UNIT 2

THE NUMBER SYSTEM

Unit Outcomes

By the end of this unit, you will be able to:

- **Describe rational numbers.**
- **Locate rational numbers on number line.**
- **Describe irrational numbers.**
- **Locate some irrational numbers on a number line.**
- **Define real numbers.**
- **Classify real numbers as rational and irrational.**
- **Solve mathematical problems involving real numbers.**

Unit Contents

KEY Words

- division algorithm
- fundamental theorem of arithmetic
- rationalizing factor
- irrational number
- rationalization
- perfect square
- greatest common factor
- principal n^{th} root
- prime factorization
- repeating decimal
- scientific notation
- bar notation
- significant digits
- significant figures
- terminating decimal
- least common multiple (LCM)
- real number
- radicand
- composite number

Introduction

In the previous grades, you learned number systems about natural numbers, integers and rational numbers. You have discussed meaning of natural numbers, integers and rational numbers, the basic properties and operations on the above number systems. In this unit, after revising those properties of natural numbers, integers and rational numbers, you will continue to learn about irrational and real numbers.

2.1 Revision on Natural Numbers and Integers

Activity 2.1

1. List five members of :-
   a. Natural numbers
   b. Integers
2. Select natural numbers and integers from the following.
   a. 6 b. 0 c. -25
3. What is the relationship between natural numbers and integers?

4. Decide if the following statements are **always true, sometimes true or never true** and provide your justification.
   a. Natural numbers are integers
   b. Integers are natural numbers
   c. -7 is a natural number
5. Draw diagram which shows the relationship of Natural numbers and Integers.

The collection of well-defined distinct objects is known as a set. The word well-defined refers to a specific property which makes it easy to identify whether the given object belongs to the set or not. The word 'distinct' means that the objects of a set must be all different.

From your grade 7 mathematics lessons, you recall that

- The set of natural numbers, which is denoted by $\mathbb{N}$ expressed as $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- The set of integers, which is denoted by $\mathbb{Z}$ is expressed as $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$.

Example

Categorize each of the following as natural numbers and integers.

$$3, -2, 11, 0, -18, 15, 7$$

Solution:

All the given numbers are integers and 3, 11, 15 and 7 are natural numbers.

Exercise 2.1

1. Categorize each of the following as natural numbers and integers.
   $8, -11, 23, 534, 0, -46, -19, 100$
2. What is the last integer before one thousand?

3. Consider any two natural numbers n_1 and n_2.
   a. Is $n_1 + n_2$ a natural number? Explain using example.
   b. Is $n_1 - n_2$ a natural number? Explain using example.
   c. What can you conclude from (a) and (b)?
4. If the perimeter of a triangle is 10 and lengths of the sides are natural numbers, find all the possible lengths of sides of the triangle.
5. Assume m and n are two positive integers and $m + n < 10$. How many different values can the product mn (m multiplied by n) have?

2.1.1 Euclid's Division lemma

Activity 2.2

1. In a book store there are 115 different books to be distributed to 8 students. If the book store shares these books equally, how many books will each student receive and how many books will be left?
2. Divide a natural number 128 by 6. What is the quotient and remainder of this process? Can you guess a remainder before performing the division process?

From your activity, using the process of dividing one positive integer by another, you will get remainder and quotient as described in the following theorem.

Theorem 2.1 Euclid's Division lemma

Given a non-negative integer a and a positive integer b, there exist unique non-negative integers q and r satisfying

$$a = b \times q + r \quad \text{with } 0 \leq r < b.$$

In theorem 2.1, a is called the dividend, q is called the quotient, b is called the divisor, and r is called the remainder.

Example

Find the unique quotient and remainder when a positive integer.

a. 38 is divided by 4 **b.** 5 is divided by 14

c. 12 is divided by 3 **d.** 2,574 is divided by 8

Solution:

a. Here, it is given that the dividend is $a = 38$ and the divisor is $= 4$. So that we need to determine the unique numbers q and r.When we divide 38 by 4, we get a quotient $q = 9$ and a remainder $r = 2$. Hence, we can write this as $38 = 4 \times 9 + 2$.

$$\begin{array}{lcrl} & & 9 & \leftarrow \textit{Quotient} \\ \textit{Divisor} \longrightarrow & 4 & \overline{)38} & \longleftarrow \textit{Dividend} \\ & & \underline{36} & \\ \textit{Remainder} \longrightarrow & & 2 & \end{array}$$

b. The number 5 is less than the divisor 14. So, the quotient is 0 and the remainder is 5.

That is, $5 = 14 \times 0 + 5$.

$$\begin{array}{lcrl} & & 0 & \leftarrow \textit{Quotient} \\ \textit{Divisor} \longrightarrow & 14 & \overline{)5} & \longleftarrow \textit{Dividend} \\ & & \underline{0} & \\ \textit{Remainder} \longrightarrow & & 5 & \end{array}$$

c. When we divide 12 by 3, we obtain 4 as a quotient and the remainder is 0. That is

$$12 = 3 \times 4 + 0.$$

$$\begin{array}{lcrl} & & 4 & \leftarrow \textit{Quotient} \\ \textit{Divisor} \longrightarrow & 3 & \overline{)12} & \longleftarrow \textit{Dividend} \\ & & \underline{12} & \\ \textit{Remainder} \longrightarrow & & 0 & \end{array}$$

d. Using long division if 2,574 is divided by 8, we get 321 as a quotient and 6 as a remainder. We can write $2{,}574 = 8 \times 321 + 6$.

$$\begin{array}{lcrl} & & 321 & \leftarrow \textit{Quotient} \\ \textit{Divisor} \longrightarrow & 8 & \overline{)2574} & \longleftarrow \textit{Dividend} \\ & & \underline{-24} & \\ & & 17 & \\ & & \underline{-16} & \\ & & 14 & \\ & & \underline{-\ 8} & \\ \textit{Remainder} \longrightarrow & & 6 & \end{array}$$

Note

For two positive integers a and b in the division lemma, we say a is divisible by b if the remainder r is zero.

In the above example (c), 12 is divisible by 3 since the remainder is 0.

Exercise 2.2

1. For each of the following pairs of numbers, let a be the first number of the pair and b be the second number. Find q and r for each pair such that $a = b \times q + r$, where $0 \leq r < b$.

   a. $14, 3$ b. $116, 7$ c. $570, 6$ d. $25, 36$ e. $987, 16$

2. Find the unique quotient and remainder when 31 is divided by 6.
3. Find four positive integers when divided by 4 leaves remainder 3.
4. A man has Birr 68. He plans to buy items such that each costs Birr 7. If he needs Birr 5 to remain in his pocket, what is the maximum number of items he can buy?
5. Find the remainder of $\frac{(5m+1)(5m+3)(5m+6)}{5}$ for m is non-negative integer.

2.1.2 Prime numbers and composite numbers

In this subsection, you will confirm important facts about prime and composite numbers. The following activity (activity 2.3) will help you to refresh your memory.

Activity 2.3

1. Fill in the blanks to make the statements correct using the numbers 3 and 12.

   a. _____________is a factor of ______________.

   b. _____________is divisible by _____________.

   c. _____________is a multiple of ______________.

2. For each of the following statements write 'true' if the statement is correct and 'false' otherwise. If your answer is false give justification why it is false.
   a. 1 is a factor of all natural numbers.
   b. There is no even prime number.
   c. 23 is a prime number.
   d. If a number is natural number, it is either prime or composite.
   e. 351 is divisible by 3.
   f. $2^2 \times 3 \times 7$ is the prime factorization of 84.
   g. 63 is a multiple of 21.
3. Write factors of :
   a. 7 b. 15

Observations

Given two natural numbers h and p, h is called a multiple of p if there is a natural number q such that $h = p \times q$. In this way we can say:

- p is called a factor or a divisor of h.
- h is divisible by p.
- q is also a factor or divisor of h.
- h is divisible by q.

Hence, for any two natural numbers h and p, h is divisible by p if there exists a natural number q such that $h = p \times q$.

Definition 2.1 Prime and composite numbers

A natural number that has exactly two distinct factors, namely 1 (one) and itself is called a **prime number** whereas a natural number that has more than two factors is called a **composite number**.

Example 1

Is 18 a prime number or a composite number? Why?

Solution:

Observe that, $18 = 1 \times 18$, $18 = 2 \times 9$ or $18 = 3 \times 6$ .This indicates $1, 2, 3, 6, 9$ and 18 are factors of 18. Hence, 18 is a composite number.

Example 2

Find a prime number(s) greater than 50 and less than 55.

Solution:

The natural numbers greater than 50 and less than 55 are $51, 52, 53$ and 54.

$51 = 3 \times 17$, so that 3 and 17 are factors of 51 .

$52 = 2 \times 26$, so that 2 and 26 are factors of 52, and

$54 = 2 \times 27$, hence 2 and 27 are factors of 54. Therefore, these three numbers are composite numbers. But $53 = 1 \times 53$. Hence, 1 and 53 are the only factors of 53 so that 53 is a prime number. Therefore, 53 is a prime number greater than 50 and less than 55.

Exercise 2.3

1. Is 21 a prime number or a composite number? Why?
2. Write true if the statement is correct and false otherwise
   a. There are 7 prime numbers between 1 and 20.
   b. $4, 6, 15$ and 21 are composite numbers.
   c. The smallest composite number is 2.
   d. 101 is the prime number nearest to 100.
   e. No prime number greater than 5 ends with 5.
3. Is 28 a composite number? If so, list all of its factors.
4. Can we generalize that 'if a number is odd, then it is prime'? Why?

5. Which one of the following is true about composite numbers?

A. They have 3 pairs of factors
B. They are always prime numbers
C. They do not have factors
D. They have more than two factors

Note

- 1 is neither prime nor composite.
- 2 is the only even prime number.
- Factors of a number are always less than or equal to the number.

2.1.3 Divisibility test

In the previous lesson, you practiced how to get the quotient and remainder while you divide a positive integer by another positive integer. Now you will revise the divisibility test to check whether a is divisible by b or not without performing division algorithm.

Activity 2.4

Check whether the first integer is divisible by the second or not without using division algorithm.

a. $2584, 2$ b. $765, 9$ c. $63885, 6$ d. $7964, 4$ e. $65475, 5$

From your activity, you may check this either by using division algorithm or by applying rules without division.

Every number is divisible by 1. You need to perform the division procedure to check divisibility of one natural number by another. The following rules can help you to determine whether a number is divisible by $2, 3, 4, 5, 6, 8, 9$ and 10. Divisibility test by 7 will not be discussed now because it is beyond the scope of this level.

Divisibility test:- It is an easy way to check whether a given number is divisible by $2, 3, 4, 5, 6, 8, 9$ or 10 without actually performing the division process.

A number is divisible by

- 2, if its unit digit is divisible by 2.
- 3, if the sum of its digits is divisible by 3.
- 4, if the number formed by its last two digits is divisible by 4.
- 5, if its unit digit is either 0 or 5.
- 6, if it is divisible by 2 and 3.
- 8, if the number formed by its last three digits is divisible by 8.
- 9, if the sum of its digits is divisible by 9.
- 10, if its unit digit is 0.

Example

Using divisibility test check whether 2,334 is divisible by 2, 3, 4, 5, 6, 8, 9 and 10.

Solution:

- 2,334 is divisible by 2 since its unit digit 4 is divisible by 2.
- 2,334 is divisible by 3 since the sum of the digits $(2 + 3 + 3 + 4)$ is 12 and it is divisible by 3.
- 2,334 is not divisible by 4 since its last two digits, that is 34 is not divisible by 4.
- 2,334 is not divisible by 5 since its unit digit is not either 0 or 5.
- 2,334 is divisible by 6 since it is divisible by 2 and 3.
- 2,334 is not divisible by 8 since its last three digits 334 is not divisible by 8.
- 2,334 is not divisible by 9 since the sum of the digits $(2 + 3 + 3 + 4 = 12)$ is not divisible by 9.
- 2,334 is not divisible by 10 since its unit digit is not zero.

Exercise 2.4

1. Using divisibility test, check whether the following numbers are divisible by

2, 3, 4, 5, 6, 8, 9 and 10:

a. 384 b. 3,186 c. 42,435

2. Given that $74{,}3x2$ is a number where x is its tens place. If this number is divisible by 8, what is (are) the possible value(s) of x?
3. Find the least possible value of the blank space so that the number 3457__40 is divisible by 4.
4. Fill the blank space with the smallest possible digit that makes the given number 81231_37 is divisible by 9.

Definition 2.2 Prime factorization

The expression of a composite number as a product of prime numbers is called prime factorization.

Consider a composite number which we need to write it in prime factorized form. Recall that a composite number has more than two factors. The factors could be prime or still composite. If both of the factors are prime, we stop the process by writing the given number as a product of these prime numbers. If one of the factors is a composite number, we will continue to get factors of this composite number till all the factors become primes. Finally, express the number as a product of all primes which are part of the process.

Example 1

Express each of the following numbers as prime factorization form.

a. 6 **b.** 30 **c.** 72

Solution:

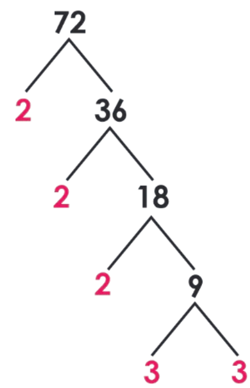
a. $6 = 2 \times 3$, since both 2 and 3 are prime, we stop the process.

b. $30 = 2 \times 15$ and $15 = 3 \times 5$.

So that, $30 = 2 \times 3 \times 5$. The right hand side of this equation is the prime factorization of 30.

The process of prime factorization can be easily visualized by using a factor tree as shown below.

c. By divisibility test, 72 is divisible by 2. So that 2 is one of its factors and it will be written as 2×36 as shown in the right side. Again 36 is divisible by 2. Hence, $36 = 2 \times 18$ and this process will continue till we get the factors as primes. Hence, the prime factorization of 72 is given as



$72 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$. Also we can observe

$72 = 8 \times 9 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$

$72 = 4 \times 18 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$

$72 = 3 \times 24 = 3 \times 3 \times 2 \times 2 \times 2 = 2^3 \times 3^2$.

The following theorem is stated without proof to generalize factors of composite numbers.

Theorem 2.2 Fundamental theorem of arithmetic

Every composite number can be expressed (factored) as a product of primes and this factorization is unique.

Example 2

Express 456 as prime factorization form.

Solution:

To determine the prime factors, we will divide 456 by 2. If the quotient is also a composite number, again using a divisibility test we will search the prime number which divides the given composite number. Repeat this process till the quotient is prime.

That is, $456 \div 2 = 228$

$228 \div 2 = 114$

$114 \div 2 = 57$

$57 \div 3 = 19$, we stop the process since 19 is prime.

