UNIT 3

SOLVING EQUATIONS

Unit Outcomes

By the end of this unit, you will be able to:

- **Revise linear equation in one variable.**
- **Use different techniques of solving systems of equations in two variables.**
- **Solve simple equations involving absolute values.**
- **Compute equations involving exponents and radicals.**
- **Solve quadratic equations.**
- **Apply equations on real life situations.**

Unit Contents

KEY Words

- exponents
- completing the square
- discriminant
- elimination method
- absolute value
- graphical method
- linear equation
- factorization
- quadratic equation
- quadratic formula
- radicals
- substitution method

INTRODUCTION

Equations are like a balance weighing machine or scale. If you have seen a weighing machine, you might have thought that an equal amount of weight has to be placed on either side for the scale to be considered “balanced”. If you add some weight to one side, the scale will point on one side and the two sides are no more in balance. Equations abide by the same reasonable judgment. Whatever is on one side of the equal sign must have exactly the same value on the other side; otherwise, it becomes an inequality.

In the earlier grades, you learned about linear equations in one variable and the methods to solve them. In the present unit, you will further discuss systems of linear equations in two variables, equations involving non-linear equations such as absolute values, exponents and radicals and quadratic equations.

3.1 Revision on linear equation in one variable

Activity 3.1

1. Determine each of the following is a linear equation in one variable or not.

a. $10x = 20$ b. $22x - 11 = 0$ c. $4x + 9 = -11$

d. $7x = 0$ e. $x^2 + x = 2$ f. $x + 2y = 1$

g. $4y + 5 = -18$ h. $2a + 3 = 4$ i. $3(n + 5) = 2(-6 - n) - 2n$

j. $3x + 6 = 0$ k. $y = x$ l. $x^2 - y = 0$

Activity 3.1 leads you to revise the rules used to solve linear equations that you learned in the previous grades.

Definition 3.1

When an equation in one variable has exponent equal to 1, it is said to be a linear equation in one variable. It is of the form $ax + b = 0$, where x is the variable, and a and b are real coefficients and $a \neq 0$. This equation has only one solution.

For solving a linear equation in one variable, the following steps are followed:

Step 1: Use LCM (Least Common Multiple) to clear the fractions if any.

Step 2: Simplify both sides of the equation.

Step 3: Isolate the variable.

Step 4: Verify your answer.

Example 1

Solve for x, if

a. $x + 2 = 0$ **b.** $3x + 4 = 10$

Solution:

a. Add -2 on both sides to isolate the variable x.

$$x + 2 + (-2) = 0 + (-2)$$

$$x = -2.$$

b. Similarly, $3x + 4 - 4 = 10 - 4$

$$3x = 6$$

$$x = \frac{6}{3} = 2. \text{ Hence, } x = 2$$

Example 2

Solve $\frac{5}{4}x + \frac{1}{2} = 2x - \frac{1}{2}$

Solution:

$\frac{5}{4}x + \frac{1}{2} = 2x - \frac{1}{2}$

$4\left(\frac{5}{4}x + \frac{1}{2}\right) = 4\left(2x - \frac{1}{2}\right)$ ……….Using LCM to clear the fraction

$5x + 2 = 8x - 2$

$5x + 2 - 8x = 8x - 2 - 8x$

$-3x + 2 = -2$

$-3x + 2 + -2 = -2 + -2$

$-3x = -4$ ………………After simplifying both sides of the equation

$x = \frac{4}{3}$ …………………..Isolating the variable

(Left hand side: LHS) = $\frac{5}{4}\left(\frac{4}{3}\right) + \frac{1}{2} = \frac{5}{3} + \frac{1}{2} = \frac{10+3}{6} = \frac{13}{6}$

(Right hand side: RHS) = $2\left(\frac{4}{3}\right) - \frac{1}{2} = \frac{8}{3} - \frac{1}{2} = \frac{16-3}{6} = \frac{13}{6}$

Thus, (LHS) = (RHS)…………...Verifying the answer

Example 3

The length of the leg of an isosceles triangle is 6 cm more than its base length. If the perimeter of the triangle is 54 cm, then find the lengths of the sides of the triangle.

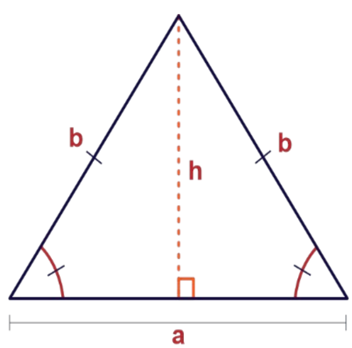


Figure 3.1

Solution:

Let us assume the base measure a cm.

Hence, each of the legs measures $b = (a + 6)\ cm$

The perimeter of a triangle is the sum of the legs of the three sides. The equations are formed and solved as follows:

$$a + 2(a + 6) = 54$$

$$a + 2a + 12 = 54$$

$$3a = 42$$

$$a = 14$$

Then, $b = 14 + 6$

$$b = 20$$

Hence, the length of the base is 14 cm and that of the two legs is 20 cm.

Exercise 3.1

1. Solve the following linear equations:
   a. $x + 7 = 0$
   b. $x + 4 = 9$
   c. $2x - 7 = 3$
   d. $3x - 7 = -10$
   e. $5x - 8 = 2x - 2$
   f. $10(x + 10) - 7 = 13$
   g. $7(y - 2) + 21 = (2y)(3)$
2. Verify whether $x = -3$ is a solution of the linear equation $10x + 7 = 13 - 5x$ or not.
3. If the sum of two consecutive numbers is 67, then find the numbers.

3.2 Systems of linear equations in two variables

Activity 3.2

Two pencils and one eraser cost Birr 5 and three pencils and two erasers cost Birr 8. Let the price of a pencil and an eraser be x and y Birr, respectively. Express the statements with x and y.

A system of linear equations consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously.

To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. This even does not guarantee a unique solution.

In this section, we will discuss systems of linear equations in two variables, which consist of two equations that contain two different variables.

To solve linear equations with two variables, there are three methods. These are solving using tables, solving by substitution method and elimination method.

Definition 3.2

An equation of the type $ax + by = c$ where a, b and c are arbitrary constants and $a \neq 0$, $b \neq 0$, is called a linear equation in two variables.

3.2.1 Solving systems of linear equations in two variables using tables

Example 1

$\begin{cases} 2x + y = 15 \\ 3x - y = 5 \end{cases}$ is a system of linear equations in two variables. Find the solution.

Solution:

Solve each equation for y. Then, find the pair of x and y that satisfies each equation as follows:

$y = -2x + 15$

x	0	1	2	3	4	5	6
y	15	13	11	9	7	5	3

$y = 3x - 5$

x	0	1	2	3	**4**	5	6
y	-5	-2	1	4	**7**	10	13

Hence, the pair of x and y that satisfies both equation is $(4, 7)$. This is the solution. The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, after filling the tables above, the ordered pair $(4, 7)$ is the solution to the system of linear equations.

Example 2

Check whether the ordered pair $(5,1)$ is a solution to the following linear system of equations or not.

$$\begin{cases} x + 3y = 8 \\ 2x - y = 9 \end{cases}$$

Solution:

Substitute the ordered pair $(5,1)$ into both equations.

First equation: $(LHS) = 5 + 3(1) = 8,\ (RHS) = 8,$

Thus, $(LHS) = (RHS)$, True

Second equation: $(LHS) = 2(5) - 1 = 9,\ (RHS) = 9$

Thus, $(LHS) = (RHS)$, True

The ordered pair $(5,1)$ satisfies both equations, so it is the solution to the system.

