凸优化与自主轨迹规划
Convex Optimization and Autonomous Trajectory Planning

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Course goals and topics

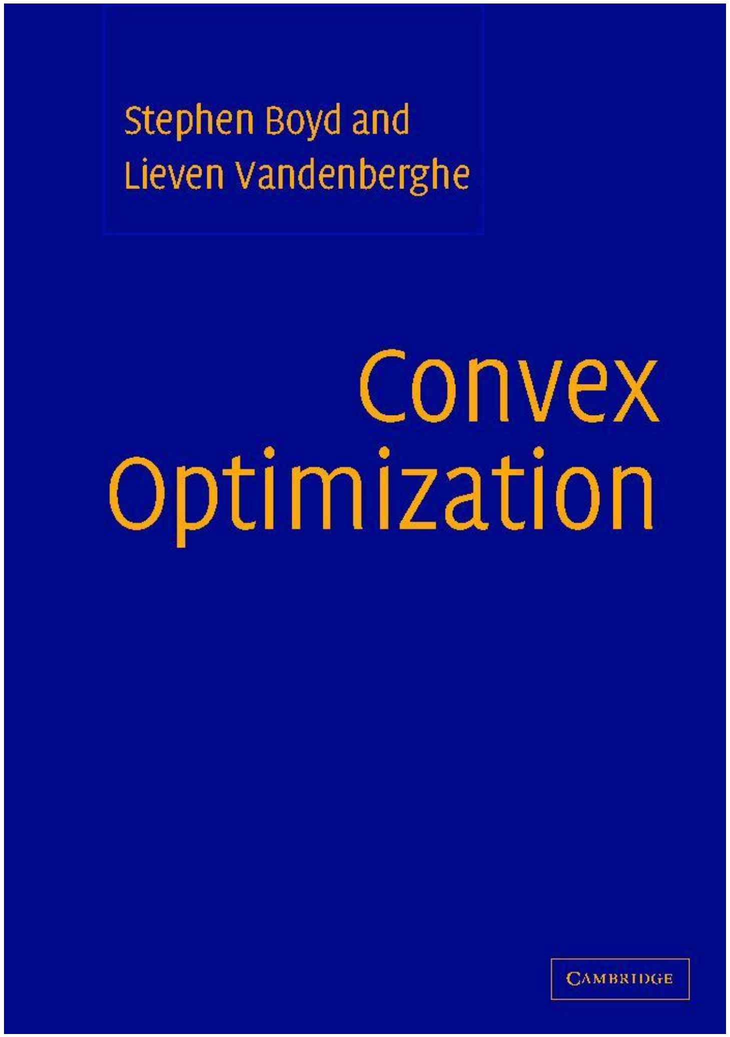
goals

1. develop a working knowledge of convex optimization
2. learn skills and background to recognize/formulate problems as convex optimization problems
3. practice to solve convex optimization problems with moderate sizes
4. master techniques on converting nonconvex optimization problems into convex ones.

topics

1. convex sets, convex functions, optimization problems, duality
2. techniques, convexification, examples and applications

References


Stephen Boyd and
Lieven Vandenberghe
Convex
Optimization
CAMBRIDGE

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1. Introduction

- mathematical optimization
- least-squares and linear programming
- convex optimization
- example
- nonlinear optimization
- brief history of convex optimization

Mathematical optimization

(mathematical) optimization problem

$$\begin{aligned} &\text{minimize} \quad f_0(x) \\ &\text{subject to} \;\; f_i(x) \leq b_i, \;\; i = 1, \dots, m \end{aligned}$$

- $x = (x_1, \dots, x_n)$: optimization variables
- $f_0 \colon R^n \to R$: objective function
- $f_i \colon R^n \to R, \; i = 1, \dots, m$: constraint functions

solution or **optimal point** x^* has smallest value of f_0 among all vectors that satisfy the constraints

Examples

portfolio optimization

- Variables: amounts invested in different assets
- Constraints: budget, max./min. investment per asset, minimum return
- Objective: overall risk or return variance

device sizing in electronic circuits

- Variables: device widths and lengths
- Constraints: manufacturing limits, timing requirements, maximum area
- Objective: power consumption

data fitting

- Variables: model parameters
- Constraints: prior information, parameter limits
- Objective: measure of misfit or prediction error, plus regularization term

