4. Convex optimization problems

Xinfu Liu

Outline

- optimization problem in standard form
- convex optimization problems
- quasiconvex optimization
- linear optimization
- quadratic optimization
- geometric programming
- generalized inequality constraints
- semidefinite programming
- vector optimization

Optimization problem in standard form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0,\ i = 1, \dots, m \\ & h_i(x) = 0,\ i = 1, \dots, l \end{array}$$

- $x \in \boldsymbol{R}^n$ is the optimization variable
- $f_0 : \boldsymbol{R}^n \to \boldsymbol{R}$ is the objective or cost function
- $f_i : \boldsymbol{R}^n \to \boldsymbol{R}, i = 1, \dots, m$ are the inequality constraint functions
- $h_i : \boldsymbol{R}^n \to \boldsymbol{R}$ are the equality constraint functions

optimal value:

$$p^* = \inf \{ f_0(x) \mid f_i(x) \le 0,\ i = 1, \dots, m,\ \ h_i(x) = 0,\ i = 1, \dots, l \}$$

- $p^* = \infty$ if problem is **infeasible** (no x satisfies the constraints)
- $p^* = -\infty$ if problem is **unbounded below**

Optimal and locally optimal points

x is **feasible** if $x \in \mathbf{dom}\, f_0$ and it satisfies the constraints

a feasible x is **optimal** if $f_0(x) = p^*$; X_{opt} is the set of optimal points

x is **locally optimal** if there is an $R > 0$ such that x is optimal for

$$\begin{array}{ll} \text{minimize (over } z \text{)} & f_0(z) \\ \text{subject to} & f_i(z) \leq 0,\ i = 1,\dots,m, \quad h_i(z) = 0,\ i = 1,\dots,l \\ & \|z - x\|_2 \leq R \end{array}$$

example: (with $n = 1, m = l = 0$)

- $f_0(x) = 1/x$, $\mathbf{dom} f_0 = \mathbf{R}_{++}$: $p^* = 0$, no optimal point
- $f_0(x) = -\log x$, $\mathbf{dom} f_0 = \mathbf{R}_{++}$: $p^* = -\infty$, unbounded
- $f_0(x) = x \log x$, $\mathbf{dom} f_0 = \mathbf{R}_{++}$: $p^* = -1/e$, $x = 1/e$ is optimal
- $f_0(x) = x^3 - 3x$, $\mathbf{dom} f_0 = \mathbf{R}$: $p^* = -\infty$, local optimum at $x = 1$

Implicit constraints

the standard form optimization problem has an **implicit constraint**

$$x \in D = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i \cap \bigcap_{i=1}^{l} \mathbf{dom}\, h_i,$$

- we call D the **domain** of the problem
- the constraints $f_i(x) \le 0, h_i(x) = 0$ are the **explicit constraints**
- a problem is **unconstrained** if it has no explicit constraints ($m = l = 0$)

example:

$$\text{minimize} \quad f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints $a_i^T x < b_i$

Feasibility problem

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & f_i(x) \leq 0,\ i = 1,\ldots,m \\ & h_i(x) = 0,\ i = 1,\ldots,l \end{array}$$

can be considered a special case of the general problem with $f_0(x) = 0$

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & f_i(x) \leq 0,\ i = 1,\ldots,m \\ & h_i(x) = 0,\ i = 1,\ldots,l \end{array}$$

- $p^* = 0$ if constraints are feasible; any feasible x is optimal
- $p^* = \infty$ if constraints are infeasible

Convex optimization problem

standard form of convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0,\ i = 1,\ldots,m \\ & a_i^T x = b_i,\ i = 1,\ldots,l \end{array}$$

- $f_0, f_1, \ldots, f_m$ are convex; equality constraints are affine
- problem is ***quasiconvex*** if f_0 is quasiconvex (and $f_1, \ldots, f_m$ convex)

a convex optimization problem is often written as

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0,\ i = 1,\ldots,m \\ & Ax = b \end{array}$$

important property: **feasible set** of a convex optimization problem is **convex**

example:

$$\begin{array}{ll} \text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = x_1/(1+x_2^2) \le 0 \\ & h_1(x) = (x_1+x_2)^2 = 0 \end{array}$$

- f_0 is convex; feasible set $\{(x_1, x_2) | x_1 = -x_2 \le 0\}$ is convex
- not a convex problem (according to our definition): f_1 is not convex, h_1 is not affine
- equivalent (but not identical) to the convex problem

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \le 0 \\ & x_1 + x_2 = 0 \end{array}$$