Exercise 3.2

1. Solve the following linear equation in two variables using table.

$$\begin{cases} 2x - y = -1 \\ x + y = 4 \end{cases}$$

2. Determine whether the ordered pair $(8,5)$ is a solution to the following linear equation in two variables or not.

$$\begin{cases} 5x - 4y = 20 \\ 2x - 3y = -1 \end{cases}$$

3.2.2 Solving systems of linear equations in two variables by substitution

One of the methods of solving a system of linear equations is substitution method. The method involves solving one of the equations in terms of the other variable and then substitute the result into the other equation to solve the second variable.

Steps in solving systems of linear equations in two variables by substitution

Step 1. Solve one of the two equations for one of the variables in terms of the other.

Step 2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.

Step 3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.

Step 4. Check the solution in both equations.

Example 1

Solve the following system of linear equations by substitution method.

$$\begin{cases} -x + y = -5 \\ 2x - 5y = 1 \end{cases}$$

Solution:

First, solve the first equation for y (or solve the second equation for y)

$y = x - 5$ (Substitute this into the second equation).

$$2x - 5(x - 5) = 1$$

$$-3x + 25 = 1$$

$$-3x = -24$$

$$x = 8$$

Now, we substitute $x = 8$ into the first equation or second equation and solve for y.

$$-8 + y = -5$$

$$y = -5 + 8 = 3.$$

Therefore, the solution of the above system of linear equations is $(x, y) = (8, 3)$.

Example 2

Solve the following system of linear equations using substitution method.

$$\begin{cases} x - 6y = -3 \\ 3x - 5y = 4 \end{cases}$$

Solution:

1st equation: $x = 6y - 3$. Substitute this into the 2nd equation.

$$3(6y - 3) - 5y = 4$$

$$18y - 9 - 5y = 4$$

$$13y = 13$$

$$y = 1$$

Then, $x = 6y - 3 = 6(1) - 3 = 3$

Exercise 3.3

1. Solve the following system of linear equations using substitution method.

a. $\begin{cases} 4x + y = 23 \\ y = 6x + 3 \end{cases}$

b. $\begin{cases} x = y - 6 \\ 5x + 2y = -2 \end{cases}$

c. $\begin{cases} x + 5y = 72 \\ x = 7y \end{cases}$

d. $\begin{cases} 2x - 7y = 2 \\ 3x + y = -20 \end{cases}$

e. $\begin{cases} 4x + y = 23 \\ y = 6x + 3 \end{cases}$

f. $\begin{cases} x = y - 6 \\ 5x + 2y = -2 \end{cases}$

g. $\begin{cases} 2x + 3y = -13 \\ 4x - 10 = 6y \end{cases}$

h. $\begin{cases} x + 3y = 22 \\ x = 4 \end{cases}$

i. $\begin{cases} \frac{x}{3} + \frac{y}{6} = 3 \\ \frac{x}{2} - \frac{y}{4} = 1 \end{cases}$

j. $\begin{cases} 8x + 3y = 9 \\ -\frac{x}{6} + \frac{y}{2} = 2 \end{cases}$

3.2.3 Solving systems of linear equations in two variables by Addition (Elimination) method

Another method of solving systems of linear equations is the addition method, also called the elimination method. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often, we must adjust by multiplying one or both of the equations so that one of the variables will be eliminated by addition.

The following are steps to solve system of equations using the addition (elimination) method.

Step 1. Write both equations with x and y variables on the left side of the equal sign and constants on the right.

Step 2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, and then add the equations to eliminate the variable.

Step 3. Solve the resulting equation for the remaining variable.

Step 4. Substitute that value into one of the original equations and solve for the second variable.

Step 5. Check the solution by substituting the values into the other equation.

Example 1

Solve the following using addition (elimination) method.

$$\begin{cases} x + 2y = 1 \\ -x + y = 5 \end{cases}$$

Solution:

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, -1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.

$$\begin{cases} x + 2y = 1 \\ -x + y = 5 \end{cases}$$

$$0 + 3y = 6$$

$$y = 2$$

Now, we have eliminated x, and we have solved for y. Then, we substitute this value for y into one of the original equations and solve for x as follows:

$$-x + 2 = 5$$

$$x = -3$$

Therefore, the solution is: $x = -3, y = 2$

Example 2

Solve the following using addition (elimination) method.

$$\begin{cases} 4x + 2y = 28 \\ x + 2y = 13 \end{cases}$$

Solution:

Note that that the coefficient of y in the first and the second equation are the same.

$$\begin{cases} 4x + 2y = 28 \\ -(x + 2y = 13) \end{cases}$$

$$3x = 15$$

$$x = 5$$

Substitute in the second equation of the original, we have

$$x + 2y = 13$$

$$5 + 2y = 13$$

$$2y = 8$$

$$y = 4$$

Therefore, the solution is: $x = 5, y = 4$

Exercise 3.4

Solve the following system of linear equations using addition (elimination) method.

a. $\begin{cases} x + 3y = 5 \\ -x + y = -1 \end{cases}$ b. $\begin{cases} 2x + y = 12 \\ 2x - y = 12 \end{cases}$

c. $\begin{cases} 2x + y = 20 \\ x - y = 4 \end{cases}$ d. $\begin{cases} 2x + 5y = 12 \\ 2x + y = 4 \end{cases}$

Addition (Elimination) method 2

Example 1

Solve the following using substitution (elimination) method.

$$\begin{cases} 3x + 4y = 2 \\ x - 2y = 4 \end{cases}$$

Solution:

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So, if we multiply the second equation by -3, the x-terms will add to zero.

$$\begin{cases} 3x + 4y = 2 \\ -3(x - 2y) = -3(4) \end{cases}$$

$$\begin{cases} 3x + 4y = 2 \\ -3x + 6y = -12 \end{cases}$$

$$10y = -10$$

$$y = -1$$

Substituting in the second equation of the original, we have

$$x - 2y = 4$$

$$x - 2(-1) = 4$$

$$x = 4 - 2 = 2$$

Example 2

Solve the following using substitution (elimination) method.

$$\begin{cases} 2x + 3y = -2 \\ 5x + 2y = 17 \end{cases}$$

Solution:

Multiply the first equation and second equation by 2 and by -3, respectively.

$$\begin{cases} 2(2x + 3y) = 2(-2) \\ -3(5x + 2y) = -3(17) \end{cases}$$

$$\begin{cases} 4x + 6y = -4 \\ -15x - 6y = -51 \end{cases}$$

$$-11x \qquad = -55$$

$$x = 5$$

Substitute in the first equation of the original, we have

$$2(5) + 3y = -2$$

$$3y = -12$$

$$y = -4$$

Therefore, the solution is $x = 5,\ y = -4$.

Exercise 3.5

Solve the following using Addition (elimination) method.

a. $\begin{cases} 2x - 3y = 5 \\ x - 2y = 6 \end{cases}$

b. $\begin{cases} 2x - 7y = 2 \\ 3x + y = -20 \end{cases}$

c. $\begin{cases} 2x + 3y = 5 \\ 3x + 5y = 7 \end{cases}$

d. $\begin{cases} 9x + 3y = 38 \\ 5x + y = 22 \end{cases}$

e. $\begin{cases} -3x + y = -9 \\ -7x + 4y = 6 \end{cases}$

Word Problems involving equations in two variables

Example

Three pens and two books cost Birr 118 and one pen and two books cost Birr 106. Find the cost of one pen and one book, separately.

Solution:

If p is the cost of one pen and b is the cost of one book, then $\begin{cases} 3p + 2b = 118 \\ p + 2b = 106 \end{cases}$

Solve the second equation for p in terms of b.