Solving optimization problems

general optimization problem

- **very difficult** to solve
- methods involve **some compromise**, e.g. very long computation time, or not always finding the solution (which may not matter in practice)

exceptions: certain problem classes can be solved reliably and efficiently

- least-squares problem
- linear programming problems
- convex optimization problems

Least-squares

$$\text{minimize} \quad \|Ax - b\|_2^2$$

solving least-squares problems

- analytical solution: $x^* = (A^T A)^{-1} A^T b$
- **reliable and efficient algorithms** and software
- computation time proportional to $n^2 k (A \in \boldsymbol{R}^{k \times n})$; **less if structured**
- a mature technology

using least-squares

- least-squares problems are easy to recognize
- a few standard techniques increase flexibility (e.g. including weights, adding regularization terms)

Linear programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i,\ i = 1, \dots, m \end{array}$$

solving linear problems

- no analytical formula for solution
- **reliable and efficient algorithms** and software
- computation time proportional to $n^2 m$ if $m \ge n$; less if structured
- a mature technology

using linear programming

- not as easy to recognize as least-squares problems
- a few standard tricks used to convert problems into linear programs

  (e.g. problems involving $\ell_1 -$ or $\ell_\infty -$norms, piecewise-linear functions)

Convex optimization

$$\text{minimize} \quad f_0(x)$$

$$\text{subject to} \quad f_i(x) \leq b_i, \; i = 1, \dots, m$$

where $f_0, \dots f_{\mathrm{m}} : R^n \to R$.

- objective and constraint functions are **convex**:

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$$

if $\alpha + \beta = 1, \; \alpha \geq 0, \; \beta \geq 0$

- includes least-squares problems and linear problems as special cases

solving convex optimization problems

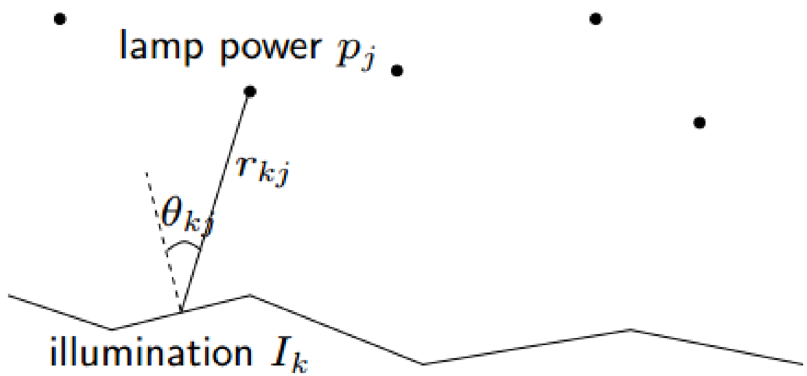
- no analytical solution
- reliable and efficient algorithms
- computation time (roughly) proportional to $max\{n^3, n^2m, F\}$, where F is cost of evaluating f_i's first and second derivatives
- almost a technology

using convex optimization

- often **difficult to recognize**
- **many more tricks** for transforming problems into convex problems than for transforming linear problems
- surprising many problems can be solved via convex optimization

Examples

m lamps illuminating n (small, flat) patches



intensity I_k at patch k depends linearly on lamp powers p_j:

$$I_k = \sum_{j=1}^{m} a_{kj} p_j \,, \qquad a_{kj} = r_{kj}^{-2} max\{cos\theta_{kj}, 0\}$$

