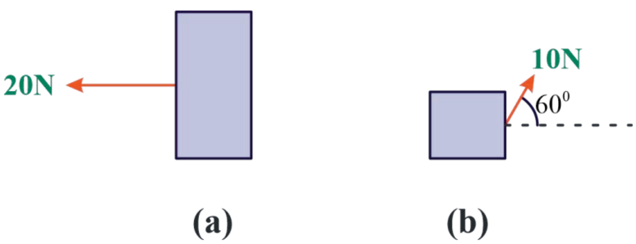


Figure 8.3

Solution:

a. The figure 8.3 (a) shows an object is pulled by a force of 20N horizontally. It is represented by a vector with magnitude 20 N to the left.

b. The figure 8.3 (b) shows an object is pulled by a force of 10 N at an angle of 60^0 with the horizontal.

Exercise 8.1

1. Which statement describes a vector?

   A. It has direction but no magnitude B. It has constant magnitude but no direction

   C. It has magnitude but no direction D. It has both magnitude and direction

2. Scalar quantities are completely described by their____________.

   A. area B. unit C. magnitude D. direction

3. Consider the following quantities and identify whether each is a scalar quantity or a vector quantity.

   i) Amount of rainfall in mm
   ii) Temperature in a room
   iii) Acceleration of a car
   iv) Area of a rectangle
   v) The speed of a car
   vi) Velocity of the boat
   vii) The height of a building
   viii) The weight of the body

8.2 Representation of a Vector

In this sub unit, we consider the vector represented by a directed line segment on a plane. This is called a vector on a plane.

How do we represent a vector?

A vector can be represented by either algebraically or geometrically. A vector is represented geometrically by a directed line segment (a line segment with direction).

In figure 8.4, the vector is represented by a directed line segment and we denote this by an arrow as $\overrightarrow{AB}$. In this case, the point A is called the *initial* and the point B is called the *terminal*

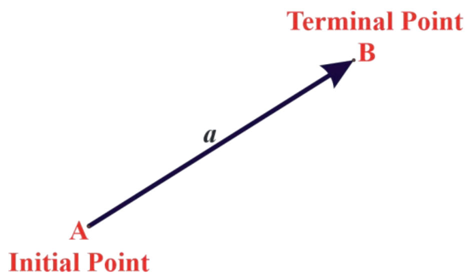


Figure 8.4

A vector also represented by using letters with an arrow bar over it such as $\vec{a}, \vec{u}, \vec{v}$ or a bold small letters like $\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}$ etc. Hence, the above vector $\overrightarrow{AB}$ can also be represented by $\vec{u}$.

Equality of vectors

Two vectors are said to be equal if and only if they have the same magnitude and direction.

In figure 8.5, the two vectors have the same magnitude (length) and the same direction.

Hence, they are equal vectors. In this case we can write this as $\overrightarrow{AB} = \overrightarrow{CD}$, or $\boldsymbol{a} = \boldsymbol{b}$.

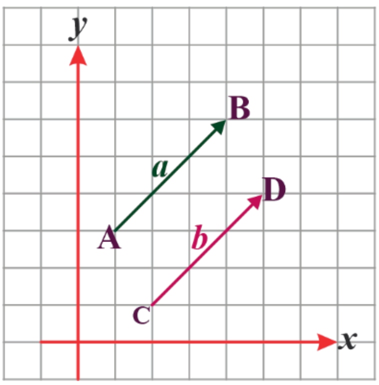


Figure 8.5

Opposite of a vector

The vector which has the same magnitude as that of a vector $\overrightarrow{AB}$, but opposite in direction is called oppositve vector of $\overrightarrow{AB}$ and is denoted by $-\overrightarrow{AB}$. From figure 8.6, you can observe that one vector is the negative of the other. Thus, $\boldsymbol{a} = -(-\boldsymbol{a})$ as shown in the first part of figure 8.6 and $\overrightarrow{AB} = -\overrightarrow{CD}$ as shown in the second part of figure 8.6.

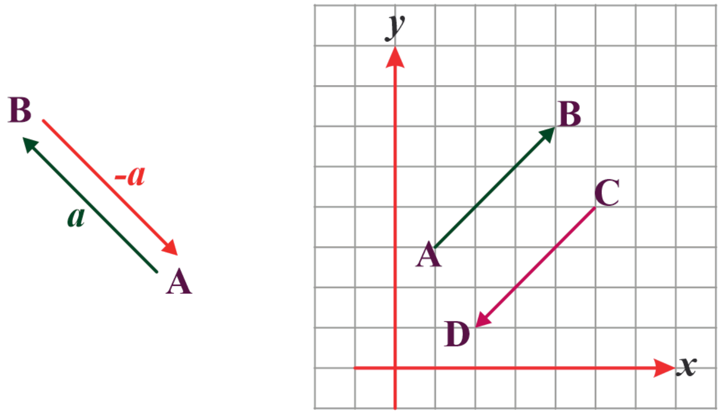


Figure 8.6

Parallel vectors

Vectors that have the same or opposite direction are called parallel vectors.

Example

Identify parallel, equal and opposite vectors from figure 8.7 below.

(a) *(b)*

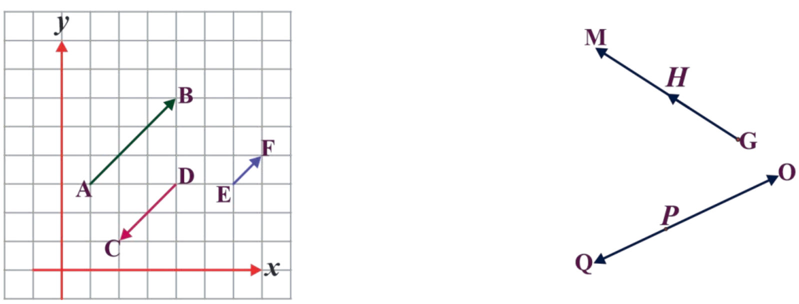


Figure 8.7

Solution::

a. $\overrightarrow{AB}$ and $\overrightarrow{EF}$ are in the same direction, $\overrightarrow{DC}$ is in opposite direction to $\overrightarrow{AB}$ and $\overrightarrow{EF}$.

Hence, these vectors $\overrightarrow{AB}$, $\overrightarrow{EF}$ and $\overrightarrow{DC}$ are parallel to each other.

There is no equal and opposite of a vector in the given figure.

b. $\overrightarrow{PO}$ and $\overrightarrow{PQ}$ are in opposite directions but on the same line, so they are parallel.

Vectors $\overrightarrow{GH}$ and $\overrightarrow{GM}$ are also parallel since they are vectors in the same direction. There is no vector which is opposite of another in the given figure. $\overrightarrow{GH} = \overrightarrow{HM}$ since they have the same direction and equal length(use a ruler to check equality of length).

Exercise 8.2

1. In the figure 8.8, identify the following with reference to ***a*.**
   a. Equal vector to $\vec{a}$
   b. Opposite to $\vec{a}$
   c. Parallel to $\vec{a}$

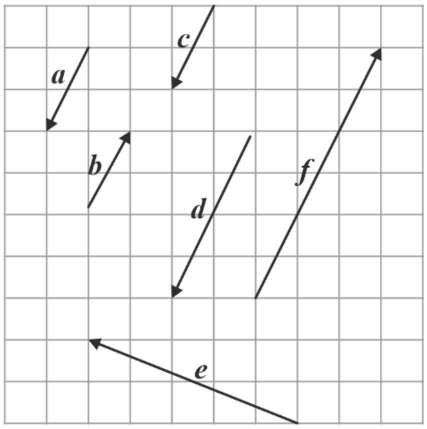


Figure 8.8

2. Write true if the statement is correct and false otherwise.
   a. Two parallel vectors may have common point.
   b. Parallel vectors have the same magnitude.
   c. The magnitude of a vector does not depend on its direction.