$p = -2b + 106$. Substitute this into the first equation, $3(-2b + 106) + 2b = 118$.

Hence, $b = 50$ and $p = 6$.

Exercise 3.6

1. Two pens and three books cost Birr 170.00 and five pens and one book cost Birr 100.00. Find the cost of one pen and one book, separately.
2. One bread and one injera cost Birr 15 and ten breads and four injera cost Birr 90. Find the cost of a bread and one injera, separately.
3. A jacket and two Ethiopian dresses cost Birr 4000.00 and five jackets and three Ethiopian dresses cost Birr 9500.00. Find the cost of one jacket and one Ethiopian dress, separately.
4. If twice the age of a son is added to age of a father, then the sum is 56. If twice the age of the father is added to the age of son, then the sum is 82. Find the ages of the father and the son.
5. In a two-digit number, the sum of the digits is 13. Twice the tens digit exceeds the units digit by two. Find the numbers.
6. I am thinking of a two-digit number. If I write 3 to the left of my number, and double this three digit number, the result is 27 times my original number. What is my original number?

3.2.4 Graphical method of solving system of linear equations in two variables

Graphical method is one of the methods of solving systems of linear equations. Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

To solve systems of linear equations using graphs, use the following steps:

Step 1. Graph the first equation.

Step 2. Graph the second equation using the same coordinate system.

Step 3. Determine whether the lines intersect, are parallel, or are the same line.

Step 4. Identify the solution to the system. If the lines intersect, identify the point of intersection.

Example 1

Solve the following system of equations graphically.

$$\begin{cases} 2x + y = -8 \\ x = y - 1 \end{cases}$$

Solution: First, solve the first equation for y, that is

$$2x + y = -8$$
$$y = -2x - 8$$

Solve the second equation for y, that is,

$$x - y = -1$$
$$y = x + 1$$

Second, graph both equations on the same set of axes:

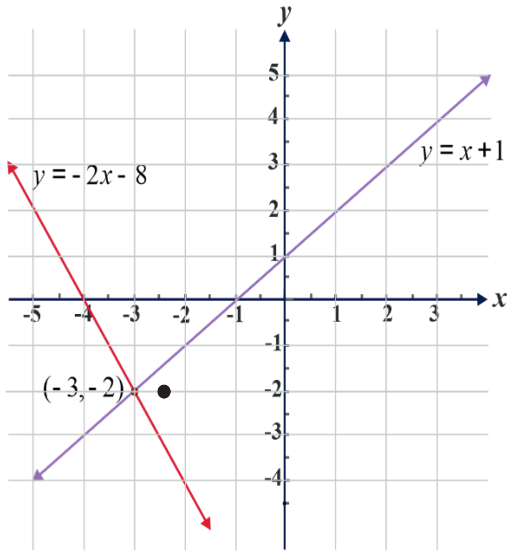


Figure 3.2

The graph above (Fig 3.2) indicates the intersection point of the lines: $y = -2x - 8$ and $y = x + 1$. The lines appear to intersect at the point$(-3, -2)$. This intersection point is the solution of the system. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

First equation: $(LHS) = 2(-3) + (-2) = -8$, $(RHS) = -8$,

Thus, $(LHS) = (RHS)$, True

Second equation: $(LHS) = -3$, $(RHS) = -2 - 1 = -3$

Thus, $(LHS) = (RHS)$, True

The solution to the system is the ordered pair $(-3, -2)$.

Example 2

Solve the following system of equations graphically.

$$\begin{cases} 2x - y = -1 \\ 4x - 2y = -8 \end{cases}$$

Solution:

First**,** solve the two equations for y.

First equation: $y = 2x + 1$

Second equation: $2y = 4x + 8$, $y = 2x + 4$

Then, draw the graphs of both equations. The graph on right (Fig 3.3) indicates that the two lines are parallel and do not intersect each other. This means that there is no solution to the system.

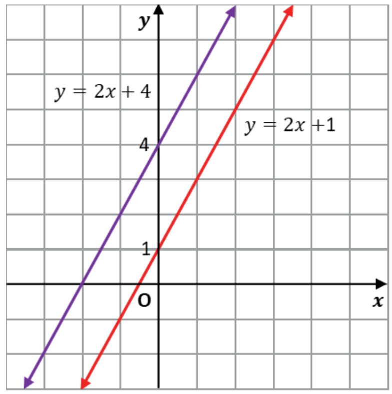


Figure 3.3

Example 3

Solve the following system of equations by graphical method.

$$\begin{cases} 3x + y = 6 \\ 6x + 2y = 12 \end{cases}$$

Solution:

When solving the two equations for y, we obtain $y = -3x + 6$

Graph the equation on the *xy*-coordinate plane.

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions.

Remark:

If a system of two equations has different slopes and intersects at one point in the plane, then the system is said to have exactly one solution. If the equations of the system have the same slope and the same y-intercepts, then the system has infinitely many solutions. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is one in which the equations represent two parallel lines. The lines have the same slope and different y-intercepts. There are no points common to both lines; hence, there is no solution to the system. Below is a comparison of graphical representations of each type of system (fig 3.4).

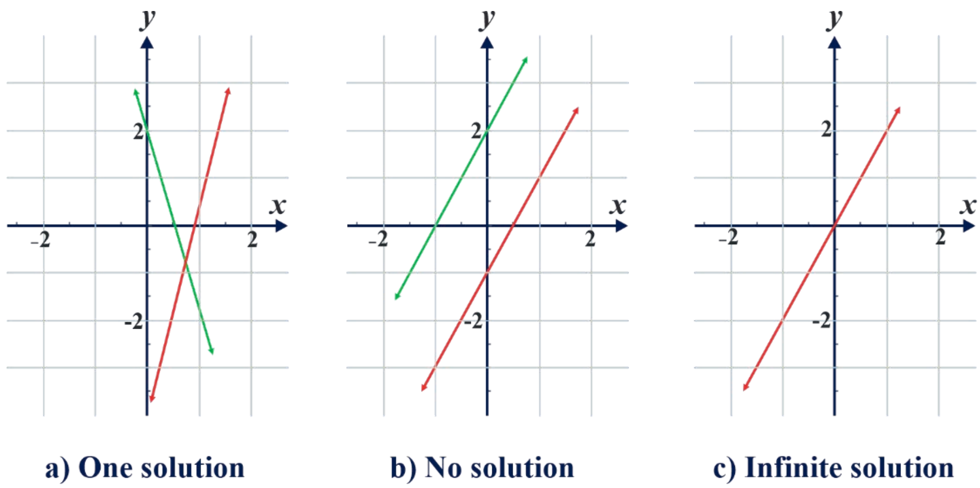


Figure 3.4

Exercise 3.7

1. Solve the following system of linear equations graphically.

a. $\begin{cases} y = 2x + 1 \\ y = 4x + 3 \end{cases}$ b. $\begin{cases} x + y = 6 \\ -x + y = 2 \end{cases}$

2. Determine whether each of the following system of equations has one solution, infinite solution or no solution.

a. $\begin{cases} 3x + y = 6 \\ 6x + 2y = 2 \end{cases}$ b. $\begin{cases} 3x + y = 6 \\ y = -3x + 6 \end{cases}$

c. $\begin{cases} 2x - y = 6 \\ -2x + 3y = 2 \end{cases}$ d. $\begin{cases} 2x + 5y = 12 \\ x - \frac{5}{2}y = 4 \end{cases}$

3. What must be the value of k so that the system of linear equations

$$\begin{cases} x - y = 3 \\ 2x - 2y = k \end{cases} \quad \text{has}$$

i. infinitely many solution?

ii. no solution?

iii. unique solution?