Local and global optimal

any **locally** optimal point of a convex problem is (**globally**) optimal

proof: suppose x is locally optimal, but there exists a feasible y with $f_0(y) < f_0(x)$

x locally optimal means there is an $R > 0$ such that

$$z \text{ feasible}, \quad \|z - x\|_2 \leq R \quad \Rightarrow \quad f_0(z) \geq f_0(x)$$

consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$

- $\|y - x\|_2 > R$, so $0 < \theta < 1/2$
- z is a convex combination of two feasible points, hence also feasible

We can get $\|z - x\|_2 = R/2$ and

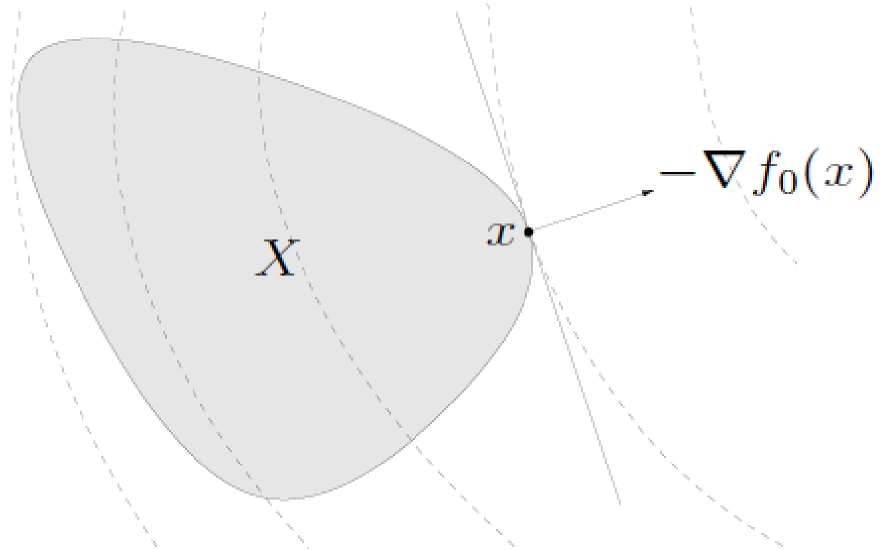
$$f_0(z) \leq \theta f_0(y) + (1 - \theta) f_0(x) < f_0(x)$$

which contradicts our assumption that x is locally optimal

Optimality criterion for differentiable f_0

x is optimal if and only if it is feasible and

$$\nabla f_0(x)^T (y - x) \geq 0 \quad \text{for all feasible } y$$



if $\nabla f_0(x) \neq 0$, $\{y | \nabla f_0(x)^T (y - x) = 0\}$ defines **a supporting hyperplane**

to the feasible set $\boldsymbol{X}$ at x

- **unconstrained problem**: x is optimal if and only if

$$x \in \textbf{dom } f_0, \; \nabla f_0(x) = 0$$

- **equality constrained problem**

$$\text{minimizes} \quad f_0(x) \quad \text{subject to} \quad Ax = b$$

x is optimal if and only if there exists a υ such that

$$x \in \textbf{dom } f_0, \quad Ax = b, \quad \nabla f_0(x) + A^T v = 0$$

- **minimization over nonnegative orthant**

$$\text{minimizes} \quad f_0(x) \quad \text{subject to} \quad x \succeq 0$$

x is optimal if and only if

$$x \in \textbf{dom } f_0, \quad x \succeq 0, \quad \begin{cases} \nabla f_0(x)_i \geq 0 & x_i = 0 \\ \nabla f_0(x)_i = 0 & x_i > 0 \end{cases}$$

Equivalent convex problems

two problems are (informally) **equivalent** if the **solution** of one is readily obtained from the solution of the other, and vice-versa

some common transformations that yield equivalent problems:

- **eliminating equality constraints**

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \ i = 1, \dots, m \\ & Ax = b \end{array}$$

is **equivalent to**

$$\begin{array}{ll} \text{minimize (over } z \text{)} & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \leq 0, \ i = 1, \dots, m \end{array}$$

where F and x_0 are such that (x_0 is a particular solution, F is a matrix whose range is the null space of A)

$$Ax = b \quad \Longleftrightarrow \quad x = Fz + x_0 \ \text{ for some } z$$

- **introducing equality constraints**

$$\begin{array}{ll} \text{minimize} & f_0(A_0 x + b_0) \\ \text{subject to} & f_i(A_i x + b_i) \le 0,\ i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize (over } x, y_i\text{)} & f_0(y_0) \\ \text{subject to} & f_i(y_i) \le 0,\ i = 1, \ldots, m \\ & y_i = A_i x + b_i,\ i = 0, 1, \ldots, m \end{array}$$

