2. Convex sets

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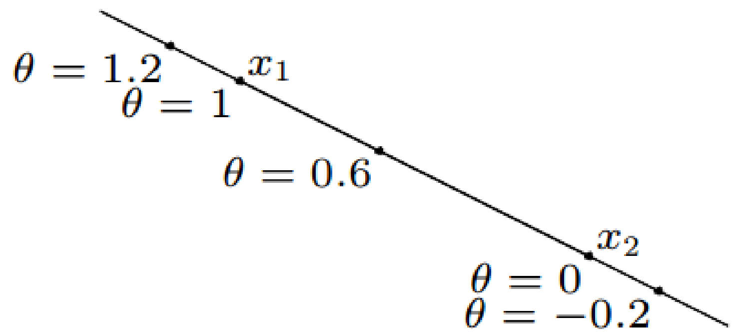
Outline

- affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities

Affine set

line through x_1, x_2: all points

$$x = \theta x_1 + (1-\theta) x_2 \qquad (\theta \in \boldsymbol{R})$$



affine set: contains the line through any two distinct points in the set

example: solution set of linear equations $\{x \mid Ax = b\}$

(conversely, every affine set can be expressed as solution set of system of linear equations)

Convex set

line segment: between x_1 and x_2: all points

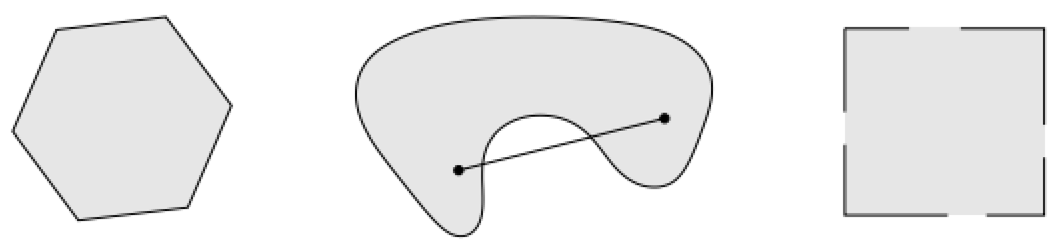
$$x = \theta x_1 + (1 - \theta) x_2$$

with $0 \leq \theta \leq 1$

convex set: contains line segment between any two points in the set

$$x_1, x_2 \in C, \qquad 0 \leq \theta \leq 1 \implies x = \theta x_1 + (1 - \theta) x_2 \in C$$

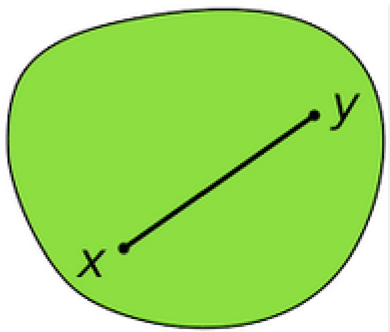
example: (one convex, two nonconvex sets)