problem: achieve desired illumination I_{des} with bounded lamp powers

minimize $max_{k=1,\ldots,n} |logI_k - logI_{des}|$

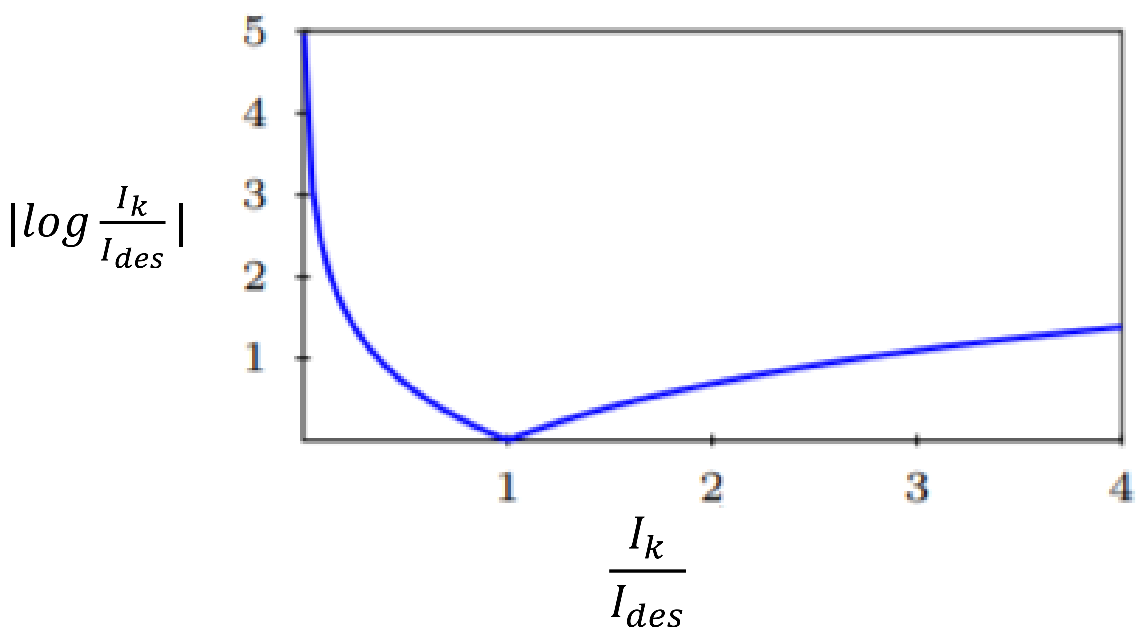
subject to $0 \leq p_j \leq p_{max},\ j = 1, \ldots, m$

The illumination problem is **difficult to solve**. Why?

The reason is that the objective function

$$|logI_k - logI_{des}| = |log\frac{I_k}{I_{des}}|$$

is **not convex**.



How to solve?

1. use uniform power: $p_j = p$, vary p
2. use **least-squares**:

$$\text{minimize} \;\; \sum_{k=1}^{n}(I_k - I_{des})^2$$

round p_j if $p_j > p_{max}$ or $p_j < 0$

3. use **weighted least-squares**:

$$\text{minimize} \;\; \sum_{k=1}^{n}(I_k - I_{des})^2 + \sum_{j=1}^{m} w_j\,(p_j - p_{max}/2)^2$$

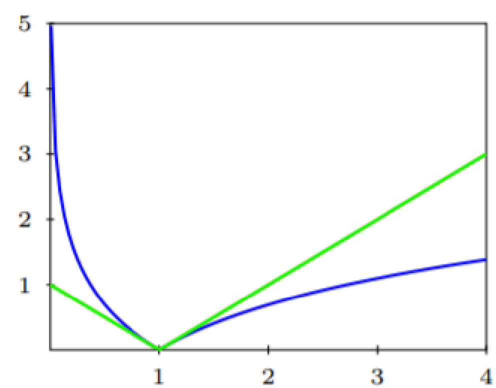
iteratively adjust weights w_j until $0 \le p_j \le p_{max}$

4. use **linear programming**:

$$\text{minimize} \quad max_{k=1,\ldots,n}|I_k - I_{des}|$$

$$\text{subject to} \;\; 0 \le p_j \le p_{max}, \;\; j = 1, \ldots, m$$

which can be solved via linear programming



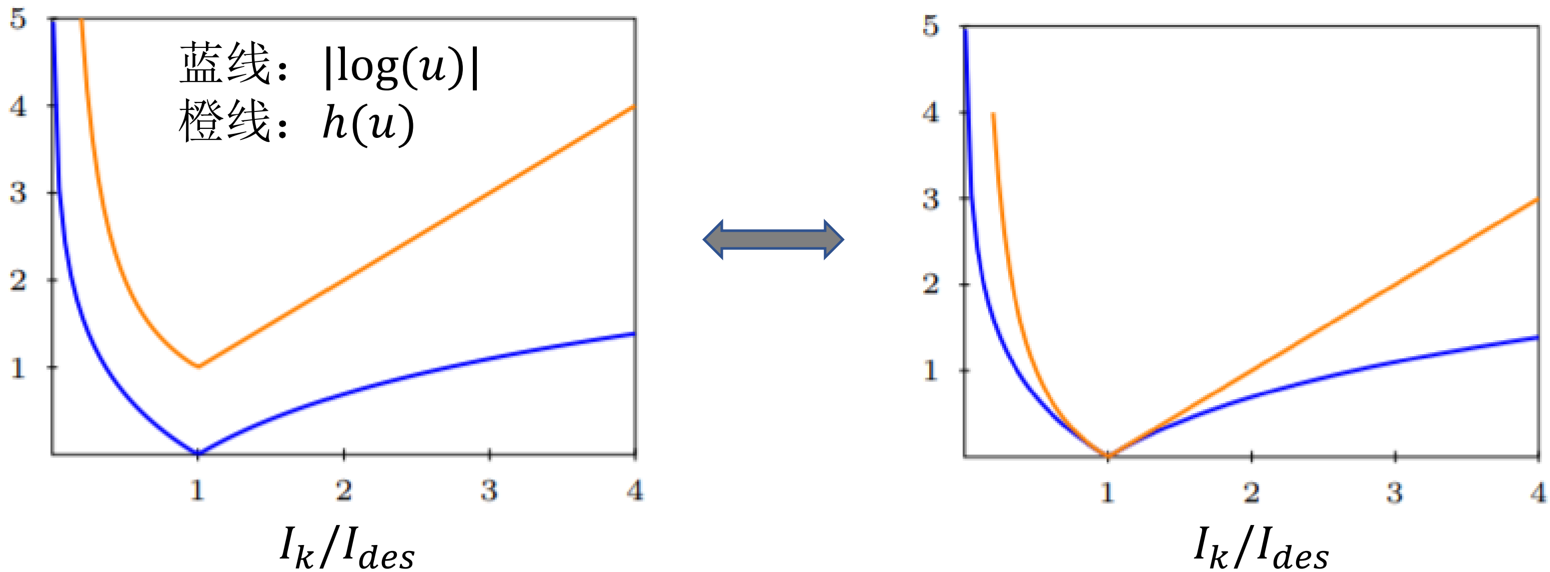
of course these are **approximates (suboptimal)** solutions

5. use convex programming: problem is **equivalent** to

$$\text{minimize } \ f_0(p) = max_{k=1,\ldots,n} h(I_k/I_{des})$$

$$\text{subject to } \ 0 \le p_j \le p_{max}, \ j = 1, \ldots, m$$

with $h(u) = max\{u, 1/u\}$



f_0 is convex because maximum of convex functions is convex

exact solution obtained with effort: **equivalent transformation of the objective function**

additional constraints: does adding 1 or 2 below complicate the problem?

1. no more than half of total power is in any 10 lamps
2. no more than half of the lamps are on ($p_j > 0$)

- answer: with (1), still easy to solve; with (2), extremely difficult
- moral: (untrained) intuition doesn't always work; without the proper background very easy problems can appear quite similar to very difficult problems

Nonlinear optimization

methods for solving general nonconvex problems involve compromises

local optimization methods (nonlinear programming)

- find a point that minimizes f_0 among feasible points near it
- fast, can handle large problems
- require initial guess
- provide no information about distance to (global) optimum

global optimization methods

- find the (global) solution
- worst-case complexity grows exponentially with problem size

these algorithms are often based on solving convex subproblems

Brief history of convex optimization

theory (convex analysis): 1900-1970

algorithms

- 1947: simplex algorithm for linear programming (Dantzig)
- 1970s: ellipsoid method and other subgradient methods
- 1980s & 90s: polynomial-time interior-point methods for convex optimization (Karmarkar 1984, Nesterov & Nemirovski 1994)
- since 2000s: many methods for large-scale convex optimization

applications

- before 1990: mostly in operations research, a few in engineering
- since 1990: many applications in engineering (control, signal processing, communications, circuit design, . . .)
- since 2000s: machine learning and statistics