1 Generalisations of the circle

Here is a short summary of some ideas outlined in [Hanson, 1994].
The equation of the circle is given by

$$x^2 + y^2 = 1 \text{ with solutions } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix},\ \theta \in [0, 2\pi). \tag{1}$$

A generalisation is given by

$$x^n + y^n = 1 \text{ with solutions } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (\cos\theta)^{2/n} \\ (\sin\theta)^{2/n} \end{pmatrix}. \tag{2}$$

This equation is already interesting for 2D graphic visualisations because the solutions for different values of $n \geq 2$ correspond to interpolations between the circle and and a square with rounded edges while $n = 1$ is a perfect square.
One may define a more general equation on the complex plane,

$$u^2 + o^2 = 1,\ u, o \in \mathbb{C}, \tag{3}$$

which is solved by

$$u = \frac{1}{2}\left(e^{\xi+i\theta} + e^{-(\xi+i\theta)}\right), \qquad o = \frac{1}{2i}\left(e^{\xi+i\theta} - e^{-(\xi+i\theta)}\right),\ \xi, \theta \in \mathbb{R} \tag{4}$$

because

$$u^2 = \frac{1}{4}\left(e^{2(\xi+i\theta)} + e^{-2(\xi+i\theta)} + 2\right) \text{ and } o^2 = -\frac{1}{4}\left(e^{2(\xi+i\theta)} + e^{-2(\xi+i\theta)} - 2\right).$$

Observe that we now actually solve two equations (because $\mathbb{C} \simeq \mathbb{R}^2$ and if we use basis $(1, i)$, then one complex equation amounts to two real equations) for two complex variables (4 real components) and thus must have 2 free real parameters in the end which amount to $\xi, \theta \in \mathbb{R}$. Thus, the solution describes a 2-dimensional surface.

The real part of the solution is

$$\Re u = \frac{e^{\xi} + e^{-\xi}}{2}\cos\theta, \qquad \Re o = \frac{e^{\xi} + e^{-\xi}}{2}\sin\theta \tag{5}$$

which restricts back to a solution of eq. (1) for $\xi = 0$.
However, by defining $s(k, n) := \exp(2\pi i k/n)$, we can solve the more general equation

$$z_1^{n_1} + z_2^{n_2} = 1 \qquad \text{with} \qquad z_1 = s(k_1, n_1)u^{2/n_1},\ z_2 = s(k_2, n_1)o^{2/n_2}, \tag{6}$$

which gives rise to the visualisation of a multitude of interesting 2D - surfaces embedded in 3-space.

References

[Hanson, 1994] Hanson, A. J. (1994). A construction for computer visualization of certain complex curves. *Notices of the Amer.Math.Soc*, pages 1156–1163. 1