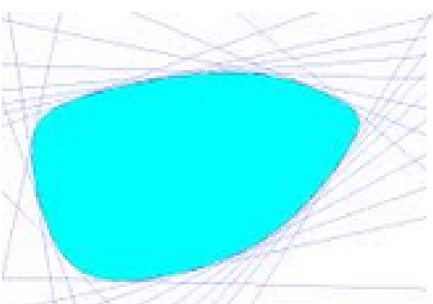


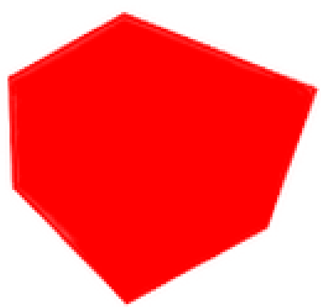
Convex set

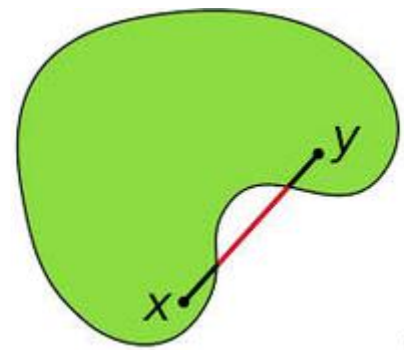
example

convex sets

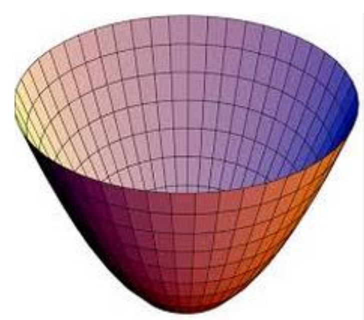


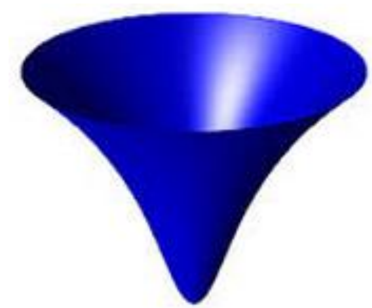






nonconvex sets





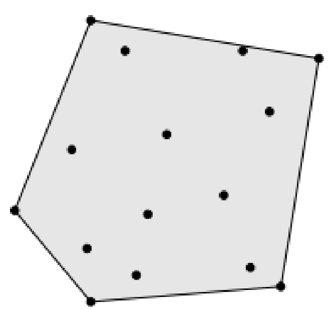
Convex combination and convex hull

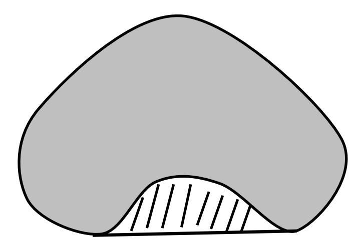
convex combination of $x_1, \dots, x_n$: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1,\ \theta_i \geq 0$

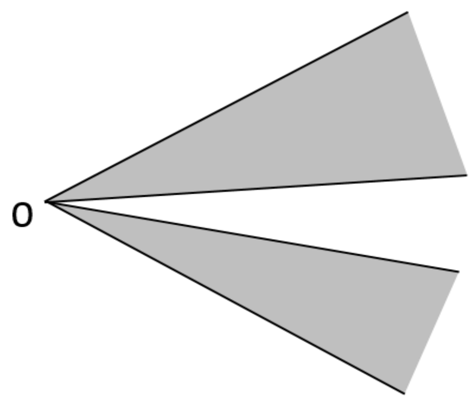
convex hull: set of all convex combinations of points in C (conv C)

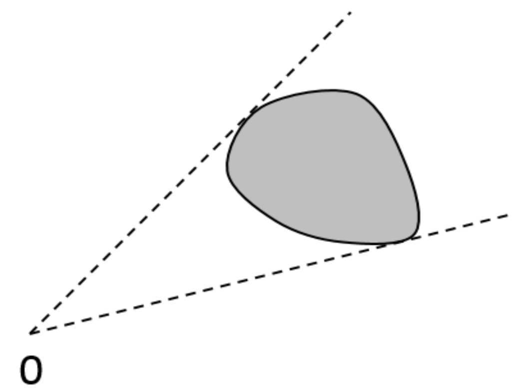




Convex cone

cone: A set C is called a cone if for every $x \in C$ and $\theta \geq 0$ we have $\theta x \in C$

