1 Introduction to PDEs

Ordinary Differential Equations (ODEs)

1st order

Let $A(t)$ satisfy $\dot{A}(t) = a(t)$, so $A(t)$ is the antiderivative of $a(t)$.

$$\begin{cases} x'(t) + a(t)x(t) = b(t) \\ x(0) = x_0 \end{cases} \qquad \boxed{x(t) = e^{A(0)-A(t)}x_0 + e^{-A(t)} \int\limits_0^t e^{A(\tau)} b(\tau) d\tau}$$

2nd order

Ansatz $y = e^{\lambda t}$: $\lambda^2 + b\lambda + c = 0 \Rightarrow \lambda_\pm = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$

$$\begin{cases} y'' + by' + cy = 0 \\ y'(0) = v_0 \\ y(0) = y_0 \end{cases}$$

Fallunterscheidung:

i) $\lambda_\pm \in \mathbb{R}$:	$y = Ae^{\lambda_+ t} + Be^{\lambda_- t}$
ii) $\lambda_+ = \lambda_- = \lambda$:	$y = (A + Bt)e^{\lambda t}$
iii) $\lambda_\pm = \alpha \pm i\omega$:	$y = e^{\alpha x}(A \sin(\omega t) + B \cos(\omega t))$

Substitutionsregel

Transformationssatz fürs Volumen
Sei $U \to R^n$ offen, $\Phi \in C_1(U, R^n)$ ein Diffeomorphismus von U auf V, $\Omega \subseteq R^n$ beschränkt und Jordan messbar, und $f : \Phi(U) \to R$ beschränkt und R-integrabel.

$$\int_{\Phi(\Omega)} f d\mu = \int_\Omega (f \circ \Phi) \cdot |det(d\Phi)| d\mu$$

$$\psi(x, y) := \begin{pmatrix} x \\ y \\ f(x, y) \end{pmatrix} \qquad \sqrt{det(D\psi(y)^T D\psi(y))} = ||D_x\psi \times D_y\psi||.$$

Definition of a Partial Differential Equation (PDE)

A PDE is an equation of the form: $F(x, u, \partial u, \ldots, \partial^k u) = 0$ where $u : U \to \mathbb{R}$ and $U \subset \mathbb{R}^n$ is open.

Properties of PDEs

- **Order of PDE**: Order of the highest derivative.
- **Linearity**: All coefficients of u and its derivatives do *not* depend on u or its derivatives. Linearity allows superposition.
- **Homogeneous**: linear, $f(x) = 0$ and contains only function itself and its derivatives.
- **Quasilinear**: linear in all its highest order derivatives (i.e. dependent only upon variables and lower order derivatives)

Diffusion/Transport Equation:

$$\frac{\partial c}{\partial t} = v\Delta c \quad j = -v\nabla c \quad \frac{d}{dt}\int_V c\, dx = -\int_{\partial V} j \cdot \nu\, dS \quad c(t, x) := g(x + tv) \tag{1.1}$$

- $c(t, x)$: Concentration of the substance
- $v > 0$: Diffusion coefficient
- $j = cv$: Particle flux density per area

For linear and homogeneous PDEs, any linear combination of solutions is also a solution.

$$Lu := -\sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} + cu \mid D = a_{11}a_{22} - a_{12}^2. \tag{1.2}$$

Classification of Second-Order PDEs

- **Elliptic:** $\Delta u = f$. $\{a_{ij}\}$ symmetric and positive definite. Solutions describe equilibrium states. $D > 0$
- **Hyperbolic:** $u_{tt} + Lu = f$. $\{a_{ij}\}$ symmetric and negative definite. Solutions propagate disturbances over time without smoothing. $D < 0$
- **Parabolic:** $u_t + Lu = f$. $\{a_{ij}\}$ symmetric and positive semi-definite. Solutions smooth out initial conditions over time. $D = 0$

1.1 Elliptic equations

Harmonic functions Let $D \subset \mathbb{R}^2$ be an open set. A function $u(x, y)$ is *harmonic* in D if it solves the Laplace equation $\Delta u = 0$ on D.

$$P_n(x, y) = \sum_{0 \le (i+j) \le n} a_{ij} x^i y^j \quad \text{i.e. } u(x, y) = x^2 - y^2$$

Compatibility condition for Neumann

A necessary condition for the existence of the Neumann problem is

$$\int_D \Delta u\, dS = \int_{\partial D} \partial_\nu u \xrightarrow[\text{(for Laplace eq.)}]{\Delta u = 0} 0 = \int_{\partial D} g(x(s), y(s)) ds$$

For circles: $\int_{\partial D} \partial_\nu u = \int_0^{2\pi} \partial_\nu u(\gamma(s))\, ||\dot\gamma(s)||_2 ds$ with $\gamma(s) = \begin{pmatrix} r\cos(s) \\ r\sin(s) \end{pmatrix}$

2 Inhomogeneous evolution equations

The Cauchy problem and d'Alembert's formula

For the 1D wave equation: $\begin{cases} u_{tt} - c^2 u_{xx} = F(x, t) & ,(x, t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x, 0) = f(x) & ,x \in \mathbb{R} \\ u_t(x, 0) = g(x) & ,x \in \mathbb{R} \end{cases}$

$$u(x, t) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y) dy + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi, \tau) d\xi d\tau$$

where f and g are the initial displacement and velocity.

Approach for simple inhomogenities

If $F(x, t)$ only depends on x or t, then it is possible to reduce the problem to a homogeneous one ($F(x, t) = 0$) by finding a particular solution v.