4. Under what conditions on a and b will the following system of linear equation have no solution, one solution or infinitely many solutions?

$$\begin{cases} 2x - 3y = a \\ 4x - 6y = b \end{cases}$$

3.3 Solving nonlinear equations

3.3.1 Equations involving absolute value

Review of absolute value

In the previous sections, you worked with equations having variables x or y that can assume any value. But sometimes it becomes necessary to consider only non-negative values.

If you consider distance, it is always non-negative. The distance of a number x is located on the real line from the origin is either positive or zero. If x is at the origin distance is 0.

Activity 3.3

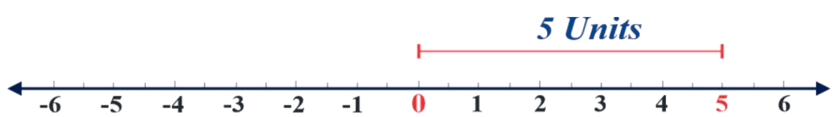
Compare each of the following numbers in absolute value.

a. 2 and -3 b. 6 and -5 c. ¼ and -1/2

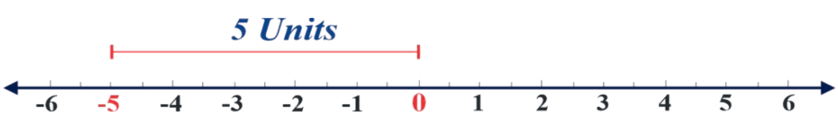
d. $-\sqrt{10}$ and 3 e. $-\frac{1}{3}$ and $-0.333\ldots$

From unit two, recall that real numbers can be represented on a line as follows:

The absolute value of 5 is 5. The distance from 5 to 0 is 5 units.



The absolute value of –5 is 5. The distance from -5 to 0 is 5 units.



From this, it is possible to determine the distance of each point, representing a number, located far away from the origin or the point representing 0.

Example 1

Let A and B be points on a number line with coordinates 5 and −5, respectively. How far are the points A and B from the origin? How many points are there equal distances from the origin on a number line?

Solution:

The distance of A and B from the origin is 5 units long on the real line. There are infinitely many points that are equal distances from the origin on a number line.

Note

If Q is a point on a number line with coordinate a real number q, then the distance of Q from the origin is called the absolute value of q and is written as $|q|$.

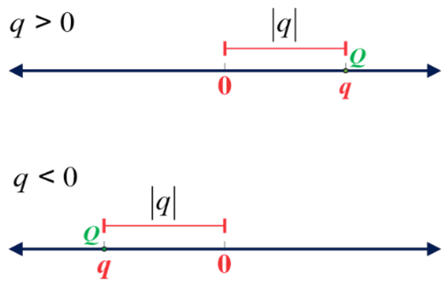


Figure 3.5

Example 2

The points represented by numbers 3 and –3 are located on the number line at an equal distance from the origin. Hence, $|3| = |-3| = 3$.

Example 3

Find the absolute value of each of the following real numbers.

a. -6 **b.** 7 **c.** -0.8

Solution:

a. $|-6| = 6$ **b.** $|7| = 7$ **c.** $|-0.8| = 0.8$

Exercise 3.8

Find the absolute value of the following:

a. $|-4|$

b. $|-\frac{1}{2}|$

c. $|0|$

Solving equations involving absolute value

Example 1

Solve the equation, $|x| = 4$

Solution:

$$|x| = 4$$
$$\mathrm{x} = \pm 4$$

Properties of absolute value.

1. For any real number p, $|p| = |-p|$.
2. For any real number p, $|p| \geq 0$.
3. For any non- negative number p, $|x| = p$ means $x = p$ or $x = -p$.

Example 2

Solve the equation, $|x + 7| = 14$

Solution:

For any non- negative number p, $|x| = p$ means $x = p$ or $x = -p$.

So, you begin by making it into two separate equations and then solve them separately.

$|x + 7| = 14$ means:

$$x + 7 = 14 \quad or \quad x + 7 = -14$$
$$x = 14 - 7 \quad or \quad x = -14 - 7$$

$$x = 7 \qquad or \qquad x = -21$$

Hence, $x = -21,\ x = 7$ are solutions of the equation.

Remark: An absolute value equation has no solution if the absolute value expression equals a negative number since an absolute value can never be negative. For example, $|x| = -1$ has no solution. (Why?)

Example 3

Solve the absolute value equation, $2|2x - 1| - 1 = 5$

Solution:

$$2|2x - 1| = 6$$
$$|2x - 1| = 3$$
$$2x - 1 = 3 \ or \ -(2x - 1) = 3$$
$$x = 2 \ or \ x = -1$$

Hence, $x = -1,\ x = 2$ are solutions to the equation.

Exercise 3.9

1. Let A and B be points on a number line with coordinates 6 and -6, respectively. How far are the points A and B from the origin? How many points are there equal distances from the origin on a number line?
2. Solve the following absolute value equations.

   a. $|x + 2| = 6$
   b. $|5 - 3x| = 7$
   c. $|x - 2| = 0$
   d. $|x + 2| = -6$
   e. $|2x - 1| + 3 = 6$
   f. $|-2x + 7| = 25$
   g. $|-8x - 2| - 6 = 10$
   h. $-4|x + 3| = -28$
   i. $\frac{|x+9|}{2} = 7$
   j. $3|5 + 3x| + 1 = 7$
   k. $4 - 6|4 + 8x| = -116$

3.3.2 Quadratic equations

Activity 3.4

Multiply the left side of each of the following. What do you get? What is the difference between these equations and linear equations?

1. $(x+3)(x-2)=0$
2. $(5x+1)(2x+4)=2$
3. $\left(\frac{1}{2}x-3\right)(x+5)=0$

Definition 3.3

An equation of the form $ax^2+bx+c=0$ where a, b, c, ∈ R and $a \neq 0$ is a quadratic equation. Here, a is called the leading coefficient, b is the middle term and c is the constant term.

Solving quadratic equations

There are three basic methods for solving quadratic equations: factorization (if possible), completing the square and the quadratic formula method.

Factorization method 1

Activity 3.5

Find two integers such that

a. the sum is 5 and the product is 6.

b. the sum is 1 and the product is -12.

How to solve a quadratic equation by factorization method?

1. Put all terms on one side of the equal sign, leaving zero on the other side.
2. Factorize the equation.

3. Set each factor equal to zero.
4. Solve each of these equations.
5. Check by inserting your answer in the original equation.

Example 1

Solve the quadratic equation $x^2 + 3x = 0$.

$$x(x+3) = 0$$

$$x = 0 \ \ or\ x + 3 = 0$$

$$x = 0 \ \ or\ x = -3$$

Hence, the solution of the quadratic equation is $x = 0,\ x = -3$.

Example 2

Solve the quadratic equation: $x^2 - 6x - 16 = 0$

Solution:

Factorizing this (find two numbers whose sum is -6 and product is -16), we have -8 and 2. Hence, $(x-8)(x+2) = 0$

$$x - 8 = 0 \text{ or } x + 2 = 0$$

Recall that $\mathrm{ab} = 0$ if and only if $\mathrm{a} = 0$ or $\mathrm{b} = 0$

$$x = 8,\ x = -2$$

Hence, the solution of the quadratic equation is $x = 8,\ x = -2$.