Hence, $456 = 2^3 \times 3 \times 19$.

Exercise 2.5

1. Express each of the following numbers as prime factorization form.
   a. 21 b. 70 c. 105
   d. 252 e. 360 f. 1,848
2. 180 can be written as $2^a \times 3^b \times 5^c$. Then, find the value of a, b and c.
3. Find the four prime numbers whose product is 462.

2.1.4 Greatest common factor and least common multiple

In this subsection, you will revise the basic concepts about greatest common factors and least common multiples of two or more natural numbers.

Greatest common factor

Activity 2.5

1. Given the numbers 12 and 16:
   a. Find the common factors of the two numbers.
   b. Find the greatest common factor of the two numbers.
2. Given the following three numbers 24, 42 and 56:
   a. Find the common factors of the three numbers.
   b. Find the greatest common factor of the three numbers.

Definition 2.3

i) Given two or more natural numbers, a number which is a factor of these natural numbers is called a common factor.

ii) The Greatest Common Factor (GCF) or Highest Common Factor (HCF) of two or more natural numbers is the greatest natural number of the common factors. $\mathrm{GCF}(a, b)$ is to mean the Greatest Common Factor of a and b.

Example 1

Find the greatest common factors of 36 and 56.

Solution:

Factors of 36 are $1, 2, 3, 4, 6, 9, 12, 18, 36$ and factors of 56 are $1, 2, 4, 7, 8, 14, 28, 56$. Here 1,2 and 4 are the common factors of both 36 and 56. The greatest one from these common factors is 4. Hence $\mathrm{GCF}(36, 56) = 4$. You can also observe this from the following Venn diagram.

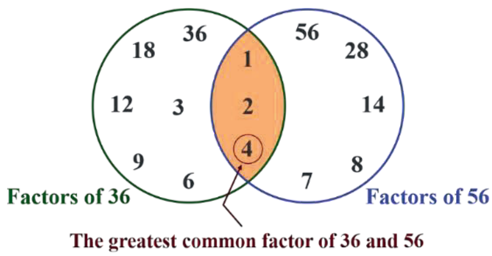


Figure 2.1 Factors of 36 and 56

Activity 2.6

Let $p = 36, q = 56$. Then, write

a. The prime factorization of p and q.

b. Write common prime factors of p and q with the least power.

c. Take the product of the common prime factors you found in the above (b) if they are two or more.

d. Compare your result in (c) with $\mathrm{GCF}(36, 56)$ you got in example 1 above.

The above activity leads you to use another alternate approach of determining GCF of two or more natural numbers using prime factorization. Using this method, the GCF of two or more natural numbers is the product of their common prime factors and the smallest number of times each power appears in the prime factorization of the given numbers.

Example 2

Use prime factorization to find $\text{GCF}(12, 18, 36)$.

Solution:

Let us write each of the given three numbers as prime factorization form.

$$12 = 2^2 \times 3^1$$
$$18 = 2^1 \times 3^2$$
$$36 = 2^2 \times 3^2$$

In the above factorizations, 2 and 3 are common prime factors of the numbers (12,18 and 36). Further the least power of 2 is 1 and least power of 3 is also 1. So that the product of these two common prime numbers with least power of each is $2 \times 3 = 6$. Hence, $\text{GCF}(12, 18, 36) = 6$.

Exercise 2.6

Find the greatest common factors (GCF) of the following numbers.

1. Using Venn diagram method
   a. $12, 18$
   b. $24, 64$
   c. $45, 63, 99$
2. Using prime factorization method
   a. $24, 54$
   b. $108, 104$
   c. $180, 270, 1{,}080$

Least common multiple

Activity 2.7

For this activity, you need to use pen and pencil.

- Write the natural numbers from 1 to 60 using your pen.
- Encircle the number from the list which is a multiple of 6 using pencil.
- Underline the number on the list which is a multiple of 8 using pencil.

Using the above task:

a. Collect the numbers from the list which are both encircled and underlined in a set.

b. What is the least common number from the set you found in (a)?

c. What do you call the number you get in (b) above for the two numbers 6

Definition 2.4

For any two natural numbers a and b, the Least Common Multiple of a and b denoted by $\mathrm{LCM}(a, b)$ is the smallest multiple of both a and b.

Intersection Method

Example 1

Find Least Common Multiple of 2 and 3, that is, $\mathrm{LCM}(2,3)$.

Solution:

$2, 4, \mathbf{6}, 8, 10, \mathbf{12}, 14, 16, \ldots$ are multiples of 2 and $3, \mathbf{6}, 9, \mathbf{12}, 15, \mathbf{18}, \ldots$ are multiples of 3. Hence, $\mathbf{6}, \mathbf{12}, \mathbf{18}, \ldots$ are common multiples of 2 and 3. The least number from the common multiples is 6.

Therefore, $\mathrm{LCM}(2,3)$ is 6.

Example 2

Find Least Common Multiple of 6 and 9, that is, $\mathrm{LCM}(6,9)$.

Solution:

$6, 12, \mathbf{18}, 24, 30, \mathbf{36}$... are multiples of 6 and $9, \mathbf{18}, 27, \mathbf{36}, 45, \ldots$ are multiples of 9. Hence, $\mathbf{18}, \mathbf{36}, \mathbf{54}, \ldots$ are common multiples of 6 and 9. The least number from the common multiples is 18.

Therefore, $\text{LCM}(6, 9)$ is 18.

Factorization Method

Example 3

Find the Least Common Multiple of 2, 3 and 5, that is, $\text{LCM}(2, 3, 5)$ using factorization method.

Solution:

2,3 and 5 are prime numbers. Taking the product of these prime numbers gives us $\text{LCM}(2,3,5) = 2 \times 3 \times 5 = 30$.

Example 4

Find the Least Common Multiple of 6,10 and 16, that is, $\text{LCM}(6, 10, 16)$ using factorization method.

Solution:

Writing each by prime factorization, we have

$$\left.\begin{aligned} 6 &= 2 \times 3 \\ 10 &= 2 \times 5 \\ 16 &= 2^4 \end{aligned}\right\} \text{The prime factors that appear in these factorization are } 2, 3 \text{ and } 5.$$

Taking the product of the highest powers gives us

$\text{LCM}(6, 10, 16) = 2^4 \times 3 \times 5 = 240$.

Activity 2.8

Consider two natural numbers 15 and 42. Then, find

a. $\text{LCM}(15,42)$ and $\text{GCF}(15, 42)$.
b. 15×42

c. What is the product of $\mathrm{GCF}(15, 42)$ and $\mathrm{LCM}(15, 42)$?

d. Compare your results of (b) and (c).

e. What do you generalize from (d)?

From the above activity you can deduce that

- For any two natural numbers a and b, $\mathrm{GCF}(a, b) \times \mathrm{LCM}(a, b) = ab$.

Example 5

Find $\mathrm{GCF}(12, 18) \times \mathrm{LCM}(12, 18)$.

Solution:

The product of $\mathrm{GCF}(a, b)$ and $\mathrm{LCM}(a, b)$ is the product of the two numbers a and b for any two natural numbers a and b. Hence

$\mathrm{GCF}(12, 18) \times \mathrm{LCM}(12, 18) = 12 \times 18 = 216$.

Exercise 2.7

1. Find the least common multiples (LCM) of the following list of numbers using both intersection method and prime factorization method.

   a. $6, 15$ b. $14, 21$ c. $4, 15, 21$ d. $6, 10, 15, 18$

2. Find GCF and LCM of each of the following numbers

   a. 4 and 9 b. 7 and 48 c. 12 and 32

   d. 16 and 39 e. $12, 16$ and 24 f. $4, 18$ and 30

3. Find $\mathrm{GCF}(16, 24) \times \mathrm{LCM}\ (16, 24)$.

4. In a school, the number of participants in Sport, Mini media and Anti Aids club are $60, 84$ and 108 respectively. Find the minimum number of rooms required. In each room the same number of participants are to be seated and all of them being in the same club.

2.2 Rational Numbers

In section 2.1, you have learnt about natural numbers and integers. In this section you will extend the set of integers to the set of rational numbers. You will also discuss how to represent rational numbers as decimals and locate them on the number line.

Activity 2.9

Given integers $7, -2, 6, 0$ and -3.

a. Divide one number by another (except division by 0).
b. From the result obtained in (a) which of them are integers?
c. What can you conclude from the above (a) and (b)?

Now let us discuss about rational numbers

Definition 2.5 Rational Numbers

Any number that can be written in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$ is called a rational number. The set of rational numbers is denoted by $\mathbb{Q}$ and is described as

$$\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\right\}.$$

Example 1

The numbers $\frac{3}{4}, -6, 0, \frac{-13}{10}$ are rational numbers. Here, -6 and 0 can be written as $\frac{-6}{1}$ and $\frac{0}{1}$, respectively. So we can conclude every natural number and integer is a rational number. You can see this relation using figure 2.2.

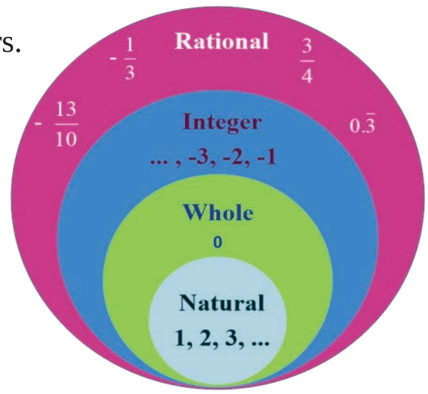


Figure 2.2

Note

Suppose $x = \frac{a}{b} \epsilon \mathbb{Q}$, x is a fraction with numerator a and denominator b,

i) If $a < b$, then x is called proper fraction.

ii) If $a \geq b$, then x is called improper fraction.

iii) If $y = c\frac{a}{b}$, where $c \epsilon \mathbb{Z}$ and $\frac{a}{b}$ is a proper fraction, then y is called a mixed fraction (mixed number).

iv) x is said to be in simplest (lowest form) if a and b are relatively prime or $\mathrm{GCF}(a, b) = 1$.

Example 2

Categorize each of the following as proper, improper or mixed fraction

$\frac{7}{8}, \frac{2}{9}, 5\frac{1}{4}, \frac{7}{3}$ and 6.

Solution: $\frac{7}{8}$ and $\frac{2}{9}$ are proper fractions,

$\frac{7}{3}$ and 6 are improper fraction (since $6 = \frac{6}{1}$) and

$5\frac{1}{4}$ is a mixed fraction.

Example 3

Express $3\frac{1}{4}$ as improper fraction.

Solution: For three integers l, m, n where $n \neq 0$: $l\frac{m}{n} = l + \frac{m}{n} = \frac{(l \times n) + m}{n}$.

So that $3\frac{1}{4} = \frac{(3 \times 4) + 1}{4} = \frac{13}{4}$.

Exercise 2.8

1. Express each of the following integers as fraction.

   a. 5 b. -3 c. 13

2. Write **true** if the statement is correct and **false** otherwise. Give justification if the answer is false.

a. Any integer is a rational number.

b. The simplest form of $\frac{35}{45}$ is $\frac{9}{7}$.

c. For $a, b \in \mathbb{Z}$, $\frac{a}{b}$ is a rational number.

d. The simplest form of $5\frac{2}{4}$ is $\frac{11}{2}$.

3. If $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers, show that $\frac{a}{b} \times \frac{c}{d}$ is also a rational number.
4. Zebiba measures the length of a table and she reads 54 cm and 4 mm. Express this measurement in terms of cm in lowest form of $\frac{a}{b}$.
5. A rope of $5\frac{1}{3}$ meters is to be cut into 4 pieces of equal length. What will be the length of each piece?

2.2.1 Representation of rational numbers by decimals

In this subsection, you will learn how to represent rational numbers by decimals and locate the rational number on the number line.

Activity 2.10

1. Perform each of the following divisions.

a. $\frac{3}{5}$ b. $\frac{5}{9}$ c. $-\frac{14}{15}$ d. $\frac{-2}{7}$

2. Write the numbers 0.2 and 3.31 as a fraction form.

From the above activity 2.10, you may observe the following:

- Any rational number $\frac{a}{b}$ can be written as decimal form by dividing the numerator a by the denominator b.
- When we change a rational number $\frac{a}{b}$ into decimal form, one of the following cases will occur
  - The division process ends when a remainder of zero is obtained. Here, the decimal is called a terminating decimal.
  - The division process does not terminate but repeats as the remainder never become zero. In this case the decimal is called repeating decimal.

In the above activity 2.10 (1), when you perform the division $\frac{3}{5}$, you obtain a decimal 0.6 which terminates, $\frac{5}{9} = 0.555\ldots$ and $\frac{-14}{15} = -0.933\ldots$ are repeating.

Notation

- To represent repetition of digit/digits we put a **bar notation** above the repeating digit/digits.

Example 1

Write the following fractions using the notation of repeating decimals.

a. $\frac{1}{3}$ **b.** $\frac{23}{99}$

Solution:

a. $\frac{1}{3} = 0.333\ldots = 0.\overline{3}$ **b.** $\frac{23}{99} = 0.232323\ldots = 0.\overline{23}$.

Converting terminating decimals to fractions

Every terminating decimal can be written as a fraction form with a denominator of a power of 10 which could be $10, 100, 1000$ and so on depending on the number of digits after a decimal point.

Example 2

Convert each of the following decimals to fraction form.

a. 7.3 **b.** -0.18

Solution:

a. The smallest place value of the digits in the number 7.3 is in the tenths column so we write this as a number of tenths. So that, $7.3 = \frac{7.3\times10}{10} = \frac{73}{10}$ (since it has 1 digit after a decimal point, multiply both the numerator and the denominator by 10).

b. The smallest place value of the digits in the number -0.18 is in the hundredths column so we write this as a number of hundredths.

Hence, $-0.18 = \frac{-0.18\times100}{100} = \frac{-18}{100}$ and after simplification, we obtain $\frac{-9}{50}$.

Exercise 2.9

1. Write the following fractions using the notation of repeating decimals.

   a. $\frac{8}{9}$ b. $\frac{1}{11}$ c. $\frac{5}{6}$ d. $\frac{14}{9}$

2. Convert each of the following as a fraction form.

   a. 0.3 b. 3.7 c. 0.77 d. −12.369

   e. 0.6 f. 9.5 g. −0.48 h. −32.125

3. Using long division, we get that $\frac{1}{7} = 0.\overline{142857}$. Can you predict the decimal expression of $\frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}$ and $\frac{6}{7}$ without doing long division? If so, how?

4. What can you conclude from question no. 3 above?

Representing rational numbers on the number line

Example 1

Locate the rational numbers $-5, 3, \frac{4}{3}$ and $\frac{-5}{2}$ on the number line.

