Signal- und Systemtheorie I

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Contents

1 Generalized Functions

A distribution is a linear functional l, which operates via x on a test function φ. Often it can be represented in the following form:

$$l_x(\varphi) := \int_{-\infty}^{\infty} x(t)\varphi(t)dt \quad \text{e.g. } l_\delta(\varphi) = \varphi(0)$$

Properties:

Concatenation: $l_{x\circ g}(\varphi) = \frac{1}{|a|} l_x(\varphi(\frac{\cdot - b}{a}))$ where $g(t) = at + b$ $\quad l_{\delta\circ g} = \frac{1}{|a|}\varphi(\frac{-b}{a})$

Convolution: $l_{x_1 * x_2}(\varphi) = l_{x_1}(\tilde{x}_2 * \varphi)$ where $\tilde{x}(t) = x(-t)$

Differentiation: $l_{x'}(\varphi) = -l_x(\varphi')$

Fourier Transform:$(\mathcal{F}l_x)(\varphi) := l_x(\mathcal{F}\varphi) = l_{Fx}(\varphi)$

2 Continuous-Time Fourier Series

A periodic signal can be expanded into a Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i \frac{k}{T} t} = c_0 + \sum_{k=1}^{\infty} (c_k + c_{-k})\cos(2\pi kt/T) + i(c_k - c_{-k})\sin(2\pi kt/T)$$

$$\hat{x}(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - \frac{k}{T} \Rightarrow y(t) = (Hx)(t) = \sum_{k=-\infty}^{\infty} c_k \hat{h}(\frac{k}{T}) e^{2\pi i \frac{k}{T} t}$$

3 Continuous-Time Fourier Transform

3.1 Representation of Aperiodic Signals

Continuous-Time Fourier Series as $T_0 \to \infty$ (needs to be bounded for convergence)

$w_0 = \dfrac{2\pi}{T_0} \to dw, kw_0 \to w, \sum \to \int \quad x_k = \dfrac{1}{T_0} x(kw_0)$

The spectrum of a periodic signal consists of discrete samples of the spectrum of its corresponding non-periodic signal. The spectrum of a non-periodic signal is the envelope of the spectrum of the corresponding periodic signal.

$T_0 x_k = \int_{-T_0/2}^{T_0/2} x(t)e^{-ikw_0 t}dt \quad \xrightarrow[T_0\to\infty]{} \quad x(w) = \int_{-\infty}^{\infty} x(t)e^{-iwt}dt$ spectral density

$x(t) = \dfrac{w_0}{2\pi}\sum_{-\infty}^{\infty} x(kw_0)e^{ikw_0 t} \quad \xrightarrow[w_0\to 0]{} \quad \dfrac{1}{2\pi}\int_{-\infty}^{\infty} x(w)e^{iwt}dw$

Dirichlet Conditions for Existence of Fourier Tranform (sufficient but not necessary)
x(t) absolute integrable, finite number of finite extreme values and discontinuities on a finite interval

3.2 Representation of Periodic Signals

Fourier Series Expansion of Periodic Signal: $x(t) = \sum_{k=-\infty}^{\infty} x_k e^{ikw_0 t}$

$x(t) = \int_{-\infty}^{\infty} \delta(w - w_0)e^{iwt}dw = e^{ikw_0 t} \to x(w) = 2\pi \sum_{k=-\infty}^{\infty} x_k \delta(w - w_0)$

The Fourier transform of a periodic signal consists of a series of impulses, each of which is located at the frequency of each harmonic of the signal, with intensity proportional to the corresponding Fourier coefficient.

3.3 Properties

- FT of periodic signal is discrete, aperiodic signal is continuous. FT of a signal with finite support is not band-limited.
- The time shift of a signal only affects its phase-frequency characteristics, by a linear phase shift
- The real part or norm of a Fourier transform are symmetric, the imaginary part or phase are antisymmetric.

3.4 Frequency Response & Domain Analysis

Eigenfunctions of LTI-Systems $x(t) = e^{2\pi i f_0 t}$

An eigenfunction is the infinite dimensional analogue of an eigenvector for a system.

$$Hx(t) = \hat{h}(f_0)e^{2\pi i f_0 t}$$

4 Discrete-Time Fourier Transform (DTFT)

We consider the Fourier Transform of a sampled time-continuous signal.

$$\mathcal{F}\{x\cdot\delta_T\} = \frac{1}{T}\sum_{k=-\infty}^{\infty}\hat{x}(f - \frac{k}{T}) \to \frac{1}{T}\text{periodic in f}$$

Its Fourier Series coefficients are $x(-lT)$, thus we can expand it as:

$$\hat{x}_d(\theta) = \mathcal{F}\{x\cdot\delta_T\} = \sum_{n=-\infty}^{\infty} x[n]e^{-2\pi i n T f} \overset{\theta=Tf}{=} \frac{1}{T}\sum_{k=-\infty}^{\infty}\hat{x}(\frac{\theta-k}{T}) \quad \theta := \frac{f}{f_s} = T_s f$$

Discrete-time Fourier Series as $N \to \infty$

$\hat{x}[k] = \dfrac{1}{N}\sum_{k=<N>} x[n]e^{-i\frac{2\pi}{N}kn} \xrightarrow[N\to\infty]{\frac{2\pi}{N}k \to w} \hat{x}(w) = \sum_{k=-\infty}^{\infty} x[n]e^{-iwn}$

$x[n] = \frac{1}{2\pi}\int_{2\pi}\hat{x}(w)e^{iwn}dw = \int_0^1 \hat{x}(\theta)e^{2\pi i n\theta}d\theta$

5 Discrete Fourier Transform (DFT)

5.1 Representation of Aperiodic Signals

We consider a bounded, time-discrete signal of length N. For such signals, N values of the DTFT $\hat{x}(\theta)$ are enough to completely characterize $x[n]$, i.e. $\hat{x}[k]$ is a uniform sample of N points of $\hat{x}(\theta)$ for $\theta \in [0, 1)$.

