CvxOpt Homework #2

姓名	学号
杨思远	3120210063

> **4.15** *Relaxation of Boolean LP.* In a *Boolean linear program*, the variable x is constrained to have components equal to zero or one:
>
> $$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i \in \{0,1\}, \quad i = 1, \ldots, n. \end{array} \tag{4.67}$$
>
> In general, such problems are very difficult to solve, even though the feasible set is finite (containing at most 2^n points).
>
> In a general method called *relaxation*, the constraint that x_i be zero or one is replaced with the linear inequalities $0 \le x_i \le 1$:
>
> $$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \le x_i \le 1, \quad i = 1, \ldots, n. \end{array} \tag{4.68}$$
>
> We refer to this problem as the *LP relaxation* of the Boolean LP (4.67). The LP relaxation is far easier to solve than the original Boolean LP.
>
> (a) Show that the optimal value of the LP relaxation (4.68) is a lower bound on the optimal value of the Boolean LP (4.67). What can you say about the Boolean LP if the LP relaxation is infeasible?
>
> (b) It sometimes happens that the LP relaxation has a solution with $x_i \in \{0,1\}$. What can you say in this case?

(a)

Let the feasible set of Boolean LP and its LP relaxation be F_{blp} and F_{lp}, formally

$$F_{blp} = \{x | Ax \preceq b, x_i \in \{0,1\}, i = 1, \ldots, n\}$$
$$F_{lp} = \{x | Ax \preceq b, 0 \le x_i \le 1, i = 1, \ldots, n\}.$$

Obviously, $F_{blp} \subset F_{lp}$, so the optimal value of the LP relxation (4.68) is a lower bound on the optimal value of the Bollean LP (4.67).

In addition, if the LP relaxation is infeasible, i.e., $F_{lp} = \emptyset$, then the Boolean LP is infeasible as well, i.e., $F_{blp} = \emptyset$.

(b)

In such case, the optimal value of the LP relaxation is also the optimal value of the Boolean LP.

5.1 *A simple example.* Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & x^2+1 \\ \text{subject to} & (x-2)(x-4) \leq 0, \end{array}$$

with variable $x \in \mathbf{R}$.

(a) *Analysis of primal problem.* Give the feasible set, the optimal value, and the optimal solution.

(b) *Lagrangian and dual function.* Plot the objective x^2+1 versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x,\lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^\star \geq \inf_x L(x,\lambda)$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g.

(c) *Lagrange dual problem.* State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution $\lambda^\star$. Does strong duality hold?

(d) *Sensitivity analysis.* Let $p^\star(u)$ denote the optimal value of the problem

$$\begin{array}{ll} \text{minimize} & x^2+1 \\ \text{subject to} & (x-2)(x-4) \leq u, \end{array}$$

as a function of the parameter u. Plot $p^\star(u)$. Verify that $dp^\star(0)/du = -\lambda^\star$.

(a)