Column vector in two dimension

Any vector $\boldsymbol{a}$ can be represented by a column vector $\boldsymbol{a} = \begin{pmatrix} x \\ y \end{pmatrix}$, where x is the horizontal and y is the vertical component of ***a*.** For example, in the figure 8.9 alongside, the illustrated vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is denoted by $\boldsymbol{a}$ or $\vec{a}$.

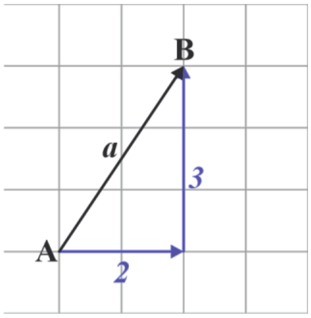


Figure 8.9

We write $\boldsymbol{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ or $\vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Magnitude of a vector

The magnitude of a given vector $\overrightarrow{AB}$ or $\vec{a}$ as shown in figure 8.9 is the length of the line segment from its initial point A to terminal point B. It is denoted by as $|\overrightarrow{AB}|$ or $|\vec{a}|$.

Example 1

Find the magnitude of the vector $\vec{u} = \binom{4}{3}$.

Solution:

$$|\vec{u}|^2 = 4^2 + 3^2 \quad \text{(Pythagoras theorem)}$$
$$= 16 + 9$$
$$= 25$$

Therefore, $|\vec{u}| = |\boldsymbol{u}| = \sqrt{25}$.

Hence, the magnitude of the vector $|\vec{u}|$ is 5 unit.

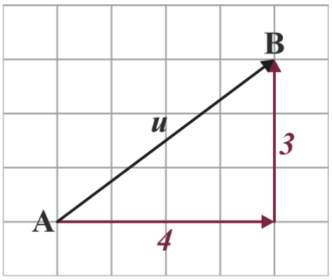


Figure 8.10

Note

- If a vector is represented on a plane sheet of paper, we can determine its magnitude (length) by measuring its length using a ruler and express it with appropriate unit.
- Any vector $\boldsymbol{a} = \binom{x}{y}$ has a magnitude (length) $|\boldsymbol{a}| = \sqrt{x^2 + y^2}$.

Example 2

Find the magnitude of each of the following vectors on the coordinate system.

Solution:

How do we get the magnitude of each vector on the coordinate system? We use distance formula. Therefore, $|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, where $P = (x_1, y_1)$, and $Q = (x_2, y_2)$ are points on the coordinate system. There are three vectors $\overrightarrow{AB}, \overrightarrow{CD}, \overrightarrow{EF}$ in figure 8.11. So, using this distance formula,

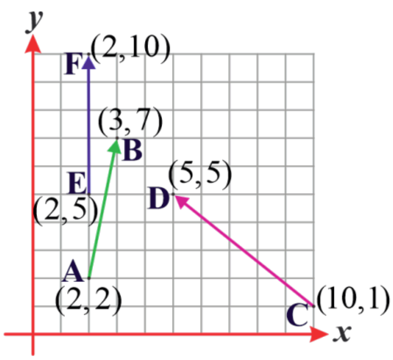


Figure 8.11

$$|\overrightarrow{AB}| = \sqrt{(3-2)^2 + (7-2)^2}$$
$$= \sqrt{1+25}$$
$$= \sqrt{26}.$$
$$|\overrightarrow{CD}| = \sqrt{(5-10)^2 + (5-1)^2}$$
$$= \sqrt{25+16}$$
$$= \sqrt{41}.$$

Similarly, $|\overrightarrow{EF}| = \sqrt{(2-2)^2 + (10-5)^2}$
$$= \sqrt{25}$$
$$= 5\ .$$

Exercise 8.3

1. Answer each of the following questions about vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ in the figure 8.12. (Hint: count square grids)
   a. Express each vector in column vector form.
   b. Find the magnitude of each vector.

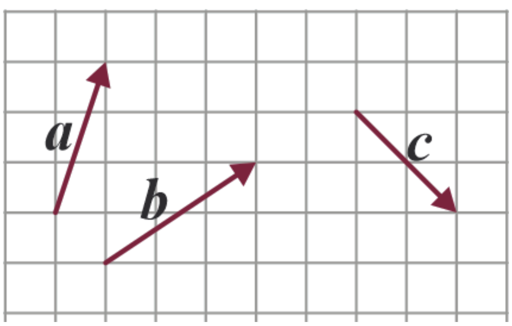


Figure 8.12

2. Find the magnitude of $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ in the figure 8.13.

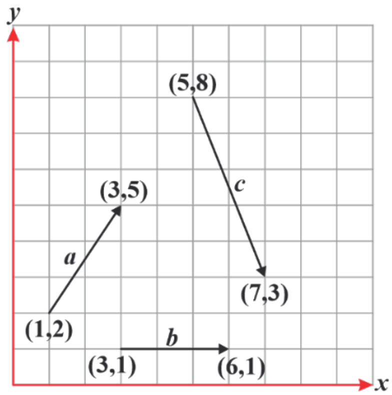


Figure 8.13

Direction of vectors

On a plane, the direction of a vector is given by the angle that the vector makes with a reference direction, often an angle with the horizontal or with the vertical.

Expressing vector direction always starts from north/south then angle measure finally east/west.

Example

The direction of the vector $\overrightarrow{AB}$ from the horizontal axis is as shown in figure 8.14.

Let y-axis is North; and x-axis is East, the complement angle of 37° is 53°.

Thus, we can say the direction of B is N 53° E.

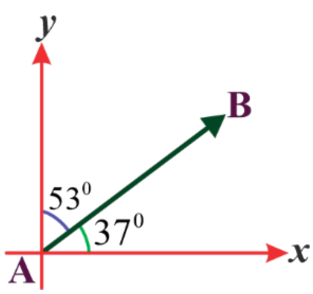


Figure 8.14

Exercise 8.4

1. Locate the vector on the coordinate system whose initial point is $A = (1,1)$ and its terminal point is $B = (5,5)$.

   a. Find the magnitude of $\overrightarrow{AB}$.

   b. Using protractor check that the angle between $\overrightarrow{AB}$ and the horizontal line passing through A is 45°.

c. What is the direction of $\overrightarrow{AB}$?

2. Locate each of the following on the coordinate system.

a. $\overrightarrow{AB}$ where point $A = (1,2)$, has length 3 cm in the direction N 60° E.

b. $\overrightarrow{OR}$ where O is the origin and it is 5 cm to the North.

c. $\overrightarrow{OS}$ whose initial is at the origin and has a length 4 cm in the direction S 30° E .

8.3 Vector Operations

In section 8.1 and section 8.2, you have studied the definition of a vector, different types of vectors and its representation. In this section, you will learn about operations on vectors.

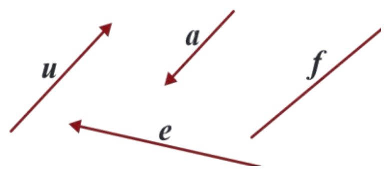
8.3.1 Addition of vectors

Activity 8.2

1. Consider the following vectors

a. Is it possible to add vectors like $\boldsymbol{a}$ and $\boldsymbol{e}$?

b. Can you subtract $\boldsymbol{e}$ from $\boldsymbol{f}$?



2. Suppose a student start walking from her home to school which is located at a distance of 800 m to the East. Then she went to the market 400 m far from the school to the North. Finally, she returned to her home.

a. Discuss the student's journey using vector representation.

b. What is the total distance covered by the student?

Your discussion in the activity leads you to the importance of addition and subtraction of vectors. There are two laws of addition of vectors:

i) Triangle law of addition of vectors

Consider two vectors as given in figure 8.15.

Now we need to add these two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. To get their sum, $\boldsymbol{v}$ is translated, keeping its direction and length unchanged, until its initial coincides with the terminal point of $\boldsymbol{u}$.