Solution:

You can easily locate the given integers -5 and 3 on a number line. But to locate a fraction, change the fraction into decimal. That is, $\frac{4}{3} = 1.\bar{3}$ and $\frac{-5}{2} = -2.5$. Now locate each of these by bold mark on the number line as shown in figure 2.3.

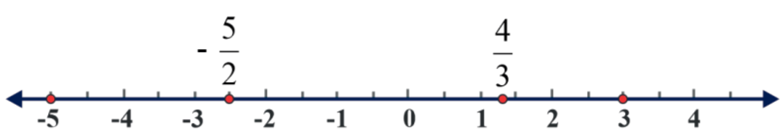


Figure 2.3

2.2.2 Conversion of repeating decimals into fractions

In section 2.2.1, you have discussed how to convert the terminating decimals into fractions. In this subsection, you will learn how repeating decimals can be converted to fractions.

Example 2

Represent each of the following decimals as a simplest fraction form (ratio of two integers).

a. $0.\overline{5}$ **b.** $2.\overline{12}$

Solution:

a. We need to write $0.\overline{5}$ as $\frac{a}{b}$ form. Now let $d = 0.\overline{5}$. Then,

$10d = 5.\overline{5}$ (multiplying both sides of $d = 0.\overline{5}$ by 10)

$\underline{-\quad d = 0.\overline{5}}$ (we use subtraction to eliminate the repeating part)

$9d = 5$.

Now dividing both sides by 9, we have $d = \frac{5}{9}$. Hence, the fraction form of $0.\overline{5}$ is $\frac{5}{9}$.

b. We need to write $2.\overline{12}$ as $\frac{a}{b}$ form. Now let $d = 2.\overline{12}$. Then,

$100d = 212.\overline{12}$ (multiplying both sides of $d = 2.\overline{12}$ by 10)

$\underline{-d = \quad 2.\overline{12}}$ (we use subtraction to eliminate the repeating part)

$99d = 210.$

Now, dividing both sides by 99 results $d = \frac{210}{99} = \frac{70}{33}$. Hence, the fraction form of $2.\overline{12}$ is $\frac{70}{33}$.

Example 3

Represent each of the following in fraction form.

a. $0.1\overline{2}$ **b.** $1.2\overline{16}$

Solution:

a. Let $d = 0.\bar{1}2$, now

$$100d = 12.\bar{2} \quad (\text{multiplying both sides of } d = 0.\bar{1}2 \text{ by } 100)$$

$$\underline{-10d = 1.\bar{2}} \quad (\text{multiplying both sides of } d = 0.\bar{1}2 \text{ by } 10)$$

$$90d = 11.$$

Solving for d, that is dividing both sides by 90 we obtain $d = \frac{11}{90}$. Hence, the fraction form of $0.\bar{1}2$ is $\frac{11}{90}$.

b. Using similar procedure of (a), let $d = 1.\bar{2}16$. Now

$$1000d = 1216.\bar{1}6 \quad (\text{multiplying both sides of } d = 1216.\bar{1}6 \text{ by } 1000)$$

$$\underline{-10d = 12.\bar{1}6} \quad (\text{multiplying both sides of } d = 12.\bar{1}6 \text{ by } 10)$$

$$990d = 1204.$$

Dividing both sides by 990, we obtain $d = \frac{1204}{990} = \frac{602}{495}$. Hence, the fraction form of $1.\bar{2}16$ is $\frac{602}{495}$.

Exercise 2.10

1. Locate the rational numbers $3, 4, \frac{4}{5}, 0.\bar{4}$ and $2\frac{1}{3}$ on the number line.
2. Represent each of the following decimals as simplest fraction form.

   a. $2.\bar{6}$ b. $0.\bar{1}4$ c. $0.\bar{7}16$ d. $1.3\bar{2}12$ e. $-0.53\bar{2}13$

2.3 Irrational Numbers

Recall that terminating and repeating decimals are rational numbers. In this subsection, you will learn about decimals which are neither terminating nor repeating

2.3.1 Neither repeating nor terminating numbers

Do you recall a perfect square?

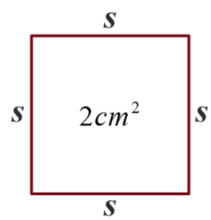
A perfect square is a number that can be expressed as a product of two equal integers. For instance, $1, 4, 9$ are perfect squares since $1 = 1^2, 4 = 2^2, 9 = 3^2$.

A square root of a number x is a number y such that $y^2 = x$. For instance, 5 and -5 are square roots of 25 because $5^2 = (-5)^2 = 25$.

Consider the following situation.

There is a square whose area is 2 cm^2. What is the length of a side of this square?

The length of a side is supposed to be s cm.



Since $1^2 = 1$, area of the square which side is 1 cm is 1 cm^2.

Likewise, $2^2 = 4$, area of the square whose side is 2 cm is 4 cm^2. From the above, $1 < s < 2$.

In a similar way,

$1.4^2 < 2 < 1.5^2 \Rightarrow 1.4 < s < 1.5$

$1.41^2 < 2 < 1.42^2 \Rightarrow 1.41 < s < 1.42$

$1.414^2 < 2 < 1.415^2 \Rightarrow 1.414 < s < 1.415$

Repeating the above method, we find $s = 1.414$ .... This number is expressed as

$s = \sqrt{2}$, and $\sqrt{2} = 1.414$ .... The sign $\sqrt{}$ is called a radical sign.

Example 1

Find the number without radical sign.

a. $\sqrt{4}$ **b.** $\sqrt{9}$ **c.** $-\sqrt{9}$ **d.** $\sqrt{0.0016}$

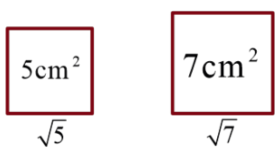
Solution:

a. $\sqrt{4} = 2$ since $2^2 = 4$ **b.** $\sqrt{9} = \sqrt{3^2} = 3 = \frac{3}{1}$

c. $-\sqrt{9} = -\sqrt{3^2} = -3$ **d.** $\sqrt{0.0016} = \sqrt{(0.04)^2} = 0.04 = \frac{4}{100} = \frac{1}{25}$

Consider the magnitude of $\sqrt{5}$ and $\sqrt{7}$. Which one is greater?

The area of square with side length $\sqrt{5}$ cm is 5 cm^2.



The area of square with side length $\sqrt{7}$ cm is 7 cm^2. From the figure, when the side length becomes greater, the area becomes greater, too.

Hence, $\sqrt{5} < \sqrt{7}$ since $5 < 7$. This leads to the following conclusion.

When $a > 0, b > 0$, if $a < b$, then $\sqrt{a} < \sqrt{b}$.

Example 2

Compare $\sqrt{5}$ and $\sqrt{6}$.

Solution:

Since $5 < 6$ then $\sqrt{5} < \sqrt{6}$.

Exercise 2.11

1. Find the numbers without radical sign.

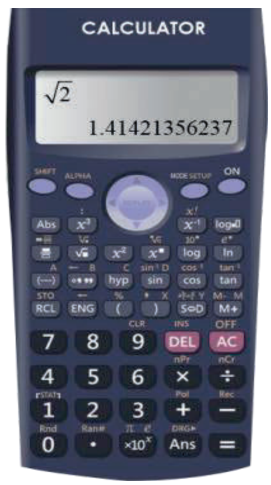
   a. $\sqrt{25}$ b. $-\sqrt{36}$ c. $\sqrt{0.04}$ d. $-\sqrt{0.0081}$

2. Compare the following pairs of numbers.

   a. $\sqrt{7}, \sqrt{8}$ b. $\sqrt{3}, \sqrt{9}$ c. $\sqrt{0.04}, \sqrt{0.01}$ d. $-\sqrt{3}, -\sqrt{4}$

What can you say about the square root of a number which is not a perfect square?

To understand the nature of such numbers, you can use a scientific calculator and practice the following.



Let us try to determine $\sqrt{2}$ using a calculator. Press 2 and then the square root button.

A value for $\sqrt{2}$ will be displayed on the screen of the calculator as $\sqrt{2} \approx 1.4142135624$.

The calculator provides a terminated result due to its capacity (the number of digit you get may be different for different calculators). But this result is not a terminating or

Figure 2.4 Scientific calculator

repeating decimal. Hence $\sqrt{2}$ is not rational number. Similarly, check $\sqrt{3}$ and $\sqrt{7}$ are not terminating

and repeating. Such types of numbers are called irrational numbers.

Definition 2.6

A decimal number that is neither terminating nor repeating is an **irrational number**.

Remark

In general, if a is natural number that is not perfect square, then $\sqrt{a}$ is an irrational number.

Example

Determine whether each of the following numbers is rational or irrational.

a. $\sqrt{25}$ **b.** $\sqrt{0.09}$ **c.** $0.12345\ldots$ **d.** $0.010110111\ldots$ **e.** $\frac{22}{7}$ **f.** π

Solution:

a. $\sqrt{25} = \sqrt{5^2} = 5$ which is rational.

b. $\sqrt{0.09} = \sqrt{0.3^2} = 0.3$ which is rational number.

c. $0.12345\ldots$ is neither terminating nor repeating, so it is irrational.

d. $0.010110111\ldots$ is also neither repeating nor terminating, so it is irrational number.

e. $\frac{22}{7}$ is a fraction form so that it is rational.

f. π which is neither terminating nor repeating, so that it is irrational.

This example (c & d) leads you to the fact that, there are decimals which are neither repeating nor terminating.

Exercise 2.12

Determine whether each of the following numbers is rational or irrational.

a. $\sqrt{36}$ b. $\sqrt{7}$ c. $\frac{6}{5}$ d. $\sqrt{0.01}$

e. $-\sqrt{13}$ f. $\sqrt{11}$ g. $\sqrt{18}$

Locating irrational number on the number line

Given an irrational number of the form $\sqrt{a}$ where a is not perfect square. Can you locate such a number on the number line?

Example

Locate $\sqrt{2}$ on the number line.

Solution:

We know that 2 is a number between perfect squares 1 and 4. That is $1 < 2 < 4$. Take a square root of all these three numbers, that is, $\sqrt{1} < \sqrt{2} < \sqrt{4}$. Therefore $1 < \sqrt{2} < 2$. Hence, $\sqrt{2}$ is a number between 1 and 2 on the number line.

(You can also show $\sqrt{2} \approx 1.4142 \ldots$ using a calculator).

To locate $\sqrt{2}$ on the number line, you need a compass and straightedge ruler to perform the following.

- Draw a number line. Label an initial point 0 and points 1 unit long to the right and left of 0. Construct a perpendicular line segment 1 unit long at 1.

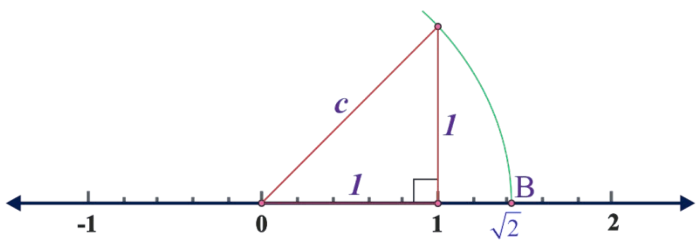


Figure 2.5 Location of $\sqrt{2}$ on the number line

- Draw a line segment from the point corresponding to 0 to the top of the 1unit segment and label its length as c.
- Using Pythagorean Theorem, $1^2 + 1^2 = c^2$ so that $c = \sqrt{2}$ unit long.
- Open the compass to the length of c. With the tip of the compass at the point corresponding to 0, draw an arc that intersects the number line at B. The distance from the point corresponding to 0 to B is $\sqrt{2}$ unit.

  The following figure 2.6 could indicate how other irrational numbers are constructed on the number line using $\sqrt{2}$.

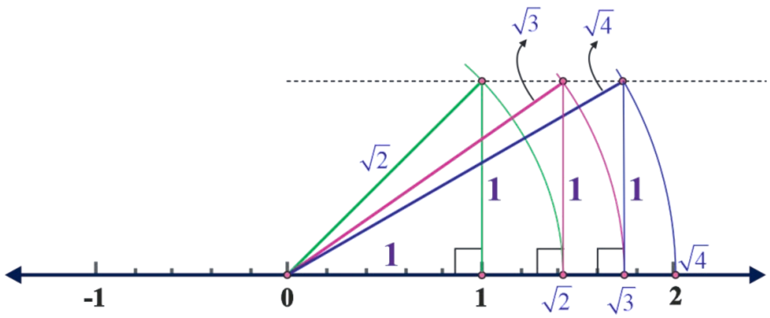


Figure 2.6

Exercise 2.13

1. Locate the following on the number line.

   a. $\sqrt{3}$ b. $\sqrt{5}$ c. $-\sqrt{3}$

2. Between which natural numbers are the following numbers?

   a. $\sqrt{3}$ b. $\sqrt{5}$ c. $\sqrt{6}$

3. Write ‘True’ if the statement is correct and ‘False’ otherwise.

   a. Every point on the number line is of the form $\sqrt{n}$, where n is a natural number.

   b. The square root of all positive integers is irrational.

2.3.2 Operations on irrational numbers

In section 2.3.1, you have seen what an irrational number is and how you represent it on the number line. In this section, we will discuss addition, subtraction, multiplication and division of irrational numbers.

Activity 2.11

1. Identify whether $1+\sqrt{2}$ is rational or irrational.
2. Is the product of two irrational numbers irrational? Justify your answer with examples.

When $a > 0, b > 0$, then $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

Example

Calculate each of the following.

a. $\sqrt{2} \times \sqrt{3}$ **b.** $\frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{3}}$ **c.** $(2+\sqrt{2}) \times (1+\sqrt{5})$ **d.** $(\sqrt{5}+\sqrt{3})^2$

Solutions:

a. $\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3} = \sqrt{6}$

b. $\frac{1}{\sqrt{3}} \times \frac{2}{\sqrt{3}} = \frac{1 \times 2}{\sqrt{3} \times \sqrt{3}} = \frac{2}{\sqrt{3 \times 3}} = \frac{2}{3}$

c. Using distribution of addition over multiplication

$$(2+\sqrt{2}) \times (1+\sqrt{5}) = (2 \times 1) + (2 \times \sqrt{5}) + (\sqrt{2} \times 1) + (\sqrt{2} \times \sqrt{5})$$
$$= 2 + 2\sqrt{5} + \sqrt{2} + \sqrt{10}$$

d. $(\sqrt{5}+\sqrt{3})^2 = (\sqrt{5}+\sqrt{3})(\sqrt{5}+\sqrt{3}) = (\sqrt{5})^2 + 2 \cdot \sqrt{5}\sqrt{3} + (\sqrt{3})^2$
$$= 8 + 2\sqrt{15}$$

Exercise 2.14

1. Calculate each of the following.

   a. $\sqrt{3} \times \sqrt{5}$
   b. $2\sqrt{5} \times \sqrt{7}$
   c. $-\sqrt{2} \times \sqrt{6}$
   d. $\frac{1}{\sqrt{5}} \times \frac{10}{\sqrt{5}}$
   e. $(2+\sqrt{3}) \times (-2+\sqrt{3})$
   f. $(\sqrt{3}+\sqrt{2})^2$
   g. $(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})$
   h. $(\sqrt{6}-\sqrt{10})^2$

2. Decide whether the following statements are **always true, sometimes true or never true** and give your justification.

   a. The product of two irrational numbers is rational.
   b. The product of two irrational numbers is irrational.
   c. Any irrational number can be written as a product of two irrational numbers.
   d. Irrational numbers are closed with respect to multiplication.