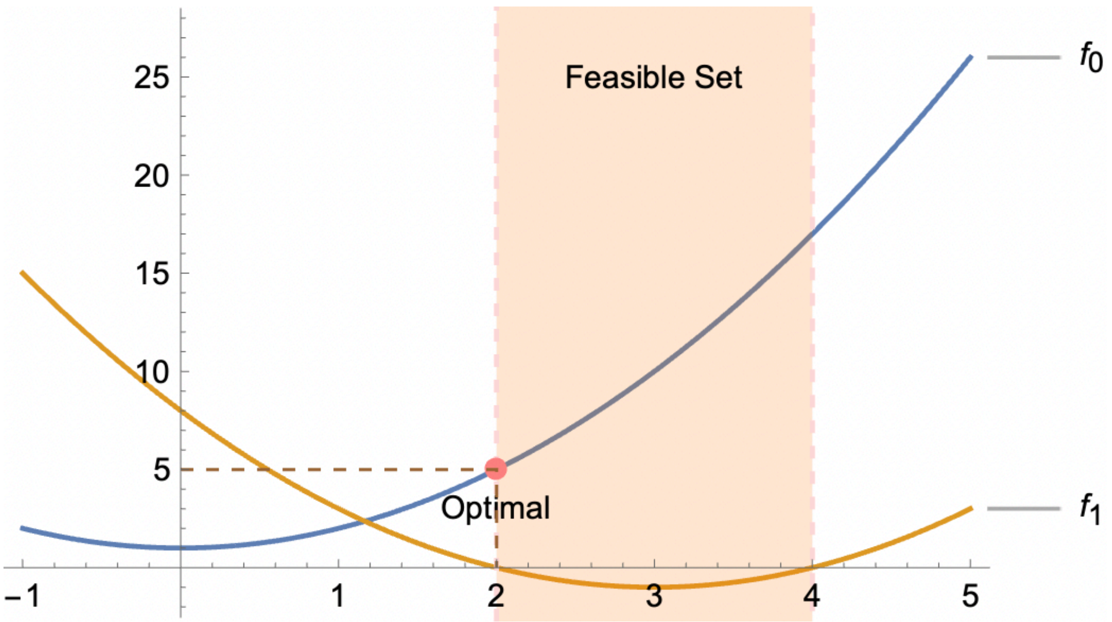
The feasible set is

$$F = \{x|(x-2)(x-4) \leq 0\} = \{x|2 \leq x \leq 4\} = [2,4].$$

The optimal value is $p^* = 5$, meanwhile the optimal solution is $x^* = 2$.

(b)

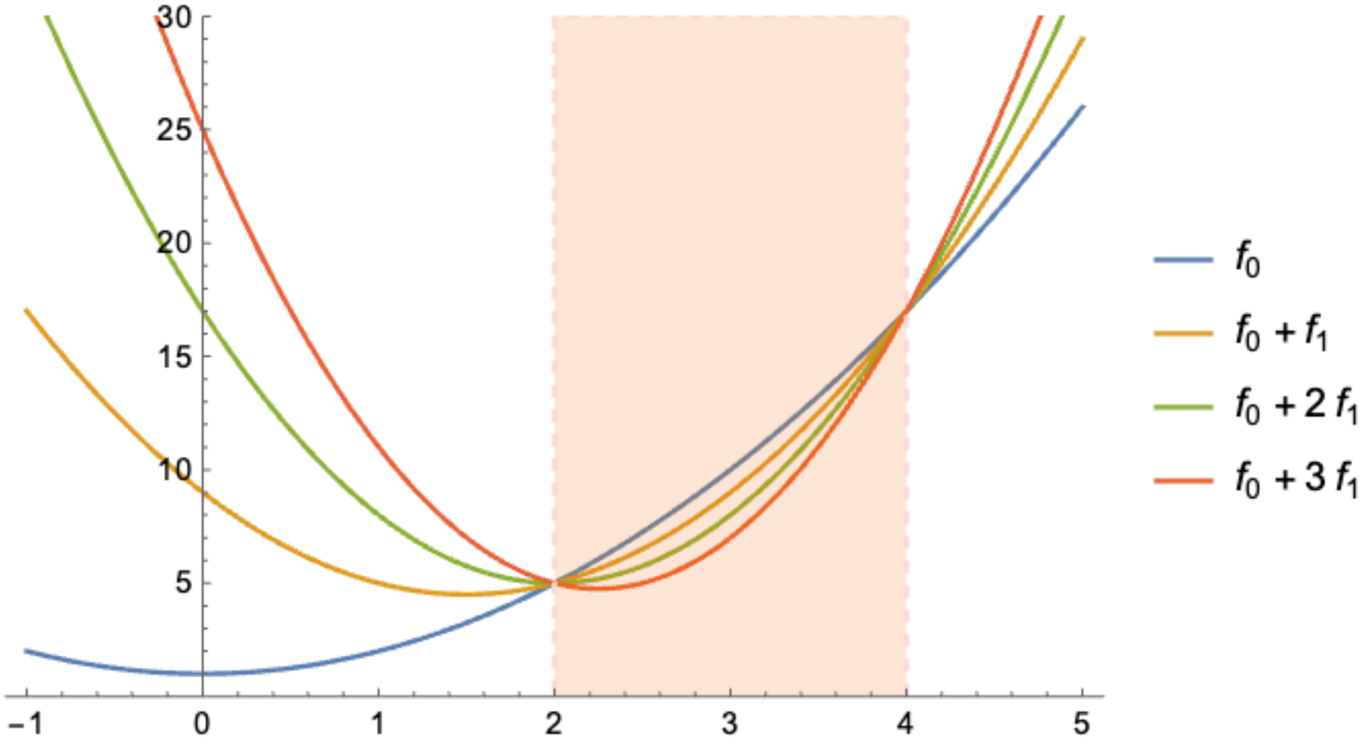
With Mathematica code (in Appendix A.1) we can get the plot as follows:



The Lagrangian is

$$
\begin{aligned}
L(x,\lambda) &= f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \\
&= x^2 + 1 + \lambda(x-2)(x-4) \\
&= (1+\lambda)x^2 - 6\lambda x + 1 + 8\lambda.
\end{aligned}
$$

Setting λ as a few values we can get the following plot (code in Appendix A.2)

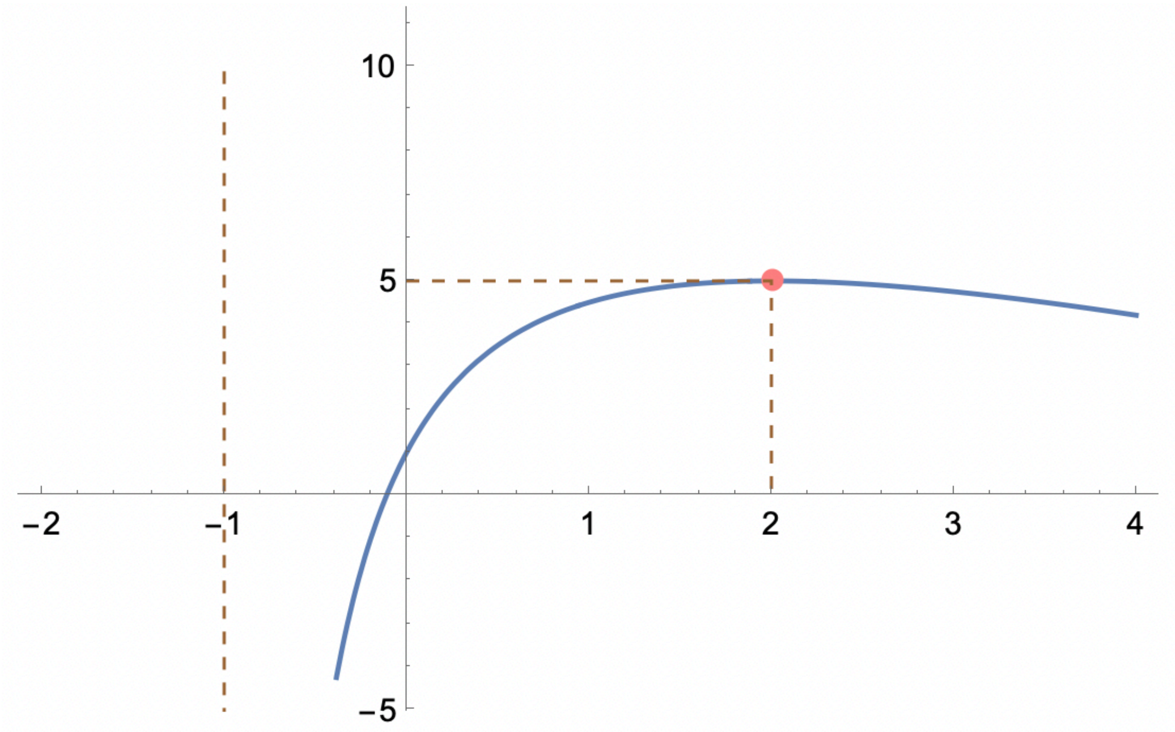


We can see that the minimum value of $L(x,\lambda)$ is always less than p^*. The mininum value reaches its maximum value at $\lambda^* = 2$, when $p^* = g(\lambda)$, where $g(\lambda) = \inf_{x\in\mathcal{D}} L(x,\lambda)$.

For $\lambda > -1$, the Lagrangian is minimal at $x^* = \frac{3\lambda}{1+\lambda}$, for $\lambda \le -1$ there is no minimum value. Hence we get

$$
g(\lambda) = \begin{cases} -\frac{9\lambda^2}{1+\lambda} + 1 + 8\lambda & \lambda > -1 \\ -\infty & \lambda \le -1, \end{cases}
$$

which is plotted below (code in Appendix A.3).



The dual function is concave, and its optimal value is $p^* = 5$ for $\lambda = 2$, while less than p^* at any other value of λ.

