CvxOpt Homework #1

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> **2.2** Show that a set is convex if and only if its intersection with any line is convex. Show that a set is affine if and only if its intersection with any line is affine.

First we proof *a set is convex iff. its intersection with any line is convex*.

$\Rightarrow$ Proof: Let set C is convex, then if the intersection of a line l and C, $l \cap C$, only contains one point, then it is obvoiusly convex; otherwise, assume $l \cap C$ has at least two points. Then for any two points A, B in $l \cap C$, because the convexity of C, for any θ with $0 \leq \theta \leq 1$ we have $\theta A + (1-\theta)B \in C$. Also because point $\theta A + (1-\theta)B$ belongs to l, so point $\theta A + (1-\theta B) \in l \cap C$. So the intersection of the set and any line is convex.

$\Leftarrow$ Proof: Let set C satisfies that for any line l, $l \cap C$ is convex, then if C contains only one point it is obviously convex; otherwise, assume C has at least two points. Then for any two points A, B in C, the intersection of line AB and C, let it be D, is convex, so for any θ with $0 \leq \theta \leq 1$, we have $\theta A + (1-\theta)B \in D$. Plus we have $D \subseteq C$, so $\theta A + (1-\theta B) \in C$, which indicates the convexity of C.

Then we proof *a set is affine iff. its intersection with any line is affine*.

$\Rightarrow$ Proof: Let set C is affine, then if the intersection of a line l and C, $l \cap C$, only contains one point, then it is obvoiusly affine; otherwise, assume $l \cap C$ has at least two points. Then for any two points A, B in $l \cap C$, because the affinity of C, for any $\theta \in \mathbf{R}$ we have $\theta A + (1-\theta)B \in C$. Also because point $\theta A + (1-\theta)B$ belongs to l, so point $\theta A + (1-\theta B) \in l \cap C$. So the intersection of the set and any line is affine.

$\Leftarrow$ Proof: Let set C satisfies that for any line l, $l \cap C$ is affine, then if C contains only one point it is obviously affine; otherwise, assume C has at least two points. Then for any two points A, B in C, the intersection of line AB and C, let it be D, is affine, so for any $\theta \in \mathbf{R}$ we have $\theta A + (1-\theta)B \in D$. Plus we have $D \subseteq C$, so $\theta A + (1-\theta B) \in C$, which indicates the affinity of C.

> **2.10** *Solution set of a quadratic inequality.* Let $C \subseteq \mathbf{R}^n$ be the solution set of a quadratic inequality,
> $$C = \{x \in \mathbf{R}^n \mid x^T A x + b^T x + c \leq 0\},$$
> with $A \in \mathbf{S}^n$, $b \in \mathbf{R}^n$, and $c \in \mathbf{R}$.
>
> (a) Show that C is convex if $A \succeq 0$.

Proof: If C only has one element then it is obviously convex; otherwise assume C has at least two elements. Then for any two elements $x_1, x_2 \in C$ and any θ with $0 \leq \theta \leq 1$, we proof $\theta x_1 + (1-\theta)x_2 \in C$.

Because $x_1, x_2 \in C$ we have

$$\begin{cases} x_1^T A x_1 + b^T x_1 + c \leq 0 \\ x_2^T A x_2 + b^T x_2 + c \leq 0 \end{cases}$$

then we get

$$\theta(x_1^T A x_1 + b^T x_1 + c) + (1-\theta)(x_2^T A x_2 + b^T x_2 + c) \leq 0$$

then

$$
\begin{aligned}
&\theta(x_1^TAx_1+b^Tx_1+c)+(1-\theta)(x_2^TAx_2+b^Tx_2+c)\\
&-((\theta x_1+(1-\theta)x_2)^TA(\theta x_1+(1-\theta)x_2)+b^T(\theta x_1+(1-\theta)x_2)+c)\\
=&\theta x_1^TAx_1+(1-\theta)x_2^TAx_2-(\theta x_1+(1-\theta)x_2)^TA(\theta x_1+(1-\theta)x_2)\\
=&\theta x_1^TAx_1+(1-\theta)x_2^TAx_2-\theta^2x_1^TAx_1-(1-\theta)\theta x_2^TAx_1-\theta(1-\theta)x_1^TAx_2-(1-\theta)^2x_2^TAx_2\\
=&\theta(1-\theta)(x_1^TAx_1+x_2^TAx_2-x_2^TAx_1-x_1^TAx_2)\\
=&\theta(1-\theta)(x_1-x_2)^TA(x_1-x_2)\\
\geq&0
\end{aligned}
$$

The last step is because the semidefiniteness of A, so that $(x_1-x_2)^TA(x_1-x_2)\geq 0$.