- **introducing slack variables for linear inequalities**

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & a_i^T x \le b_i,\ \ i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize (over } x, s\text{)} & f_0(x) \\ \text{subject to} & a_t^T x + s_i = b_i,\ i = 1, \ldots, m \\ & s_i \ge 0,\ i = 1, \ldots, m \end{array}$$

- **epigraph form:**

  the standard form of **convex problem** is equivalent to

$$
\begin{array}{ll}
\text{minimize (over } x, t) & t \\
\text{subject to} & f_0(x) - t \le 0 \\
 & f_i(x) \le 0, \quad i = 1, \dots, m \\
 & Ax = b
\end{array}
$$

Is this optimization problem convex?

- **minimizing over some variables**

$$
\begin{array}{ll}
\text{minimize} & f_0(x_1, x_2) \\
\text{subject to} & f_i(x_1) \le 0, \quad i = 1, \dots, m
\end{array}
$$

  is equivalent to

$$
\begin{array}{ll}
\text{minimize} & \tilde{f}_0(x_1) \\
\text{subject to} & f_i(x_1) \le 0, \quad i = 1, \dots, m
\end{array}
$$

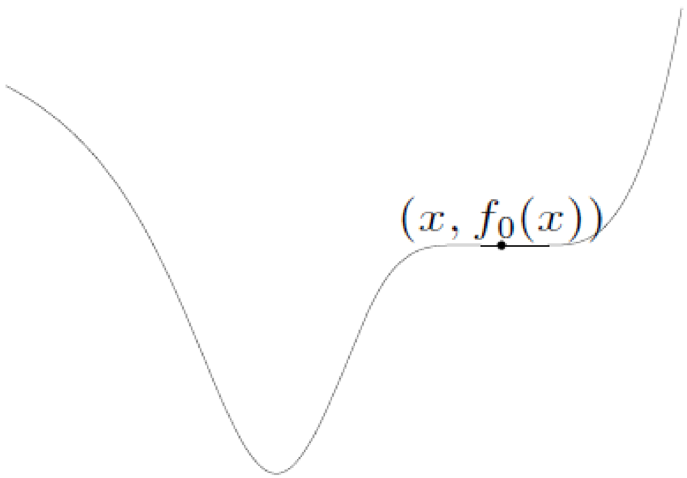
  where $\tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$

Quasiconvex optimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0,\ i = 1, \ldots, m \\ & Ax = b \end{array}$$

with $f_0 : \mathbf{R}^n \longrightarrow \mathbf{R}$ **quasiconvex**, $f_1, \ldots, f_m$ **convex**

can have locally optimal points that are not (globally) optimal



convex representation of sublevel sets of f_0

if f_0 is quasiconvex, there exists a family of functions ϕ_t such that:

- $\phi_t(x)$ is convex in x for fixed t
- t-sublevel set of f_0 is 0-sublevel set of ϕ_t, i.e.,

$$f_0(x) \leq t \quad \Leftrightarrow \quad \phi_t(x) \leq 0$$

example

$$f_0(x) = \frac{p(x)}{q(x)}$$

with p convex, q concave, and $p(x) \geq 0, q(x) > 0$ on $\mathbf{dom} f_0$

can take $\phi_t(x) = p(x) - tq(x)$:

- for $t \geq 0$, ϕ_t convex in x
- $p(x)/q(x) \leq t$ if and only if $\phi_t(x) \leq 0$

quasiconvex optimization via convex feasibility problems

$$\phi_t(x) \le 0, \quad f_i(x) \le 0, \quad i = 1, \ldots, m, \quad Ax = b \qquad (1)$$

- for fixed t , a convex feasibility problem in x
- if **feasible**, we can conclude that $t \ge p^*$; if **infeasible**, $t \le p^*$

Bisection method for quasiconvex optimization

given $l \le p^\star$, $u \ge p^\star$, tolerance $\epsilon > 0$.
repeat
1. $t := (l + u)/2$.
2. Solve the convex feasibility problem (1).
3. **if** (1) is feasible, $u := t$; **else** $l := t$.

until $u - l \le \epsilon$.