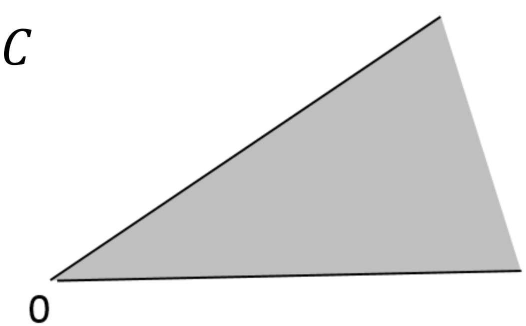




convex cone: A set that is convex and a cone, which means that for any

$x_1, x_2 \in C,\ \theta_1, \theta_2 \geq 0$, we have

$$\theta_1 x_1 + \theta_2 x_2 \in C$$

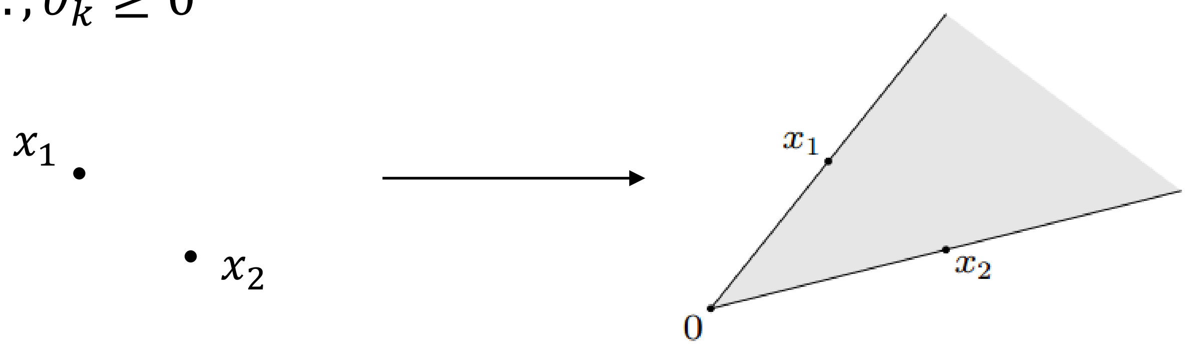


Convex cone

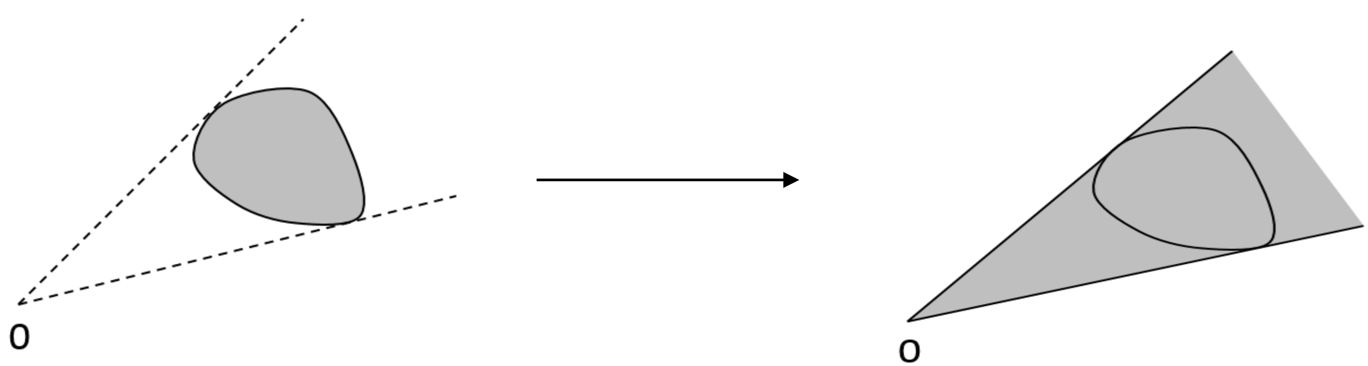
conic (nonnegative) combination of $x_1, \dots, x_n$: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1, \theta_2, \dots, \theta_k \geq 0$