Activity 2.12

1. Consider irrational numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{8}$. Divide each of these numbers by $\sqrt{2}$.
2. What do you conclude from the result you obtained in (1) above.

When $a > 0, b > 0$, then $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.

Example 1

Calculate

a. $\frac{\sqrt{2}}{\sqrt{3}}$

b. $\frac{-\sqrt{18}}{\sqrt{2}}$

Solutions:

a. $\frac{\sqrt{2}}{\sqrt{3}} = \sqrt{\frac{2}{3}}$

b. $\frac{-\sqrt{18}}{\sqrt{2}} = -\sqrt{\frac{18}{2}} = -\sqrt{9} = -3$

When $a > 0, b > 0$, then $a\sqrt{b} = \sqrt{a^2b}$, $\sqrt{a^2b} = a\sqrt{b}$.

Example 2

Convert each of the following in $\sqrt{a}$ form.

a. $3\sqrt{2}$

b. $5\sqrt{5}$

Solutions:

a. $3\sqrt{2} = \sqrt{3^2 \times 2} = \sqrt{18}$

b. $5\sqrt{5} = \sqrt{5^2 \times 5} = \sqrt{125}$

Exercise 2.15

1. Calculate.

   a. $\frac{\sqrt{27}}{\sqrt{3}}$ b. $\frac{\sqrt{12}}{\sqrt{3}}$ c. $\frac{\sqrt{14}}{\sqrt{7}}$ d. $\frac{-\sqrt{15}}{\sqrt{3}}$

2. Convert each of the following in $\sqrt{a}$ form.

   a. $4\sqrt{2}$ b. $5\sqrt{3}$ c. $-3\sqrt{7}$ d. $7\sqrt{6}$

Activity 2.13

Give an example which satisfies:

a. Two neither repeating nor terminating decimals whose sum is rational.

b. Two neither repeating nor terminating decimals whose sum is irrational.

c. Any two irrational numbers whose difference is rational.

d. Any two irrational numbers whose difference is irrational.

Example 1

Simplify each of the following.

a. $0.131331333\ldots + 0.535335333\ldots$

b. $0.4747747774\ldots - 0.252552555\ldots$

Solutions:

a. $0.131331333\ldots + 0.535335333\ldots = 0.666\ldots$

b. $0.4747747774\ldots - 0.252552555\ldots. = 0.222\ldots$

Example 2

Simplify each of the following.

a. $2\sqrt{3} + 4\sqrt{3}$ **b.** $3\sqrt{5} - 2\sqrt{5}$

Solutions:

a. $2\sqrt{3} + 4\sqrt{3} = (2+4)\sqrt{3} = 6\sqrt{3}$

b. $3\sqrt{5} - 2\sqrt{5} = (3-2)\sqrt{5} = \sqrt{5}$

Example 3

Simplify each of the following.

a. $\sqrt{8} + \sqrt{2}$ **b.** $\sqrt{12} - 5\sqrt{3}$

c. $\sqrt{18} + \sqrt{50}$ **d.** $\sqrt{72} - \sqrt{8}$

Solutions:

First, convert $\sqrt{a^2 b}$ into $a\sqrt{b}$ form.

a. $\sqrt{8} + \sqrt{2} = \sqrt{4 \times 2} + \sqrt{2} = \sqrt{2^2 \times 2} + \sqrt{2} = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$

b. $\sqrt{12} - 5\sqrt{3} = \sqrt{2^2 \times 3} - 5\sqrt{3} = 2\sqrt{3} - 5\sqrt{3} = -3\sqrt{3}$

c. $\sqrt{18} + \sqrt{50} = \sqrt{3^2 \times 2} + \sqrt{5^2 \times 2} = 3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$

d. $\sqrt{72} - \sqrt{8} = \sqrt{2^2 \times 3^2 \times 2} - \sqrt{2^2 \times 2}$

$$= \sqrt{(2 \times 3)^2 \times 2} - \sqrt{2^2 \times 2} = 6\sqrt{2} - 2\sqrt{2} = 4\sqrt{2}$$

Exercise 2.16

Simplify each of the following.

a. $0.56456445644456\ldots - 0.111\ldots$

b. $2.1010010001\ldots + 1.0101101110\ldots$

c. $2\sqrt{5} - 4\sqrt{5}$

d. $\sqrt{18} + 2\sqrt{2}$

e. $\sqrt{3} + 4\sqrt{3}$

f. $\sqrt{80} - \sqrt{20}$

g. $5\sqrt{8} + 6\sqrt{32}$

h. $\sqrt{8} + \sqrt{72}$

i. $\sqrt{12} - \sqrt{48} + \sqrt{\frac{3}{4}}$

j. $\sqrt{5} - \sqrt{45}$

2.4 Real Numbers

In the previous two sections, you have learnt about rational numbers and irrational numbers. Rational numbers are either terminating or repeating. You can locate these numbers on the number line. You have also discussed that it is possible to locate irrational numbers which are neither repeating nor terminating on the number line.

Activity 2.14

1. Can you think of a set which consists both rational numbers and irrational numbers?
2. What can you say about the correspondence between the points on the number line with rational and irrational numbers?

Based on the reply for the above questions and recalling the previous lessons, you can observe that every decimal number (rational or irrational) corresponds to a point on the number line. So that there should be a set which consists both rational and irrational numbers. This leads to the following definition.

Definition 2.7 Real numbers

A number is called a real number if and only if it is either a rational number or an irrational number. The set of real numbers is denoted by $\mathbb{R}$, and is described as the union of the sets of rational and irrational numbers. We write this mathematically as

$$\mathbb{R} = \{x : x \text{ is a rational number or irrational number}\}.$$

The following diagram indicates the set of real number, $\mathbb{R}$, is the union of rational and irrational numbers

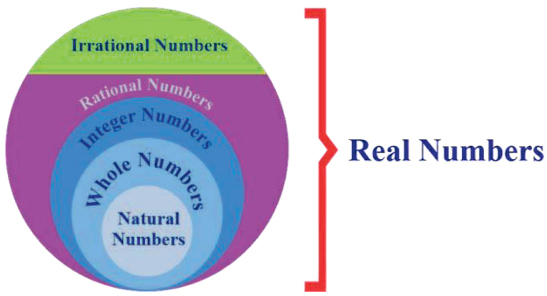


Figure 2.7: Real numbers

Comparing real numbers

You have seen that there is a one to one correspondence between a point on the number line and a real number.

- Suppose two real numbers a and b are given. Then one of the following is true $a < b$ or $a = b$ or $a > b$ (This is called trichotomy property).
- For any three real numbers a, b and c, if $a < b$ and $b < c$, then $a < c$ (called transitive property order)

Applying the above properties we have

- For any two non-negative real numbers a and b if $a^2 < b^2$, then $a < b$.

Example

Compare each pair (you can use Scientific calculator whenever it is necessary).

a. $-2,\ -8$ **b.** $\frac{5}{8}, 0.8$

c. $\frac{\sqrt{2}}{3},\ 0.34$ **d.** $\frac{\sqrt{3}}{6}, \frac{2}{3}$

Solution:

a. The location of these two numbers on the number line help us to compare the given numbers. Here, -8 is located at the left of -2 so that $-8 < -2$.

b. Find the decimal representation of $\frac{5}{8}$, that is $\frac{5}{8} = 0.625$.

Hence, $\frac{5}{8} < 0.8$.

c. Using a scientific calculator we approximate $\frac{\sqrt{2}}{3} \approx 0.4714$. Then, $\frac{\sqrt{2}}{3} > 0.34$.

d. Here use the third property given above. Square the two numbers

$$\left(\frac{\sqrt{3}}{6}\right)^2 = \frac{3}{36} \text{ and } \left(\frac{2}{3}\right)^2 = \frac{4}{9} = \frac{16}{36}.$$

Hence, it follows $\frac{\sqrt{3}}{6} < \frac{2}{3}$.

Exercise 2.17

Compare each of the following pairs (using $>$ or $<$, or $=$).

a. $-7, -3$ b. $\frac{2}{5}, 0.6$

c. $\frac{\sqrt{5}}{2}, 0.123$ d. $\frac{\sqrt{2}}{3}, \frac{4}{9}$

e. $0.236,\ 0.256$ f. $-7\sqrt{145}\ ,\sqrt{7}$

g. $-0.135, -0.1351$ h. $6\sqrt{5}\ , 5\sqrt{7}$

i. $\frac{\sqrt{5}}{3}, 0.234$ j. $2 + \sqrt{3}, 4$

k. $\frac{22}{7}, \pi$

Determining real numbers between two numbers

Example

Find a real number between 2 and 4.2.

Solution:

Take the average of the two numbers, that is, $\frac{2+4.2}{2} = \frac{6.2}{2} = 3.1$.

This number is between 2 and 4.2. That is, $2 < 3.1 < 4.2$. Also take the average of 2 and 3.1. That is $\frac{2+3.1}{2} = 2.55$. Again take the average of 3.1 and 4.2, that is $\frac{3.1+4.2}{2} = 3.65$. These numbers 2.55 and 3.65 are also between 2 and 4.2.

Therefore, we conclude that there are infinitely many real numbers between two real numbers.

Exercise 2.18

1. Find at least two real numbers between:
   a. -0.24 and -0.246
   b. $\frac{3}{5}$ and $\frac{2}{7}$
   c. $\sqrt{2}$ and $\sqrt{3}$
2. How many real numbers are there between two real numbers?

2.4.1 Intervals

Activity 2.15

1. List some of the real numbers between 2 and 6 including 2.
2. List some of the real numbers between $\sqrt{2}$ and 3 excluding the two numbers.
3. Can you denote real numbers in (1) and (2) symbolically?

From activity 2.15 above, you observed that there should be a notation to describe such real numbers.

Case I (Representing real numbers between two points)

A real interval is a set which contains all real numbers between two numbers. We have the following.

Consider a real number x between two real numbers a and b such that $a \leq x \leq b$.

Types of intervals are shown in the table 2.1 below.

Table 2.1

Inequality	Interval Notation	Graph on Number Line	Description
$a < x < b$	(a,b)		x strictly between a and b
$a \leq x < b$	$[a,b)$		x between a and b, including a
$a < x \leq b$	$(a,b]$		x between a and b, including b
$a \leq x \leq b$	$[a,b]$		x between a and b, including a and b

Case II (Representing real numbers in numbers with one end point)

Notation:

The symbol '∞' read as 'infinity' means endlessness or absence of end to the right and '$-\infty$' read as 'negative infinity' means endlessness or absence of end to the left.

Using a point a or b, ∞ and $-\infty$, the intervals contain real numbers as shown below in table 2.2.

Table 2.2

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a

Example

Represent each of the following using interval notation.

a. Real numbers between 3 and 5 including both points.

b. Real numbers between -2 and $\frac{3}{2}$ including -2 and excluding $\frac{3}{2}$.

c. Real numbers on the left of 2 excluding 2.

d. Real numbers on the right of 0 including 0.

Solution:

a. The interval [3,5] represents a set which contains infinitely many real numbers between 3 and 5 including both end points. This is shown below on the number line in figure 2.7(a), where

- the filled circle ● on the number line is to mean <u>included</u> and the empty circle ○ is <u>not included</u> in the interval
- [ ] indicates the interval include the end points and ( ) indicates the interval do not include the end points.

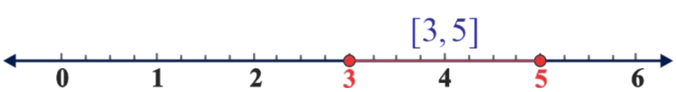


Figure 2.8(a)

b. An interval $[-2, \frac{3}{2})$ represent a set which contain infinitely many real numbers between -2 and $\frac{3}{2}$ including -2 in figure 2.8(b) below.

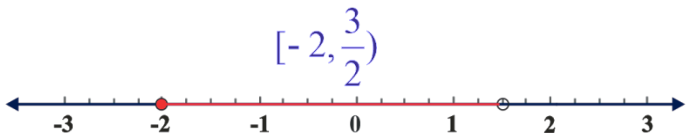


Figure 2.8(b)

c. The real numbers on the left of 2 excluding 2 is denoted by $(-\infty, 2)$ as shown in figure 2.8 (c) below.

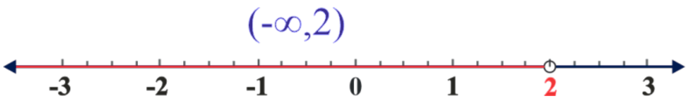


Figure 2.8(c)

d. The real numbers on the right of 0 including 0 is denoted by $[0, \infty)$ as shown in figure 2.8 (d) below.

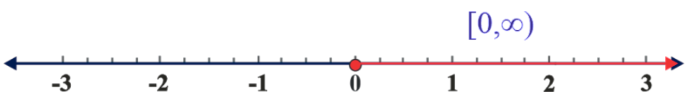


Figure 2.8(d)

Exercise 2.19

1. Represent each of the following using interval notations.
   a. Real numbers between -3 and 8 including both end points.
   b. Real numbers between 4 and 6 excluding the end points.
   c. Real numbers on the right of -1 including -1.
2. Represent each of the following intervals on the number line.
   a. $[1,3)$
   b. $(-\infty, -5]$
   c. $\left(\frac{1}{2}, 3\right)$
   d. $(-2,4) \cup [5, \infty)$

2.4.2 Absolute values

Definition 2.8

The absolute value of a real number, denoted by $|x|$, is defined as

$$|x| = \begin{cases} x \text{ if } x \geq 0, \\ -x \text{ if } x < 0 \end{cases}.$$

Example

Find the absolute value of each of the following real numbers.

a. 4 **b.** -2 **c.** $\frac{1}{2}$

d. -0.4 **e.** $-\sqrt{2}$ **f.** $2-\sqrt{8}$

Solution:

a. So using the definition,

$|4| = 4$, since 4 is a positive number and its absolute value is the number itself.

b. The given number is -2 which is less than zero. But its distance from zero is 2 units. Hence, $|-2| = 2$.

c. Following similar procedure as that of (a) and (b), $\left|\frac{1}{2}\right| = \frac{1}{2}$ (since $\frac{1}{2}$ is a positive number).

d. $|-0.4| = 0.4$ (its distance from zero is 0.4).

e. $|-\sqrt{2}| = \sqrt{2}$. (its distance from zero is $\sqrt{2}$).

f. $|2-\sqrt{8}| = \sqrt{8}-2$ ($\sqrt{8} = \sqrt{2^3} = 2\sqrt{2}$, also $\sqrt{2} \approx 1.414$ so that $2\sqrt{2} > 2$. Hence, $2-\sqrt{8} < 0$ and $\sqrt{8}-2 > 0$).