Exercise 3.10

Solve the following quadratic equations using factorization method.

a. $x^2 - 5x = 0$
b. $x^2 + 7x + 10 = 0$
c. $x^2 + x - 6 = 0$
d. $x^2 - 4x + 3 = 0$

Factorization Method 2

Example 1

Solve the quadratic equation $x^2 + 6x + 9 = 0$

Solution:

$(x + 3)(x + 3) = (x + 3)^2 = 0$ (Factorizing: sum $= 6$ and product $= 9$).

$$x = -3$$

Hence, $x = -3$ is the only solution.

Example 2

Solve the quadratic equation $x^2 - 9 = 0$

Solution:

$(x - 3)(x + 3) = 0$ (Factorizing: sum $= 0$ and product$= 9$)

$$x = 3 \text{ or } x = -3$$

Hence, the solution of the quadratic equation is $x = 3,\ x = -3$.

Example 3

Solve the quadratic equation $4x^2 + 4x + 1 = 0$

Solution:

Re-writing $4x^2 + 2x + 2x + 1 = 0$

$$2x(2x + 1) + 1(2x + 1) = 0$$

$$(2x + 1)(2x + 1) = 0$$

$$(2x + 1)^2 = 0$$

$$x = -\frac{1}{2}$$

Hence, the solution of the quadratic equation is $x = -\frac{1}{2}$.

Exercise 3.11

Solve the following quadratic equations using factorization method.

a. $x^2 + 10x + 25 = 0$

b. $x^2 - 8x + 16 = 0$

c. $x^2 - 4 = 0$

d. $9x^2 - 6x + 1 = 0$

Completing the square method

Example 1

Is it possible to solve $x^2 + 6x + 4 = 0$ using factorization method?

Solution:

Since there are no two integers whose sum is equal to 6 and product is equal to 4, this quadratic equation may not be solved using factorization method. Hence, we need another method to solve the equation.

$$x^2 + 6x + 9 + 4 - 9 = 0$$

$$(x+3)^2 - 5 = 0$$

$$(x+3)^2 = 5$$

$$x + 3 = \pm\sqrt{5}$$

$$x = -3 \pm \sqrt{5}$$

Completing the square is where we take the quadratic equation

$ax^2 + bx + c = 0, a \neq 0$ and convert it into $\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2} = 0$ as follows:

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ ($since\ a \neq 0$)

$(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) + \frac{c}{a} - \frac{b^2}{4a^2} = 0.$ Taking half of the coefficient of the middle term and squaring it and adding its opposite). The expression in the bracket is a perfect square. Hence, after simplifying, we have

$\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2} = 0 \ldots\ldots (*)$

Equivalently, $ax^2 + bx + c = 0$ if and only if $\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2} = 0$

Example 2

Solve $x^2 + 4x + 4 = 0$ using completing the square method.

Solution:

$(x^2 + 4x + 4) + 4 - 4 = 0$ (Adding the square of half of the coefficient of the middle term (4) and its opposite.

$$(x + 2)^2 - 0 = 0. \text{ (Writing as a perfect square)}$$

$$(x + 2)^2 = 0$$

$$x = -2$$

Example 3

Solve $x^2 + 6x + 7 = 0$ using completing the square method.

Solution:

$x^2 + 6x + 9 + 7 - 9 = 0$ (Adding the square of half of the coefficient of the middle term (6) and its opposite)

$(x^2 + 6x + 9) + 7 - 9 = 0$ (Colleting those terms which sum up as a perfect square)

$$(x + 3)^2 - 2 = 0. \text{ (Writing as a perfect square)}$$

$$(x + 3)^2 = 2$$

$$x + 3 = \pm\sqrt{2} \quad \text{(Taking the square root)}$$

$$x = -3 \pm \sqrt{2}$$

Therefore, $x = -3 + \sqrt{2}$ and $x = -3 - \sqrt{2}$ are the solutions.

Exercise 3.12

Use completing the square method to solve the following.

a. $x^2 + 4x + 1 = 0$

b. $x^2 - 6x - 5 = 0$

c. $2x^2 + 5x + 3 = 0$

The quadratic formula method

There are quadratic equations that cannot be solved by factorization method. This is generally true when the roots are not rational numbers. Consider the quadratic equation $ax^2 + bx + c = 0,$ where $a \neq 0$.

From the completing the square method, we have $\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2} = 0$

Solving for x, $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Quadratic formula

For the quadratic equation $ax^2 + bx + c = 0, a \neq 0,$ $\boldsymbol{x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}}$

This is the quadratic formula

Example

Solve $x^2 - 6x + 3 = 0$ using the quadratic formula.

Solution:

$1, -6$ and 3 are the values for a, b, and c, respectively in the quadratic formula.

Thus,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(3)}}{2(1)}$$

$$x = \frac{6 \pm \sqrt{36 - 12}}{2} = \frac{6 \pm \sqrt{24}}{2} = 3 \pm \sqrt{6}$$

Hence, $3 + \sqrt{6}$ and $3 - \sqrt{6}$ are the solutions. Can you solve this using other methods?

Exercise 3.13

Use quadratic formula to solve the following.

a. $x^2 + 3x + 1 = 0$ b. $x^2 + 5x - 2 = 0$ c. $2x^2 - 3x - 1 = 0$

Discriminant

Remark: When using the quadratic formula, you should be aware of three possibilities. These three possibilities are distinguished by a part of the formula called the discriminant. The discriminant is the value under the radical sign which is $b^2 - 4ac$. A quadratic equation with real numbers as coefficients can have the following:

The roots of the quadratic equation are $x = \frac{-b \pm \sqrt{D}}{2a}$, where $D = b^2 - 4ac$.

$If\ D > 0$, then the roots are real and distinct (unequal); the quadratic equation has two distinct roots.

$If\ D = 0$, then the roots are real and equal (coincident); the quadratic equation has exactly one real root.

$If\ D < 0$, then there are no real roots.

Example 1

Check whether $x^2 + 2x + 2 = 0$ has distinct real roots, one real root or no real roots. If root exists, find it.

Solution:

Here, $a = 1$, $b = 2$ and $c = 2$.

Since $b^2 - 4ac = 4 - 8 = -4 < 0$, no need to find the roots. The following is to show that the roots are not real number.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(2)\pm\sqrt{(2)^2-4(1)(2)}}{2(1)}$$

$$x = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm\sqrt{-4}}{2} \text{ are not real numbers.}$$

Example 2

Check whether $x^2 + 18x + 81 = 0$ has distinct real roots, one real root or no real roots. If root exists, find it.

Solution:

Here, $a = 1, b = 18 \; and \; c = 81,$

$D = b^2 - 4ac = 18^2 - 4(1)(81) = 324 - 324 = 0$. Since $D = 0$, the equation has exactly one root. Solving the equation,

$$(x+9)^2 = 0$$

$$x = -9$$

Example 3

Check whether $x^2 + 4x + 3 = 0$ has distinct real roots, one real root or no real roots. If root exists, find it.

Solution:

Here, $a = 1, b = 4$ and $c = 3$, and then

$D = b^2 - 4ac = 4^2 - 4(3) = 16 - 12 = 4$

Since $D > 0$, the equation has two distinct real roots. Solving the equation,

$$(x+1)(x+3) = 0$$

$$x = -1 \text{ and } x = -3$$

Exercise 3.14

Check whether the following quadratic equations have two real roots, one real root or no real roots. If any real root exists, find it.

a. $x^2 + 2x + 3 = 0$ b. $x^2 + 12x + 36 = 0$ c. $x^2 + 8x + 7 = 0$

Relationships between roots and coefficients of a quadratic equation

If $r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$, then

$$r_1 + r_2 = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-b}{a} \text{ and}$$

$$r_1 \cdot r_2 = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) = \frac{c}{a}$$

Example 1

Let $ax^2 + 6x + c = 0$. If the sum of the roots is $-\frac{6}{5}$ and the product is $\frac{1}{5}$, then find the values of a and c.