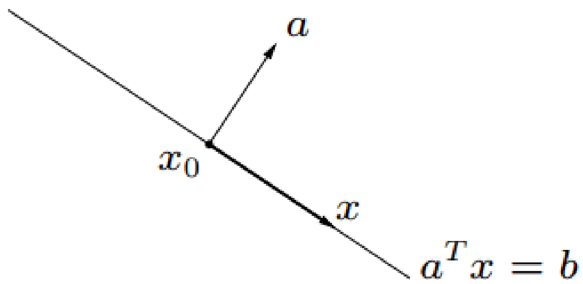


conic hull: A set that contains all conic combinations of points in C

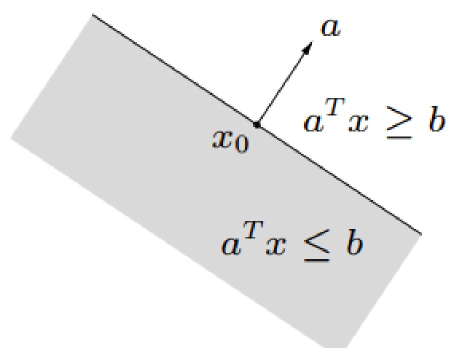


Hyperplanes and halfspaces

hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$



halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$



- hyperplanes are affine and convex; halfspaces are convex

Euclidean balls and ellipsoids

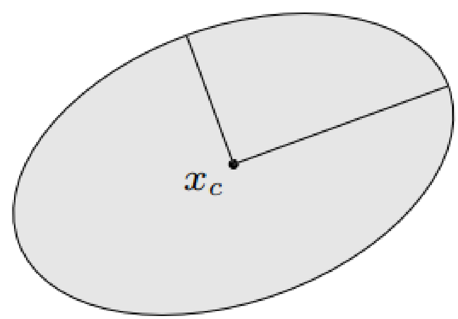
(Euclidean) ball with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$

ellipsoid: set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\}$$

with $P \in \boldsymbol{S}^n_{++}$ (i.e. P symmetric positive definite)



other representation: $\{x_c + Au \mid \|u\|_2 \le 1\}$ with A square and nonsingular

Norm balls and norm cones

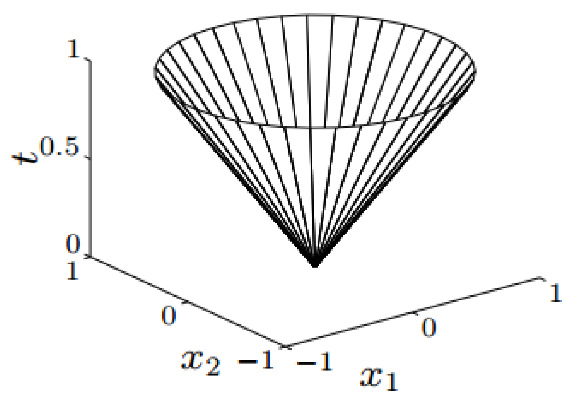
norm: a function $\|\cdot\|$ that satisfies

- $\|x\| \geq 0$
- $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t|\|x\|$ for $t \in \boldsymbol{R}$
- $\|x + y\| \leq \|x\| + \|y\|$

notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{symb}$ is particular norm

norm ball: with center x_c and radius r: $\{x \mid \|x - x_c\| \leq r\}$

norm cone: $\{(x, t) \mid \|x\| \leq t\}$



$\|x\|_2 \leq t$

Euclidean norm cone is called **second-order cone**

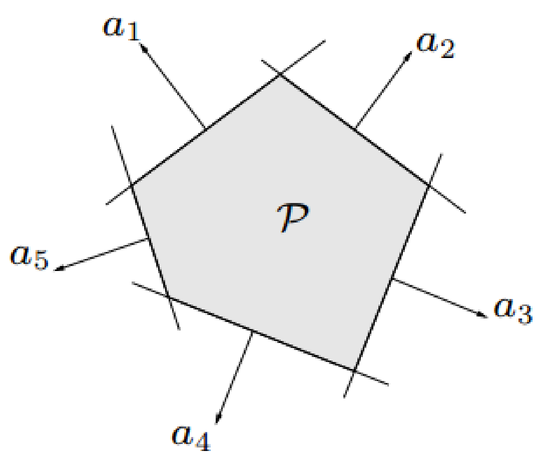
norm balls and norm cones are convex

Polyhedra

solution set of finitely many linear inequalities and equalities

$$Ax \preccurlyeq b, \qquad Cx = d$$

($A \in R^{m \times n}, C \in R^{p \times n}, \preccurlyeq$ is componentwise inequality)



polyhedron is intersection of finite number of halfspaces and hyperplanes

Positive semidefinite cone

notation:

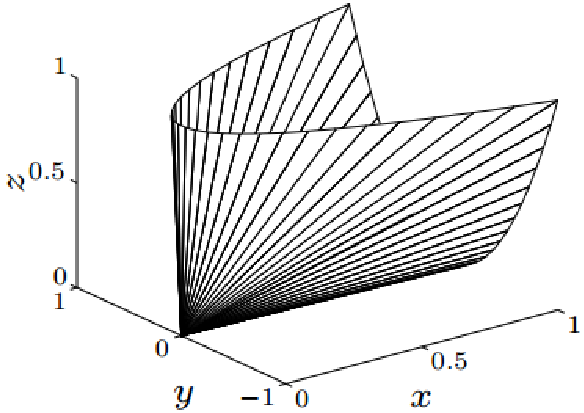
- $\boldsymbol{S}^n$ is set of symmetric $n \times n$ matrices
- $\boldsymbol{S}_+^n = \{X \in S^n | X \succcurlyeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \boldsymbol{S}_+^n \Leftrightarrow z^T X z \geq 0 \text{ for all } z$$

  $\boldsymbol{S}_+^n$ is **convex cone**

- $\boldsymbol{S}_{++}^n = \{X \in \boldsymbol{S}^n | X \succ 0\}$: positive definite $n \times n$ matrices

example:



$$\boldsymbol{S}_+^2 = \{X \in \boldsymbol{S}^2 | \begin{bmatrix} x & y \\ y & z \end{bmatrix} \succcurlyeq 0\}$$

it is equivalent to (Sylvester's Criterion):

$$x \geq 0 \text{ and } \begin{vmatrix} x & y \\ y & z \end{vmatrix} \geq 0 \quad \Longrightarrow \quad x \geq 0 \text{ and } xz \geq y^2$$

Operation that preserve convexity

practical methods for establishing convexity of a set C

1. apply definition

$$x_1, x_2 \in C, \qquad 0 \leq \theta \leq 1 \implies \theta x_1 + (1-\theta) x_2 \in C$$

2. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . .) by operations that preserve convexity
   - Intersection
   - affine functions
   - perspective function
   - linear-fractional functions

Intersection

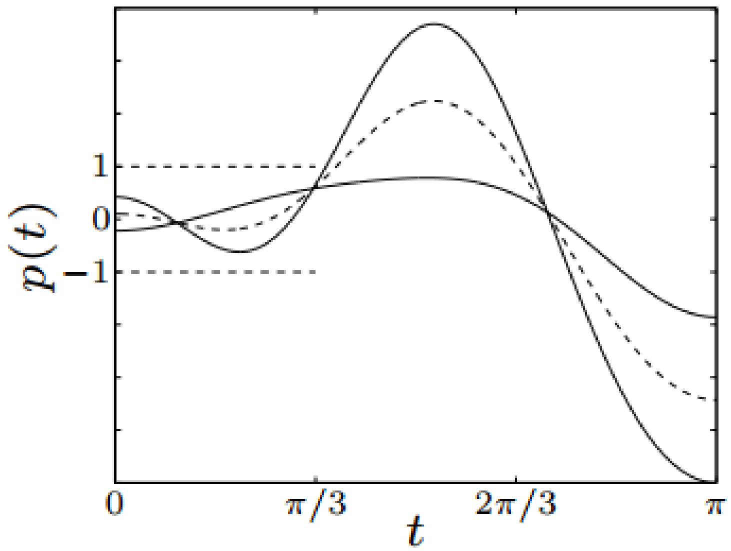
the intersection of (any number of) convex sets is convex