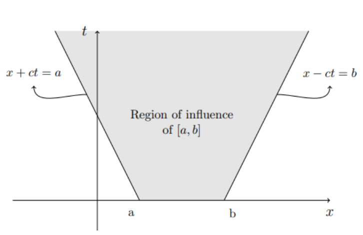
1. Find a $v(t)$ (or $v(x)$) that solves $F(t) = v_{tt}$ (or $F(x) = -c^2 v_{xx}$).
2. Use the superposition $w(x, t) := u(x, t) - v(x, t)$:

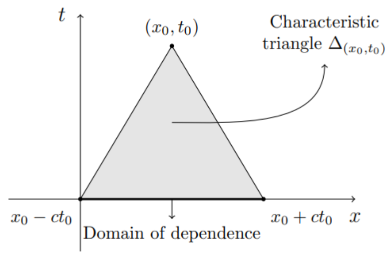
$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, & (x, t) \in \mathbb{R} \times (0, \infty) \\ w(x, 0) = f(x) - v(x, 0), & x \in \mathbb{R} \\ w_t(x, 0) = g(x) - v_t(x, 0), & x \in \mathbb{R} \end{cases}$$

- **Uniqueness**: The 1D-wave equation has a unique solution.
- **Symmetry**: If the initial data $f(x), g(x)$ and the inhomogenity $F(x, t)$ are even/odd/periodic with respect to x, then the solution is even/odd/periodic with respect to x as well.

Domain of dependence and region of influence

- **The region of influence** is the set of points influenced by the values of $f(x), g(x)$ with $x \in [a, b]$. This is the area given by the two lines $x + c \cdot t \ge a$ and $x - c \cdot t \le b$.
- **The domain of dependence** of $u(x, t)$ at a point (x_0, t_0) is given by $[x_0 - c \cdot t_0, x_0 + c \cdot t_0]$. The solution only changes if values inside this interval are changed.





Remark: If the initial conditions are smooth on the interval, then the solution itself is smooth in $\Delta_{(x_0, t_0)}$.

For a local disturbance the initial conditions u_0 and v_0 satisfy:

$$u_0 = u_\infty \quad \text{on } \mathbb{R} \setminus [a, b] \quad v_0 = 0 \quad \text{on } \mathbb{R} \setminus [a, b] \quad \int_a^b v_0(y)\, dy = 0.$$

For a fixed point x, the solution becomes constant (u_∞) after a time:

$$t_x = \max\left(\frac{x - a}{c}, \frac{b - x}{c}\right).$$

2.1 Method of Separation of Variables

Only applicable to PDEs that contain derivatives of one variable. $X''(X) = -\lambda X(x)$

$$X(x) = \begin{cases} A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) & ,\lambda > 0 \quad C_\pm \in \mathbb{C} \quad (C_- = \overline{C_+}) \\ C + Dx & ,\lambda = 0 \\ A\sinh(\sqrt{-\lambda}x) + B\cosh(\sqrt{-\lambda}x) & ,\lambda < 0 \quad C_\pm \in \mathbb{R} \end{cases}$$

The boundary conditions are then used to find the non-trivial solutions:

Dirichlet:	$u(0, t) = u(L, t) = 0$	$X_n(x) = \sin\left(\sqrt{\lambda_n}x\right)$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$
Neumann:	$u_x(0, t) = u_x(L, t) = 0$	$X_n(x) = \cos\left(\sqrt{\lambda_n}x\right)$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$
Mixed 1:	$u(0, t) = u_x(L, t) = 0$	$X_n(x) = \sin\left(\sqrt{\lambda_n}x\right)$	$\lambda_n = \left(\frac{(n+1/2)\pi}{L}\right)^2$
Mixed 2:	$u_x(0, t) = u(L, t) = 0$	$X_n(x) = \cos\left(\sqrt{\lambda_n}x\right)$	$\lambda_n = \left(\frac{(n+1/2)\pi}{L}\right)^2$

The Homogeneous 1D heat equation

Let $k \in \mathbb{R}^+$ be the constant of diffusivity. Diffusion in a 1D structure is

$$\begin{cases} u_t - ku_{xx} = 0 & ,(x, t) \in (0, L) \times (0, \infty) \\ \text{BC} & ,t \in \mathbb{R}^+ \\ u(x, 0) = f(x) & ,x \in (0, L) \end{cases} \begin{cases} X'' = -\lambda X \\ T' = -\lambda k T \end{cases} \Rightarrow \boxed{T_n(t) = A_n e^{-k\lambda_n t}}$$

$$u(t, x) = \sum_{n=1}^\infty X_n(x) \cdot T_n(t) = \sum_{n=1}^\infty e^{-k\lambda_n t}\left(C_+ e^{\sqrt{\lambda_n}x} + C_- e^{-\sqrt{\lambda_n}x}\right)$$

Homogeneous 1D wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & (x,t) \in [0,L] \times [0,\infty) \\ u(0,t) = u(L,t) = 0 & t > 0 \\ u(x,0) = f(x), & x \in \mathbb{R} \quad f(-x) = -f(x) \\ u_t(x,0) = g(x), & x \in \mathbb{R} \quad g(-x) = -g(x) \end{cases} \quad \begin{cases} X'' = -\lambda X & , x \in (0,L) \\ T'' = -\lambda c^2 T & , t > 0 \end{cases}$$

$$T_n(t) = \hat{f}_n \cos(ct\sqrt{\lambda_n}) + \frac{\hat{g}_n}{c\sqrt{\lambda_n}} \sin(ct\sqrt{\lambda_n}) \qquad u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} X_n(x) \cdot T_n(t)$$

Non homogeneous boundary conditions

If the boundary condition isn't $= 0$, then the idea is to find a new function $w(x,t)$ that satisfies the same boundary conditions and subtract it $v(x,t) := u(x,t) - w(x,t)$.