So we have

$$
\begin{aligned}
&((\theta x_1+(1-\theta)x_2)^TA(\theta x_1+(1-\theta)x_2)+b^T(\theta x_1+(1-\theta)x_2)+c)\\
\leq&\ \theta(x_1^TAx_1+b^Tx_1+c)+(1-\theta)(x_2^TAx_2+b^Tx_2+c)\\
\leq&\ 0
\end{aligned}
$$

Which indicates $\theta x_1+(1-\theta)x_2\in C$, so C is convex.

> **2.24** *Supporting hyperplanes.*
>
> (a) Express the closed convex set $\{x\in\mathbf{R}_+^2 \mid x_1x_2\geq 1\}$ as an intersection of halfspaces.

Let $C=\{x\in\mathbf{R}_+^2|x_1x_2\geq 1\}$, then $\mathbf{bd}C=\{x\in\mathbf{R}_+^2|x_1x_2=1\}$, the for any point $x=(t,\frac{1}{t})$ in $\mathbf{bd}C$, the supporting hyperplane is

$$\{x\in\mathbf{R}^2|\frac{x_1}{t^2}+x_2\geq\frac{2}{t}\}$$

then C can be expressed as

$$C=\bigcap_{t>0}\{x\in\mathbf{R}^2|\frac{x_1}{t^2}+x_2\geq\frac{2}{t}\}$$

> **2.32** Find the dual cone of $\{Ax \mid x\succeq 0\}$, where $A\in\mathbf{R}^{m\times n}$.

The dual cone can be expressed as

$$\{y|Ay\succeq 0\}$$

> **2.35** *Copositive matrices.* A matrix $X\in\mathbf{S}^n$ is called *copositive* if $z^TXz\geq 0$ for all $z\succeq 0$. Verify that the set of copositive matrices is a proper cone. Find its dual cone.

Let the set of all *copositive* matrices be K in $\mathbf{S}^n$. So K can be expressed as infinite halfspaces defined by

$$z^TXz=\sum_i\sum_j z_iz_jX_{ij}\geq 0,$$

so K is closed convex cone. K has nonempty interior because it includes $\mathbf{S}_+^n$. K is pointed because $z^TXz\geq 0$ and $z^T(-X)z\geq 0$ gives $z^TXz=0$ for all $z\succeq 0$, hence $X=0$. Hence K is a proper cone.

The dual cone of K is the set of normal vectors of all halfspaces supporting K. It can be expressed as

$$K = \{zz^T | z \succeq 0\}$$

> **3.6** *Functions and epigraphs.* When is the epigraph of a function a halfspace? When is the epigraph of a function a convex cone? When is the epigraph of a function a polyhedron?

a) When is the epigraph of a function a halfspace?

Let the epigraph of $f : \mathbf{R}^n \to \mathbf{R}$ be $E = \{s = (x,t) | f(x) \leq t\}$. Because E is a halfspace, so $\exists A = (a, a^*)$ and b, where $a \in \mathbf{R}^n$, $a^* \in \mathbf{R}$, and $b \in \mathbf{R}$, $s.t.$

$$\begin{aligned} E &= \{s | A^T s \leq b\} \\ &= \{(x,t) | a^T x + a^* t \leq b\} \\ &= \{(x,t) | a^T x - b \leq -a^* t\} \end{aligned}$$

Because E has the form of epigraph, so the coefficient of t must be positive, $i.e.$, $-a^* > 0$.

So $f(x) = \dfrac{a^T}{-a^*} x + \dfrac{b}{a^*}$, which is a affine function.

b) When is the epigraph of a function a convex cone?

Let the epigraph of $f : \mathbf{R}^n \to \mathbf{R}$ be $E = \{s = (x,t) | f(x) \leq t\}$. By definition for any $x \in \mathbf{R}^n$ and any $\alpha \in \mathbf{R}^+$, $(x, f(x)) \in E$ and $(\alpha x, f(\alpha x)) \in E$. Then because E is a cone, $(\alpha x, \alpha f(x)) \in E$ and $(\dfrac{1}{\alpha}(\alpha x), \dfrac{1}{\alpha} f(\alpha x)) \in E$, which indicates

$$\begin{cases} f(\alpha x) \leq \alpha f(x) \\ f(\dfrac{1}{\alpha}(\alpha x)) \leq \dfrac{1}{\alpha} f(\alpha x) \end{cases}$$

which is $f(\alpha x) \leq \alpha f(x)$ and $f(\alpha x) \geq \alpha f(x)$. So $f(\alpha x) = \alpha f(x)$. In addition, because the epigraph E of $f(x)$ is convex, $f(x)$ is convex function.

So, $f(x)$ is a convex function which holds the property of positively homogeneous, $i.e.$, for any $x \in \mathbf{R}^n$ and $\alpha \in \mathbf{R}^+$, $f(\alpha x) = \alpha f(x)$.

c) When is the epigraph of a function a polyhedron?