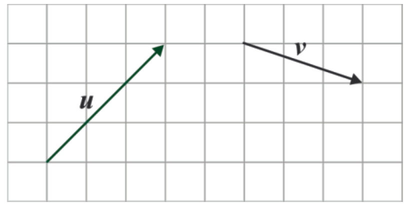


Figure 8.15

Then, the sum of $\boldsymbol{u} + \boldsymbol{v}$ is defined by the vector represented by the third side of the completed triangle, that is $\boldsymbol{w}$ as shown in figure 8.16.

Note that, from figure 8.16, we can write $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ since going along vector $\boldsymbol{u}$ and then along vector $\boldsymbol{v}$ is equivalent to going along vector $\boldsymbol{w}$.

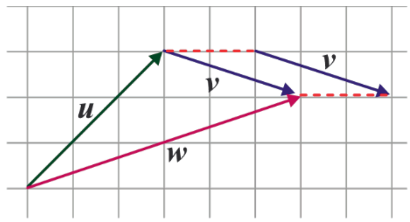
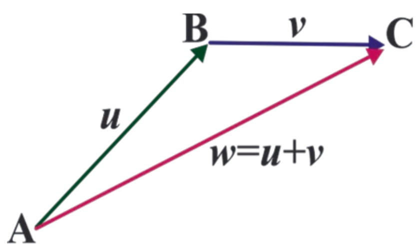


Figure 8.16

Definition 8.3 Triangle law of vector addition

Consider two vectors $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{BC} = \vec{v}$ in a coordinate system. The sum of $\overrightarrow{AB} + \overrightarrow{BC} = \vec{u} + \vec{v}$ is a directed line segment connecting A to C say $\vec{w} = \overrightarrow{AC}$ such that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ or $\vec{u} + \vec{v} = \vec{w}$. Such a vector $\overrightarrow{AC}$ is called a ***resultant vector***.



Addition of vectors can be done graphically, and addition of vector can be done using components. Now, we will find the sum of vectors by graphical method. The second method will be presented after section 8.4.

Example

By using figure 8.17 , determine each of the following.

a. $\overrightarrow{FH} + \overrightarrow{HG}$

b. Compare $\overrightarrow{EF} + \overrightarrow{FG}$ and $\overrightarrow{GH} + \overrightarrow{HE}$

Solution:

a. $\overrightarrow{FH} + \overrightarrow{HG} = \overrightarrow{FG}$

b. $\overrightarrow{EF} + \overrightarrow{FG} = \overrightarrow{EG}$ and $\overrightarrow{GH} + \overrightarrow{HE} = \overrightarrow{GE}$, the two vectors are parallel and opposite.

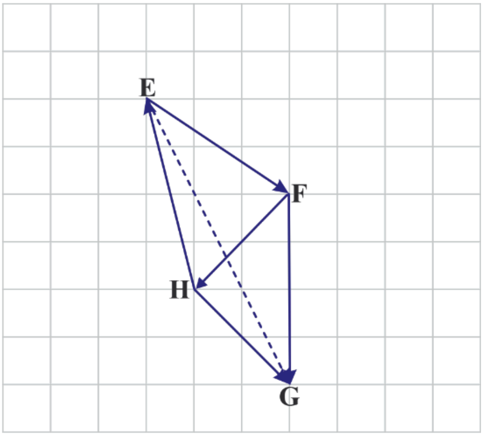
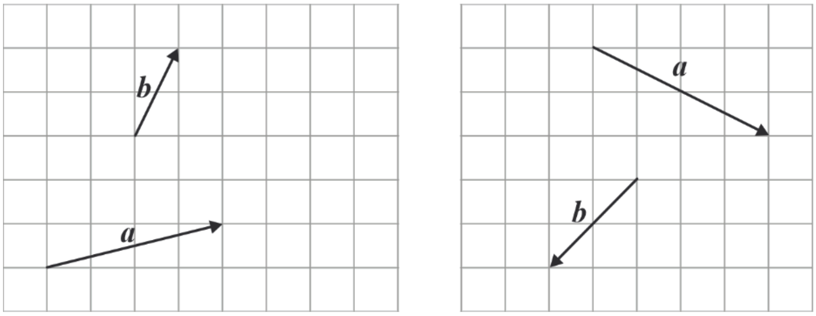


Figure 8.17

Exercise 8.5

1. Draw the vectors $\boldsymbol{a} + \boldsymbol{b}$ using the triangle law.

(a) *(b)*

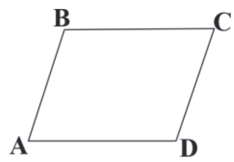


ii) Parallelogram law of vector addition:

For two or more free vectors with different initial points, you have learned how to determine the resultant vector (sum of two vectors) by joining the initial point of the first vector to the terminal point of the second. But, there are co-initial vectors where you may be asked to determine their sum.

Activity 8.3

1. What is a parallelogram?
2. What are the properties of corresponding sides of parallelogram?



Let us consider two vectors with the same initial point (A) and make a non-zero angle between them as shown in figure 8.18.

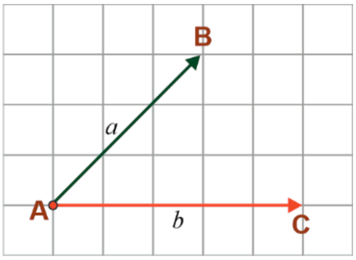


Figure 8.18

Can you construct a parallelogram using these two intersecting vectors?

Recall, the definition of equal vectors. Construct a broken line passing through B and parallel to vector $\vec{b}$. Similarly construct a broken line through C and parallel to vector $\vec{a}$. The two broken lines intersect at a point, say D, as shown in figure 8.19.

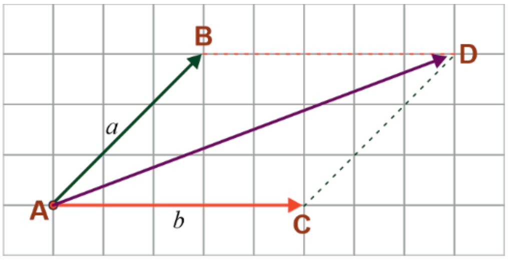


Figure 8.19

Did you observe a parallelogram and two pairs of equal vectors in this figure?

$\overrightarrow{AB}$ & $\overrightarrow{CD}$, $\overrightarrow{AC}$ & $\overrightarrow{BD}$ are equal vectors. By triangle law of vector addition, the resultant vector will be $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$. That is, $\boldsymbol{a} + \boldsymbol{b} = \overrightarrow{AD}$. This method of determining the sum of vectors is called ***parallelogram law of vector addition***.

Note

If two vectors have different initial points, you need to bring the two initial points placed at a point by construction before applying parallelogram method.

Subtraction of vectors

How do you subtract a vector from another vector?

Subtraction of vectors can be treated as addition of vector and a negative vector. Consider two vectors which are to be subtracted as shown in figure 8.20. To determine the resultant vector (the difference of the two vectors) , we follow the following procedures.