(c)

The Lagrange dual problem is

$$\begin{array}{ll} \text{maximize} & -9\lambda^2/(1+\lambda)+1+8\lambda \\ \text{subject to} & \lambda \geq 0. \end{array}$$

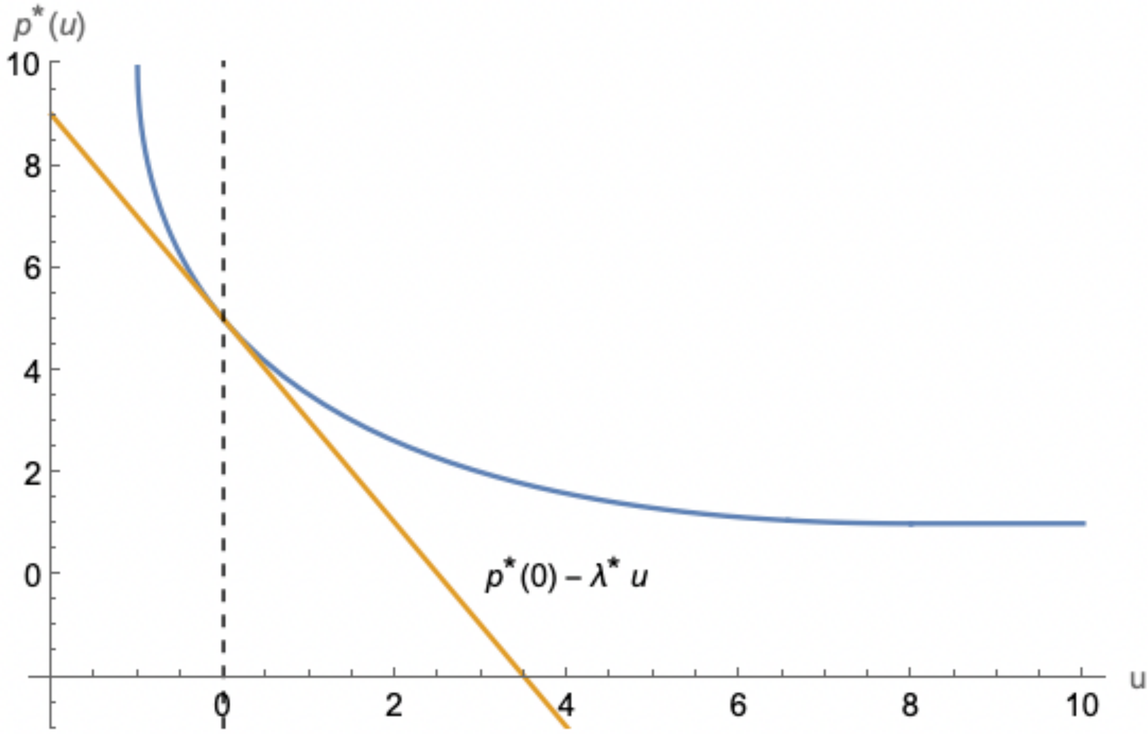
The dual reaches its optimum $d^* = 5$ at $\lambda = 2$, hence strong duality holds.

(d)

The feasible set is notempty when $u \geq 1$, and is the interval $[3-\sqrt{1+u}, 3+\sqrt{1+u}]$. So when the interval contains the unconstrained optimal point $x^* = 0$, i.e., $u \geq 8$, the optimum is as so; otherwise, the optimum is $x^*(u) = 3 - \sqrt{1+u}$. Formally,

$$p^*(u) = \begin{cases} \infty & u < -1 \\ 11 + u - 6\sqrt{1+u} & -1 \leq u \leq 8 \\ 1 & u \geq 8. \end{cases}$$

With code in Appendix A.4 we can plot the function as



The derivative of $p^*(u)$ at $u=0$ is

$$\frac{\mathrm{d}p^*(0)}{\mathrm{d}u} = 1 - \frac{3}{\sqrt{1+u}}\bigg|_{u=0} = -2 = -\lambda^*.$$

> **5.5** *Dual of general LP.* Find the dual function of the LP
>
> $$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b. \end{array}$$
>
> Give the dual problem, and make the implicit equality constraints explicit.