Activity 2.16

Consider two numbers $a = -3$ and $b = 4$, then

i. Find the number of units between a and b by counting on the number line.

ii. Find $|a - b|$

iii. Find $|b - a|$.

iv. What do you conclude from ii and iii.

You will learn more applications of absolute value in unit three.

Exercise 2.20

1. Find the absolute value of each of the following.

   a. 8 b. -5 c. $-\frac{2}{3}$ d. 0.7

   e. $\sqrt{3}$ f. $1 - \sqrt{2}$ g. $3 - \sqrt{5}$ h. $-0.\bar{1}2$

2. Find the distance between the given numbers on the number line.

   a. 2 and 10 b. 7 and -9 c. -49 and -100 d. -50 and 50

3. Determine the unknown 'x' for each.

   a. $|x| = 8$ b. $|x| = 0$ c. $|x| = -4$

   d. $|x| + 3 = 4$ e. $|x| - 3 = -2$

4. The coldest temperature on the Earth, $-89°C$, was recorded in 1983 at Vostok station, Antarctica. The hottest temperature on the Earth, $58°C$, was recorded at Al'Aziziyah, Libya. Calculate the temperature range on the Earth.

2.4.3 Exponents and radicals

You have learnt about multiplication of two or more real numbers in lower grades. So that you can easily write

$$3 \times 3 \times 3 = 27.$$
$$4 \times 4 \times 4 = 64.$$
$$2 \times 2 \times 2 \times 2 \times 2 = 32.$$

Think of the situation when 14 is multiplied 20 times. It takes more time and space to write.

$$\underbrace{14 \times 14 \times 14 \times ... \times 14}_{20\ times}$$

This difficulty can be overcome by introduction of exponential notations. In this subsection, you shall learn the meaning of such notations, laws of exponents, radicals, simplification of expressions using the laws of exponents and radicals.

Definition 2.9

If a is a real number and n is a positive integer, then

$\underbrace{a \times a \times a \times a \times a \times ... \times a}_{n\ times} = a^n$ is an exponential expression, where a is the base and n is the exponent or power.

For instance: **a)** 2^3 means $2 \times 2 \times 2 = 8$

b) $(-3)^4$ means $(-3) \times (-3) \times (-3) \times (-3) = 81$

c) -2^3 means $-(2 \times 2 \times 2) = -8$.

Activity 2.17

1. Complete the following table.

Power form of a number	2^4	2^3	2^2	2^1	2^0	2^{-1}	2^{-2}	2^{-3}	2^{-4}
A number	16	8	4						

Hint:- To fill the incomplete boxes of the table, observe that the first row of the table is moving to the right by decreasing the power by 1. So that, the boxes on the second row will be determined by dividing the previous number by 2.

2. From the above pattern, find the relationship between 2^n and 2^{-n}.
3. What can you generalize about the relationship between a^n and a^{-n} for a real number $a \neq 0$ and positive integer n.

Note

1. Zero exponent: If $a \neq 0$, then $a^0 = 1$.
2. Negative exponent: If $a \neq 0$ and n is positive integer, then $a^{-n} = \frac{1}{a^n}$.

The above activity leads to the following definition.

Definition 2.10

If $a^2 = b$, then a is a square root of b. If $a^3 = b$, then a is a cube root of b. If n is a positive integer and $a^n = b$, then a is called the n$^{\text{th}}$ root of b.

Definition 2.11 Principal n^{th} root

If b is any real number and n is a positive integer greater than 1, then the principal n^{th} root of b is denoted by $\sqrt[n]{b}$ is defined as

$$\sqrt[n]{b} = \begin{cases} \text{the positive } n^{th} \text{ root of } b \text{, if } b > 0, \\ \text{the negative } n^{th} \text{ root of } b \text{, } if\ b < 0 \text{ and } n \text{ is odd,} \\ 0, \text{ if } b = 0. \end{cases}$$

Notations

- $\sqrt[n]{b}$ called radical expression where $\sqrt{\ }$ is radical sign, n is index and b is radicand. When the index is not written, the radical sign indicates the principal square root.

Definition 2.12 The $\left({}^{1}/{}_{n}\right)^{th}$ power

i) If $b \in \mathbb{R}$ and n is an odd positive integer greater than 1, then $b^{\frac{1}{n}} = \sqrt[n]{b}$.

ii) If $b \geq 0$ and n is an even positive integer greater than 1, then $b^{\frac{1}{n}} = \sqrt[n]{b}$.

Example 1

Find a. $\sqrt{9}$ b. $\sqrt{0.01}$

c. $\sqrt[4]{81}$ d. $\sqrt[5]{-100{,}000}$.

Solution:

a. 3

b. $\sqrt{0.01} = 0.1$ because $(0.1)^2 = 0.01$.

c. $\sqrt[4]{81} = 3$ because $3^4 = 81$.

d. $\sqrt[5]{-100{,}000} = -10$ because $(-10)^5 = -100{,}000$.

Example 2

Express each of the following in exponential form.

a. $\sqrt{3}$ **b.** $\sqrt[3]{6}$ **c.** $\frac{1}{\sqrt[4]{10}}$

Solution:

a. $\sqrt{3} = 3^{\frac{1}{2}}$ **b.** $\sqrt[3]{6} = 6^{\frac{1}{3}}$ **c.** $\frac{1}{\sqrt[4]{10}} = 10^{-\frac{1}{4}}$

Exercise 2.21

1. Find

   a. $\sqrt{36}$ b. $\sqrt{0.016}$ c. $\sqrt[4]{16}$ d. $\sqrt[3]{-1000}$.

2. Express each of the following in exponential form

   a. $\sqrt{5}$ b. $\sqrt[7]{7}$ c. $\frac{1}{\sqrt[3]{3^4}}$

   d. $\left(\sqrt[4]{81}\right)^2$ e. $\sqrt[3]{\frac{2}{5}}$

3. Simplify each of the following

   a. $(-27)^{\frac{1}{3}}$ b. $32^{\frac{1}{5}}$ c. $\frac{\sqrt[3]{125}}{\sqrt{625}}$

   d. $\sqrt{0.09}$ e. $\sqrt[4]{(-5)^4}$

Law of exponents

Activity 2.18

For the given table below,

i) Determine the simplified form of each term.

ii) Compare the value you obtained for (I) and (II) for each row.

No	I	II
1	$2^3 \times 3^3$	$(2 \times 3)^3$
2	$\frac{2^5}{2^3}$	2^{5-3}
3	$(3^2)^3$	$(3^3)^2$
4	$(3 \times 4)^2$	$3^2 \times 4^2$
5	$2^2 \times 2^3$	2^{2+3}

The above activity 2.17 leads you to have the following laws of exponents.

For any $a, b \in \mathbb{R}$ and $n, m \in \mathbb{N}$, the following holds:

- ✓ $a^n \times a^m = a^{n+m}$
- ✓ $\frac{a^n}{a^m} = a^{(n-m)}$, where $a \neq 0$
- ✓ $(a^n)^m = (a^m)^n = a^{nm}$
- ✓ $(a \times b)^n = a^n \times b^n$
- ✓ $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$, for $b \neq 0$.

Note that these rules also applied for $\left(\frac{1}{n}\right)^{th}$ power and rational power of the form $a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m$.

Example 1

Use the above rules and simplify each of the following.

a. $7^{\frac{1}{2}} \times 7^{\frac{3}{2}}$ **b.** $(25 \times 9)^{\frac{1}{2}}$ **c.** $\frac{3^{\frac{5}{2}}}{3^{\frac{1}{2}}}$ **d.** $\left(\frac{8}{27}\right)^{\frac{1}{3}}$ **e.** $(81)^{\frac{1}{4}}$

Solution:

a. Using $a^n \times a^m = a^{n+m}$, $7^{\frac{1}{2}} \times 7^{\frac{3}{2}} = 7^{\left(\frac{1}{2}+\frac{3}{2}\right)} = 7^2 = 49$.

b. Using $(a \times b)^n = a^n \times b^n$,

$(25 \times 9)^{\frac{1}{2}} = 25^{\frac{1}{2}} \times 9^{\frac{1}{2}} = (5^2)^{\frac{1}{2}} \times (3^2)^{\frac{1}{2}} = 5 \times 3 = 15$.

c. Using $\frac{a^n}{a^m} = a^{(n-m)}$, $\frac{3^{\frac{5}{2}}}{3^{\frac{1}{2}}} = 3^{\left(\frac{5}{2}-\frac{1}{2}\right)} = 3^2 = 9$.

d. Using $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$, for $b \neq 0$, $\left(\frac{8}{27}\right)^{\frac{1}{3}} = \frac{8^{\frac{1}{3}}}{27^{\frac{1}{3}}} = \frac{(2^3)^{\frac{1}{3}}}{(3^3)^{\frac{1}{3}}} = \frac{2}{3}$.

e. Using $(a^n)^m = (a^m)^n = a^{nm}$, $(81)^{\frac{1}{4}} = (3^4)^{\frac{1}{4}} = 3$.

Example 2

Simplify each of the following.

a. $\sqrt{50}$ **b.** $\frac{32^{\frac{1}{4}}}{162^{\frac{1}{4}}}$ **c.** $\sqrt[3]{16} \times \sqrt[3]{4}$ **d.** $\left(8 + \sqrt[3]{27}\right)^{\frac{1}{3}}$

Solution:

a. To simplify this, we use different rules of radicals and exponent. 50 can be written as a product of a square number 25 and another number 2.

That is, $\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = \sqrt{5^2} \times \sqrt{2} = 5\sqrt{2}$

b. Both numbers 32 and 162 are multiples of 2 so that we can express these numbers as a power of 2, that is $\frac{32^{\frac{1}{4}}}{162^{\frac{1}{4}}} = \frac{(2 \times 16)^{\frac{1}{4}}}{(2 \times 81)^{\frac{1}{4}}} = \frac{2^{\frac{1}{4}} \times 16^{\frac{1}{4}}}{2^{\frac{1}{4}} \times 81^{\frac{1}{4}}} = \frac{16^{\frac{1}{4}}}{81^{\frac{1}{4}}} = \frac{(2^4)^{\frac{1}{4}}}{(3^4)^{\frac{1}{4}}} = \frac{2}{3}$.

c. Express 16 as a product of 4, so that $\sqrt[3]{16} \times \sqrt[3]{4} = \sqrt[3]{16 \times 4} = \sqrt[3]{4 \times 4 \times 4} = (4^3)^{\frac{1}{3}} = 4$.

d. First express 27 as a multiple of 3 and apply the rule of exponents. This results,

$$\left(8+\sqrt[3]{27}\right)^{\frac{1}{3}}=\left(8+\sqrt[3]{3^3}\right)^{\frac{1}{3}}=\left(8+3^{\frac{3}{3}}\right)^{\frac{1}{3}}=(8+3)^{\frac{1}{3}}=11^{\frac{1}{3}}=\sqrt[3]{11}.$$

Exercise 2.22

1. Use the laws of exponents and simplify each of the following.

   a. $5^{\frac{1}{3}}\times 5^{\frac{2}{3}}$ b. $(16\times 49)^{\frac{1}{2}}$ c. $\dfrac{2^{\frac{10}{3}}}{2^{\frac{1}{3}}}$ d. $32^{\frac{2}{5}}$

2. Simplify each of the following.

   a. $\sqrt{125}$ b. $\dfrac{9^{\frac{1}{3}}}{243^{\frac{1}{3}}}$ c. $\sqrt[5]{0.00032}$

   d. $2^{\frac{3}{2}}\times\sqrt{2}$ e. $\dfrac{5\sqrt{24}\div 2\sqrt{50}}{3\sqrt{3}}$ f. $\sqrt[3]{0.001}+\sqrt[4]{0.0081}$

   g. $(5^{-1})^3\times 5^{\frac{5}{4}}\times\sqrt{25}\times\left(\sqrt[3]{125}\right)^3$ h. $\left(\sqrt{0.64}\right)\times\left(\sqrt[3]{\dfrac{1}{64}}\right)^2\times 32^{\frac{1}{5}}$

   i. $3^{\frac{1}{4}}\times 27^{\frac{1}{4}}$ j. $\sqrt{5+2\sqrt{6}}+\sqrt{8-2\sqrt{15}}$

Addition and subtraction of radicals

Activity 2.19

What do you say about these statements? Which one is correct?

a. $\sqrt{2}-\sqrt{2}=0$ b. $\sqrt{16}-\sqrt{4}=\sqrt{12}$

c. $\sqrt{2}+\sqrt{3}=\sqrt{5}$ d. $4\sqrt{2}-\sqrt{2}=3\sqrt{2}$

Definition 2.13

Radicals that have the same index and the same radicand are said to be like radicals.

For instance:- **i)** $2\sqrt{5}, -\frac{2}{3}\sqrt{5}$ and $\sqrt{5}$ are like radicals

ii) $\sqrt{2}$ and $\sqrt{3}$ are not like radicals

iii) $\sqrt{3}$ and $\sqrt[3]{3}$ are not like radicals

By following the same procedure of addition and subtraction of similar terms, we can add or subtract like radicals. In some cases we got like radicals after simplification.