Duhamel's Principle

Duhamel's principle is a general method to solve inhomogeneous linear evolution equations with homogeneous initial conditions. These equations are of the form:

$$\partial_t^k u + L(u) = f,$$

$$u(t=0,x) = \partial_t^0 u(t=0,x) = 0, \quad \partial_t^1 u(t=0,x) = 0, \quad \ldots, \quad \partial_t^{k-1} u(t=0,x) = 0.$$

where $k \in \mathbb{N}$, L is a differential operator involving only spatial derivatives $x_1, \ldots, x_n$, and f is the inhomogeneity (source term). This principle reduces the problem to solving the corresponding homogeneous PDE:

$$\partial_t^k v + L(v) = 0,$$

$$u(t=s,x) = 0, \quad \partial_t^j u(t=s,x) = 0 \quad \text{for } j = 0, \ldots, k-2 \quad \partial_t^{k-1} u(t=s,x) = f(s,x).$$

$$u(t,x) = \int_0^t v(s,t,x)\, ds.$$

3 Fourier Theory for Periodic Functions

The Fourier series of a periodic piecewise continuous function f is defined as:

$$\sum_{k=-\infty}^{\infty} f_k e^{ik\omega x} = f(x) := \frac{1}{2}\left(\lim_{y \to x^-} f(y) + \lim_{y \to x^+} f(y)\right) \qquad \omega = \frac{2\pi}{P}$$

Let f be a P-periodic, piecewise continuous function. For $k \in \mathbb{N}_0$, we define:

$$a_k := \frac{2}{P}\int_0^P f(x)\cos(k\omega x)\, dx, \quad b_k := \frac{2}{P}\int_0^P f(x)\sin(k\omega x)\, dx.$$

The N-th partial Fourier sum is then given by:

$$s_f^N(x) = \frac{\mathbf{a_0}}{\mathbf{2}} + \sum_{k=1}^{N}\left(a_k\cos(k\omega x) + b_k\sin(k\omega x)\right).$$

If $\lim_{N\to\infty} |s_f^N(x)| < \infty$, then the Fourier series converges absolutely and uniformly to f(x). (otherwise pointwise for a piecewise continuously differentiable function)

Complex Fourier Series

$$f(x) = c_0 + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} c_k \cdot e^{ik\omega x} \quad \Big| \quad \mathbf{c_0} = \frac{1}{P}\int_0^P f(x)dx \quad \Big| \quad c_k = \frac{1}{P}\int_0^P f(x) \cdot e^{-ik\omega x}dx$$

$$a_k = c_k + \overline{c_k} = 2\Re(c_k) \;\Big|\; c_k = \tfrac{1}{2}(a_k - ib_k) \;\Big|\; b_k = i(c_k - \overline{c_k}) = -2\Im(c_k) \;\Big|\; c_{-k} = \overline{c_k}$$

The heat equation $u_t - au_{xx}$ can be expressed as an infinite system of ordinary differential equations (ODEs) for Fourier coefficients:

$$\frac{d}{dt}u(t,x) = a\frac{d^2}{dx^2}u(t,x) \to \frac{d}{dt}\hat{u}_k(t) = -ak^2w^2\hat{u}_k(t),$$

with the solution:

$$\hat{u}_k(t) = \hat{v}_k e^{-ak^2w^2t}.$$

Here, $\hat{v}_k$ are the Fourier coefficients of the initial condition $v(x)$, and the overall solution is reconstructed via:

$$u(t,x) = \sum_{k=-\infty}^{\infty} \hat{u}_k(t)e^{ikwx} = \sum_{k=-\infty}^{\infty} \hat{v}_k e^{-ak^2w^2t} e^{ikwx} \qquad \lim_{t\to\infty} u(t,x) = \hat{u}_0$$

4 Fourier Transformation

For every function $f \in L^1_{\text{pc}}(\mathbb{R}, \mathbb{C})$, the Fourier transform $\hat{f} = \mathcal{F}(f)$ is continuous and converges to 0 as $\xi \to \pm\infty$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i\xi\cdot x}\mathrm{d}x, \quad \check{f}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(\xi)e^{i\xi\cdot x}\mathrm{d}\xi.$$

Elementary Properties *Let $f, g, h \in L^1$, $h \in C_0^1(\mathbb{R}^n)$, $\alpha, \beta \in \mathbb{C}$, $\lambda \in \mathbb{R} \setminus \{0\}$*

1. $\widehat{f\left(\frac{x}{a}\right)} = \widehat{\delta_a(f)}(\xi) = |a|\hat{f}(a\xi)$
2. $f\widehat{(x-v)} = \widehat{\tau_v(f)}(\xi) = e^{-iv\xi}\hat{f}(\xi)$
3. $e^{\widehat{-ivx}f(x)} = \hat{f}(\xi + v)$
4. $\int_{\mathbb{R}^n} \hat{f}g\mathrm{d}x = \int_{\mathbb{R}^n} f\hat{g}\mathrm{d}x$
5. $\overline{\hat{f}(\xi)} = 2\pi\hat{\overline{f}}(-\xi)$
6. $\overline{f(x)} = \frac{1}{2\pi}\hat{\overline{f}}$
7. $\mathcal{F}(x^n f) = i^n \frac{\partial^n \hat{f}}{\partial \xi^n}$
8. $\mathcal{F}(f^{(n)}) = (i\xi)^n \hat{f}(\xi)$