Solution:

Let r_1 and r_2 be the roots. From the relationships above,

$$r_1 + r_2 = -\frac{6}{a} = -\frac{6}{5}, \text{ which implies } a = 5$$

$$r_1 \cdot r_2 = \frac{c}{a} = \frac{1}{5} \qquad \text{Since } a = 5,\ \frac{c}{5} = \frac{1}{5}, \text{ which also implies } c = 1$$

Therefore, the quadratic equation is $5x^2 + 6x + 1 = 0$

Example 2

If the difference between the roots of the equation $x^2 - 13x + k = 0$ is 7, find the value of k.

Solution:

Let r_1 and r_2 be the roots. From the relationships above,

$$r_1 + r_2 = -\frac{b}{a} = -\frac{-13}{1} = 13 \text{ and}$$

$$r_1 - r_2 = 7$$

$$\begin{cases} r_1 + r_2 = 13 \\ r_1 - r_2 = 7 \end{cases}$$

Solving this, $r_1 = 10$ and $r_2 = 3$

$r_1 \cdot r_2 = (10)(3) = 30$, while $r_1 \cdot r_2 = \frac{c}{a} = \frac{k}{1}$ from the above relationship.

Therefore, $k = 30$.

Exercise 3.15

1. Let $x^2 + bx + c = 0$ If the sum of the roots of the quadratic equation is 10 and the product of the roots is 16, find b and c.
2. Find the quadratic equation: $ax^2 + 3x + c = 0$ when the sum of the roots is -3, the product is 2.
3. Solve the following using quadratic formula and check using any other methods of solving quadratic equation.

a. $x^2 - 16 = 0$	b. $x^2 - 9x = 0$
c. $3x^2 - 75 = 0$	d. $x^2 - 6x + 5 = 0$
e. $x^2 + 6x + 8 = 0$	f. $x^2 + 3x - 28 = 0$
g. $2x^2 + 7x + 3 = 0$	h. $-x^2 + 2x + 15 = 0$
i. $x^2 + 10x + 7 = 0$	j. $-3x^2 + 10x - 7 = 0$
k. $3x^2 = 6x - 1$	l. $5x^2 + x + 2 = 0$
m. $-3x^2 + 8x - 5 = 0$	n. $2x^2 + 5x = 12$
o. $8x^2 - 10x = -3$	p. $2x^2 + 5x + 2 = 0$
q. $-3x^2 + 7x - 6 = 0$	r. $10x^2 - 9x + 6 = 0$
s. $3x^2 = -x + 14$	t. $9x^2 - 7 = 9x$
u. $\frac{1}{3}x^2 - x - 2 = 0$	w. $-4x(x - 2) = 6(x + 3) - 11x^2$
x. $x(x - 3) = -7 - 10x$	

3.3.3 Equations involving exponents and radicals

Equations involving exponents 1

Activity 3.6

Consider the following equations and discuss on their solutions:

a. $x^2 = 4$ b. $x^3 = -64$

How many solution(s) does each equation have? What are they? What can you generalize?

Note

For a > 0, $a^x = a^y$ if and only if $x = y$.

Example 1

Solve the followings.

a. $2^x = 8$

b. $3^x = \frac{1}{27}$

Solution:

a. $2^x = 2^3$ (Making the same base by writing $8 = 2^3$)

$x = 3$ (Equating the exponents because the bases are the same)

b. $3^x = \frac{1}{3^3}$

$3^x = 3^{-3}$

$x = -3$

Some rules of exponential equations.

(a, m, and n are real number and $a \neq 0$, and $b \neq 0$)

a. $\frac{1}{a^n} = a^{-n}$

b. $a^m a^n = a^{m+n}$

c. $(a^m)^n = a^{mn}$

d. $(ab)^n = a^n b^n$

e. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

f. $\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}$

Exercise 3.16.1

Solve the following equations.

a. $3^x = 81$ b. $5^x = 125$ c. $2^x = \frac{1}{8}$ d. $4^x = \frac{1}{64}$

Example 2

Solve the following equations.

a. $8^x = 16$ b. $2^{x+4} = 32$ c. $9^{x+4} = 27$

Solution:

a. $2^{3x} = 2^4$

$3x = 4$ (Equating the exponents)

$x = \frac{4}{3}$

b. $2^{x+4} = 32$

$2^{x+4} = 2^5$

$x + 4 = 5$

$x = 1$

c. $9^{x+4} = 27$

$3^{2(x+4)} = 3^3$

$2x + 8 = 3$

$x = -\frac{5}{2}$

Exercise 3.16.2

Solve the following.

a. $4^x = 32$

b. $3^{2x-1} = 243$

c. $32^x = 2^{x+4}$

Equations involving exponents 2

Example 1

Solve: $(x^2 + 6x)^{\frac{1}{4}} = 2$

Solution:

Remove the 4th root by raising each side of the equation to the 4th power.

$$\left((x^2 + 6x)^{\frac{1}{4}}\right)^4 = 2^4$$
$$x^2 + 6x = 16$$
$$x^2 + 6x - 16 = 0$$
$$(x + 8)(x - 2) = 0$$

Therefore, $x = -8$, $x = 2$.

These are possible solution. Verify them as follows.

If $x = -8$,

$$(LHS) = [(-8)^2) + 6(-8)]^{\frac{1}{4}} = (64 - 48)^{\frac{1}{4}} = (16)^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = 2$$

Thus, $(LHS) = (RHS)$. Therefore $x = -8$ is a solution.

Similarly, if $x = 2$, then

$$(LHS) = [(2)^2) + 6(2)]^{\frac{1}{4}} = (4 + 12)^{\frac{1}{4}} = (16)^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = 2 = (RHS)$$

Therefore, $x = 2$ is also a solution.

Example 2

Solve for y if $5y^{\frac{2}{3}} + 3 = 23.$

Solution:

Isolate the y-term on the left side of the equation. Subtract 3 from each side of the equation.

$$5y^{\frac{2}{3}} = 23 - 3$$
$$5y^{\frac{2}{3}} = 20$$
$$y^{\frac{2}{3}} = 4.$$

First, raise both sides by 3rd power,

$$\left(y^{\frac{2}{3}}\right)^{3} = 4^{3}$$

Then $y^{2} = 4^{3} = (2^{2})^{3} = (2^{3})^{2}$,

$$y = \pm\sqrt{(2^{3})^{2}} = \pm 2^{3} = \pm 8$$

Exercise 3.17

1. Solve the following equations involving exponents.

   a. $(x^{2} + 2x)^{\frac{1}{2}} = 4$

   b. $\left(16x^{\frac{1}{2}}\right)^{\frac{1}{4}} = 2$

2. Solve the following equations involving exponents.

   a. $2^{x} = 2$

   b. $3^{x} = 9$

   c. $25^{2x-1} = 125^{3x+4}$

   d. $\left(\frac{1}{2}\right)^{x} = 16$

   e. $9^{2x-5} = 27$

   f. $\left(\frac{3}{2}\right)^{x} = \frac{81}{16}$

   g. $2^{x+6} = 32$

   h. $x^{\frac{2}{5}} = 4$

   i. $8^{2x-3} = \left(\left(\frac{1}{16}\right)^{x-2}\right)$

Equations involving radicals

Definition 3.4

Radical equations are equations that contain variables in the radicand (the expression under a radical symbol).

For example, $\sqrt{3x+18} = x$, $\sqrt{x+3} = x - 3$ and $\sqrt{x+5} - \sqrt{x-3} = 2$ are radical equations.