Example

Simplify each of the following.

a. $\sqrt{3}+\sqrt{12}$ **b.** $2\sqrt{18}-\sqrt{2}+\sqrt{8}$

Solution:

a. $\sqrt{3}+\sqrt{12}=\sqrt{3}+\sqrt{4\times 3}=\sqrt{3}+2\sqrt{3}=(1+2)\sqrt{3}=3\sqrt{3}.$

b. $2\sqrt{18}-\sqrt{2}+\sqrt{8}=2\sqrt{9\times 2}-\sqrt{2}+\sqrt{4\times 2}$

$=6\sqrt{2}-\sqrt{2}+2\sqrt{2}$

$=(6-1+2)\sqrt{2}$ (taking common term)

$=7\sqrt{2}$

Exercise 2.23

Simplify each of the following.

a. $\sqrt{12}-\sqrt{3}$ b. $\sqrt[3]{54}-\sqrt[3]{2}$ c. $3\sqrt[3]{27}-\sqrt[3]{125}+\sqrt{169}$

Operations on real numbers

You discussed the rules of exponents, operations on radicals and how to simplify numbers related to radicals. Recall that a real number system is a collection of rational and irrational numbers. You studied operations on rational numbers on section 2.2 and operations on irrational numbers on section 2.3.2. We can summarize operations on rational and irrational numbers as follows:

- The rational numbers satisfy the commutative, associative and distributive laws for addition and multiplication. Moreover, if we add, subtract, multiply or divide (except by zero) two rational numbers, we still get a rational number (that is, rational numbers are 'closed' with respect to addition, subtraction, multiplication and division).
- Irrational numbers also satisfy the commutative, associative and distributive laws for addition and multiplication. However, the sum, difference, quotients and products of irrational numbers are not always irrational.

Activity 2.20

1. Identify whether each of the following is rational or irrational.

   a. $\pi + 2$ b. $\sqrt{2} - 1$ c. $(1-\sqrt{3}) \times (1+\sqrt{3})$ d. $\frac{\sqrt{5}}{2}$

   e. $\frac{22}{7} \times \frac{22}{7}$ f. $\sqrt{7}(4-\sqrt{7})$ g. $0.2323\ldots + 0.3232\ldots$ h. $\frac{8}{\sqrt{2}}$

2. Give an example for each (if possible).

   a. Two irrational numbers where their product is rational.

   b. A rational and irrational numbers and their sum is rational.

   c. A rational and irrational numbers and their difference is rational.

   d. A rational and irrational numbers and their quotient is irrational.

Now the above activity leads to draw the following conclusions.

- The sum and difference of rational and irrational numbers is irrational.
- The product, quotient of non-zero rational number and an irrational number is an irrational number.

Let us see some examples of four operations on the set of real numbers.

Example 1

If $x = 5\sqrt{3} + 2\sqrt{2}$ and $y = \sqrt{2} - 4\sqrt{3}$, then find

a. the sum of x and y. **b.** the difference between y and x

Solution:

a. $x + y = 5\sqrt{3} + 2\sqrt{2} + \sqrt{2} - 4\sqrt{3}$

$= 5\sqrt{3} - 4\sqrt{3} + 2\sqrt{2} + \sqrt{2}$ (rearranging terms)

$= (5 - 4)\sqrt{3} + (2 + 1)\sqrt{2}$ (taking out common factor)

$= \sqrt{3} + 3\sqrt{2}$

b. $x - y = 5\sqrt{3} + 2\sqrt{2} - (\sqrt{2} - 4\sqrt{3})$

$= 5\sqrt{3} + 4\sqrt{3} + 2\sqrt{2} - \sqrt{2}$

$= (5 + 4)\sqrt{3} + (2 - 1)\sqrt{2}$

$= 9\sqrt{3} + \sqrt{2}$

Example 2

Find the product of

a. $3\sqrt{3}$ and $2\sqrt{2}$

b. $5\sqrt{2}$ and $\frac{\sqrt{2}}{5}$.

Solution:

a. $3\sqrt{3} \times 2\sqrt{2} = 6 \times \sqrt{3} \times \sqrt{2} = 6\sqrt{6}$

b. $5\sqrt{2} \times \frac{\sqrt{2}}{5} = \frac{5\times\sqrt{2}\times\sqrt{2}}{5} = 2$

Example 3

Divide

a. $6\sqrt{8}$ by $3\sqrt{2}$

b. $\sqrt{3}$ by $\left(2\sqrt{2} \times 3\sqrt{3}\right)$.

Solution:

a. $6\sqrt{8} \div 3\sqrt{2} = \frac{6\sqrt{8}}{3\sqrt{2}} = 2\sqrt{\frac{8}{2}} = 2 \times 2 = 4$

b. $\sqrt{3} \div \left(2\sqrt{2} \times 3\sqrt{3}\right) = \frac{\sqrt{3}}{(2\sqrt{2}\times3\sqrt{3})} = \frac{\sqrt{3}}{6\sqrt{6}} = \frac{1}{6} \times \sqrt{\frac{3}{6}} = \frac{1}{6\sqrt{2}}$

Since a real number is a union of rational and irrational numbers, it is closed under addition, subtraction, multiplication and division (excluding division by zero).

Exercise 2.24

1. If $x = 4\sqrt{2} + 7\sqrt{5}$ and $y = \sqrt{2} - 3\sqrt{5}$, then find
   a. the sum of x and y.
   b. the difference between y and x
2. Find the product of
   a. $2\sqrt{5}$ and $4\sqrt{3}$
   b. $3\sqrt{3}$ and $\frac{\sqrt{3}}{3}$
   c. $\sqrt{3} - \sqrt{2}$ and $3\sqrt{3} - 4\sqrt{2}$
3. Divide
   a. $4\sqrt{6}$ by $2\sqrt{2}$
   b. $10\sqrt{2}$ by $5\sqrt{18}$
   c. $\sqrt{5}$ by $(3\sqrt{2} \times 4\sqrt{5})$
4. Compute each of the following.
   a. $5\sqrt{2} + 2\sqrt{3} + 3\sqrt{3} - \sqrt{2}$
   b. $\sqrt{145} - \sqrt{232} + \sqrt{261}$

Activity 2.21

1. Determine the opposite of each of the following real numbers
   a. $\sqrt{3}$
   b. $-\pi$
   c. $0.\bar{6}1$
   d. $\frac{1}{2\sqrt{5}}$
   e. 0
   f. $\sqrt{3} - \sqrt{2}$
2. Determine the reciprocal of each of the following real numbers.
   a. $\frac{-\sqrt{3}}{2}$
   b. $3.1\bar{4}34$
   c. $3\sqrt{3} - 1$
   d. $\frac{\sqrt[3]{4}}{1-\sqrt{6}}$
   e. $3^{\frac{2}{3}}$

From the above examples and activities, we have the following basic properties with addition and multiplication of real numbers.

Closure property:

The set of real numbers $\mathbb{R}$ is closed under addition and multiplication. This means that the sum and product of any two real numbers is always a real number. In other words, for all $a, b \in \mathbb{R}$, $a + b \in \mathbb{R}$ and ab or $a \times b \in \mathbb{R}$.

Commutative property

Addition and multiplication are commutative in $\mathbb{R}$: That is, for all $a, b \in \mathbb{R}$,

i) $a + b = b + a$

ii) $ab = ba$

Associative property

Addition and multiplication are associative in $\mathbb{R}$: That is, for all $a, b, c \in \mathbb{R}$,

i) $(a + b) + c = a + (b + c)$

ii) $(ab)c = a(bc)$.

Existence of additive and multiplicative identities:

There are real numbers 0 and 1 such that:

i) $a + 0 = 0 + a = a$ for all $a \in \mathbb{R}$.

ii) $a \times 1 = 1 \times a = a$ for all $a \in \mathbb{R}$.

Existence of additive and multiplicative inverses:

i. For each $a \in \mathbb{R}$ there exists $-a \in \mathbb{R}$ such that $a + (-a) = 0 = (-a) + a$, and $-a$ is called the additive inverse of a.

ii. For each non-zero $a \in \mathbb{R}$, there exists $\frac{1}{a} \in \mathbb{R}$ such that $\frac{1}{a} \times a = 1 = a \times \frac{1}{a}$ and $\frac{1}{a}$ is called the multiplicative inverse or reciprocal of a.

Distributive property:

Multiplication is distributive over addition; that is, if $a, b, c \in \mathbb{R}$,

i) $a(b + c) = ab + ac$.

ii) $(b + c)a = ba + ca$.

Example

Find the additive and multiplicative inverse for each of the following:

a. $\sqrt{2}$ **b.** $\frac{3}{4}$ **c.** 0.5

Solution:

a. Additive inverse of $\sqrt{2}$ is $-\sqrt{2}$ and multiplicative inverse of $\sqrt{2}$ is $\frac{1}{\sqrt{2}}$.

b. Additive inverse of $\frac{3}{4}$ is $\frac{-3}{4}$ and multiplicative inverse of $\frac{3}{4}$ is $\frac{4}{3}$.

c. Additive inverse of 0.5 is -0.5 and multiplicative inverse of 0.5 is 2 (since $0.5 = \frac{1}{2}$).

Exercise 2.25

1. Find the additive and multiplicative inverse of each of the following.
   a. $\sqrt{3}$
   b. $-\frac{2}{5}$
   c. 1.3
   d. $0.\overline{1}$
2. Identify the property of real numbers that is applied in the following statements.
   a. $\sqrt{3}\left(\sqrt{3}-2\right) = 3 - 2\sqrt{3}$
   b. $5 + (-8 + 2) = (5 - 8) + 2$
   c. $6 \times 4 \times \frac{1}{6} = 6 \times \frac{1}{6} \times 4$
3. Does every real number have a multiplicative inverse? Explain.
4. You define a new mathematical operation using the symbol (*). This operation is defined as $a * b = 2a + b$.
   a. Is this operation commutative? Explain.
   b. Is this operation associative? Explain.

2.4.4 Limit of accuracy

Measurements play an important role in daily life because they are useful to do basic tasks, such as take a child's temperature with a thermometer, make time estimations, measure out medicine and find weights, areas and volumes of different materials or substances. In the process of measurement, exact value may not be obtained so that you will be forced to take approximate value.

In this subtopic you will learn certain mathematical concepts related to approximation like rounding of numbers, significant figures (s.f.), decimal place (d.p.) and accuracy.

Activity 2.22

1. Round off the number 45,676 to the nearest.

   a. 100 b. 1000

2. Write the number 8.426

   a. to one decimal place

   b. to two decimal places.

3. Write the number 28.79 to three significant figures.

Rounding

Rounding off is a type of estimation. Estimation is used in everyday life and also in subjects like Mathematics and Physics. Many physical quantities like the amount of money, distance covered, length measured, etc., are estimated by rounding off the actual number to the nearest possible whole number and for decimals at various places of hundreds, tens, tenths, etc.

Rule for rounding whole numbers

When rounding numbers, you need to know the term "rounding digit". When working with whole numbers and rounding to the closest 10, the rounding digit is the second number from the right or the tens place. When rounding to the nearest hundred, the third place from the right is the rounding digit or hundreds place. To perform rounding:-

First, determine what your rounding digit is and then look for the digit at the right side.

- If the digit is 0, 1, 2, 3, or 4, do not change the rounding digit. All digits that are in the right hand side of the requested rounding digit become zero.
- If the digit is 5, 6, 7, 8, or 9 the rounding digit rounds up by one number. All digits that are on the right hand side of the requested rounding digit become zero.

Example 1

38,721 people live in a town. Round this number to various level of accuracy.

a. Nearest 100

b. Nearest 1,000

c. Nearest 10,000.

Solution:

a. To the nearest 100 the number of people would be rounded up to 38,700, since the rounding digit is 7, 2 is the number at the right side of it and keep the rounding digit as it is and all digits at the right hand side of it become zero.

b. To the nearest 1,000 this number would be rounded up to 39,000. The fourth number from the right, that is 8 is the rounding digit. 7 is the number at the right of this rounding digit, so that the rounding digit rounds up by one number so that it would be 9. All digits at the right hand side of this rounding digit become zero.

c. Similarly to the nearest 10,000 this number would be rounded up to 40,000, since 8 is at the right of the rounding digit(3), the rounding digit rounds up by one number so that it would be 4. All digits at the right hand side of this rounding digit become zero.

In this type of situation, it is unlikely that the exact number would be reported. In other words, the result is less accurate but easier to use.

Decimal place

A number can also be approximated to a given number of decimal places (d.p.). This refers to the number of figures written after a decimal point.

Example 2

a. Write 10.673 to 1 d.p. **b.** Write 3.173 to 2 d.p.

Solution:

a. The answer needs to be written with one number after the decimal point. However, to do this, the second number after the decimal point also needs to be considered. If it is 5 or more, then the first number is rounded up.

That is,10.673 is written as 10.7 to 1 d.p.

b. The answer here is to be given with two numbers after the decimal point. In this case, the third number after the decimal point needs to be considered. As the third number after the decimal point is less than 5, the second number is not rounded up. That is, 3.173 is written as 3.17 to 2 d.p.

Note

To approximate a number to 1 decimal place means to approximate the number to the nearest tenth. Similarly approximating a number to 2 decimal places means to approximate to the nearest hundredth.

Significant figures

Numbers can also be approximated to a given number of significant figures (s.f.). In the number 43.25, the 4 is the most significant figure as it has a value of 40. In contrast, the 5 is the least significant as it only has a value of 5 hundredths. When we desire to use significant figures to indicate the accuracy of approximation, we count the number of digits in the number from left to right, beginning at the first non-zero digit. This is known as the number of significant figures.

Example 3

a. Write 2.364 to 2 s.f. **b.** Write 0.0062 to 1 s.f. **c.** Write 0.3041 to 2 s.f.

Solution:

a. We want to write only the two most significant digits. The third digit (6) needs to be considered to see whether the second digit (3) is to be rounded up or not. That is, 2.364 is written as 2.4 to 2 s.f.

b. Notice that in this case 6 and 2 are the only significant digits. The number 6 is the most significant digit and we will consider 2 to be rounded up or not. That is,0.0062 is written as 0.006 to 1 s.f. (since 2 is less than 5 we ignore its effect).

c. Here all digits after the decimal point are significant. So moving two steps from a decimal point to the right we will consider 4 to be rounded up or not. That is , 0.3041 is written as 0.30 to 2 s.f. (since 4 is less than 5 we ignore its effect).

Exercise 2.26

1. Round 86,343 to various level of accuracy

   a. Nearest 100 b. Nearest 1,000 c. Nearest 10,000.

2. Round each of the following to the nearest whole number.

   a. 35.946 b. 45.1999 c. $\frac{7}{8}$ d. $\sqrt{5}$

3. Express the following decimals to 1 d.p. and 2 d.p.

   a. 1.936 b. 4.752 c. 12.998

4. Write each of the following to the number of significant figures indicated in brackets.

   a. 28,645 (1 s.f.) b. 41,909 (3 s.f.) c. 4.5568 (3 s.f.)

Accuracy

In this lesson you will learn how to approximate upper and lower bounds for data to a specified accuracy (for example, numbers rounded off or numbers expressed to a given number of significant figures).

Activity 2.23

1. Round each of the following to 1 d.p.

   3.51, 3.48, 3.53, 3.4999, 3.49, 3.45, 3.47, 3.42, 3.57, 3.41, 3.59, 3.54.

2. Collect the numbers in (1) above which gives 3.5 after rounding and determine the maximum and minimum value from the list.