The Lagranian is

$$\begin{aligned} L(x,\lambda,\nu) &= c^T x + \lambda^T(Gx-h) + \nu^T(Ax-b) \\ &= (C^T + \lambda^T G + \nu^T A)x - (\lambda^T h + \nu^T b), \end{aligned}$$

which clearly a affine function of x. Hence the dual function is

$$g(\lambda,\nu) = \inf_{x\in\mathcal{D}} L(x,\lambda,\nu) = \begin{cases} -\lambda^T h - \nu^T b & c^T + \lambda^T G + \nu^T A = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

The dual problem

$$\begin{array}{ll} \text{maximize} & g(\lambda,\nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

can be stated as

$$\begin{array}{ll} \text{maximize} & -\lambda^T h - \nu^T b \\ \text{subject to} & c^T + \lambda^T G + \nu^T A = 0 \\ & \lambda \succeq 0. \end{array}$$

5.13 *Lagrangian relaxation of Boolean LP.* A *Boolean linear program* is an optimization problem of the form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i \in \{0,1\}, \quad i = 1,\dots,n, \end{array}$$

and is, in general, very difficult to solve. In exercise 4.15 we studied the LP relaxation of this problem,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \le x_i \le 1, \quad i = 1,\dots,n, \end{array} \tag{5.107}$$

which is far easier to solve, and gives a lower bound on the optimal value of the Boolean LP. In this problem we derive another lower bound for the Boolean LP, and work out the relation between the two lower bounds.

(a) *Lagrangian relaxation.* The Boolean LP can be reformulated as the problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i(1-x_i) = 0, \quad i = 1,\dots,n, \end{array}$$

which has quadratic equality constraints. Find the Lagrange dual of this problem. The optimal value of the dual problem (which is convex) gives a lower bound on the optimal value of the Boolean LP. This method of finding a lower bound on the optimal value is called *Lagrangian relaxation.*

(b) Show that the lower bound obtained via Lagrangian relaxation, and via the LP relaxation (5.107), are the same. *Hint.* Derive the dual of the LP relaxation (5.107).

(a)

The Lagrangian is

$$\begin{aligned} L(x,\mu,\nu) &= c^T x + \mu^T(Ax-b) - \nu^T x + x^T \mathbf{diag}(\nu) x \\ &= x^T \mathbf{diag}(\nu) x + (c^T + \mu^T A - \nu^T) x - \mu^T b, \end{aligned}$$

which minimum value can be given by

$$g(\mu,\nu) = \begin{cases} -b^T\mu - \sum_{i=1}^n (c_i + a_i^T\mu - \nu_i)/(4\nu_i) & \nu \succeq 0 \\ -\infty & \text{otherwise} \end{cases}$$

where c_i, ν_i represents the $i\text{th}$ element of c, ν and a_i is the $i\text{th}$ column of A.

So the dual problem is

$$\begin{array}{ll} \text{maximize} & -b^T\mu - \sum_{i=1}^n (c_i + a_i^T\mu - \nu_i)^2/(4\nu_i) \\ \text{subject to} & \nu \succeq 0. \end{array}$$

To simplify, we can observe that

$$\begin{aligned} \sup_{\nu_i \ge 0} \left(-\frac{(c_i + a_i^T\mu - \nu_i)^2}{\nu_i} \right) &= \begin{cases} (c_i + a_i^T\mu) & c_i + a_i^T\mu \le 0 \\ 0 & c_i + a_i^T\mu \ge 0 \end{cases} \\ &= \min\{0, (c_i + a_i^T\mu)\}. \end{aligned}$$

So the dual problem can be simplified as

$$\begin{array}{ll} \text{maximize} & -b^T\mu + \sum_{i=1}^n \min\{0, c_i + a_i^T\mu\} \\ \text{subject to} & \mu \succeq 0. \end{array}$$

(b)

The Lagrangian and dual function of the LP relaxation are

$$\begin{aligned} L(x,u,v,w) &= c^Tx + u^T(Ax-b) - v^Tx + w^T(x-\mathbf{1}) \\ &= (c + A^T - v + w)^Tx - b^Tu - \mathbf{1}^Tw \\ g(u,v,w) &= \begin{cases} -b^Tu - \mathbf{1}^Tw & A^Tu - v + w + c = 0 \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

So the dual problem is

$$\begin{array}{ll} \text{maximize} & -b^Tu - \mathbf{1}^Tw \\ \text{subject to} & A^Tu - v + w + c = 0 \\ & u \succeq 0, v \succeq 0, w \succeq 0, \end{array}$$

which is clearly equivalent to the Lagrange relaxation problem derived before, hence gives the same value as well.