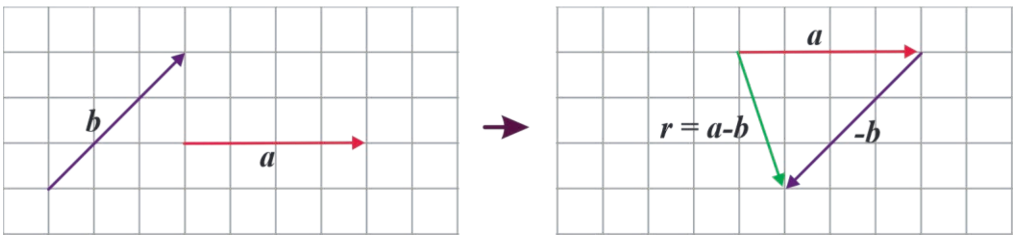
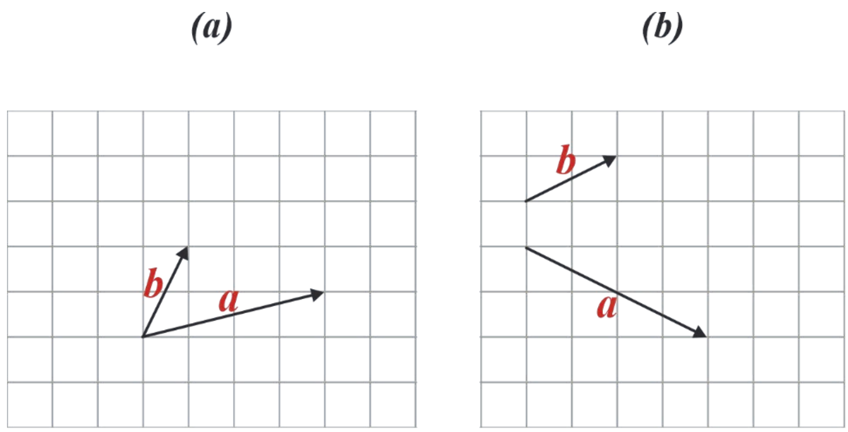


Figure 8.20

- The first vector is drawn with proper scale and in a given direction.
- Then, from the terminal point of the first vector ($\vec{a}$), a vector is drown with the same scale and in the opposite direction of the second vector (-$\vec{b}$).
- Then, the vector joining the initial point of the first vector and the terminal pint of the second vector represent the resultant vector ($\vec{r}$) with direction and magnitude.

Exercise 8.6

1. Draw the vectors $\boldsymbol{a}+\boldsymbol{b}$ and $\boldsymbol{a}-\boldsymbol{b}$ using the parallelogram law or the triangle law of vector addition.



Addition and subtraction of column vectors

For two vectors in component form $\begin{pmatrix} p \\ q \end{pmatrix}$ and $\begin{pmatrix} r \\ s \end{pmatrix}$, we have

$$\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p+r \\ q+s \end{pmatrix} \text{ and } \begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p-r \\ q-s \end{pmatrix}.$$

Example 1

Consider column vectors $\boldsymbol{a} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ .Then find

a) $\boldsymbol{a}+\boldsymbol{b}$ **b)** $\boldsymbol{a}-\boldsymbol{b}$

Solution:

a. $\boldsymbol{a}+\boldsymbol{b} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} -4+5 \\ -3+0 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. This can also be observed from the following figure 8.21.

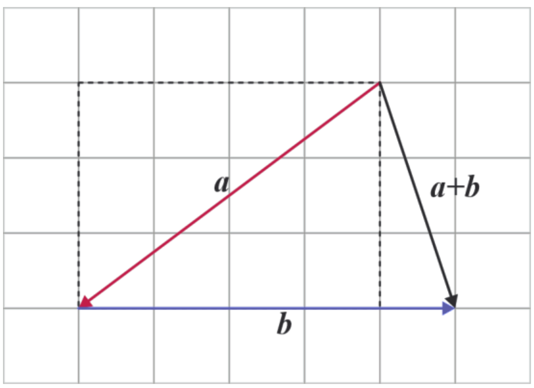


Figure 8.21

b. $\boldsymbol{a}-\boldsymbol{b}=\binom{-4}{-3}-\binom{5}{0}=\binom{-4-5}{-3-0}=\binom{-9}{-3}$ as shown in figure 8.22.

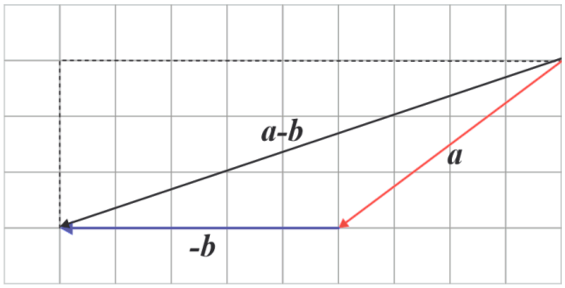


Figure 8.22

Example 2

Consider the quadrilateral as shown in the following figure 8. 23. Given that $\overrightarrow{AB}=\boldsymbol{a}, \overrightarrow{DC}=\boldsymbol{b}$ and $\overrightarrow{DB}=\boldsymbol{c}$, then find the following in terms of $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$.

i) $\overrightarrow{AC}$ ii) $\overrightarrow{AB}$ iii) $\overrightarrow{BC}$

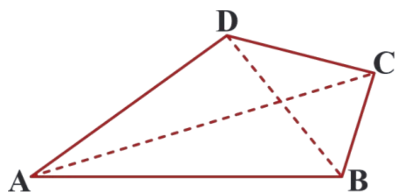


Figure 8.23

Solution:

i. From the figure $\overrightarrow{AC}=\boldsymbol{a}+\boldsymbol{b}$

ii. $\overrightarrow{AB}=\boldsymbol{a}+\boldsymbol{c}$

iii. $\overrightarrow{BC}=\boldsymbol{b}-\boldsymbol{c}$

Exercise 8.7

1. For the vectors $\boldsymbol{a}=\binom{1}{3}$, $\boldsymbol{b}=\binom{2}{-1}$ and $\boldsymbol{c}=\binom{-4}{5}$, find

a) $\boldsymbol{a}+\boldsymbol{b}$ b) $\boldsymbol{a}-\boldsymbol{b}$

c) $\boldsymbol{a}+\boldsymbol{c}$ d) $\boldsymbol{b}-\boldsymbol{c}$

e) $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$

2. $ABCD$ is a rhombus , $\triangle DBC$,$\triangle ABD$ and $\triangle BCE$ are equilateral triangles.

Suppose $\overrightarrow{AD}=\boldsymbol{a}$, $\overrightarrow{AB}=\boldsymbol{b}$ and $\overrightarrow{BD}=\boldsymbol{c}$.

Find each in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$

i) $\overrightarrow{AC}$ ii) $\overrightarrow{ED}$

iii) $\overrightarrow{BC}$ iv) $\overrightarrow{CE}$

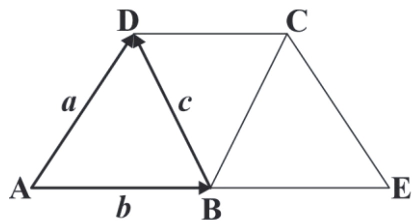


Figure 8.24

3. Using the following figure 8.25,
   a. Write $\overrightarrow{AC}$ as a sum of two vector.
   b. Describe $\overrightarrow{BD}$ as a difference of two vectors.
   c. What is the sum of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CD}$?

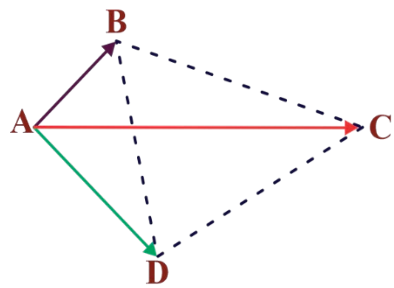


Figure 8.25

4. Given $ABDC$ is a parallelogram
   a. By construction show that $\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB}$.
   b. Construct the diagonals of the parallelogram and show that these vectors are the sum and difference of the adjacent side vectors of the parallelogram

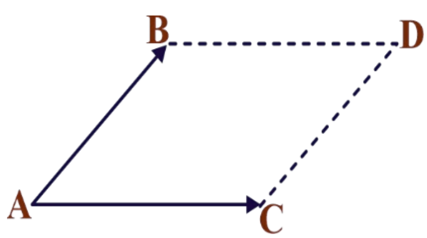


Figure 8.26

8.3.2 Multiplication of a vector by a scalar

In the previous section, you have studied how to get the sum and difference of vectors geometrically. In this section, you will learn scalar multiplication of vectors

Activity 8.4

Using figure 8.27 below, express vectors $\boldsymbol{v}, \boldsymbol{m}, \boldsymbol{n}$ and $\boldsymbol{p}$ in terms of a vector $\boldsymbol{u}$.