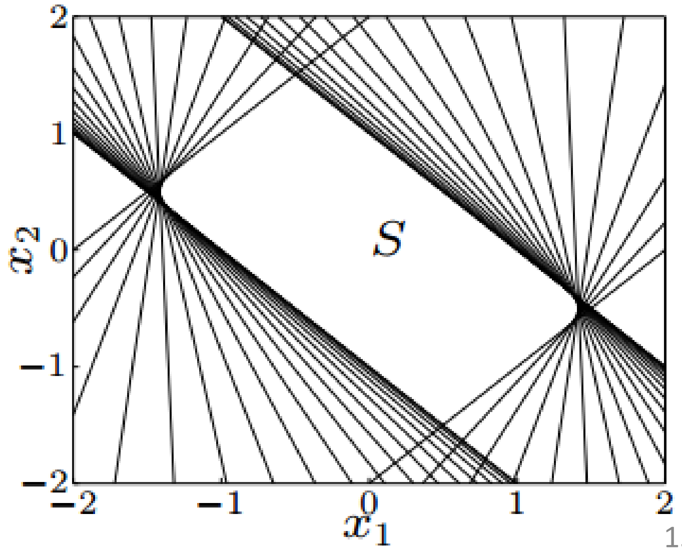
example:

$$S = \{x \in \boldsymbol{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

where $p(t) = x_1 cos t + x_2 cos 2t + \cdots + x_m cos mt$

For $m = 2$:





Affine function

suppose $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$ is **affine** ($f(x) = Ax + b$ with $A \in \boldsymbol{R}^{m \times n}, b \in \boldsymbol{R}^m$)

- the image of a convex set under f is convex

$$S \subseteq \boldsymbol{R}^n \text{ convex } \implies f(S) = \{f(x) | x \in S\} \text{ convex}$$

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \boldsymbol{R}^m \text{ convex } \implies f^{-1}(C) = \{x \in \boldsymbol{R}^n | f(x) \in C\} \text{ convex}$$

example

- scaling, translation, projection
- solution set of linear matrix inequality $\{x | x_1 A_1 + \cdots + x_m A_m \preccurlyeq B\}$ ($A_i, B \in \boldsymbol{S}^p$)
- hyperbolic cone $\{x | x^T P x \le (c^T x)^2, c^T x \ge 0\}$ (with $P \in \boldsymbol{S}_+^n$)

Perspective and linear-fractional function

perspective function $P: \boldsymbol{R}^{n+1} \to \boldsymbol{R}^n$:

$$P(x, t) = x/t, \qquad \mathbf{dom}\, P = \{(x, t) | t > 0\}$$

images and inverse images of convex sets under perspective are convex

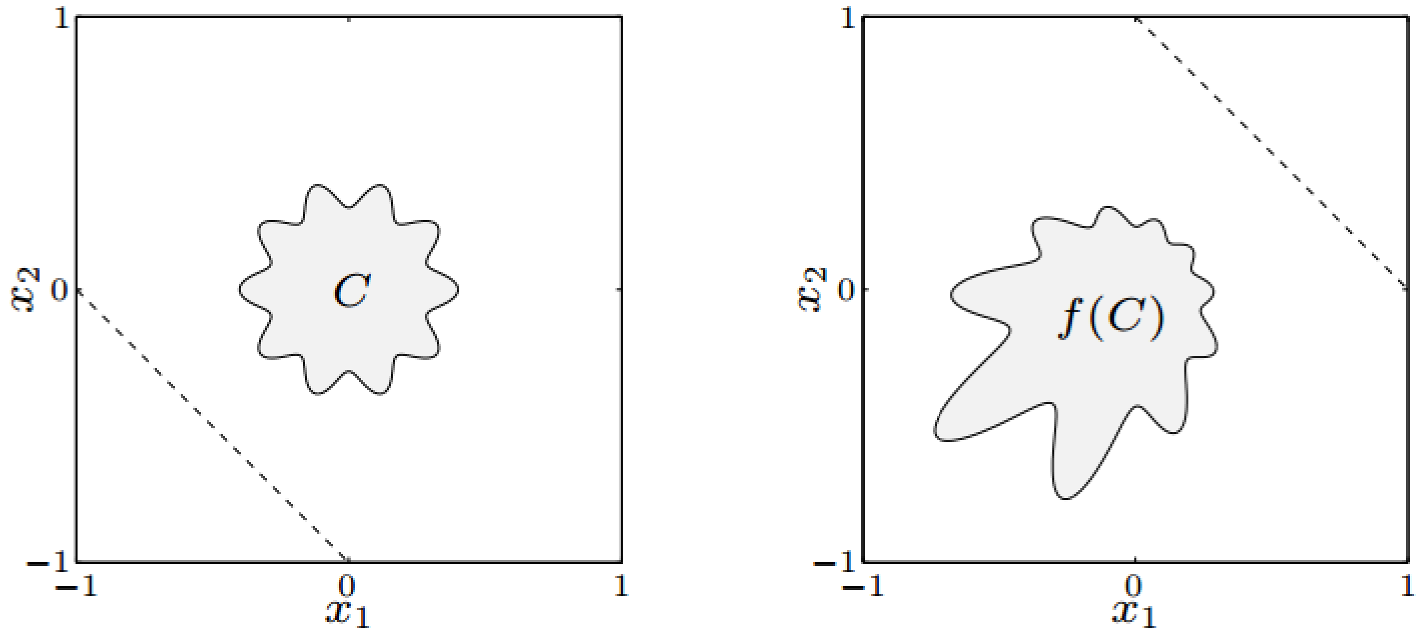
linear-fractional function $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$:

$$f(x) = \frac{Ax + b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x | c^T x + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex

example of a linear-fractional function

$$f(x) = \frac{1}{x_1 + x_2 + 1} x$$



Generalized inequalities

A cone $K \subseteq \boldsymbol{R}^n$ is a **proper cone** if

- K is convex
- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)

Examples

- Nonnegative orthant $K = \boldsymbol{R}_+^n = \{x \in \boldsymbol{R}^n | x_i \geq 0, i = 1, \ldots, n\}$
- Positive semidefinite cone $K = \boldsymbol{S}_+^n$
- Nonnegative polynomials on $[0,1]$:

$$K = \{x \in \boldsymbol{R}^n | x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0,1]\}$$

generalized inequality defined by a proper cone K:

$$x \preccurlyeq_K y \iff y - x \in K, \qquad x \prec_K y \iff y - x \in \operatorname{int} K$$

examples

- component wise inequality ($K = \boldsymbol{R}_+^n$)

$$x \preccurlyeq_{R_+^n} y \iff x_i \le y_i, \qquad i = 1, \dots, n$$

- matrix inequality ($K = \boldsymbol{S}_+^n$)

$$X \preccurlyeq_{S_+^n} Y \iff Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in $\preccurlyeq_K$

properties: many properties of $\preccurlyeq_K$ are similar to $\le$ on $\boldsymbol{R}$, e.g.

$$x \preccurlyeq_K y, u \preccurlyeq_K v \Rightarrow x + u \preccurlyeq_K y + v$$

Minimum and minimal elements

$\preccurlyeq_K$ is not in general a linear ordering: we can have $x \not\preccurlyeq_K y$ and y $\not\preccurlyeq_K x$

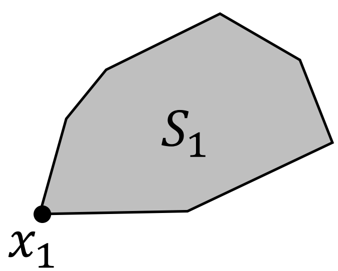
$x \in S$ is **the minimum element** of S with respect to $\preccurlyeq_K$ if

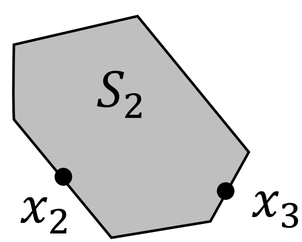
$$x \preccurlyeq_K y \quad \text{for every } y \in S$$