Examples

$$(e^{-\frac{x^2}{a}})^\wedge(\xi) = \sqrt{a\pi}e^{-\frac{a\xi^2}{4}} \text{ and } (e^{-\frac{\xi^2}{a}})^\vee(x) = \tfrac{1}{2}\sqrt{\tfrac{a}{\pi}}e^{-\frac{ax^2}{4}}$$

$$\left(e^{-a|x|}\right)^\wedge(\xi) = \tfrac{2a}{a^2+\xi^2} \in L^1(\mathbb{R}) \quad Re(a) > 0 \quad \int_{-\infty}^{\infty} e^{-(ak^2+bk+c)}dk = \sqrt{\tfrac{\pi}{a}}e^{\frac{b^2}{4a}-c}$$

$$\hat{\chi}_{[-1,1]} = 2\frac{sin(x)}{x} \quad \mathcal{F}\left(xe^{-ax^2}\right)(\xi) = \tfrac{i\xi}{(2a)^{3/2}}e^{-\frac{\xi^2}{4a}} \quad \mathcal{F}^{-1}\left(i\xi e^{-b\xi^2}\right) = \tfrac{x}{(2b)^{3/2}}e^{-\frac{x^2}{4b}}$$

Fourier Transform of the Gaussian Function

L^2-Normalized a-Rescaled Gaussian Function

Let $a \in (0,\infty)$. Define the L^2-normalized a-rescaled Gaussian function as:

$$\psi_a(x) := \frac{1}{\sqrt[4]{\pi}\sqrt{a}}e^{-\frac{x^2}{2a^2}} \quad \|\psi_a\| := \int_{-\infty}^{\infty} |\psi_a(x)|^2 dx = 1 \quad \mathcal{F}(\psi_a) = \sqrt{2\pi}\psi_{1/a}$$

The standard deviation of the random variable x with probability density $|\psi_a|^2$ is:

$$\sigma_x = \frac{a}{\sqrt{2}}, \quad \sigma_\xi = \frac{1}{\sqrt{2}a} \quad f(x) = e^{-\frac{1}{2}\langle Ax, x\rangle} \quad \hat{f}(k) = \frac{(2\pi)^{n/2}}{(\det A)^{1/2}}e^{-\frac{1}{2}\langle A^{-1}k, k\rangle}$$

Heat Kernel The heat kernel $K(t, x-y)$ acts as a propagator, evolving the initial state $v(x)$ into the solution $u(t,x)$. It provides a deterministic description of heat conduction and diffusion processes.

$$K(t,x) := \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t}} = (e^{-\xi^2 t})^\vee(x), \quad t > 0,\, x \in \mathbb{R}$$

Heat Equation on $\mathbb{R}$ (Without Periodic Boundary Conditions)

Let $v : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous and bounded function:

$$|v(x)| \leq C_\varepsilon e^{\varepsilon x^2}, \quad \forall \varepsilon > 0,$$

Define the function $u : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ as:

$$u(t,x) := \int_{-\infty}^{\infty} K(t, x-y)v(y)\, dy = (K_t * v)(x)$$

which solves the heat equation:

$$u_t = u_{xx}, \quad u(t,y) \to v(x) \quad \text{as } (t,y) \to (0,x), \quad \forall x \in \mathbb{R} \text{ (continuity points of } v)$$

(Uniqueness) If v is continuous and $u : (0,T] \times \mathbb{R} \to \mathbb{R}$ is a twice differentiable solution satisfying:

$$|u(t,x)| \leq Ce^{ax^2}, \quad \text{for some constants } C, a \geq 0,$$

then $u(t,x)$ is unique.

5 Laplace and Poisson Equations

Poisson's Formula for a Unit Disk

The Dirichlet problem for the Laplace equation on the unit disk $B_2 \subset \mathbb{R}^2$ is given by:

$$\Delta u = u_{xx} + u_{yy} = 0 \quad \text{on } B_2,$$

with boundary condition:

$$u(x,y) \to g(x_0,y_0) \quad \text{for } (x,y) \to (x_0,y_0), \quad \forall (x_0,y_0) \in \partial B_2.$$

To solve this problem, the **Poisson kernel** for the unit disk is defined as:

$$K(r,\varphi) := \frac{1-r^2}{1-2r\cos\varphi + r^2}, \quad \text{for } r \in [0,1),\, \varphi \in \mathbb{R}.$$

Using the Poisson kernel, the solution $u(x,y)$ is given by:

$$u(x,y) = v(r,\varphi) = \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{g(\cos\psi, \sin\psi)}{1-2r\cos(\varphi-\psi)+r^2}\, d\psi,$$

where the polar coordinates $(x,y) = (r\cos\varphi, r\sin\varphi)$ are used.

(i) **Smoothness:** The solution $u(x,y)$ is infinitely differentiable (C^∞) within the unit disk.

(ii) **Uniqueness:** If $u : B_2 \to \mathbb{R}$ is a twice continuously differentiable function that solves the Dirichlet problem, then u is unique.

For periodic boundary conditions:

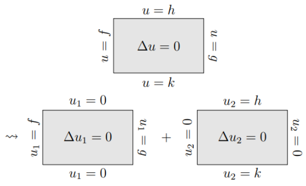
$$v(r,\varphi) = \sum_{k=-\infty}^{\infty} \hat{h}_k r^{|k|} e^{ik\varphi}.$$

$$\hat{h}_k = \frac{1}{2\pi}\int_0^{2\pi} h(\psi)e^{-ik\psi}\, d\psi.$$

where $v(1,\varphi) = h(\varphi)$.