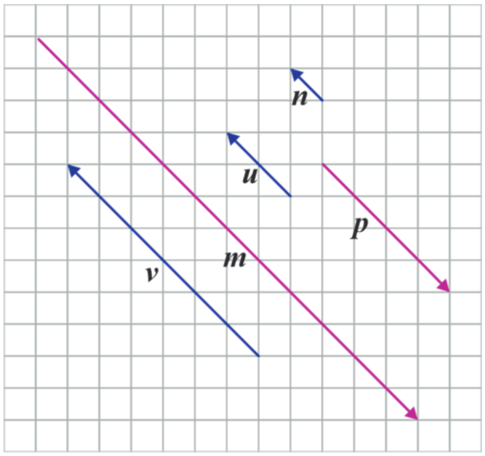


Figure 8.27

What do you conclude from your activity? All the given vectors are parallel and you can express the length of each vector interms of the length of $\boldsymbol{u}$. For example, $\boldsymbol{v}$ is in the same direction as $\boldsymbol{u}$ and its magntude is 3 times that of length of $\boldsymbol{u}$.Hence, it is possible to write $\boldsymbol{v} = \boldsymbol{3u}$.

Definition 8.4

Let $\overrightarrow{AB}$ be any given vector and k be any real number. The scalar multiple vector $k\overrightarrow{AB}$ is the vector whose magnitude (length) is k times the magnitude of $\overrightarrow{AB}$ and

i. the direction of $k\overrightarrow{AB}$ is the same as the direction of $\overrightarrow{AB}$, if $k > 0$,

ii. the direction of $k\overrightarrow{AB}$ is the opposite of $\overrightarrow{AB}$, if $k < 0$.

Example

Given a vector $\overrightarrow{AB}$ as shown in Figure 8.28 . Draw a vector $2\overrightarrow{AB}$, $\frac{1}{2}\overrightarrow{AB}$, $-2\overrightarrow{AB}$ and $\frac{-3}{2}\overrightarrow{AB}$.

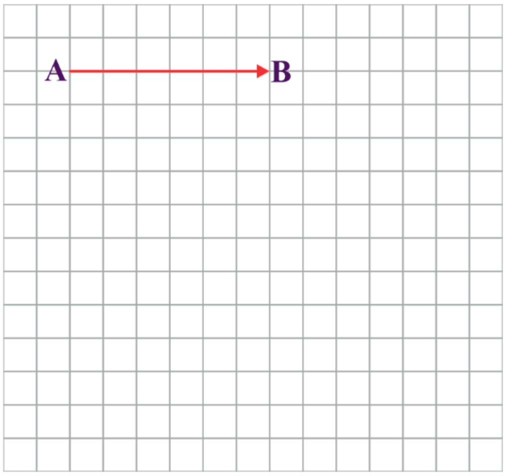


Figure 8.28

Solution:

Using definition 8.4, we can draw the required vectors as follows.

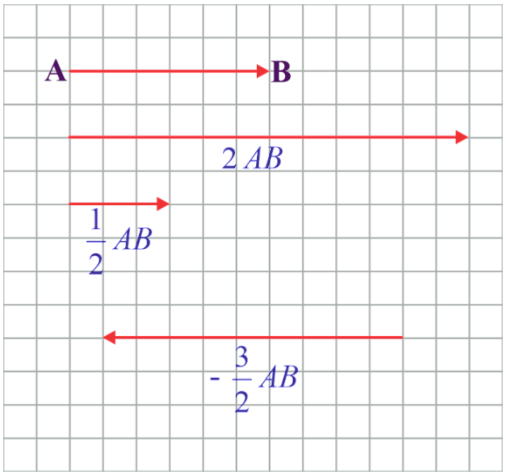


Figure 8.29

Note

Two vectors are said to be parallel if one can be written as a scalar multiple of the other. If $\vec{b}$ is parallel to $\vec{a}$, then it can be expressed as $\vec{b} = k\vec{a}$.

Exercise 8.8

1. Given $\overrightarrow{AB}$ as shown in figure 8.30, draw vector

   a. $2\overrightarrow{AB}$ b. $\frac{1}{2}\overrightarrow{AB}$ c. $-3\overrightarrow{AB}$

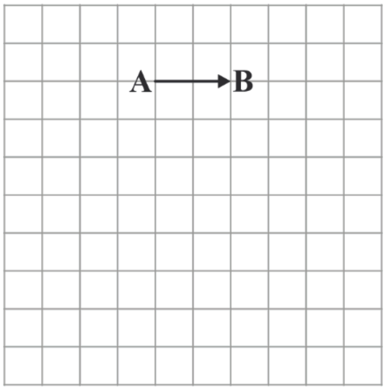


Figure 8.30

Example 2

Consider column vector $\boldsymbol{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.Then find $-\boldsymbol{a}$, $\mathbf{2}\boldsymbol{a}$ and $\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{a}$

Solution:

For the vector in component form $\begin{pmatrix} a \\ b \end{pmatrix}$,

$k\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix}$ where k is a scalar.

So that for $\boldsymbol{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $-\boldsymbol{a} = -1\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$,

$\mathbf{2}\boldsymbol{a} = 2\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$ and $\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{a} = \frac{1}{2}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. This is illustrated in the following Figure 8.31. From the figure, you can observe that

$\boldsymbol{a}, -\boldsymbol{a}$, $\mathbf{2}\boldsymbol{a}$ and $\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{a}$ are parallel to each other.

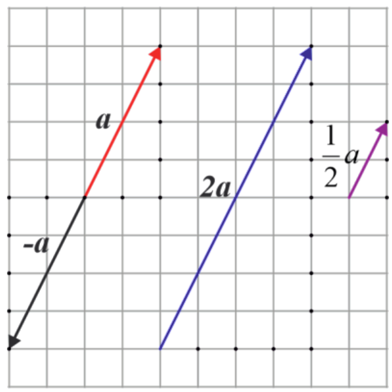


Figure 8.31

Exercise 8.9

1. Given vectors $\boldsymbol{a} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$. Then find

   a. $4\boldsymbol{a}$ b. $-3\boldsymbol{a}$ c. $\frac{1}{3}\boldsymbol{b}$ d. $-\frac{1}{3}\boldsymbol{b}$

2. Write true if the statement is correct and false otherwise.

   a. A vector is a scalar multiple of its opposite vector.

   b. For a given vector $\vec{u}$ and scalar k, $k\vec{u}$ has larger magnitude than $\vec{u}$ for $k > 0$.

c. If the magnitude of two vectors is the same, then they have the same direction.

d. A vector is parallel to itself.

3. From the following figure 8.32, describe vectors $\overrightarrow{AC}, \overrightarrow{AD}, \overrightarrow{AF}$ and $\overrightarrow{AG}$, each vector as a scalar multiple of $\overrightarrow{AB}$.

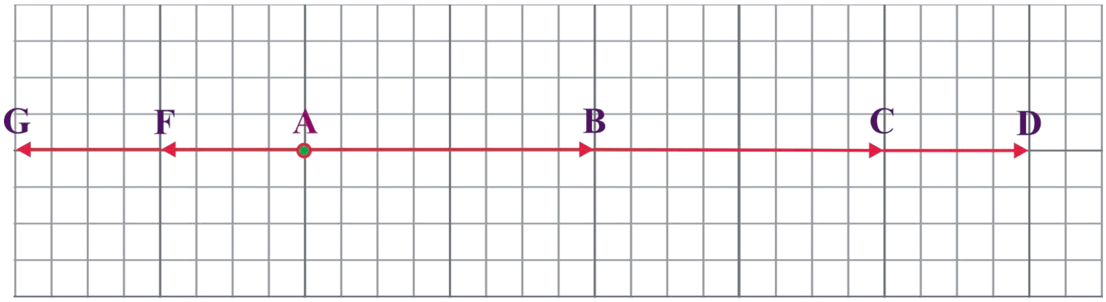


Figure 8.32

8.4 Position Vector

In the previous 3 sections you practiced how to represent vectors geometrically and vector operations. In this section you will discuss representing a vector with a fixed initial point and the use of component of a vector to determine its magnitude and position.