In solving radical equations, the following are steps to follow.

1. Positive radical number and radicand (the expression under a radical symbol) should be positive. Thus, at first, specify the domain of x. That is, $\sqrt{a} > 0,\ a > 0$

2. Isolate the radical expression on one side of the equal sign. Put all remaining terms on the other side.
3. If the radical is a square root, then square both sides of the equation.
4. Solve the resulting equation.
5. If a radical term still remains, repeat steps 1–2.
6. Check solutions by substituting them into the original equation.

Example 1

Solve the equation $\sqrt{15-2x}=x$.

Solution:

First specify the domain of x.

Since the radicand term $(15-2x)$ should be non-negative, that is $15-2x\geq 0$.

Thus, the right side should also be non-negative too. $x\geq 0$

From the 1st inequality, $2x\leq 15$ then $x\leq\frac{15}{2}$

Hence, $0\leq x\leq\frac{15}{2}$

Squaring both sides of the given equation,

$$15-2x=x^2$$

$$x^2+2x-15=0$$

$$(x-3)(x+5)=0$$

$$x-3=0 \text{ or } x+5=0$$

$$x=3 \text{ or } x=-5$$

Observe that $x=\ -5$ cannot be a solution. (Why? Because $0\leq x\leq\frac{15}{2}$)

Example 2

Solve the equation: $\sqrt{x+3}=3x-1$

Solution:

First specify the domain of x.

$$x+3 \geq 0 \quad \text{and} \quad 3x-1 \geq 0$$

$$x \geq -3 \;\; \text{and}\; x \geq \frac{1}{3}, \qquad \text{Thus,}\;\; x \geq \frac{1}{3}$$

Squaring both sides of the given equation,

$$x+3 = 9x^2 - 6x + 1$$

$$9x^2 - 7x - 2 = 0$$

$$(x-1)(9x+2) = 0 \quad \text{(after factorizing)}$$

$$x = 1 \;\; \text{or} \;\; x = -\frac{2}{9}$$

Observe that $x = 1$ is the only solution, as $x \geq \frac{1}{3}$.

Example 3

Solve the equation: $\sqrt{2x+3} + \sqrt{x-2} = 4$

Solution:

First specify the domain of x.

$2x+3 \geq 0$ and $x-2 \geq 0$ From the two inequalities, $x \geq 2$……(i)

In addition, $4 - \sqrt{x-2} \geq 0$ Then, $\sqrt{x-2} \leq 4$……(ii)

The equation becomes $\sqrt{2x+3} = 4 - \sqrt{x-2}$

Squaring both sides of the equation,

$$2x+3 = 16 - 8\sqrt{x-2} + x - 2$$

$$x - 11 = -8\sqrt{x-2}$$

Squaring both sides of the equation again,

$$x^2 - 22x + 121 = 64(x-2)$$

$$x^2 - 86x + 249 = 0$$

$$(x-3)(x-83) = 0 \quad \text{(After factorizing)}$$

$$x = 3 \;\; \text{or} \;\; x = 83$$

Observe that $x = 3$ is the only solution, while $x =$ 83 cannot be a solution, because $x = 3$ meet both conditions (i) and (ii), however, $x = 83$ does not.

Exercise 3.18

Solve the following equations involving radicals.

a. $\sqrt{3x-2} = x$ b. $\sqrt{3x-2} = \sqrt{4x+2}$

c. $\sqrt{3x+7} + \sqrt{x+2} = 1$

3.4 Some Applications of Solving Equations

Application of the system of linear equations in two variables

Activity 3.7

Meal tickets at a wedding ceremony cost Birr 4.00 for children and Birr 12.00 for adults. If 1,650 meal tickets were bought for a total of Birr 14,200, how many children and how many adults bought meal tickets?

Example

The cost of a ticket to a circus is Birr 25.00 for children and Birr 50.00 for adults. On a certain day, attendance at the circus is 2,000 and the total gate revenue is Birr 70,000. How many children and how many adults bought tickets?

Solution:

Let c = the number of children and a = the number of adults in attendance.

The total number of people is 2,000. We can use this to write an equation for the number of people at the circus that day.

$$c + a = 2{,}000$$

The money collected from all children can be found by multiplying Birr 25.00 by the number of children, $25c$. The money collected from all adults can be found by multiplying Birr 50.00 by the number of adults, $50a$. The total revenue is Birr 70,000. We can use this to write an equation for the revenue.

$$25c + 50a = 70{,}000$$

We now have a system of linear equations in two variables.

$$c + a = 2{,}000$$

$$25c + 50a = 70{,}000$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either c *and* a. Let us solve for a.

$$a = 2{,}000 - c$$

Substitute the expression $2{,}000 - c$ in the second equation for a and solve for c.

$$25c + 50(2{,}000 - c) = 70{,}000$$

$$-25c = -30000$$

$$c = 1200$$

Substitute $c = 1200$ into the first equation to solve for a.

$$1200 + a = 2000$$

$$a = 800$$

We find that 1,200 children and 800 adults bought tickets to the circus that day.

Exercise 3.19

1. The cost of a ticket to a football match is Birr 50.00 for children and Birr 75.00 for adults. On a certain day, attendance at the football match is 25,000 and the total gate revenue is Birr 1,375,000. How many children and adults bought tickets?
2. The cost of two tables and three chairs is Birr 705. If the table costs Birr 40 more than the chair, find the cost of the table and the chair.
3. Abdullah is choosing between two car-rental companies. The first, "Keep on carefully driving", charges an up-front fee of Birr 15 and 61 cents a kilometer. The second, "Atiften Tidesaleh", charges an up-front fee of Birr 12 and 65 cents a kilometer. When will "Keep on carefully driving" be better choice for Abdullah?

Application of quadratic equations

Example 1

An object dropped from a height of 600m has a height, h, in meter after t seconds have elapsed, such that $h = 600 - 16t^2$. Find the time taken to reach a height of $200m$ by first finding an expression for $t > 0$.

Solution:

We are asked two things. First, we will solve the height equation for t, and then we will find how long it takes for the object's height above the ground to be $200m$. Solve for t:

$$h = 600 - 16t^2$$

$$t^2 = \frac{600-h}{16}$$

$$t = \sqrt{\frac{600-h}{16}} = \frac{\sqrt{600-h}}{4}$$

We want time to be only positive since we are talking about a measurable quantity, so we will restrict our answers to just $t = \frac{\sqrt{600-h}}{4}$

We want to know at what time the height will be $200m$. So we can substitute 200 for h.

$$t = \frac{\sqrt{600-200}}{4} = \frac{\sqrt{400}}{4} = \frac{20}{4} = 5$$

Therefore, it takes 5 seconds for the object to be at the height of $200m$.

Example 2

Suppose a rocket projectile is launched from a tower into the air with an initial velocity of 48 meters per second. Its height, h, in meter above the ground is modeled by the equation $h = -16t^2 + v_0t + 64$ where $t > 0$ is the time in second, since the projectile was launched and v_0 is the initial velocity. How long was the projectile in the air?

Solution:

The rocket projectile was in the air till the height $h = 0$, with initial velocity $v_0 = 48m/s$

$$0 = -16t^2 + 48t + 64$$

$$0 = 16(t^2 - 3t - 4)$$

$$0 = 16(t + 1)(t - 4)$$

Hence, $t = 4$ seconds. , $t = -1$ cannot be taken as a solution. Why? This means that the rocket projectile hits the ground after 4 seconds.

Exercise 3.20

A ball is shot into the air from the ground. Its initial velocity is 32 meter per second. The equation $h = -16t^2 + 32t + 48$ can be used to model the height of the ball after t seconds. About how long does it take for the ball to hit the ground?