> **5.17** *Robust linear programming with polyhedral uncertainty.* Consider the robust LP
>
> $$\begin{array}{ll} \text{minimize} & c^Tx \\ \text{subject to} & \sup_{a\in\mathcal{P}_i} a^Tx \le b_i, \quad i = 1,\ldots,m, \end{array}$$
>
> with variable $x \in \mathbf{R}^n$, where $\mathcal{P}_i = \{a \mid C_ia \preceq d_i\}$. The problem data are $c \in \mathbf{R}^n$, $C_i \in \mathbf{R}^{m_i\times n}$, $d_i \in \mathbf{R}^{m_i}$, and $b \in \mathbf{R}^m$. We assume the polyhedra $\mathcal{P}_i$ are nonempty. Show that this problem is equivalent to the LP
>
> $$\begin{array}{ll} \text{minimize} & c^Tx \\ \text{subject to} & d_i^Tz_i \le b_i, \quad i = 1,\ldots,m \\ & C_i^Tz_i = x, \quad i = 1,\ldots,m \\ & z_i \succeq 0, \quad i = 1,\ldots,m \end{array}$$
>
> with variables $x \in \mathbf{R}^n$ and $z_i \in \mathbf{R}^{m_i}$, $i = 1,\ldots,m$. *Hint.* Find the dual of the problem of maximizing a_i^Tx over $a_i \in \mathcal{P}_i$ (with variable a_i).

The LP can be expressed as

$$\begin{array}{ll} \text{minimize} & c^Tx \\ \text{subject to} & f_i(x) \le b_i, \quad i = 1,\ldots,m \end{array}$$

where $f_i(x)$ is the optimal value of the following LP

$$\begin{array}{ll} \text{maximize} & x^Ta \\ \text{subject to} & C_ia \preceq d_i \end{array}$$

with variable a_i in each problem. So the dual problem of such LP is

$$\begin{array}{ll} \text{minimize} & d_i^Tz \\ \text{subject to} & C_i^Tz - x = 0 \\ & z \succeq 0. \end{array}$$

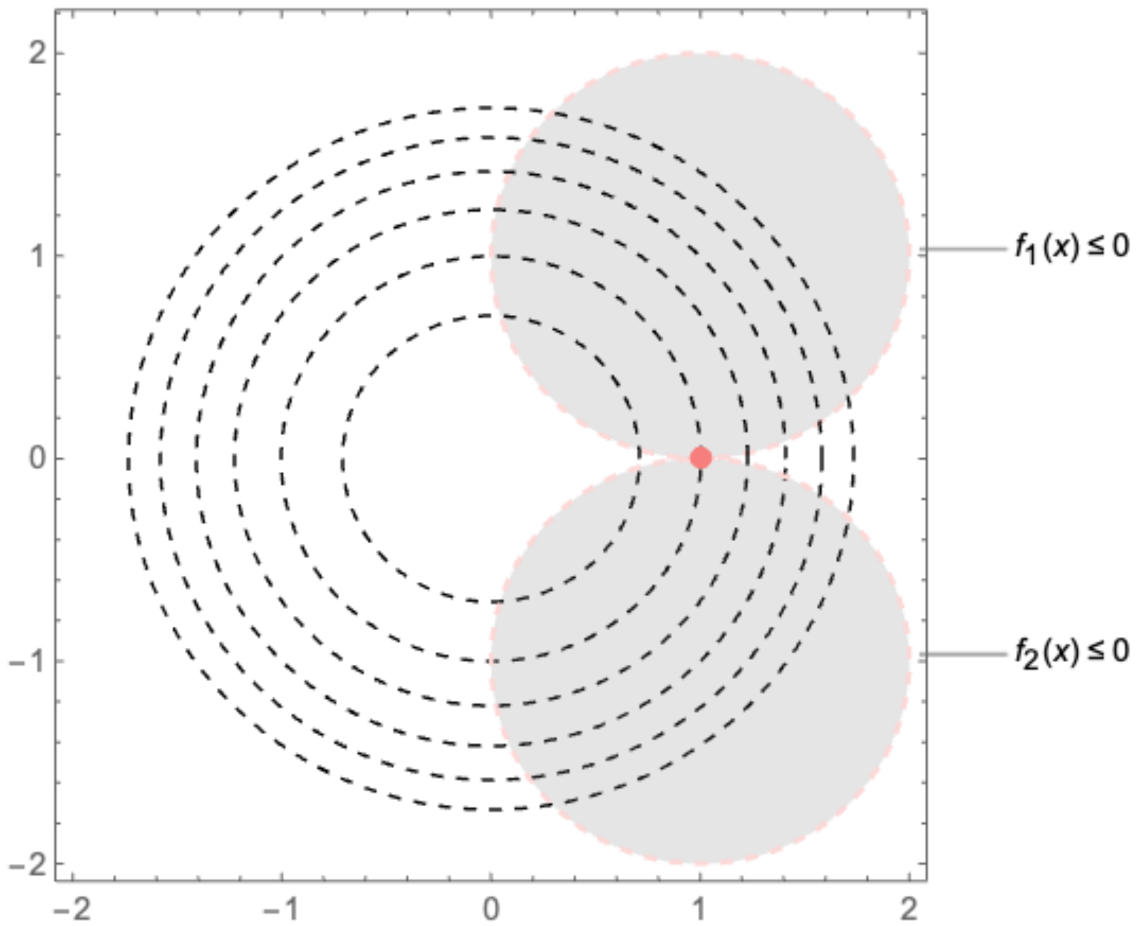
Altogether we get the equivalent form of the original LP, which is