Activity 8.5

1. On the coordinate system, there are different vectors as shown in figure 8.33.
   a. What can you say about the two vectors q and p?
   b. What can you say about vectors v and $\vec{w}$?
   c. Is it possible to bring (move) the initial of r to the initial of v ?
   d. What makes vectors v and $\vec{w}$ differ from other vectors?

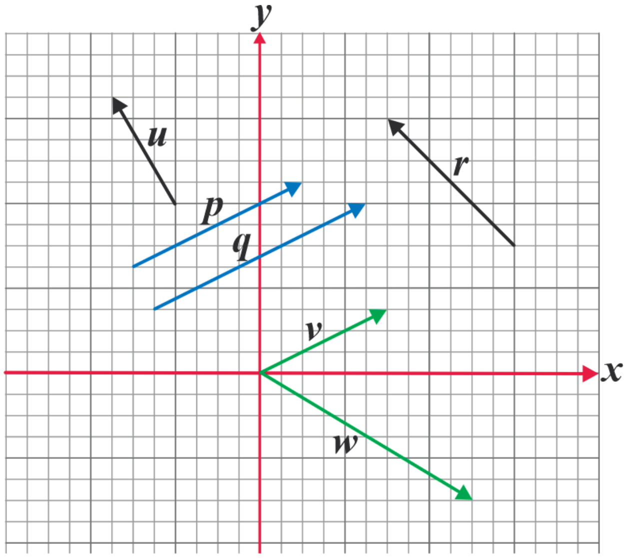


Figure 8.33

2. Suppose $\overrightarrow{AB}$ be a vector with initial point at $A = (1,2)$ and the terminal point $B = (3,5)$. If the initial point moves to the point $O = (0,0)$.
   a. What will be the location of the terminal point (say, point C)?
   b. What is the relationship between $\overrightarrow{AB}$ and $\overrightarrow{OC}$?

From your discussion in the activity, you have seen how to move the initial point of a vector to the origin (a point on a plane with order pair $O = (0,0)$).

Now, let us define what a position vectors mean.

Definition 8.5

Vectors that start at the origin (O) are called position vectors.

The position vectors are also said to be vectors which are written in the standard form. These are used to determine the position of a point with reference to the origin.

How can we determine the position vector of any vector with initial point $A = (x_1, y_1)$ and terminal point $B = (x_2, y_2)$?

To find the position vector corresponding to vector $\overrightarrow{AB}$, subtract the corresponding components of A from B as $(x_2 - x_1, y_2 - y_1)$. Let this point be $P = (x,\ y)$. Now, we have two equal vectors $\overrightarrow{AB}$ *and* $\overrightarrow{OP}$ with the same component, where $\overrightarrow{OP}$ is the position vector of $\overrightarrow{AB}$.

Let $(x_2 - x_1,\ y_2 - y_1) = (x, y)$, so that $\overrightarrow{OP}$ has components x and y as shown in the following figure 8.34.

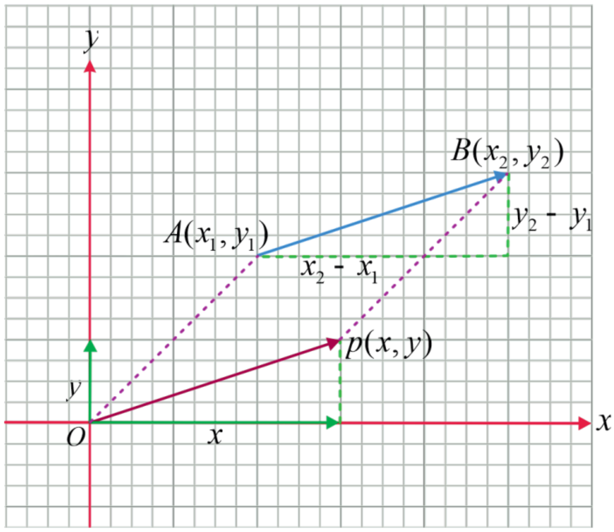


Figure 8.34

Notation

A position vector $\overrightarrow{OP}$ can be represented by $\overrightarrow{OP} = \vec{u} = \binom{x}{y}$, $\vec{u}$ is expressed as a sum of two vectors $\vec{x}$ and $\vec{y}$ as $\vec{u} = \vec{x} + \vec{y}$ from triangle or parallelogram law where $\vec{x} = x\vec{\imath}$ *and* $\vec{y} = y\vec{\jmath}$. The vectors $\vec{\imath} = \binom{1}{0}$ *and* $\vec{\jmath} = \binom{0}{1}$ are vectors with 1 unit long in the x and y directions, respectively. Hence, the given vector $\vec{u}$ can be written as $\vec{u} = x\vec{\imath} + y\vec{\jmath}$, in this case x and y are called components of $\vec{u}$.

The magnitude of the position vector $|\vec{u}| = \sqrt{x^2 + y^2}$.

Example

Consider the position vector $\vec{u} = \binom{3}{4}$.

a. Draw this vector on the coordinate plane.

b. Find $|\vec{u}|$.

c. Determine the direction of $\vec{u}$.

Solution:

a. Since this is a position vector, it starts from the origin to its terminal point (3,4). It can be written as $\vec{u} = 3\vec{\imath} + 4\vec{\jmath}$. This is to mean, move 3 units to the right and then 4 units up as shown in figure 8.34.

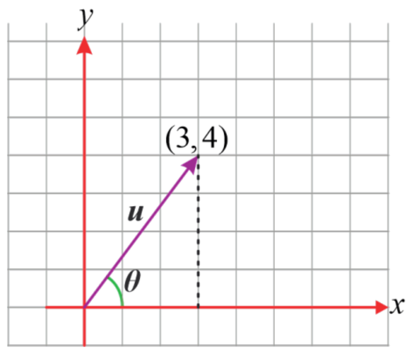


Figure 8.35

b. The magnitude of the vector is

$$\sqrt{3^2 + 4^2} = \sqrt{25} = 5.$$

c. By definition $\tan\theta = \frac{4}{3}$. From trigonometric table, we can read that $\theta \approx 53°$.

Exercise 8.10

1. Draw the following vector on a coordinate plane. Then, find the magnitude.

   a. $\vec{u} = \binom{2}{3}$ b. $\vec{v} = \binom{-1}{2}$ c. $\vec{w} = \binom{4}{-3}$ d. $\vec{n} = \binom{-1}{-5}$

Example

Determine the position vector formed by the given initial and terminal point.

a. Initial point $(1,3)$, terminal pint $(2,2)$.

b. Initial point $(-1,-2)$, terminal point $(3,1)$.

Solution:

a. The position vector formed by the given initial point $(1,3)$ and terminal point $(1,2)$ is $(2-1, 2-3) = (1,-1)$. This position vector can be written as $\vec{u} = 1\vec{\imath} - 1\vec{\jmath}$ which

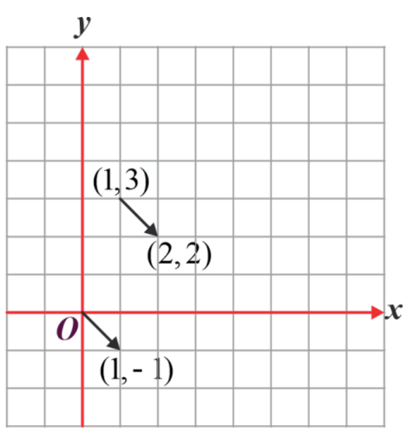


Figure 8.36 (a)

could be represented as $\vec{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. (See figure 8.36 (a)).

b. The position vector formed by the given initial point $(-1, -2)$ and terminal point $(3,1)$ will be $(3+1, 1+2) = (4,3)$. This position vector can be written as $\vec{u} = 4\vec{i} + 3\vec{j}$ which can be written as $\vec{u} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. (see figure 8.36 (b)).