Summary

1. Equations are expressions involving equality.
2. Linear equation in one variable is of the form $ax + b = 0$, where x is the variable and a and b are real coefficients and $a \neq 0$.
3. An equation of the type $cx + dy = 0$, where c and d are arbitrary constants $c \neq 0$ and $d \neq 0$ is said to be a linear equation in two variables and its solutions are infinitely many points and the graph is a straight line.
4. A system of linear equation is a set of two or more linear equations, and a system of two linear equations in two variables are equations that can be represented as

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2 = c_2 \end{cases}$$

5. Solution to a system of linear equation in two variables is the set of ordered pairs (x, y) that satisfy both linear equations.
   a. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the system has infinite solutions and is called dependent system.
   b. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the system has no solution and is called inconsistent system.
   c. If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then the system has one solution and is called independent system.
6. Geometrically,
   a. The system has infinite solution if the two lines coincide (or identical).
   b. The system is said to have no solution if the lines are parallel, i.e. no intersection point.
   c. The system is said to have one solution if the lines intersect at one point.

7. A system of linear equation in two variables can be solved by one of the following methods: graphically, or by substitution or by elimination.
8. The absolute value of a number x, denoted by $|x|$, is defined as the distance $|x|$ from zero on a number line.
9. For any non- negative number p, $|x| = p$ means $x = p$ or $x = -p$.
10. An equation of the form $ax^2 + bx + c = 0$ where $a, b, c, \in R$ and $a \neq 0$ is a quadratic equation.
11. Factorization, completing the square and quadratic formula are methods which can be used to solve a quadratic equation $ax^2 + bx + c = 0$
12. If the quadratic equation, $ax^2 + bx + c = 0$, has roots $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, then the sum and product of the roots are $-\frac{b}{a}$ and $\frac{c}{a}$, respectively.
13. The discriminant of quadratic equations, $D = b^2 - 4ac$,

    $If\ D > 0$, then the quadratic equation has two distinct roots.

    $If\ D = 0$, then the quadratic equation has exactly one real root.

    $If\ D < 0$, then there are no real roots.
14. Let $x^n = k$. If n is even, $x = \pm\sqrt[n]{k}$. If n is odd, $x = \sqrt[n]{k}$.
15. For $a > 0, a^x = a^y$ if and only if $x = y$.
16. For solving radical equations, specify the domain of x, as the radicand (the expression under a radical symbol) should be positive.

Review Exercise

For questions 1-3, choose the correct answer from the given alternatives.

1. Which of the following linear equation passes through the origin?

   a) $y = 3x + 0$ b) $y = 3x + 1$ c) $y = 3x - 10$ d) $y = 3x - \frac{1}{2}$

2. Which of the following is the solution of the linear equation $x + 2 = -x - 4$

   a) $x = 3$ b) $x = -3$ c) $x = -10$ d) $x = -\frac{1}{2}$

3. Which of the following is the solution of $-2x + 2 = x + 4 - 3x$

a) $x = 1$ b) $x = 0$ c) $x = -1$ d) No solution

For questions 4-9 use any Method to find the solution to the given system or to determine if the system is inconsistent or dependent

4. $\begin{cases} x - 7y = -11 \\ 5x + 2y = -18 \end{cases}$

5. $\begin{cases} -4x + 2y = 3 \\ 2x - y = -12 \end{cases}$

6. $\begin{cases} 3x + 9y = -6 \\ -4x - 12y = 8 \end{cases}$

7. $\begin{cases} 6x - 5y = 8 \\ -12x + 2y = 0 \end{cases}$

8. $\begin{cases} -x + 5y = 2 \\ 5x - 25y = -10 \end{cases}$

9. $\begin{cases} 2x + 3y = 20 \\ x + \frac{3}{2}y = 10 \end{cases}$

For questions 10-39, solve each of the following non-linear equations.

10. $|4x - 7| = 5$
11. $|3 - 4x| = 5$
12. $|2 - 4x| = 2$
13. $|\frac{2}{3}x - 7| = 0$
14. $|x^2 - 1| = 3$
15. $|x^2 + 9| = 1$
16. $x^2 - 3x = 0$
17. $x^2 + 9x = 0$
18. $x^2 + 8x + 15 = 0$
19. $x^2 - 5x + 7 = 0$
20. $-3x^2 + x + 4 = 0$
21. $x^2 + 6x + 9 = 0$
22. $2x^2 + 5x + 3 = 0$
23. $2x^2 - 5x + 2 = 0$
24. $6x^2 - x - 1 = 0$
25. $-5x^2 + 6x - 1 = 0$
26. $2x^2 - 11x + 14 = 0$
27. $2x^2 - 7x - 1 = 0$
28. $16x - 7x^2 = 79$
29. $4^{2x+5} = 64$
30. $9^{x+1} = 81$
31. $64 \cdot 4^{3x} = 16$
32. $2^{-2x+5} \cdot \frac{1}{4} = 2^{x+2}$
33. $6^{2x-3} = 6$
34. $(x - 4)^{2/3} = 25$
35. $(x + 5)^{3/2} = 8$
36. $(x + 12)^{3/2} = 8$
37. $\sqrt{3x + 18} = x$
38. $\sqrt{x + 3} = x - 3$
39. $\sqrt{x + 5} - \sqrt{x - 3} = 2$

For questions 40-54, solve the following word problems.

40. If $x = 2$ and $y = 3$ is a solution of the equation $8x - ay + 2a = 0$, find the value of a.

41. If the sum of two numbers is 13 and their product is 42 find the numbers.

42. The present age of Gaddisa is 4 years more than twice the present age of his cousin Chala. Chala's present age is 2 years more than $\frac{1}{3}$ of the present age of Gaddisa. Find the difference of their ages.

43. We want to fence a field whose length is three times the width and we have 100 meters of fencing material. If we use all the fencing material, what would the dimensions of the field be?

44. Three coffees and two makiatos cost a total of Birr 17. Two coffees and four makiatos cost a total of Birr 18. What is the individual price for a single coffee and a single makiato?

45. Which of the following is the solution for the system of linear equations

$$\begin{cases} 3x + y = 11 \\ y = x + 2 \end{cases} ?$$

A) $x = 3,\ y = 2$ B) $x = 3,\ y = 5$

C) $x = \frac{9}{4},\ y = \frac{17}{4}$ D) $x = \frac{17}{4},\ y = \frac{9}{4}$

46. The solution for the quadratic equation $x^2 + 10x = -25$ is:

A) 10 B) -5 C) -10 D) 5

47. The solution for the absolute value equation $|-2x + 8| = 12$ is:

A) $x = 2,\ x = 10$ B) $x = -2,\ x = 10$

C) $x = 2,\ x = -10$ D) $x = -2,\ x = -10$

48. Find the values of k for which the quadratic expression $(x - k)(x - 10) + 1 = 0$ has integral roots.

49. Find the values of k such that the equation $\frac{p}{x+r} + \frac{q}{x-r} = \frac{k}{2x}$ has two equal roots.

50. Find the quadratic equation with rational coefficients when one root is $\frac{1}{2+\sqrt{5}}$.

51. If the coefficient of x in the quadratic equation $x^2 + bx + c = 0$ was taken as 17 in place of 13, its roots were found to be -2 and -15. Find the roots of the original quadratic equation.

52. For what value of k, both the quadratic equations $6x^2 - 17x + 12 = 0$ and $3x^2 - 2x + k = 0$ will have a common root.

53. Solve $|x^2 + 2x - 4| = 4$