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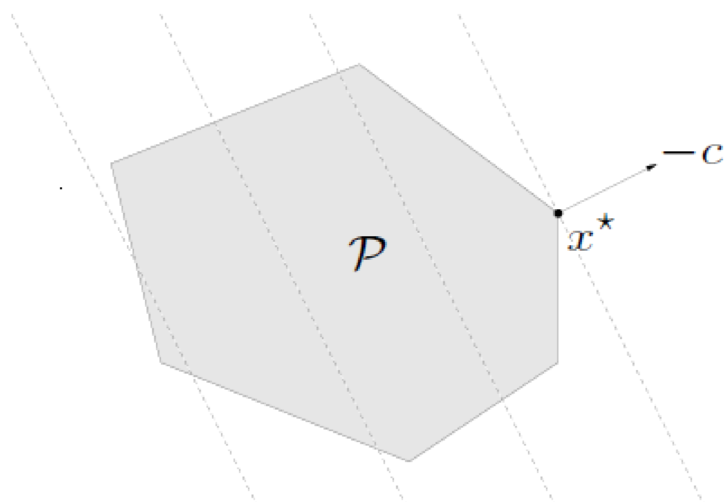
requires exactly $[\log_2((u - l)/\epsilon)]$ iterations (where u, l are initial values)

Linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



Examples

diet problem: choose quantities $x_1, \dots, x_n$ of n foods

- one unit of food j costs c_j , contains amount a_{ij} of nutrient i
- healthy diet requires nutrient i in quantity at least b_i

to find cheapest healthy diet,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \succeq b, \quad x \succeq 0 \end{array}$$

piecewise-linear minimization

$$\text{minimize} \quad \max_{i=1,\dots,m}(a_i^T x + b_i)$$

equivalent to an LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \le t, \; i = 1, \dots, m \end{array}$$

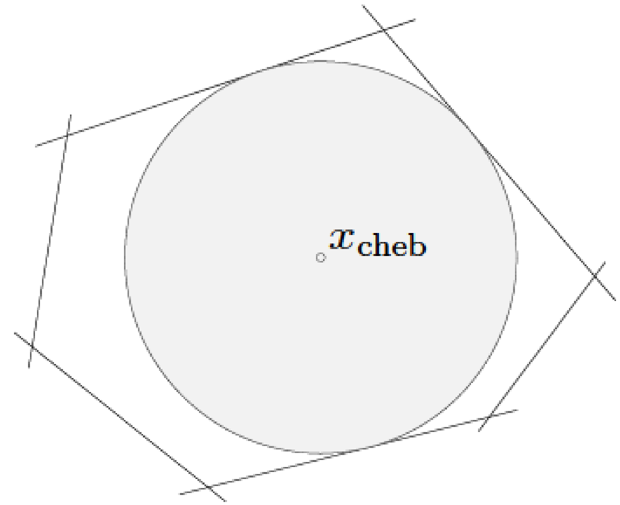
Chebyshev center of a polyhedron

Chebyshev center of

$$\boldsymbol{P} = \{x \mid a_i^T x \le b_i,\ i = 1, \ldots, m\}$$

is center of largest inscribed ball

$$\boldsymbol{B} = \{x_c + u \mid \|u\|_2 \le r\}$$



- $a_i^T x \le b_i$ for all $x \in \boldsymbol{B}$ if and only if

$$\sup\left\{a_i^T(x_c + u) \mid \|u\|_2 \le r\right\} = a_i^T x_c + r\|a_i\|_2 \le b_i$$

- hence, x_c, r can be determined by solving the LP

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & a_i^T x_c + r\|a_i\|_2 \le b_i,\ i = 1, \ldots, m \end{array}$$

Linear-fractional program

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

linear-fractional program

$$f_0(x) = \frac{c^T x + d}{e^T x + f} \qquad \mathbf{dom}\ f_0(x) = \{x \mid e^T x + f > 0\}$$

- a quasiconvex optimization problem; can be solved by **bisection**
- also equivalent to the LP (variables y, z)

$$\begin{array}{ll} \text{minimize} & c^T y + dz \\ \text{subject to} & Gy \preceq hz \\ & Ay = bz \\ & e^T y + fz = 1 \\ & z \geq 0 \end{array}$$

generalized linear-fractional program

$$f_0(x) = \max_{i=1,\dots,r} \frac{c_i^T x + d_i}{e_i^T x + f_i} \quad \textbf{dom}\ f_0(x) = \{x \mid e_i^T x + f_i > 0, i = 1, \dots, r\}$$

a quasiconvex optimization problem; can be solved by bisection

example: Von Neumann model of a growing economy

$$\begin{array}{ll} \text{maximize (over } x, x^+) & \min_{i=1,\dots,n} x_i^+ / x_i \\ \text{subject to} & x^+ \succeq 0, \quad Bx^+ \preceq Ax \end{array}$$