The above activity leads us to define upper and lower bound of a number.

Definition 2.14 Lower and upper bound

The upper and lower bounds are the maximum and minimum values that a number could have been before it was rounded.

Consider numbers 1.5, 1.50 and 1.500. They may appear to represent the same number but they actually do not. This is because they are written to different degrees of accuracy. 1.5 is rounded to one decimal place (or to the nearest tenths) and therefore any number from 1.45 up to but not including 1.55 would be rounded to 1.5. On the number line this would be represented as

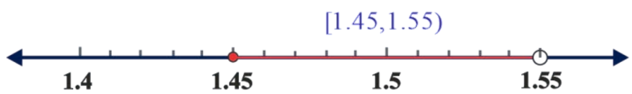


Figure 2.9

As an inequality, it would be expressed as $1.45 \leq 1.5 < 1.55$.

Here, 1.45 is known as the lower bound of 1.5, while 1.55 is known as the upper bound.

In order to find the lower and upper bounds of a rounded number, follow the following steps:

1) Identify the place value of the degree of accuracy stated.
2) Divide this place value by 2.
3) Add this amount (value in step2) to the given value to obtain the upper bound and subtract this value (value in step2) from the given value to obtain the lower bound.

Example 1

The weight of the tree is 872 kilograms to the nearest kilogram.

a. Find the upper and lower bounds where the weight of the tree lies.

b. If the tree's weight is wt kilograms, describe this range as inequality.

Solution:

a. 872 kg is rounded to the nearest kilogram,

Step 1. The place value is 1.

Step 2. Divide 1 by 2 and we get 0.5.

Step 3. Lower bound is $872 - 0.5 = 871.5$ and the upper bound is

$872 + 0.5 = 872.5$.

Therefore, the lower bound is 871.5 kg and the upper bound is 872.5.

b. When the tree's weight is kg, using the answer we got in (a), it could be expressed using inequality as $871.5 \leq wt < 872.5$.

Example 2

A number was given as 10.7 to 1 d.p. Find the lower and upper bounds of a number.

Solution:

Step 1. Place value of the degree of accuracy is $\frac{1}{10}$ or 0.1.

Step 2. Dividing 0.1 by 2 results 0.05.

Step 3. The lower bound is $10.7 - 0.05 = 10.65$ and upper bound is

$10.7 + 0.05 = 10.75$.

Exercise 2.27

1. The speed of a car is given as 45 m/s to the nearest integer.
   a. Find the lower and upper bounds within which the car speed can lie.
   b. If the car's speed is v m/s, express this range as inequality.
2. Find the lower and upper bounds of :
   a. 45 rounded to the nearest integer.
   b. 12.6 rounded to 1 d.p.
   c. 4.23 rounded to 2 d.p.
3. Express each of the following in between the upper and lower bounds.
   a. $x = 34.7$ b. $y = 21.36$ c. $z = 154.134$

Effect of operation on accuracy

Activity 2.24

1. Given $a = 5$ and $b = 2$, then write:-
   i. lower bound of a + lower bound of b
   ii. lower bound of a + upper bound of b
   iii. upper bound of a + lower bound of b
   iv. upper bound of a + upper bound of b

   Which of these gives the lowest value? Is this lowest value the sum of the lowest numbers?

   Which of these gives the highest value? Is this highest value the sum of the highest numbers?
2. Given $a = 9$ and $b = 6$, then write:-
   i. lower bound of a − lower bound of b
   ii. lower bound of a − upper bound of b
   iii. upper bound of a − lower bound of b
   iv. upper bound of a − upper bound of b

   Which of these gives the lowest value? Is this lowest value the difference of the lowest numbers?

   Which of these gives the highest value? Is this highest value the difference of the highest numbers?

When approximated numbers are added, subtracted and multiplied, their sum, difference and product give a range of possible answers.

Example

Given that a rectangular farmland has length 12 meters and width 6 meters.

a. Find the upper and lower bounds of the sum of length and width of the farmland.

b. Find the range of the area of the farmland (upper and lower bounds of the area).

Solution:

a. Let l be the length and w be the width of the farmland. That is, $l = 12$ meters and $w = 6$ meters. Then, $11.5 \leq l < 12.5$ and $5.5 \leq w < 6.5$.

The lower bound of the sum is obtained by adding the two lower bounds.

Therefore, the minimum sum is $11.5 + 5.5 = 17.0$

The upper bound of the sum is obtained by adding the two upper bounds.

Therefore, the maximum sum is $12.5 + 6.5 = 19.0$.

So, the sum lies between 17.0 meters and 19.0 meters.

b. The area of a rectangle A is determined by $A = l \times w$. So, let us determine the upper and lower bounds of the product of l and w. The lower bound of the product is obtained by multiplying the two lower bounds. Therefore, the minimum product is $11.5 \times 5.5 = 63.25$.

The upper bound of the product is obtained by multiplying the two upper bounds.

Therefore, the maximum product is $12.5 \times 6.5 = 81.25$.

So, the product lies between 63.25 and 81.25. Hence the area of the farmland ranges between $63.25\ \text{m}^2$ and $81.25\ \text{m}^2$.

Exercise 2.28

1. Consider two numbers, 15 and 7.
   a. Find the upper and lower bounds of the sum of the two numbers.
   b. Find the range of the product of the two numbers.
2. Calculate the upper and lower bounds for the following calculations if each of the numbers is given to 1 decimal place.

   a. $5.4 + 6.2$ b. 4.6×2.7 c. $14.3 - 5.7$ d. $\frac{10.8}{3.1}$
3. Calculate upper and lower bounds for the area of a school football field shown below if its dimensions are correct to 1 decimal place.

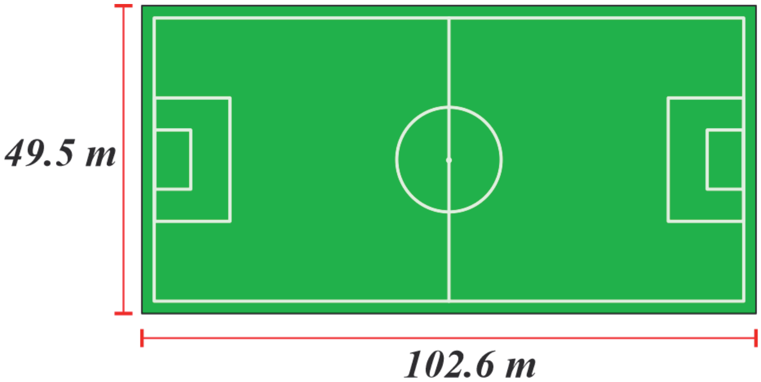


Figure 2.10

2.4.5 Standard notation (Scientific notation)

In science and technology, it is usual to see very large and very small numbers. For instance:

- Abay River travels 1,450,000 meters through Ethiopia and Sudan.
- The average distance from the earth to the moon is 382,500 kilometers.
- The mass of the hydrogen atom is 0.00000000000000000000000000167 kilograms.

Very large numbers and very small numbers may sometimes be difficult to work with or write. Hence you often write very large or very small numbers in scientific notation, also called standard notation.

Activity 2.25

Express each of the following as a multiple of 10.

i) 486.00017 ii) 14580,000,000,000

iii) 0.0006504 iv) 0.00000000000078436

As you have seen from the above activity, you might give different answer for the same question. This will lead us to have the following definition.

Definition 2.15

A number is said to be in scientific notation (or standard notation) if it is written as a product of the form $a \times 10^n$ where $1 \leq a < 10$ and n is an integer.

To convert numbers into scientific notation, we follow the following steps.

1. Move decimal point until there is one (non-zero) number in front of the decimal point.
2. The exponent of the power of 10 is determined by the number of places you moved the decimal point.
   a. If the original number is large, you moved the decimal point to the left and the number
   of places you moved the decimal point is exponent of the power of 10.
   b. If the original number is small, you moved the decimal point to the right, and the
   exponent of the power of 10 is negative integer of the number of places you moved the decimal point.
3. Put the number in the correct pattern for scientific notation.

Example 1

Express each of the following numbers into scientific notation

a. 1,856,700,000 **b.** 0.000000314

Solution:

9 8 7 6 5 4 3 2 1

a. **1856700000.**

1. First let us find a number between 1 and 10, that is a number which has 1 digit in front of the decimal point from the given number. The number is 1.8567.

2. The decimal point has been moved 9 places to the left so the exponent of the power of 10 is 9.
3. Here, the decimal points will move to the left and we will count the number to find n. That is, 1.8567×10^{9}.

1 2 3 4 5 6 7

b. **0.000000314**

1. First let us find a number between 1 and 10, that is a number which has 1 digit in front of the decimal point from the given number. The number is 3.14.
2. The decimal point has been moved 7 places to the right so the exponent of the power of 10 is -7.
3. Here, the decimal points will move to the right and we will count the number to find n. That is, 3.14×10^{-7}.

Example 2

Express each of the following in decimal notation.

a. 5.37×10^{6} **b.** 1.7×10^{-7}

Solution:

a. To change the given number in to ordinary decimal, the decimal point moves to the right 6 units. That is, $5.37 \times 1000000 = 5{,}370{,}000$.

b. Similarly, $1.7 \times 10^{-7} = 0.00000017$.

Exercise 2.29

1. Express each of the following in standard notation.

   a. 426,000 b. 158.762 c. 0.000089 d. 56,897.00547

2. Write each of the following in ordinary decimal notation

   a. 1.34×10^{6} b. 3.3×10^{-3} c. $\frac{2}{5} \times 10^{-7}$

3. Find the simplified expression in standard notation form.

a. $(4.2 \times 10^3) + (1.6 \times 10^3)$

b. $(2.1 \times 10^3)(1.3 \times 10^4)$

c. $(1.5 \times 10^{-3})(3.1 \times 10^3)$

d. $\frac{(5.0\times10^5)}{(2\times10^{-2})}$

2.4.6 Rationalization

Whenever we have a ratio of numbers where the denominator is irrational, determining the quotient might be difficult. So, there is an intention of changing the denominator as rational.

For instance: $\frac{1}{\sqrt{3}}$ has an irrational denominator, $\sqrt{3}$. Here, our aim is changing the number $\frac{1}{\sqrt{3}}$ to an equivalent number with a rational denominator. What shall we do?

The technique of transferring the radical expression from the denominator to the numerator is called rationalizing the denominator (changing the denominator into a rational number).

Do you recall the existence of multiplicative identity 1 for the set of real numbers?

The number that can be used as a multiplier to rationalize the denominator is called the rationalizing factor which is equivalent to 1.

For example: If you have an irrational number $\sqrt{a}$ and if you need to rationalize the denominator of $\frac{1}{\sqrt{a}}$, the rationalizing factor will be $\frac{\sqrt{a}}{\sqrt{a}} = 1$.

Example

Rationalize the denominator for each of the following.

a. $\frac{2}{\sqrt{3}}$ **b.** $\frac{\sqrt{2}}{\sqrt{5}}$ **c.** $\frac{1}{\sqrt[3]{2}}$ (to change in to 2 as its denominator)

Solution:

a. The rationalizing factor for this number is $\frac{\sqrt{3}}{\sqrt{3}}$. So that $\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$. (why?)

b. The rationalizing factor is $\frac{\sqrt{5}}{\sqrt{5}}$. So, $\frac{\sqrt{2}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{10}}{5}$. (why?)

c. We can take $\left(\sqrt[3]{2^2}/\sqrt[3]{2^2}\right)$ as rationalizing factor, $\frac{1}{\sqrt[3]{2}} = \frac{\sqrt[3]{2^2}}{\sqrt[3]{2}\times\sqrt[3]{2^2}} = \frac{\sqrt[3]{2^2}}{2} = \frac{\sqrt[3]{4}}{2}$.

Exercise 2.30

Rationalize the denominator of each of the following and write in simplest form.

a. $\frac{3}{\sqrt{5}}$ b. $\frac{2\sqrt{2}}{\sqrt{7}}$ c. $5\sqrt{\frac{1}{3}}$

More on rationalizations of denominators

Activity 2.26

1. What is the product of $\left(\sqrt{a}-\sqrt{b}\right)$ and $\left(\sqrt{a}+\sqrt{b}\right)$?
2. What is the product of $\left(a+\sqrt{b}\right)$ and $\left(a-\sqrt{b}\right)$?
3. What is the product of $\left(\sqrt{a}-b\right)$ and $\left(\sqrt{a}+b\right)$?

You might have got a rational number as a product for the above expressions.

This will lead you to have the following conclusion.

Table 2.3

No	Given number	Rationalizing factor
1	$\frac{1}{\sqrt{a}-\sqrt{b}}$	$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}$
2	$\frac{1}{a+\sqrt{b}}$	$\frac{a-\sqrt{b}}{a-\sqrt{b}}$
3	$\frac{1}{\sqrt{a}-b}$	$\frac{\sqrt{a}+b}{\sqrt{a}+b}$
4	$\frac{1}{\sqrt{a}+\sqrt{b}}$	$\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}$

Example

Rationalize the denominator of

a. $\frac{1}{\sqrt{2}+1}$ **b.** $\frac{3}{\sqrt{5}-\sqrt{3}}$ **c.** $\frac{1}{\sqrt{3}-\sqrt{2}+1}$.

Solution:

a. The rationalizing factor is $\frac{\sqrt{2}-1}{\sqrt{2}-1}$, so that $\frac{1}{\sqrt{2}+1} = \frac{1}{\sqrt{2}-1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{\sqrt{2}-1}{2-1} = \sqrt{2} - 1$.

b. The rationalizing factor is $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$, so that $\frac{3}{\sqrt{5}-\sqrt{3}} = \frac{3}{\sqrt{5}-\sqrt{3}} \times \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}} = \frac{3(\sqrt{5}+\sqrt{3})}{2}$.

c. Consider a rationalizing factor $\frac{\sqrt{3}-\sqrt{2}-1}{\sqrt{3}-\sqrt{2}-1}$. Now multiply both the numerator and denominator by this rationalizing factor.

$\frac{1}{\sqrt{3}-\sqrt{2}+1} \times \frac{\sqrt{3}-\sqrt{2}-1}{\sqrt{3}-\sqrt{2}-1} = \frac{\sqrt{3}-\sqrt{2}-1}{2(2-\sqrt{6})}$ (using multiplication and collecting like terms).

The denominator is still not rational. Hence, again we will take $\frac{2+\sqrt{6}}{2+\sqrt{6}}$ as a rationalizing factor. Multiplying both the numerator and the denominator by $\frac{2+\sqrt{6}}{2+\sqrt{6}}$, we obtain $\frac{\sqrt{3}-\sqrt{2}-1}{2\,(2-\sqrt{6})} \times \frac{(2+\sqrt{6})}{(2+\sqrt{6})} = \frac{\sqrt{2}-\sqrt{6}-2}{-4}$.