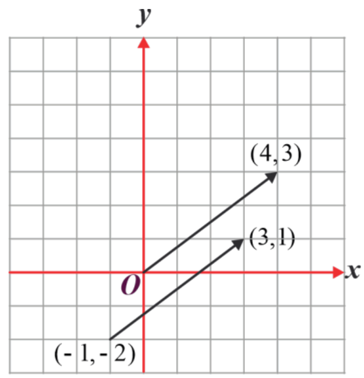


Figure 8.36 (b)

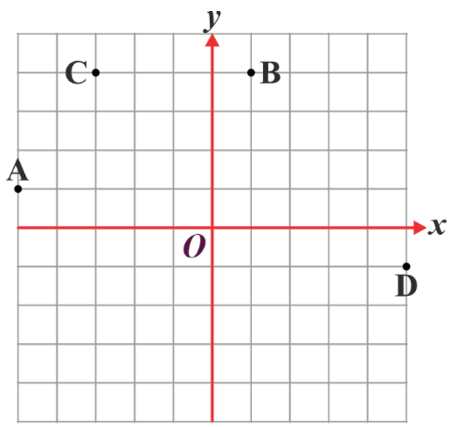
Exercise 8.11

Given points $A = (-5,1)$, $B = (1,4)$, $C = (-3,4)$ *and* $D = (5,-1)$ be points on the coordinate system.

1. i. Find the position vector whose initial point is A and terminal point B.

   ii. Find the position vector whose initial point is C and terminal point is D.

2. Determine the magnitude and direction of $\overrightarrow{BC}$ and $\overrightarrow{AB}$ using the components of the position vectors corresponding to the given vectors.



8.5 Applications of Vectors in Two Dimensions

Until now, you have studied vector operation and expressed a vector as a position vector and determining its components. In this section, we will see some application of vectors in two dimensions.

Example 1

Suppose a boat start moving on the sea 8 km to the South and then 8 km to the West. What is the displacement of the boat from its starting position to its last destination?

Solution:

Let the starting point of the boat be A and the first destination be B, and its last destination be C as shown in figure 8.37.

The magnitude $AC = \sqrt{(AB)^2 + (BC)^2}$

$= \sqrt{64 + 64} = 8\sqrt{2}$

and $\tan\alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{8}{8} = 1.$

So that, $\alpha = 45°$. Hence, the displacement of $\overrightarrow{AC}$ is $8\sqrt{2}$ km in the S 45° W direction.

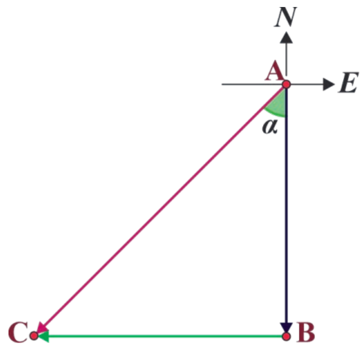


Figure 8.37

Example 2

$ABCD$ is a rectangle, point E is midpoint of AB, point F is midpoint of BC, point G is midpoint of CD, point H is midpoint of AD. $\overrightarrow{AE} = \boldsymbol{a}$ and $\overrightarrow{AH} = \boldsymbol{b}$ as shown in figure 8.38 below.

a. Express each of the following interms of $\boldsymbol{a}$ and $\boldsymbol{b}$

i) $\overrightarrow{AF}$ ii) $\overrightarrow{BD}$

iii) $\overrightarrow{EC}$

b. Show that $\overrightarrow{BD}$ is parallel to $\overrightarrow{EH}$

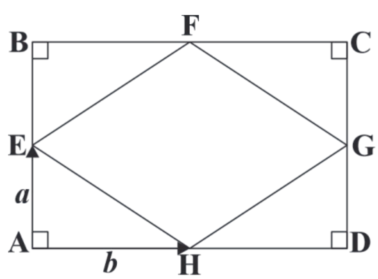


Figure 8.38

Solution:

a. i) $\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{BF} = 2\boldsymbol{a} + \boldsymbol{b}$

ii) $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$ so that

$\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB} = 2\boldsymbol{b} - 2\boldsymbol{a}$

iii) $\overrightarrow{EC} = \overrightarrow{EB} + \overrightarrow{BC} = \boldsymbol{a} - 2\boldsymbol{b}$

b. From above part a. (ii) , $\overrightarrow{BD} = 2\boldsymbol{b} - 2\boldsymbol{a}$. $\overrightarrow{AB} + \overrightarrow{EH} = \overrightarrow{AH}$, so that

$\overrightarrow{EH} = \overrightarrow{AH} - \overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$. $\overrightarrow{BD} = 2(\boldsymbol{b} - \boldsymbol{a}) = 2\overrightarrow{EH}$.

Therefore $\overrightarrow{BD}$ is parallel to $\overrightarrow{EH}$.

Example 3

Two forces, one 10 N and the other $10\sqrt{3}$ N, are acting on the body. Given that the two forces are acting perpendicularly to each other, find the magnitude of the third force which would just counter the two forces.

Solution:

Let us say, the first force be $F_1 = 10$ N and the second be $F_2 = 10\sqrt{3}$ N as shown in the following figure 8.39.

Using parallelogram law, the resultant force $\vec{R} = \vec{F_1} + \vec{F_2}$. Since the two forces are acting perpendicularly, the magnitude of the resultant force is determined by $|\vec{R}| = \sqrt{|\vec{F_1}|^2 + |\vec{F_2}|^2}$.

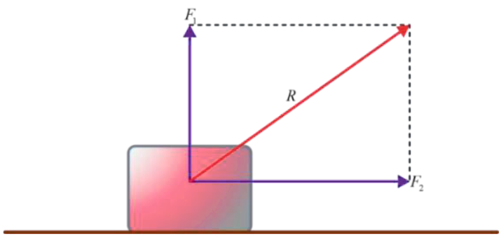


Figure 8.39

That is $|\vec{R}| = \sqrt{10^2 + (10\sqrt{3})^2}$

$= \sqrt{400} = 20.$

Hence, the magnitude of the resultant force is 20 N.

Exercise 8.12

1. Express each of the following in terms of $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$, using the following figure 8.40

   a. $\overrightarrow{AC}$ b. $\overrightarrow{AD}$

   c. $\overrightarrow{AE} + \overrightarrow{ED}$ d. $\overrightarrow{BD}$

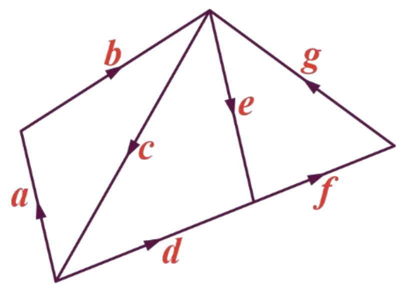


Figure 8.40

2. A wagon is being pulled by a rope that makes a $60°$ with the ground. A person is pulling with a force of $100\,N$ along the rope. Determine the horizontal and vertical components of the vector.
3. A swimmer heads directly to the north across a river swimming at 1.6 m/s relative to still water which moves to the east. She arrives at a point 40 m downstream from the point directly across the river, which is 80 m wide. Determine the displacement of the swimmer.