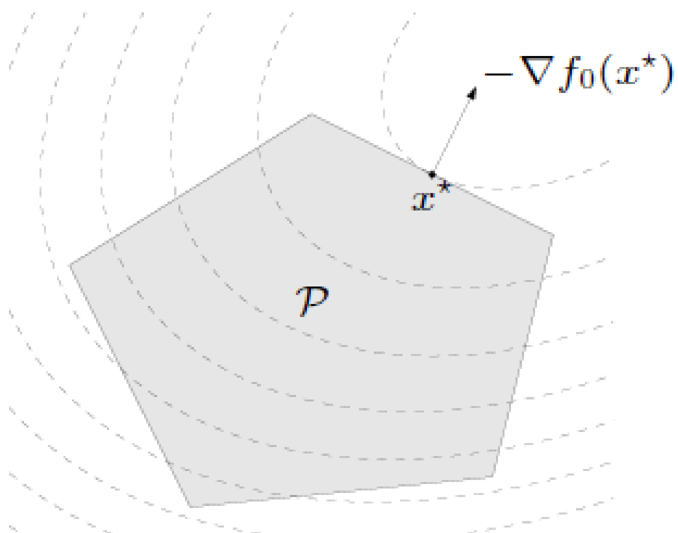
- $x, x^+ \in \mathbf{R}^n$:activity levels of n sectors, in current and next period
- $(Ax)_i, (Bx^+)_i$:produced, resp. consumed, amounts of good i
- x_i^+/x_i :growth rate of sector i

allocate activity to maximize growth rate of slowest growing sector

Quadratic program (QP)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- $P \in \mathbf{S}_+^n$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



Examples

least-squares

$$\text{minimize} \quad \|Ax - b\|_2^2$$

- analytical solution $x^\star = A^\dagger b$ ($A^\dagger$ is pseudo-inverse)
- can add linear constraints, e.g., $l \preceq x \preceq u$

linear program with random cost

$$\begin{array}{ll} \text{minimize} & \mathbf{E}c^T x + \gamma \operatorname{var}(c^T x) = \bar{c}^T x + \gamma x^T \Sigma x \\ \text{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

- c is a random vector with mean $\bar{c}$ and covariance Σ
- hence, $c^T x$ is a random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$
- $\gamma > 0$ is risk aversion parameter; controls the trade-off between expected cost and variance (risk)

Quadratically constrained quadratic program (QCQP)

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \le 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

- $P_i \in \mathbf{S}_+^n$; objective and constraints **are convex quadratic**

- if $P_1, \ldots, P_m \in \mathbf{S}_{++}^n$, **feasible region** is intersection of m **ellipsoids** and an **affine set**

Second-order cone program (SOCP)

$$
\begin{array}{ll}
\text{minimize} & f^T x \\
\text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \\
& Fx = g
\end{array}
$$

$(A_i \in \mathbf{R}^{n_i \times n}, F \in \mathbf{R}^{p \times n})$

- inequalities are called **second-order cone (SOC) constraints**:

$$(A_i x + b_i, c_i^T x + d_i) \in \text{second-oder cone in } \mathbf{R}^{n_i+1}$$

- for $n_i = 0$, reduces to an LP; if $c_i = 0$, reduces to a QCQP

- **more general** than QCQP and LP

Robust linear programming

the **parameters** in optimization problems are often **uncertain**, e.g., in an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

there can be **uncertainty** in c, a_i, b_i

two common approaches to handling uncertainty (in a_i, for simplicity)

- **deterministic model**: constraints must **hold for all** $a_i \in \mathcal{E}_i$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \ \text{ for all } a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m, \end{array}$$

- **stochastic model**: a_i is random variable; constraints must **hold with probability** η

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \text{prob}(a_i^T x \leq b_i) \geq \eta, \ i = 1, \ldots, m \end{array}$$

deterministic approach via SOCP

- choose an ellipsoid as $\mathcal{E}_i$:

$$\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \le 1\} \qquad (\bar{a}_i \in \mathbf{R}^n,\ P_i \in \mathbf{R}^{n\times n})$$

center is $\bar{a}_i$, semi-axes determined by singular values/eigenvectors of P_i

- robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i,\ \forall a_i \in \mathcal{E}_i, \quad i = 1,\dots,m \end{array}$$

is **equivalent to the SOCP**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \le b_i, \quad i = 1,\dots,m \end{array}$$

(follows from $\sup_{\|u\|_2 \le 1} (\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2$)

stochastic approach via SOCP

- assume a_i is Gaussian with mean $\bar{a}_i$, covariance $\Sigma_i (a_i \sim \mathrm{N}(\bar{a}_i, \Sigma_i))$
- $a_i^T x$ is Gaussian r.v. with mean $\bar{a}_i^T x$, variance $x^T \sum x$; hence

$$\mathbf{prob}(a_i^T x \le b_i) = \Phi\left(\frac{b_i - \bar{a}_i^T x}{\left\|\Sigma_i^{1/2} x\right\|_2}\right)$$

where $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2} dt$ is CDF of $\mathrm{N}(0,1)$

- robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \le b_i) \ge \eta_i,\ i = 1, \ldots, m, \end{array}$$

with $\eta \ge 1/2$, is **equivalent to the SOCP**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta) \left\|\Sigma_i^{1/2} x\right\|_2 \le b_i, \quad i = 1, \ldots, m \end{array}$$

Geometric programming

monomial function

$$f(x) = c x_1^{a_1} \dots x_n^{a_n}, \ \mathbf{dom}\ f = \mathbf{R}_{++}^n$$

with $c > 0$; exponent a_i can be any real number

posynomial function: sum of monomials

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} \dots x_n^{a_{nk}}, \ \mathbf{dom}\ f = \mathbf{R}_{++}^n$$

geometric program (GP)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 1, \ i = 1, \dots, m \\ & h_i(x) = 1, \ i = 1, \dots, p \end{array}$$

with f_i are posynomials, h_i are monomials

Geometric program in convex form

change of variables to $y_i = \log x_i$, and take the logarithm of cost/constraints

- monomial $f(x) = c x_1^{a_1} \dots x_n^{a_n}$ transforms to

$$\log f(e^{y_1}, \dots, e^{y_n}) = a^T y + b \qquad (b = \log c)$$

- posynomial $f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \dots x_n^{a_{nk}}$ transforms to

$$\log f(e^{y_1}, \dots, e^{y_n}) = \log(\textstyle\sum_{k=1}^{K} e^{a_k^T y + b_k}) \qquad (b_k = \log c_k)$$

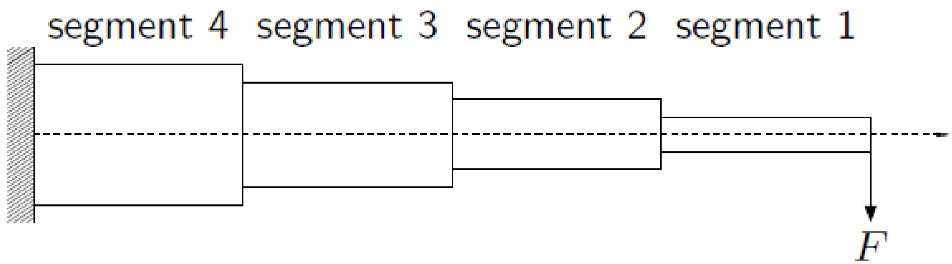
- geometric program transforms to **convex problem**

$$\begin{array}{ll} \text{minimize} & \log(\sum_{k=1}^{K} \exp(a_{0k}^T y + b_{0k})) \\ \text{subject to} & \log(\sum_{k=1}^{K} \exp(a_{ik}^T y + b_{ik})) \le 0, \quad i = 1, \dots, m \\ & Gy + d = 0 \end{array}$$

- when the objective and constraint functions all are monomials, then the

above problem reduces to **a linear program**

Design of cantilever beam



- N segments with unit lengths, rectangular cross-sections of size $\omega_i \times h_i$

- given vertical force F applied at the right end

design problem

minimize	total weight
subject to	upper & lower bounds on ω_i, h_i
	upper bound & lower bounds on aspect ratios h_i/ω_i
	upper bound on stress in each segment
	upper bound on vertical deflection at the end of the beam

variables: ω_i, h_i for $i = 1, \dots, N$

objective and constraint functions

- total weight $\omega_1 h_1 + \cdots + \omega_{\mathrm{N}} h_N$ is posynomial
- aspect ratio h_i/ω_i and inverse aspect ratio ω_i/h_i are monomials
- maximum stress in segment i is given by $6iF/(\omega_i h_i^2)$, a monomial
- the vertical deflection y_i and slope v_i of central axis at the right end of segment i are defined recursively as