5.0.1 Dirichlet problem with Rectangular Domain

$$\begin{cases} \Delta u = 0 & ,(x,y) \in (a,b) \times (c,d) \\ u_1(a,y) = f(y) & ,(x,y) \in \{a\} \times [c,d] \\ u_1(b,y) = g(y) & ,(x,y) \in \{b\} \times [c,d] \\ u_2(x,d) = h(x) & ,(x,y) \in [a,b] \times \{d\} \\ u_2(x,c) = k(x) & ,(x,y) \in [a,b] \times \{c\} \end{cases}$$



Seperation of variables gives us two ODEs for u_1 (for u_2 switch x & y):

$$X''(x) - \lambda x(x) = 0 \qquad \begin{cases} Y''(y) + \lambda Y(y) = 0 \\ Y(c) = Y(d) = 0 \end{cases}$$

1. The general solutions to this problem are given by:

$$u_1(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{d-c}(y-c)\right)\left[A_n \sinh\left(\frac{n\pi}{d-c}(x-a)\right) + B_n \sinh\left(\frac{n\pi}{d-c}(x-b)\right)\right]$$

$$u_2(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{b-a}(x-a)\right)\left[C_n \sinh\left(\frac{n\pi}{b-a}(y-c)\right) + D_n \sinh\left(\frac{n\pi}{b-a}(y-d)\right)\right]$$

2. Use the Initial conditions to then find all coefficients and $u = u_1 + u_2$.

5.0.2 Laplace equation in circular domains

Let the domain be $D := \{r \in [a,b], \Theta \in [0, 2\pi)\}$. With separation of variables $w(r,\theta) = R(r) \cdot \Theta(\theta)$ we get the following ODE system:

$$\begin{cases} r^2 R''(r) + rR'(r) = \lambda R(r) \\ \Theta''(\theta) = -\lambda\Theta(\theta) \end{cases} \qquad \begin{aligned} &\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta),\ n \in \mathbb{N} \\ &R_n(r) = \begin{cases} C_0 + D_0 \log r, & \text{for } n = 0 \\ C_n r^n + D_n r^{-n}, & \text{for } n \neq 0 \end{cases} \end{aligned}$$

$$w(r,\Theta) = A_0 + B_0 \log(r) + \sum_{n=1}^{\infty} r^{\pm n}\left[A_n \cos(n\Theta) + B_n \sin(n\Theta)\right]$$

$$u(R,\theta)/u_r(R,\theta) = f(\theta) \text{ on } \partial D \to C_n = \frac{1}{R^n\pi / nR^{n-1}\pi} \int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta$$

Attention: The functions r^{-n} and $\log(r)$ are singular at $r = 0$ inside the domain D, so discard them if $(0,0)$ is inside the domain (disks)!

Volumes of Balls and Spheres

The volume of an n-dimensional ball of radius r centered at x_0 is given by:

$$\mathrm{Vol}_n(B_r^n(x_0)) = \alpha_n r^n$$

The $(n-1)$-dimensional volume of the boundary sphere $S_r^{n-1}(x_0)$ is:

$$\mathrm{Vol}_{n-1}(S_r^{n-1}(x_0)) = n\alpha_n r^{n-1}$$

$$\mathrm{Vol}_{n-1}(S^{n-1}) = n\,\mathrm{Vol}_n(B^n) = n\alpha_n$$

Mean Value Principle for Harmonic Functions

Let $u : B_r^n(x_0) \to \mathbb{R}$ be a continuous function that is C^2 and harmonic on $B_r^n(x_0)$. Then:

$$u(x_0) = \frac{1}{\mathrm{Vol}_{n-1}(S_r^{n-1}(x_0))} \int_{S_r^{n-1}(x_0)} u\,dA = \frac{1}{\mathrm{Vol}_n(B_r^n(x_0))} \int_{B_r^n(x_0)} u\,dx.$$

In 2D:

$$u(x_0,y_0) = \frac{1}{2\pi R} \int_{\partial B_r(x_0,y_0)} u(x(s),y(s))ds = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + R\cos\Theta, y_0 + R\sin\Theta)d\Theta$$

Weak maximum principle

Let D be a **bounded** domain and let $u(x,y) \in C^2(D) \cap C(\overline{D})$ be a harmonic function in D. The maximum (and minimum) of u in $\overline{D}$ is achieved on the boundary ∂D:

$$\max_{\overline{D}} u = \max_{\partial D} u \qquad \min_{\overline{D}} u = \min_{\partial D} u$$

Strong maximum principle

Let u be harmonic in D and D is *a connected subset* of $\mathbb{R}^2$. If u attains its maximum (or its minimum) at an interior point of D, then u **is constant**.

Fundamental Solution of the Laplace Equation

For $n \geq 2$, the fundamental solution $\Phi_n(x)$ is defined as:

$$\Phi_n(x) = \begin{cases} -\frac{1}{2\pi}\log\|x\|, & \text{if } n = 2, \\ \frac{1}{n(n-2)\alpha_n}\frac{1}{\|x\|^{n-2}}, & \text{if } n \geq 3, \end{cases}$$

where α_n is the volume of the unit ball in $\mathbb{R}^n$.

- $\Delta\Phi_n = 0$ for $x \neq 0$.
- Heuristically, Φ_n satisfies: $-\Delta\Phi_n = \delta_x$

Solution of the Poisson Equation on $\mathbb{R}^n$

For any open subset $U \subset \mathbb{R}^n$ and any function $\varphi : U \to \mathbb{R}$, the **support** of φ is defined as:

$$\mathrm{supp}\,\varphi := \overline{\{x \in U \mid \varphi(x) \neq 0\}} \quad C_c^2(U,\mathbb{R}) := \{\varphi \in C^2(U,\mathbb{R}) \mid \mathrm{supp}\,\varphi \text{ is compact} \in U\}.$$

For a given $f \in C_c^2(\mathbb{R}^n, \mathbb{R})$, the solution $u(x)$ of the Poisson equation is given by:

$$-\Delta u = f \quad \text{on } \mathbb{R}^n \quad u(x) = \int_{\mathbb{R}^n} \Phi_n(x-y)f(y)\,dy,$$

Formally:

$$-\Delta u = -\Delta(\Phi_n * f) = (-\Delta\Phi_n) * f = \delta * f = f.$$

Green's Second Identity

Let $U \subset \mathbb{R}^n$ be a bounded C^1-domain, and let ν denote the outward unit normal vector on ∂U. For $\varphi, \psi \in C^2(U)$, the boundary normal derivative is:

$$\partial_\nu\varphi := \nabla\varphi \cdot \nu : \partial U \to \mathbb{R}. \quad \int_U ((\Delta\varphi)\psi - \varphi\Delta\psi)dx = \int_{\partial U} ((\partial_\nu\varphi)\psi - \varphi\partial_\nu\psi)dA$$

The boundary terms on the rhs vanish if ψ or φ has compact support on U.