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & d_i^T z_i \leq b_i, \quad i = 1, \ldots, m \\
& C_i^T z_i = x, \quad i = 1, \ldots, m \\
& z_i \succeq 0, \quad i = 1, \ldots, m.
\end{array}
$$

> **5.26** Consider the QCQP
>
> $$
> \begin{array}{ll}
> \text{minimize} & x_1^2 + x_2^2 \\
> \text{subject to} & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1 \\
> & (x_1 - 1)^2 + (x_2 + 1)^2 \leq 1
> \end{array}
> $$
>
> with variable $x \in \mathbf{R}^2$.
>
> (a) Sketch the feasible set and level sets of the objective. Find the optimal point $x^\star$ and optimal value $p^\star$.
>
> (b) Give the KKT conditions. Do there exist Lagrange multipliers $\lambda_1^\star$ and $\lambda_2^\star$ that prove that $x^\star$ is optimal?
>
> (c) Derive and solve the Lagrange dual problem. Does strong duality hold?

(a)

Using the code in Appendix A.5 we can sketch the feasible set and level sets of the objective.



The feasible set only contains one point, hence the optimal point $x^* = (1, 0)$, the optimal value $p^* = 1$.

(b)

The KKT conditions are

$$(x_1-1)^2+(x_2-1)^2 \le 1, \quad (x_1-1)^2+(x_2+1)^2 \le 1,$$
$$\lambda_1 \ge 0, \quad \lambda_2 \ge 0$$
$$2x_1+2\lambda_1(x_1-1)+2\lambda_2(x_1-1)=0$$
$$2x_2+2\lambda_1(x_2-1)+2\lambda_2(x_2+1)=0$$
$$\lambda_1((x_1-1)^2+(x_2-1)^2-1)=\lambda_2((x_1-1)^2+(x_2+1)^2-1)=0$$

At $x^*=(1,0)$ the conditions are

$$\lambda_1 \ge 0, \quad \lambda_2 \ge 0, \quad 2=0, \quad -2\lambda_1+2\lambda_2=0$$

which obviously have no solution.

(c)

The Lagrange dual function is

$$g(\lambda_1,\lambda_2)=\inf_{x_1,x_2} L(x_1,x_2,\lambda_1,\lambda_2)$$

where

$$\begin{aligned} L(x_1,x_2,\lambda_1,\lambda_2) &= x_1^2+x_2^2+\lambda_1((x_1-1)^2+(x_2-1)^2-1)+\lambda_2((x_1-1)^2+(x_2+1)^2-1) \\ &= (1+\lambda_1+\lambda_2)x_1^2+(1+\lambda_1+\lambda_2)x_2^2-2(\lambda_1+\lambda_2)x_1-2(\lambda_1-\lambda_2)x_2+\lambda_1+\lambda_2. \end{aligned}$$

When $x_1=\frac{\lambda_1+\lambda_2}{1+\lambda_1+\lambda_2}$ and $x_2=\frac{\lambda_1-\lambda_2}{1+\lambda_1+\lambda_2}$,

$$g(\lambda_1,\lambda_2)=\begin{cases} -\frac{(\lambda_1+\lambda_2)^2+(\lambda_1-\lambda_2)^2}{1+\lambda_1+\lambda_2} & 1+\lambda_1+\lambda_2 \ge 0 \\ -\infty & \text{otherwise.} \end{cases}$$

So the dual problem is

$$\begin{array}{ll} \text{maximize} & \frac{\lambda_1+\lambda_2-(\lambda_1-\lambda_2)^2}{1+\lambda_1+\lambda_2} \\ \text{subject to} & \lambda_1 \ge 0, \lambda_2 \ge 0 \end{array}$$

The maximum is reached when $\lambda_1, \lambda_2 \to \infty$, and $d^*=p^*=1$. So the strong duality does not hold.