$$v_i = 12(i-1/2)\frac{F}{E\omega_i h_i^3} + v_{i+1}$$

$$y_i = 6(i-1/3)\frac{F}{E\omega_i h_i^3} + v_{i+1} + y_{i+1}$$

for $i = N, N-1, \ldots, 1$,with $v_{N+1} = y_{N+1} = 0$ (E is Young's modulus)

v_i and y_i are posynomial functions of w, h

formulation as a GP

$$
\begin{array}{ll}
\text{minimize} & \omega_1 h_1 + \ldots + \omega_N h_N \\
\text{subject to} & \omega_{\max}^{-1}\omega_i \le 1, \quad \omega_{\min}\omega_i^{-1} \le 1, \quad i = 1, \ldots, N \\
& h_{\max}^{-1} h_i \le 1, \quad h_{\min} h_i^{-1} \le 1, \quad i = 1, \ldots, N \\
& S_{\max}^{-1}\omega_i^{-1} h_i \le 1, \quad S_{\min}\omega_i h_i^{-1} \le 1, \quad i = 1, \ldots, N \\
& 6iF\sigma_{\max}^{-1}\omega_i^{-1} h_i^{-2} \le 1, \quad i = 1, \ldots, N \\
& y_{\max}^{-1} y_1 \le 1
\end{array}
$$

note

- we write $\omega_{min} \le \omega_i \le \omega_{max}$ and $h_{min} \le h_i \le h_{max}$

$$\omega_{\min} / \omega_i \le 1, \qquad \omega_i / \omega_{\max} \le 1, \qquad h_{\min} / h_i \le 1, \qquad h_i / h_{\max} \le 1,$$

- we write $S_{min} \le h_i/\omega_i \le S_{max}$ as

$$S_{\min}\omega_i / h_i \le 1, \qquad h_i / S_{\max}\omega_i \le 1$$

Generalized inequality constraints

convex problem with generalized inequality constraints

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} 0, \quad i=1,\ldots,m \\ & Ax = b \end{array}$$

- $f_0 : \mathbf{R}^n \to \mathbf{R}$ convex; $f_i : \mathbf{R}^n \to \mathbf{R}^{k_i}$ K_i-convex w.r.t. proper cone K_i
- same properties as standard convex problem (convex feasible set, local optimum is global, etc.)

conic form problem: special case with affine objective and constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Fx + g \preceq_K 0 \\ & Ax = b \end{array}$$

extends linear programming $(K = \mathbf{R}^{\mathrm{m}}_+)$ to nonpolyhedral cones

Semidefinite program (SDP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \ldots + x_n F_n + G \preceq 0 \\ & Ax = b \end{array}$$

with $F_i, G \in \mathbf{S}^k$

- inequality constraint is called linear matrix inequality (LMI)
- includes problems with **multiple LMI constraints**: for example,

$$x_1 \hat{F}_1 + x_2 \hat{F}_2 + \ldots + x_n \hat{F}_n + \hat{G} \preceq 0, \quad x_1 \tilde{F}_1 + x_2 \tilde{F}_2 + \ldots + x_n \tilde{F}_n + \tilde{G} \preceq 0$$

is equivalent to single LMI

$$x_1 \begin{bmatrix} \hat{F}_1 & 0 \\ 0 & \tilde{F}_1 \end{bmatrix} + x_2 \begin{bmatrix} \hat{F}_2 & 0 \\ 0 & \tilde{F}_2 \end{bmatrix} + \ldots + x_n \begin{bmatrix} \hat{F}_n & 0 \\ 0 & \tilde{F}_n \end{bmatrix} + \begin{bmatrix} \hat{G} & 0 \\ 0 & \tilde{G} \end{bmatrix} \preceq 0$$

LP and SOCP as SDP

LP and equivalent SDP

LP： minimize $c^T x$
subject to $Ax \preceq b$

SDP： minimize $c^T x$
subject to $\mathbf{diag}(Ax - b) \preceq 0$

(note different interpretation of generalized inequality $\preceq$)

SOCP and equivalent SDP

SOCP： minimize $f^T x$
subject to $\|A_i x + b_i\|_2 \le c_i^T x + d_i, \ i = 1, \ldots, m$

SDP： minimize $f^T x$
subject to
$$\begin{bmatrix} (c_i^T x + d_i) I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, \ldots, m$$