Laplace Equation on a Half-Space

Let $n \in \mathbb{N}$ and define the half-space:

$$\mathbb{R}_+^n := \mathbb{R}^{n-1} \times (0,\infty) = \{x \in \mathbb{R}^n \mid x_n > 0\}.$$

Let $g : \mathbb{R}^{n-1} \times \{0\} \to \mathbb{R}$ be a continuous and bounded function. We consider the following boundary value problem for $u \in C^2(\mathbb{R}_+^n, \mathbb{R})$:

$$\Delta u = 0 \quad \text{on } \mathbb{R}_+^n, \quad u(x) \to g(x_0) \quad \text{as } x \to x_0,\ \forall x_0 \in \partial\mathbb{R}_+^n = \mathbb{R}^{n-1} \times \{0\}.$$

The **Poisson kernel** for $\mathbb{R}_+^n$ is defined as:

$$K_{\mathbb{R}_+^n}(x,y) := \frac{2}{n\alpha_n}\frac{x_n}{\|x-y\|^n}, \quad x \in \mathbb{R}_+^n,\ y \in \mathbb{R}^{n-1} \times \{0\}$$

$$u(x) := \int_{\mathbb{R}^{n-1}\times\{0\}} K_{\mathbb{R}_+^n}(x,y)g(y)\,dy.$$

Laplace Equation on the Unit Ball

Let $B^n := B_1^n(0)$ and $S^{n-1} := S_1^{n-1}(0)$ denote the unit ball and unit sphere in $\mathbb{R}^n$, respectively. Let $g : S^{n-1} \to \mathbb{R}$ be a continuous function. We consider the following boundary value problem for $u \in C^2(B^n, \mathbb{R})$:

$$\Delta u = 0 \quad \text{on } B^n, \quad u(x) \to g(x_0) \quad \text{as } x \to x_0,\ \forall x_0 \in \partial B^n = S^{n-1}.$$

The **Poisson kernel** for B^n is defined as:

$$K_{B^n}(x,y) := \frac{1 - \|x\|^2}{n\alpha_n\|x-y\|^n}, \quad x \in B^n,\ y \in S^{n-1}.$$

$$u(x) := \int_{S^{n-1}} K_{B^n}(x,y)g(y)\,dA(y).$$

Green's Function for General Domains

Let $U \subset \mathbb{R}^n$ be a bounded C^1-domain, and let $f : U \to \mathbb{R}$, $g : \partial U \to \mathbb{R}$ be continuous functions. Consider the boundary value problem:

$$-\Delta u = f \quad \text{on } U, \quad u = g \quad \text{on } \partial U.$$

A **Green's function** for the domain U is a function:

$$G : (x,y) \in U \times U \setminus \{x = y\} \to \mathbb{R},$$

such that for each $x \in U$, the function $G_x := G(x,\cdot) \in C^2(U \setminus \{x\}, \mathbb{R})$ satisfies:

$$\int_U -G_x(y)\Delta\varphi(y)\,dy = \varphi(x), \quad \forall\varphi \in C_c^2(U,\mathbb{R}), \quad G_x = 0 \quad \text{on } \partial U.$$

Solution via Green's Function

Assume G is a Green's function for U, and $u \in C^2(U,\mathbb{R})$ solves the boundary value problem $(-\Delta u = f, u = g)$. Then:

$$u(x) = \int_{\partial U} g(y)\partial_\nu G_x(y)\,dA(y) + \int_U f(y)G(x,y)\,dy, \quad \forall x \in U.$$

Green's function for $\mathbb{R}_+^n$

$$G(x,y) := \Phi_n(y-x) - \Psi_x(y) = \Phi_n(y-x) - \Phi_n(y-\bar{x}).$$

where a harmonic correction function Ψ_x ensures that G vanishes on ∂U.

$$\Psi_x(y) := \Phi_n(y - \bar{x})$$

Define the reflection in the hyperplane $\mathbb{R}^{n-1} \times \{0\}$ as:

$$\mathbb{R}^n \ni x = (x_1, \ldots, x_n) \to \bar{x} = (x_1, \ldots, x_{n-1}, -x_n).$$

Green's Function for B^n

The Green's function for B^n is defined as:

$$G(x,y) := \Phi_n(y-x) - \Psi(x,y) = \Phi_n(y-x) - \|x\|^{2-n}\Phi_n(y-\bar{x}).$$

Define the reflection at the unit sphere (or inversion) in n dimensions as:

$$\mathbb{R}^n \setminus \{0\} \ni x \to \bar{x} := \frac{x}{\|x\|^2} \in \mathbb{R}^n \setminus \{0\}.$$

The correction function for the unit ball is:

$$\Psi(x,y) := \Phi_n(\|x\|(y-\bar{x})) := \|x\|^{2-n}\Phi_n(y-\bar{x}).$$

The Poisson kernel is then respectively derived as:

$$K(x,y) := -\partial_\nu G^x(y) = -\nabla G^x(y)\cdot\nu.$$

6 Calculus of Variations

The principle of least action states that the trajectory of a physical system is such that the action S is stationary (typically minimized or maximized). The action S is defined as the integral of the Lagrangian function L, which encodes the system's dynamics.