Exercise 2.31

Rationalize each of the following numbers.

a. $\frac{3}{2+\sqrt{5}}$ b. $\frac{1+\sqrt{2}}{\sqrt{3}-1}$ c. $\frac{2}{\sqrt{3}-\sqrt{2}}$ d. $\frac{2}{\sqrt{2}+\sqrt{3}+1}$

2.5 Applications

In the last 4 subsections, you have discussed number systems. In this subsection you will learn some application problems based on the covered topics.

Example 1

A small town has 530 flower pots. The gardener wants to arrange them in rows such that each row contains 21 flower pots. Find the number of completed rows and how many flower pots are left over.

Solution:

Total number of flower pots is 530. Number of flower pots in each row is 21.

Now, we have to find how many groups of 21 in 530. So, we use Euclid's Division Lemma, that is dividing 530 by 21 as shown in the right.

The result indicates, the gardener can arrange the flower pots in 25 rows with each row consisting of 21 pots. Also the remaining number of flower pots is 5.

$$\begin{array}{r} 25 \\ 21\overline{)530} \\ \underline{42} \\ 110 \\ \underline{105} \\ 5 \end{array}$$

Example 2

The teacher wants to paste a square pieces of equal size colored papers on a white board measuring 20 cm by 50 cm. If only squares of length with natural number be considered, and the board is to be completely covered, find the largest possible length of the side of each square pieces.

Solution:

The dimension of the white board is 50 cm by 20 cm. There are different alternatives to construct the square pieces, in figure 2.11(a), we have 1 cm by 1 cm square pieces.

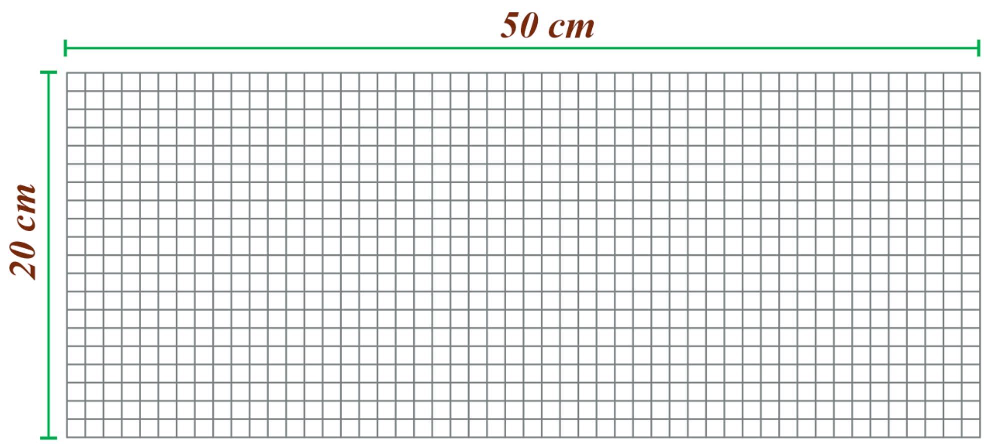


Figure 2.11 (a)

The following figure 2.11(b) is a 5 cm by 5 cm square pieces.

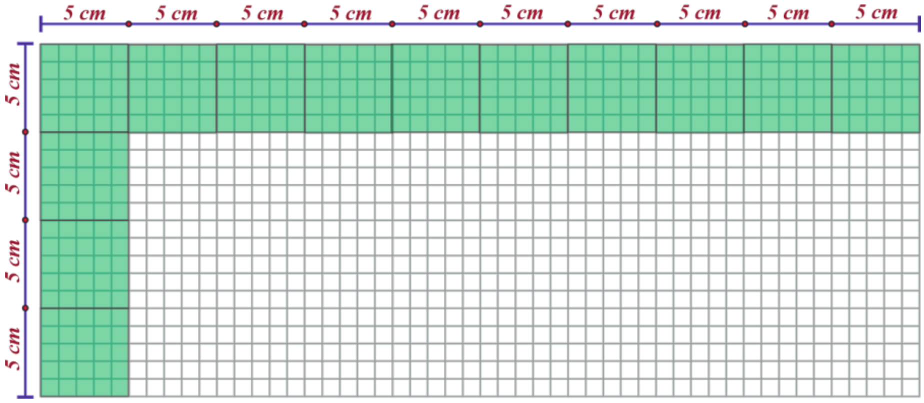


Figure 2. 11 (b)

And figure 2.11 (c) below, we have 10 cm by 10 cm.

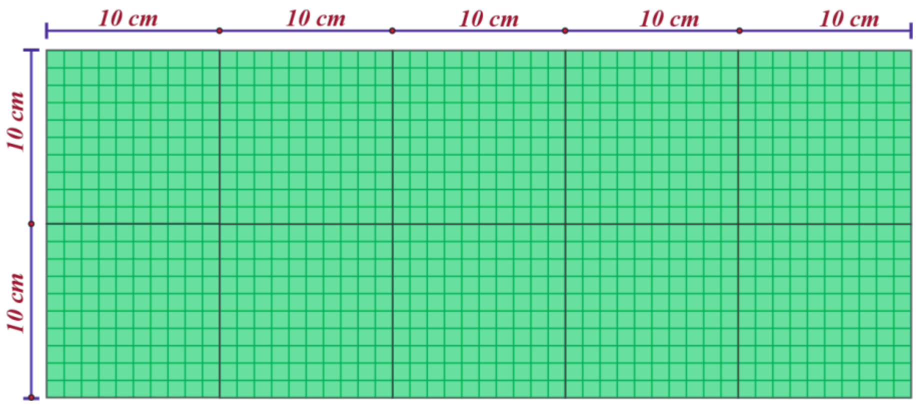


Figure 2.11 (c)

There are also other options to construct such square pieces with different lengths. Since we need only natural number length, we will try to find the GCF of 50 and 20. Using prime factorization $50 = 2 \times 5^2$ and $20 = 2^2 \times 5$. So, $\mathrm{GCF}(50,20) = 10$. Hence, the largest possible length of the side of each square piece is 10 cm.

Exercise 2.32

1. Assume three strings of different lengths 78 cm, 117 cm and 351 cm cut into equal lengths. Find the greatest possible length of each piece.
2. Two 9th grade students A and B start running around the sport field together. Student A completes one round in 6 minutes while B takes 14 minutes to

complete the same round. After how many minutes will they meet again at the starting point for the second time?

3. There are 340 notebooks. A teacher is thinking to distribute them to 24 students equally as much as possible. Find number of notebooks each student gets and how many notebooks are left.

Summary

1. The set of Natural numbers, Integers and Rational numbers are denoted by $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$, respectively and described as

   $\mathbb{N} = \{1, 2, 3, \ldots\}$,

   $\mathbb{Z} = \{\ \ldots, -3, -2, -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\}$ and

   $\mathbb{Q} = \left\{\frac{a}{b} : \text{a, b} \in \mathbb{Z} \text{ and } b \neq 0\right\}$.

2. Euclid's Division algorithm

   Given non-negative integer a and a positive integer b, there exist unique non-negative integers q and r satisfying:

   $a = (b \times q) + r$ with $0 \leq r < b$.

   From the above equation, a is called the <u>dividend</u>, q is called the <u>quotient</u>, b is called the <u>divisor</u>, and r is called the <u>remainder.</u>

3. **a.** A natural number that has exactly two distinct factors, namely 1 and itself is called a **prime number.**

   b. A natural number that has more than two factors is called a **composite number.**

   c. Prime factorization is a process of expressing a natural number as a product of prime numbers.

   d. **Fundamental theorem of arithmetic** states that every composite number can be expressed (factorized) as a product of primes, and this factorization is unique, apart from the order in which the prime factors occur.

4. **a.** Given two or more natural numbers, a number which is a factor of all of them is called a **common factor**.

a. The Greatest Common Factor (GCF) or Highest Common Factor (HCF) of two or more natural numbers is the greatest natural number of the common factors.

b. Two or more natural numbers where their GCF is 1 are called relatively prime.

c. For any two natural numbers a and b, the Least Common Multiple of a and b denoted by $\text{LCM}(a, b)$, is the smallest multiple of both a and b.

d. For any two natural numbers a and b, $\text{LCM}(a, b) \times \text{GCF}(a, b) = a \times b$.

5. a. Any rational number can be expressed as decimal by dividing the numerator a by the denominator b.

a. To change a rational number $^{a}/_{b}$ in to decimal form, one of the following cases will occur.

- The division process ends when a remainder of zero is obtained. Here, the decimal is called a terminating decimal.
- The division process does not terminate but repeats as the remainder never becomes zero. In this case the decimal is called a repeating decimal.

c. A repeating decimal also can be converted in to fractions.

6. Any terminating decimal or repeating decimal is a rational number where as a decimal number that is neither terminating nor repeating is an irrational number.
7. The sum of an irrational and a rational number is always an irrational number.
8. The set of irrational numbers is not closed with respect to addition, subtraction, multiplication and division.
9. A number is called a real number, if and only if it is either a rational number or an irrational number, that is $\mathbb{R} = \{x : x \text{ is rational or } x \text{ is irrational}\}$.

10. The absolute value of a real number x, denoted by $|x|$, is defined as

$$|x| = \begin{cases} x \text{ if } x \geq 0, \\ -x \text{ if } x < 0. \end{cases}$$

11. For any real number b and positive integer > 1, $b^{\frac{1}{n}} = \sqrt[n]{b}$ (whenever $\sqrt[n]{b}$ is a real number).

12. For all real numbers a and $b \neq 0$ for which the radicals are defined and for all integers $n \geq 2$.

i. $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$ **ii.** $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

13. A number is said to be in scientific notation (or standard form) if it is written as the form $a \times 10^n$ where $1 \leq a < 10$ for an integer n.

Review Exercise

1. Identify whether each of the following numbers is divisible by $2, 3, 4, 5, 6, 8, 9$ or 10.

a. 657 **b.** 10,222 **c.** 64,916

2. A four digit number has digit form '$a4b2$'. If it is divisible by 3, what are the possible values of $a + b$?

3. Fill in the blank space.

a. ________is the sum of the smallest prime number and the smallest composite number.

b. ________ is the smallest composite number.

4. Write each as prime factorization form.

a. 57 **b.** 168 **c.** 536

5. Find the GCF and LCM of the numbers given below.

a. 36 , 60 **b.** 84,224 **c.** 15,39,105 **d.** 16,20,48

6. The GCF of two numbers is 9 and the LCM of these two numbers is 54. If one of the numbers is 27, what is the other number?

7. Express each of the following rational numbers as decimals.

a. $-2\frac{1}{5}$ **b.** $\frac{11}{7}$ **c.** $\frac{17}{33}$

8. Express each of the following decimals as a fraction in simplest form.

a. 0.38 **b.** $2.\bar{1}2$ **c.** $-0.\bar{1}23$

9. Prove that there exists a real number between any two real numbers.

10. If $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers, show that $\frac{a}{b} \times \frac{c}{d}$ is also rational number.

11. Find the absolute value of each of the following.

a. $-10.1\bar{4}18$ **b.** $\sqrt[3]{6} - 2$ **c.** $\frac{a}{b}$, where $a, b \in \mathbb{R}$ and $a \geq 0$ and $b < 0$.

12. Give equivalent expression, containing fractional exponents, for each of the following

a. $\sqrt{14}$ **b.** $\sqrt{x+y}$ **c.** $\sqrt[5]{7}$ **d.** $\sqrt[3]{\frac{5}{6}}$

13. Describe the following numbers as fractions with rational denominators.

a. $-\frac{2}{\sqrt{5}}$ **b.** $\frac{4}{3-\sqrt{2}}$ **c.** $\frac{\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ **d.** $\frac{1}{\sqrt{2}+\sqrt{5}+1}$

14. Find an irrational number between $\frac{1}{7}$ and $\frac{2}{7}$.

15. Given $\frac{\sqrt{2}+\sqrt{3}}{4+\sqrt{3}} = a\sqrt{2} + b\sqrt{3} + c\sqrt{6} + d$. Then, find a, b, c and d.

16. Simplify each of the following.

a. $7\sqrt{2} + \sqrt{8} - \sqrt{18}$ **b.** $\frac{1}{3} \times (\sqrt{5} - \sqrt{2}) \times (\sqrt{5} + \sqrt{2})$ **c.** $3\sqrt{28} \div 8\sqrt{7}$

d. $\sqrt[4]{b^2 \times \sqrt[3]{b^2}}$, for $b \geq 0$ **e.** $\frac{\sqrt{72} - 3\sqrt{24}}{\sqrt{2}}$ **f.** $\frac{2\sqrt{72}}{3} - \frac{3\sqrt{128}}{4} + 5\sqrt{\frac{1}{2}}$

17. Each of the following numbers is correct to one decimal place.

- Give the upper and lower bounds of each.
- Using x as the number, express the range in which the number lies as an inequality.

a. 2.4 **b.** 10.6 **c.** 2.0 **d.** -0.5

18. Express each of the following numbers in scientific notation.

a. 567,200,000 **b.** 0.00000774 **c.** 154.6×10^5

19. On the number line which one of the following real numbers is found farthest from the origin?

A. $\sqrt{1.44}$ **B.** $-\sqrt{3}$ **C.** $\frac{\sqrt{5}}{7}$ **D.** $-\sqrt{0.0001}$

20. According to the division algorithm, what should be the value of q and r respectively so that $736 = 12q + r$?

A. 60 and -4 **B.** 54 and 8 **C.** 60 and 4 **D.** 61 and 4

21. Suppose x is a rational number and y is an irrational number, then which one of the following is necessarily true?

A. xy is rational. **B.** $\frac{x}{y}$ is irrational number.

C. $\frac{y}{x}$ is real number. **D.** $x - y$ is irrational for any x.

22. Which one of the following statements is true about the set of real numbers $\mathbb{R}$?

A. Every real number has a multiplicative inverse.

B. Subtraction is associative over $\mathbb{R}$.

C. $-a$ is the additive inverse of a for every $a \epsilon \mathbb{R}$.

D. Division is commutative over $\mathbb{R}$.

23. When the positive integers a, b and c are divided by 13, the respective remainders are $9, 7$ and 10, respectively. Show that $a + b + c$ is divisible by 13.

24. There are 14 girls and 21 boys in a school. These students want to give a voluntary traffic safety service. Their school director assigned a task to be done in a team. Each team must have an equal number of boys and girls on it. What is the greatest number of teams the director can make if every student should be in a team? How many boys and girls will be in each team?

25. Floor of an ICT room of your school is to be fitted with a square marble of the largest possible size. The size of the room is 10 m by 7 m. What should be the size of the tiles required that has to be cut and how many such tiles are required?