$x \in S$ is **the minimal element** of S with respect to $\preccurlyeq_K$ if

$$\text{for } y \in S,\ \ y \preccurlyeq_K x \text{ only if } y = x$$

example ($K = \boldsymbol{R}_+^2$)





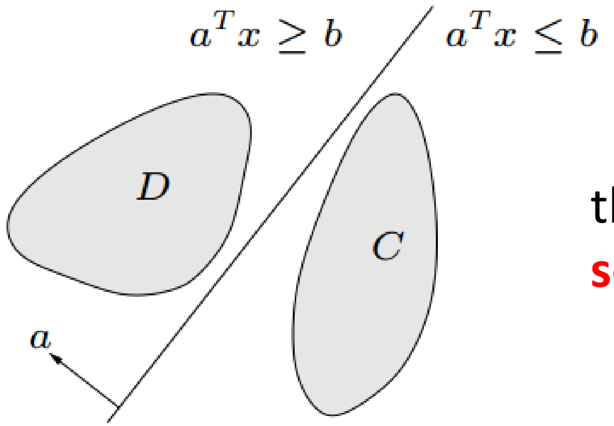
x_1 is the **minimum** element of S_1

x_2 is the **minimal** element of S_2

Separating hyperplane theorem

if C and D are nonempty convex sets that **do not intersect**, there exist $a \neq 0$ and b such that

$$a^T x \leq b \text{ for all } x \in C \quad \text{and} \quad a^T x \geq b \text{ for } x \in D$$



the hyperplane $\{x | a^T x = b\}$ **separates** C and D

strict separation requires the stronger condition that $a^T x < b$ for all $x \in C$ and $a^T x > b$ for $x \in D$

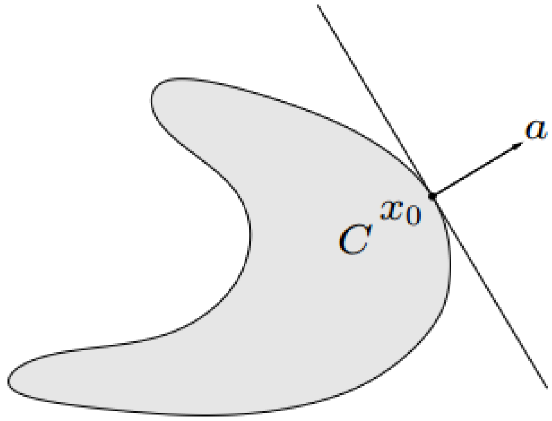
it is possible that disjoint convex sets **cannot be strictly separated**. example?

Supporting hyperplane theorem

supporting hyperplane: for set $C \in \mathrm{R}^n$ and a point x_0 in its boundary, if $a \neq 0$ satisfies $a^T x \leq a^T x_0$ for all $x \in C$, then the hyperplane

$$\{x \mid a^T x = a^T x_0\}$$

is called a supporting hyperplane at the point x_0



supporting hyperplane theorem:

if C is convex, then there exists a supporting hyperplane at every boundary point of C

Dual cones and generalized inequalities

dual cone of a cone K:

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$$

examples

- $K = \boldsymbol{R}_+^n \quad \rightarrow \quad K^* = \boldsymbol{R}_+^n$
- $K = \boldsymbol{S}_+^n \quad \rightarrow \quad K^* = \boldsymbol{S}_+^n$
- $K = \{(x,t) \mid \|x\|_2 \leq t\} \quad \rightarrow \quad K^* = \{(x,t) \mid \|x\|_2 \leq t\}$
- $K = \{(x,t) \mid \|x\|_1 \leq t\} \quad \rightarrow \quad K^* = \{(x,t) \mid \|x\|_\infty \leq t\}$

First three examples are **self-dual cones**

Dual cones of proper cones are proper, hence define generalized inequalities:

$$y \succcurlyeq_{K^*} 0 \quad \Longleftrightarrow \quad y^T x \geq 0 \text{ for all } x \succcurlyeq_K 0$$