Appendix A. Mathematica Codes

A.1

```
func = Plot[{x^2 + 1, (x - 2) (x - 4)}, {x, -1, 5},
   PlotLabels -> {ToExpression["f_0", TeXForm],
     ToExpression["f_1", TeXForm]}];
feas = RegionPlot[(x - 2) (x - 4) <= 0, {x, 1, 5}, {y, -30, 30},
   BoundaryStyle -> {LightRed, Dashed},
   PlotStyle -> {Directive[Orange, Opacity[0.2]]}];
opti = Graphics[{PointSize[Large], Pink, Point[{2, 5}]}];
text = Graphics[{Text["Optimal", {2, 4}, {Center, Top}],
    Text["Feasible Set", {3, 25}]}];
line = Graphics[{Dashed, Brown, Line[{{0, 5}, {2, 5}, {2, 0}}]}];
Show[func, feas, opti, text, line]
```

A.2

```
feas = RegionPlot[(x - 2) (x - 4) <= 0, {x, 1, 5}, {y, -30, 30},
   BoundaryStyle -> {LightRed, Dashed},
   PlotStyle -> {Directive[Orange, Opacity[0.2]]}];
lagr = Plot[
   Evaluate[Table[(1 + l) x^2 - 6 l x + 1 + 8 l, {l, 0, 3}]], {x, -1,
    5}, PlotRange -> {0, 30},
   PlotLegends -> {ToExpression["f_0", TeXForm],
     ToExpression["f_0+f_1", TeXForm],
     ToExpression["f_0+2f_1", TeXForm],
     ToExpression["f_0+3f_1", TeXForm]}];
Show[lagr, feas]
```

A.3

```
func = Plot[-9*l^2/(1 + l) + 1 + 8 l, {l, -2, 4}];
line = Graphics[{Dashed, Brown, Line[{{-1, -10}, {-1, 10}}]}];
opti = Graphics[{PointSize[Large], Pink, Point[{2, 5}]}];
dsline = Graphics[{Dashed, Brown, Line[{{0, 5}, {2, 5}, {2, 0}}]}];
Show[func, line, opti, dsline]
```

A.4

```
f[x_] = If[x >= 8, 1, 11 + x - 6 Sqrt[1 + x]];
func = Plot[{f[x], f[0] + f'[0] x}, {x, -2, 10},
   PlotRange -> {-3, 10},
   AxesLabel -> {"u", ToExpression["p^*(u)", TeXForm]},
   AxesOrigin -> {-2, -2}];
text = Text[ToExpression["p^*(0)-\lambda^*u", TeXForm], {4, 0}];
line = Line[{{0, -10}, {0, 20}}];
Show[func, Graphics[text], Graphics[{Dashed, line}]]
```

A.5

```
func = Graphics[
   Table[{Dashed, Circle[{0, 0}, Sqrt[r]]}, {r, 0.5, 3, 0.5}]];
fea1 = RegionPlot[(x - 1)^2 + (y - 1)^2 <= 1, {x, -2, 2}, {y, -2, 2},
   BoundaryStyle -> {LightRed, Dashed},
   PlotStyle -> {Directive[Gray, Opacity[0.2]]},
   PlotLabels -> ToExpression["f_1(x)\le0", TeXForm]];
fea2 = RegionPlot[(x - 1)^2 + (y + 1)^2 <= 1, {x, -2, 2}, {y, -2, 2},
   BoundaryStyle -> {LightRed, Dashed},
   PlotStyle -> {Directive[Gray, Opacity[0.2]]},
   PlotLabels -> ToExpression["f_2(x)\le0", TeXForm]];
opti = Graphics[{PointSize[Large], Pink, Point[{1, 0}]}];
Show[fea1, func, fea2, opti]
```