Dirichlet Principle

The solutions of the Poisson equation correspond to the minima of the Dirichlet functional:

$$S(u) = \int_U \left(\frac{1}{2}\|\nabla u\|^2 - uf\right)dx, \quad u \in A,\ u = g \text{ on } \partial U.$$

Euler-Lagrange Equation

For a functional $S_{L,g}[u] = \int_U L(x,u,u')\,dx$:

$$\frac{\partial S}{\partial u} - \frac{d}{dx}\frac{\partial S}{\partial u'} = 0. \quad -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}L_{\xi_i}(x,u,\nabla u) + L_y(x,u,\nabla u) = 0. \tag{6.1}$$

(i) A C^2-variation of u is a family $u_\bullet$ of real-valued functions on U such that: $u^0 = u$, and $u^a = u$ on ∂U, $\forall a \in J$.

(ii) A *critical point* (or *stationary point*) of S is a function $u \in A$ such that: $\frac{d}{da}\Big|_{a=0} S(u^a) = 0, \quad \forall$ variation $u^\bullet$ of u.

Lagrangian Mechanics

$$L(t,q,\dot{q}) = \frac{1}{2}m\dot{q}^2 - U_t(q). \quad S(q) = \int_{t_0}^{t_1} L(t,q,\dot{q})dt \quad m\ddot{q} + \partial_q U_t = 0 \quad \text{or} \quad m\ddot{q} = F_t \circ q.$$

Lagrangian for an LC Circuit

$$L(t,Q,\dot{Q}) = \frac{1}{2}L\dot{Q}^2 - \frac{1}{2C}Q^2 \quad S(Q) = \int_{t_0}^{t_1} L(t,Q,I)dt \quad L\ddot{Q} + \frac{1}{C}Q = 0$$

The Lagrangian for electromagnetic fields is given by:

$$L = -\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} - A_\mu J^\mu = L_{\text{field}} + L_{\text{int}}$$

where:

- $x = (x_0 = ct, x_i)_{i=1,2,3}$: Point in space-time
- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$: Electromagnetic field tensor
- $j^\mu = (\rho c, \mathbf{j})$: Current density
- $A_\mu = (\frac{\varphi}{c} A_i)_{i=1,2,3} : \mathbb{R}^4 \to \mathbb{R}^{1\times 4}$ Electromagnetic potential

Electric and Magnetic Fields

$$\mathbf{E} = -\nabla\varphi - \partial_t\mathbf{A}, \quad \mathbf{B} = \nabla\times\mathbf{A} = \begin{bmatrix}\partial_{x_2}A_3 - \partial_{x_3}A_2 \\ \partial_{x_3}A_1 - \partial_{x_1}A_3 \\ \partial_{x_1}A_2 - \partial_{x_2}A_1\end{bmatrix}$$

Euler-Lagrange Equations for the Four-Potential

$$\delta S = -\partial_\alpha\left(\frac{\partial L}{\partial(\partial_\alpha A_\beta)}\right) + \frac{\partial L}{\partial A_\beta} = 0 \quad \Rightarrow \quad \frac{1}{\mu_0}\partial_\alpha F^{\alpha\beta} = J^\beta.$$

Relativistic Formulation

$$(F_{\mu\nu}) = \begin{bmatrix} 0 & \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} \\ -\frac{E_1}{c} & 0 & -B_3 & B_2 \\ -\frac{E_2}{c} & B_3 & 0 & -B_1 \\ -\frac{E_3}{c} & -B_2 & B_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\mathbf{E}^\top}{c} \\ \frac{\mathbf{E}}{c} & \mathbf{B} \end{bmatrix} \quad \xi^{\mu\nu} := \eta^{\nu\sigma}\xi^\mu_\sigma = \eta^{\mu\rho}\eta^{\nu\sigma}\xi_{\rho\sigma}.$$

Maxwell's Equations

$$-\Delta\Phi = \nabla\cdot E = \frac{\rho}{\varepsilon_0} \quad \text{(Gauss's law for electricity)} \tag{6.2}$$

$$\nabla\cdot B = 0 \quad \text{(Gauss's law for magnetism)} \tag{6.3}$$

$$\nabla\times E + \frac{\partial B}{\partial t} = 0 \quad \text{(Faraday's law of induction)} \tag{6.4}$$

$$\nabla\times B - \varepsilon_0\mu_0\frac{\partial E}{\partial t} = \mu_0 j \quad \text{(Ampère's law with Maxwell's correction)} \tag{6.5}$$

7 Appendix

7.1 Trigonometry

$$\sin(ax) = \begin{cases} |\cdot|\max & ax = (2n+1)\frac{\pi}{2} \qquad 0 \\ 0 & ax = (2n+0)\frac{\pi}{2} = n\pi \quad |\cdot|\max \end{cases} = \cos(ax)$$

$\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, $\sinh x = \frac{e^x-e^{-x}}{2}$, $\cosh x = \frac{e^x+e^{-x}}{2}$

$A\cos x + B\sin x = \frac{A-iB}{2}e^{ix} + \frac{A+iB}{2}e^{-ix}$

$\sin^2 x + \cos^2 x = 1 \qquad \sin x + \cos x = \frac{1}{1+i}e^{ix} + \frac{1}{1-i}e^{-ix} \qquad \cosh^2 x - \sinh^2 x = 1$

$\sin(x\pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y), \quad \sin 2x = 2\sin x\cos x$

$\cos(x\pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y), \quad \cos 2x = \cos^2 x - \sin^2 x$

$\sin x\sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y)), \quad \sin^2(x) = \frac{1}{2}(1-\cos(2x))$

$\cos x\cos y = \frac{1}{2}(\cos(x-y) + \cos(x+y)), \quad \cos^2(x) = \frac{1}{2}(1+\cos(2x))$