Let the epigraph of $f : \mathbf{R}^n \to \mathbf{R}$ be $E = \{s = (x,t) | f(x) \leq t\}$. Because E is a polyhedron, so $\exists A_j = (a_j, a_j^*)$ and $b_j (j = 1, \ldots, m)$, where $a_j \in \mathbf{R}^n$, $a_j^* \in \mathbf{R}$, and $b \in \mathbf{R}$, $s.t.$

$$\begin{aligned} E &= \{s | A_j^T s \leq b_j, j = 1, \ldots, m\} \\ &= \{(x,t) | a_j^T x + a_j^* t \leq b_j, j = 1, \ldots, m\} \\ &= \{(x,t) | a_j^T x - b_j \leq -a_j^* t, j = 1, \ldots, m\} \end{aligned}$$

Because E has the form of epigraph, so the coefficient of t must be positive, $i.e.$, $-a_j^* > 0$ for $j = 1, \ldots, m$. So

$$\begin{aligned} E &= \{(x,t) | \frac{a_j^T}{-a_j^*} x + \frac{b_j}{a_j^*} \leq t, j = 1, \ldots, m\} \\ &= \{(x,t) | \max_{j=1,\ldots,m} \left(\frac{a_j^T}{-a_j^*} x + \frac{b_j}{a_j^*} \right) \leq t\} \end{aligned}$$

So $f(x) = \max_{j=1,...,m}\left(\frac{a_j^T}{-a_j^*}x + \frac{-b_j}{-a_j^*}\right)$, which is a piecewise affine function.

> **3.20** *Composition with an affine function.* Show that the following functions $f : \mathbf{R}^n \to \mathbf{R}$ are convex.
>
> (a) $f(x) = \|Ax - b\|$, where $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, and $\|\cdot\|$ is a norm on $\mathbf{R}^m$.
>
> (b) $f(x) = -\left(\det(A_0 + x_1A_1 + \cdots + x_nA_n)\right)^{1/m}$, on $\{x \mid A_0 + x_1A_1 + \cdots + x_nA_n \succ 0\}$, where $A_i \in \mathbf{S}^m$.
>
> (c) $f(X) = \mathbf{tr}\,(A_0 + x_1A_1 + \cdots + x_nA_n)^{-1}$, on $\{x \mid A_0+x_1A_1+\cdots+x_nA_n \succ 0\}$, where $A_i \in \mathbf{S}^m$. (Use the fact that $\mathbf{tr}(X^{-1})$ is convex on $\mathbf{S}^m_{++}$; see exercise 3.18.)
>
> Using the conclusion of:
>
> **3.18** Adapt the proof of concavity of the log-determinant function in §3.1.5 to show the following.
>
> (a) $f(X) = \mathbf{tr}\left(X^{-1}\right)$ is convex on $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.
>
> (b) $f(X) = (\det X)^{1/n}$ is concave on $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.

a) $f(x)$ is a composition of a norm and an affine function, hence convex;

b) $f(x)$ is a composition of a convex function in 3.18 b) and an affine transformation, hence convex;

c) $f(X)$ is a composition of a convex function in 3.18 a) and an affine transformation, hence convex.

> **3.21** *Pointwise maximum and supremum.* Show that the following functions $f : \mathbf{R}^n \to \mathbf{R}$ are convex.
>
> (a) $f(x) = \max_{i=1,\ldots,k} \|A^{(i)}x - b^{(i)}\|$, where $A^{(i)} \in \mathbf{R}^{m\times n}$, $b^{(i)} \in \mathbf{R}^m$ and $\|\cdot\|$ is a norm on $\mathbf{R}^m$.
>
> (b) $f(x) = \sum_{i=1}^r |x|_{[i]}$ on $\mathbf{R}^n$, where $|x|$ denotes the vector with $|x|_i = |x_i|$ (*i.e.*, $|x|$ is the absolute value of x, componentwise), and $|x|_{[i]}$ is the ith largest component of $|x|$. In other words, $|x|_{[1]}, |x|_{[2]}, \ldots, |x|_{[n]}$ are the absolute values of the components of x, sorted in nonincreasing order.

a) Let $f(x) = \max_{i=1,...,k} f_i(x)$. By 3.20 a) we get $f_i(x) = \|A^{(i)}x - b^{(i)}\|$ is convex. Hence as pointwise maximum of k convex function, $f(x)$ is also convex.

b) Write $f(x)$ as

$$f(x) = \sum_{i=1}^{r} |x|^{[i]} = \max_{1\le i_1<i_2<\cdots<i_r\le n} |x_{i_1}| + |x_{i_2}| + \cdots + |x_{i_r}|$$

which is the pointwise maximum of $n!/r!(n-r)!$ convex functions, hence convex.