Summary

1. Scalars are quantities that are fully described by a magnitude (or numerical value) alone.
2. Vectors are quantities that are fully described by both a magnitude and a direction.
3. A vector is represented geometrically by a directed line segment (a line segment with direction) denoted by an arrow$\overrightarrow{AB}$. In this case, the point A is called the *initial* and the point B is called the *terminal.* A vector also represented by using letters with an arrow bar over it such as $\vec{u}, \vec{v}, \vec{a}$ etc.
4. The magnitude of a given vector $\overrightarrow{AB}$ or $\vec{u}$ is the length of the line segment from its initial point A to its terminal point. It is denoted by absolute value sign as $|\overrightarrow{AB}|$or $|\vec{u}|$.
5. On a plane, the direction of a vector is given by the angle the vector makes with a reference direction, often an angle with the horizontal or with the vertical.
6. Vectors that have the same or opposite direction are called parallel vectors.
7. Two vectors are said to be equal if they have the same magnitude and the same direction.
8. For any two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (the Triangle Law)
9. A vector that has no magnitude and direction is called a zero vector or null vector
10. The diagonal of a parallelogram is the sum of the side vectors. This is called Parallelogram Law.
11. Any vector can be enlarged or shortened, by multiplying the vector by a scalar. If $\vec{u}$ is a vector and k be a scalar
    a. if $|k| > 1$, it enlarges the vector
    b. if $0 < |k| < 1$, it shortens the vector

c. if $k > 0$, a scalar multiple of $\vec{u}$ $(k\vec{u})$ is in the same direction to $\vec{u}$.

d. if $k < 0$, a scalar multiple of $\vec{u}$ $(k\vec{u})$ is in the opposite direction to $\vec{u}$.

12. A vector that starts from the origin (O) is called a *position vector*. The position vector $\vec{u}$ can be written as $\vec{u} = x\vec{i} + y\vec{j} = \begin{pmatrix} x \\ y \end{pmatrix}$ where $\vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are vectors whose magnitude is 1 in the direction of x *and* y axis respectively, x *and* y are called components of the vector $\vec{u}$. The magnitude of $\vec{u}$ is determined by $|\vec{u}| = \sqrt{x^2 + y^2}$.

13. Let the initial and terminal points of a vector are (x_1, y_1) and (x_2, y_2) then its position vector can be calculated as $P = (x_2 - x_1, y_2 - y_1) = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$.

Review Exercise

1. Define scalar and vector quantities.
2. List out some scalar and vector quantities.
3. Find the magnitude of each of the following vectors where each grid is of square unit.

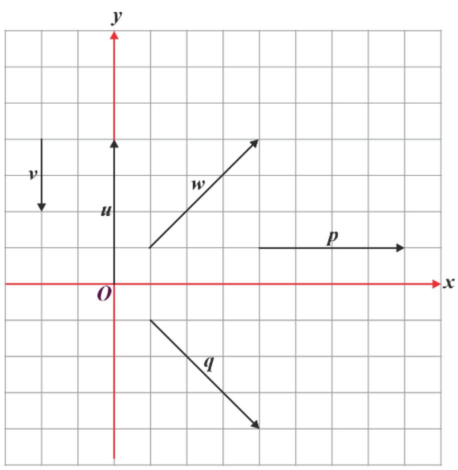


Figure 8.41

4. Consider the points $A = (2,1), B = (10,6), C = (13,4)$ and $D = (16,-2)$. Determine the components form of vector $\overrightarrow{AD}, \overrightarrow{BC}, \overrightarrow{CD}$ *and* $\overrightarrow{AB}$.

5. Simplify (using Figure 8.42)

a. $\boldsymbol{c}+\boldsymbol{f}$ **c.** $\boldsymbol{g}-\boldsymbol{f}$

b. $\boldsymbol{c}+\boldsymbol{d}+\boldsymbol{e}$ **d.** $\boldsymbol{d}+\boldsymbol{e}$

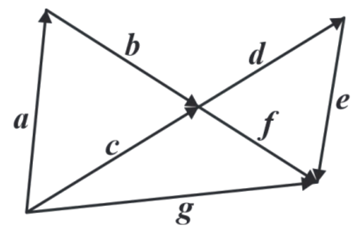


Figure 8.42

6. Simplify

a. $\overrightarrow{AB}+\overrightarrow{BD}$ **b.** $\overrightarrow{PQ}+\overrightarrow{QM}+\overrightarrow{MN}$ **c.** $\overrightarrow{LM}+\overrightarrow{MN}+\overrightarrow{NS}+\overrightarrow{SK}$

7. Use the figure 8.43 to name the directed line segment

i. $\boldsymbol{a}+\boldsymbol{b}$

ii. $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{e}$

iii. $\boldsymbol{g}+\boldsymbol{e}$

iv. $\boldsymbol{c}+\boldsymbol{d}$

v. $\boldsymbol{e}+\boldsymbol{f}$

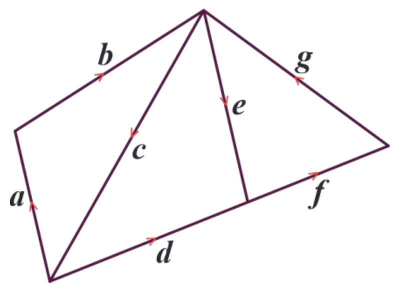


Figure 8.43

8. Suppose $A=(0,1), B=(0,3)$ be points on the coordinate system and $\vec{u}$ be a position vector whose terminal point is $(6,8)$. Compare the magnitude of $\overrightarrow{AB}$ and $\vec{u}$.

9. Sketch a vector of length 3cm in the direction of

a. South **b.** South 30° West **c.** North 45° East

10. Let A and B be two vectors given by $A=3i-2j$ and $B=5i-\frac{1}{2}j$ then the vector $\frac{1}{2}A-\frac{2}{3}B=$ ___________.

A. $-\frac{11}{6}i-\frac{4}{3}j$ **B.** $8i-\frac{3}{2}j$ **C.** $-\frac{11}{6}i-\frac{2}{3}j$ **D.** $-\frac{29}{6}i-\frac{4}{3}j$

11. Seven vectors are drawn as shown in the figure below (figure 8.44), where $PQRS$ is a parallelogram and $\overrightarrow{PT}=-\overrightarrow{QS}$. Which one of the following is true about these vectors?

A. $\overrightarrow{PR} - \overrightarrow{QR} = \overrightarrow{QP}$

B. $\overrightarrow{PQ} - \overrightarrow{PS} = \overrightarrow{PT}$

C. $\overrightarrow{PS} - \overrightarrow{PT} = \overrightarrow{RS}$

D. $\overrightarrow{PS} - \overrightarrow{SR} = \overrightarrow{PR}$

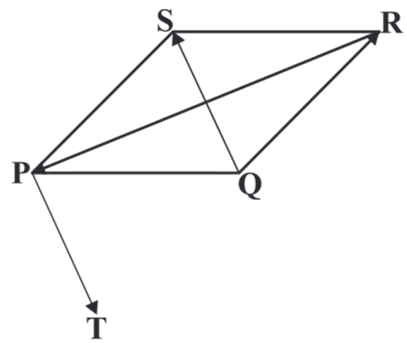


Figure 8.44

12. If the position vector V is given by $V = 5i + j$, then which one of the following is equal to V?

A. $\overrightarrow{EF}$ where $E(-1,-2)$ and $F(0,3)$ C. $\overrightarrow{PQ}$ where $P(-3,-4)$ and $Q(2,-1)$

B. $\overrightarrow{GH}$ where $G(3,-1)$ and $H(2,2)$ D. $\overrightarrow{RS}$ where $R(5,6)$ and $S(10,7)$

13. The vector $\vec{v}$ has initial point $P = (1,0)$ and terminal point Q that is on the $y - axis$ and above the initial point. Find coordinates of terminal point Q such that the magnitude of the vector $\vec{v}$ is $\sqrt{10}$.

14. Let $\vec{a}$ be a standard-position vector with terminal point$(-2,-4)$. Let $\vec{b}$ be a vector with initial point $(1,2)$ and terminal point $(-1,4)$. Find the magnitude of $-3\vec{a} + \vec{b} - 4\vec{i} + \vec{j}$.

15. A merchant start to move in a city. He drives to the bookstore which is 17 km to the north. Then he moves to the music shop 6 km to east and finally he drives 5km to south. Find the displacement of the merchant.

16. A vector $\vec{u}$ is directed along 30° west of north and another vector $\vec{v}$ along 75° east of south. In which of the direction of the resultant of the two vectors could not exist? Justify your answer.

A. North B. North-East C. East D. South

17. A plane flies 100 km, N30°W and after a brief stopover flies150 km, N60°E. Determine the plane's displacement.