$\sin x\cos y = \frac{1}{2}(\sin(x-y) + \sin(x+y)), \quad \sin x\cos x = \frac{1}{2}\sin(2x)$

$1+\tan^2 x = \frac{1}{\cos^2 x}, \quad \tan(x+y) = \frac{\tan x + \tan y}{1-\tan x\tan y}, \quad \langle x,y\rangle = \cos(\gamma)|x||y|$

7.2 Inequalities, Estimates and others

$|\langle u,v\rangle| \le \|u\|\|v\|, \quad 2|ab| \le |a|^2 + |b|^2, \quad ||x|-|y|| \le |x\pm y| \le |x|+|y|$

$|\int_E f(x)\mathrm{d}x| \le \int_E |f(x)|\mathrm{d}x, \quad (1+x)^n \ge 1+nx,\ x \ge -1,\ n\in\mathbb{N}$

Factorization $ax^2 + bx = a(x+\frac{b}{2a})^2 - \frac{b^2}{4a}$ $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

7.3 Integrals and Series

$\sum_{k=0}^n q^k = \begin{cases}\frac{1-q^{n+1}}{1-q} & q\ne 1 \\ n+1 & q=1\end{cases}, \quad \sum_{k=0}^\infty q^k = \begin{cases}\frac{1}{1-q} & |q|<1 \\ \infty & \text{sonst}\end{cases}, \quad \prod_{k=0}^n (2k+1) = \frac{(2k)!}{2^k k!}$

$e^x = \sum_{n=0}^\infty \frac{x^n}{n!}, \quad \log(x) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}(x-1)^n, \quad \frac{1}{1-x} = \sum_{n=0}^\infty x^n\ |x|<1$

$\sin x = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{6} + \ldots \quad \cos x = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2} + \ldots$

$$\int\sin(nx)\cos(mx)\mathrm{d}x = \begin{cases} -\frac{\cos((n+m)x)}{2(n+m)} - \frac{\cos((n-m)x)}{2(n-m)} + C & n\ne\pm m \\ -\frac{1}{4n}\cos(2nx) + C & n = m\end{cases}$$

$$\int\sin(nx)\sin(mx)\mathrm{d}x = \begin{cases} \frac{\sin((n-m)x)}{2(n-m)} - \frac{\sin((n+m)x)}{2(n+m)} + C & n\ne\pm m \\ \pm\frac{x}{2} \mp \frac{1}{3n}\sin(2nx) + C & n = \pm m\end{cases}$$

$$\int\cos(nx)\cos(mx)\mathrm{d}x = \begin{cases} \frac{\sin((n+m)x)}{2(n+m)} + \frac{\sin((n-m)x)}{2(n-m)} + C & n\ne\pm m \\ \frac{x}{2} + \frac{1}{4n}\sin(2nx) + C & n = \pm m\end{cases}$$

$\int_a^{a+2\pi l/n}\sin(nx)\mathrm{d}x = \int_a^{a+(2l+1)\pi/n}\cos(nx)\mathrm{d}x = 0,\ a\in\mathbb{R},\ l\in\mathbb{N}$

$\int_0^{l\pi}\sin(nx)\cos(mx)\mathrm{d}x = C\delta_{nm},\ l\in\mathbb{N}$

$\int_{-\pi}^{\pi}\sin(nx)\cos(mx)\mathrm{d}x = \int_0^{2\pi}\sin(nx)\cos(mx)\mathrm{d}x = 0$

$\int_{\mathbb{R}} e^{-x^2/a}\mathrm{d}x = \sqrt{a\pi} \quad \int_0^\infty e^{-x^2/a}\mathrm{d}x = \frac{1}{2}\sqrt{a\pi}$

$\int x sin(ax)dx = \frac{1}{a^2}sin(ax) - \frac{x}{a}cos(ax)$ $\int x cos(ax)dx = \frac{1}{a^2}cos(ax) + \frac{x}{a}sin(ax)$

7.4 Coordinate Systems

Polar Coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, $r = \sqrt{x^2+y^2}$, $\varphi = \arctan(y/x)$ *für* $x > 0$, $0 \le r \le \infty$, $0 \le \varphi < 2\pi$. $\triangle = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\varphi^2$.

Spherical Coordinates $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$
$r = \sqrt{x^2+y^2+z^2}$, $\theta = \arccos\left(\frac{z}{r}\right)$, $\varphi = \arg(x,y)$, $0\le r\le\infty$, $0\le\varphi<2\pi$, $0\le\theta\le\pi$.
$\mathrm{d}V = r^2\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\varphi$. $\triangle = \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\varphi^2$

Invariances Δ, ∇ *under euclidean transformations*
D under bijective and affine transformations

Vector Calculus $rot(\nabla f) = 0, div(rot(X)) = 0, div(f\cdot rotX)) = \nabla f\cdot rot(X), rot(rot(X)) = \nabla(div(X)) - \Delta X$

Rotationsinvarianz: $g(Rx) = g(x) = f(|x|) \quad \forall R(R^TR = 1) \Rightarrow \hat{g}$ rotationsinvariant
Gamma-Funktion: $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,\mathrm{d}t$

$\Gamma(s+1) = s \quad \Gamma(s)\Gamma(n+1) = n!$, für $n\in\mathbb{N}$ $\quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$

7.5 FT Properties

$f(t)$	$c_n, \widehat{f}(w)$
reell ($\overline{f(t)} = f(t)$)	konjugiert symmetrisch ($c_n = \overline{c_{-n}}$) Kosinus-Reihe mit reellen Koeffizienten
imaginär ($\overline{f(t)} = -f(t)$)	konjugiert antisymmetrisch ($c_n = -\overline{c_{-n}}$) Sinus-Reihe mit reellen Koeffizienten
r/i + gerade	r/i + gerade
r/i + ungerade	i/r + ungerade