Eigenvalue minimization

$$\text{minimize} \quad \lambda_{\max}(A(x))$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ (with given $A_I \in \boldsymbol{s}^K$)

equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A(x) \preceq tI \end{array}$$

- variables $x \in \mathbf{R}^n, t \in \mathbf{R}$
- follows from

$$\lambda_{\max}(A) \leq t \iff A \preceq tI$$

Matrix norm minimization

$$\text{minimize} \quad \|A(x)\|_2 = (\lambda_{\max}(A(x)^T A(x)))^{1/2}$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ (with given $A_i \in \mathbf{R}^{p\times q}$)

equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \succeq 0 \end{array}$$

- variables $x \in \mathbf{R}^n, t \in \mathbf{R}$
- constraint follows from

$$\|A\|_2 \le t \iff A^T A \preceq t^2 I, \ t \ge 0$$

$$\iff \begin{bmatrix} tI & A \\ A^T & tI \end{bmatrix} \succeq 0$$

Vector optimization

general vector optimization problem

$$\begin{array}{ll} \text{minimize (w.r.t. } K) & f_0(x) \\ \text{subject to} & f_i(x) \leq 0,\ i=1,\ldots,m \\ & h_i(x) = 0,\ i=1,\ldots,p \end{array}$$

vector objective $f_0 : \mathbf{R}^n \to \mathbf{R}^q$, minimized w.r.t. proper cone $K \in \mathbf{R}^q$

convex vector optimization problem

$$\begin{array}{ll} \text{minimize (w.r.t. } K) & f_0(x) \\ \text{subject to} & f_i(x) \leq 0,\ i=1,\ldots,m \\ & Ax = b \end{array}$$

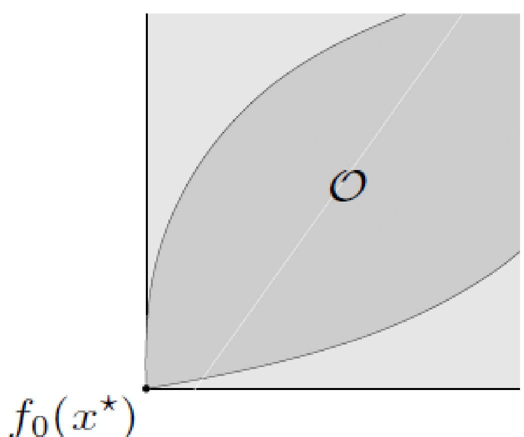
with f_0 K-convex, $f_1, \ldots, f_m$ convex

Optimal and Pareto optimal points

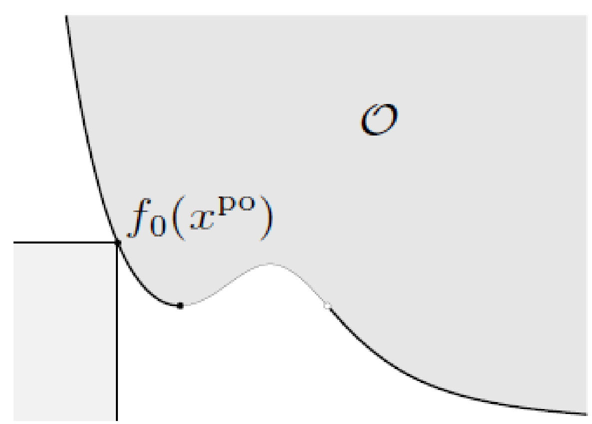
set of achievable objective values

$$\mathcal{O} = \{f_0(x) \mid x \text{ feasible}\}$$

- a feasible x is **optimal** if $f_0(x)$ is the minimum value of $\mathcal{O}$
- a feasible x is **Pareto optimal** if $f_0(x)$ is a minimal value of $\mathcal{O}$



$x^\star$ is optimal



x^{po} is Pareto optimal

Multicriterion optimization

vector optimization problem with $K = \mathbf{R}_+^q$

$$f_0(x) = (F_1(x), \ldots, F_q(x))$$

- q different objectives F_i; roughly speaking we want all $F_i{}'s$ to be small
- feasible x^* is **optimal** if

$$y \text{ feasible} \quad \Rightarrow \quad f_0(x^*) \preceq f_0(y)$$

if there exists an optimal point, the objectives are **noncompeting**

- feasible x^{po} is **Pareto optimal** if

$$y \text{ feasible}, \quad f_0(y) \preceq f_0(x^{PO}) \quad \Rightarrow \quad f_0(x^{PO}) = f_0(y)$$

if there are **multiple** Pareto optimal values, there is a **trade-off** between the objectives