$\hat{x}[k] = \hat{x}(\theta)|_{\theta=\frac{k}{N}} = \sum_{k=<N>} x[n]e^{-i\frac{2\pi}{N}kn} \qquad x[n] = \dfrac{1}{N}\sum_{k=<N>}\hat{x}[k]e^{i\frac{2\pi}{N}kn}$

$$\begin{pmatrix}\hat{x}[0]\\ \vdots\\ \vdots\\ \vdots\\ \hat{x}[N-1]\end{pmatrix} = \underbrace{\begin{pmatrix}1 & 1 & 1 & \dots & 1\\ 1 & w_N & w_N^2 & \dots & w_N^{N-1}\\ \vdots & w_N^2 & w_N^4 & \dots & w_N^{2(N-1)}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & w_N^{(N-1)} & w_N^{2(N-1)} & \dots & w_N^{(N-1)^2}\end{pmatrix}}_{[F_N]_{n,k} = w_N^{kn}} \begin{pmatrix}x[0]\\ \vdots\\ \vdots\\ \vdots\\ x[N-1]\end{pmatrix}$$

The columns of F_N are orthogonal, i.e.

$$< f_i, f_j >= \begin{cases} N & i = j\\ 0 & \text{otherwise}\end{cases} \qquad F_N^H F_N = N I_N$$

Thus

$$x[n] = \frac{1}{N}F_N^H F_N x = \frac{1}{N}\left[f_0 f_1 \cdots f_{N-1}\right]\begin{bmatrix}f_0^H\\ f_1^H\\ \vdots\\ f_{N-1}^H\end{bmatrix} x = \frac{1}{N}\sum_{k=0}^{N-1}\langle x, f_k\rangle f_k$$

Since F_N^H or F_N^{-1} is unique, $x[n]$ is uniquely determined by $\hat{x}[k]$. The determinant of the Vandemonde matrix F_N can be computed as $\prod_{i=0\dots N-1, j=0\dots N-1, i<j}(\omega_N^j - \omega_N^i)$.

5.2 Convolution

5.2.1 Cyclic

Let x_1, x_2 be 2 N-periodic signals with N-point DFTs $\hat{x}_1, \hat{x}_2$.

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m]x_2[n-m]$$

in Matrix-form: $x_3 = X_2 x_1$ where X_2 is a $N \times N$ circular matrix with entries $[X_2]_{ij} = x_2[i-j] \quad i, j \in \{0, \dots, N-1\}$. It can be shown that the eigenvectors of a circular matrix are the columns of the normalised DFT matrix $1/\sqrt{N}F_N^H$ and the associated eigenvalues are given by the DFT of the first column.

$$X = \frac{1}{N}F_N^H \Lambda_{\hat{x}[k]} F_N$$

For a LTI-system H with impulse response h it holds:

$$y = Hx = h * x \qquad \hat{y} = F_N y = \underbrace{\frac{1}{N}F_N F_N^H}_{I_N}\Lambda_{\hat{h}[k]}\underbrace{F_N x}_{\hat{x}} = \sum_k \hat{h}[k]\hat{x}[k]$$

5.2.2 Linear

Let x_1, x_2 be finite, non-periodic signals of length L/P. We define $\tilde{x}_i[n]$ as $x_i[n]$, if $n \le P, L-1$ and 0 otherwise, i.e. zero-padded up to $N \ge P + L - 1$. Then we compute the N-point DFTs of $\tilde{x}_i$. Thus, we can compute the linear convolution of arbitrary bounded signals over cyclic convolution.

5.3 FFT

The computational complexity of the DFT can be reduced from $\mathcal{O}(N^2)$ to $\mathcal{O}(4N\log_2(N))$ through the FFT-Algorithm of Cooley & Tukey, where $w_N = e^{-2\pi i/N}$ with $N = 2^n$.

$$F_2 = \begin{bmatrix}1 & 1\\ 1 & w_1\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \Rightarrow \begin{matrix}\hat{x}[0] = x[0] + x[1]\\ \hat{x}[1] = x[0] - x[1]\end{matrix}$$

6 Z-Transform

$\hat{x}(\theta) = X(e^{2\pi i\theta}) \to$ Z-transform applied on the unit circle

$X(z) = DTFT\{x[k]r^{-k}\} = \sum_{k=-\infty}^{\infty} x[k]r^{-k}e^{-iwk} = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$

$x(k) = \dfrac{1}{2\pi}\int_{2\pi} X[re^{iw}]r^k e^{iwk}dw \xrightarrow[dz=ire^{iw}dw=izdw]{} x(k) = \dfrac{1}{2\pi i}\oint_c X[z]z^{k-1}dz$

Region Of Convergence ROC $X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k} < \infty$

Since $|z| = r$, the convergence areas are connected circular disks around the origin that must not contain any poles. Z-transform does not exist for any signal and does not converge for any z. X(z) and ROC uniquely determine a signal.

- X rational funtion: $X(z) = \dfrac{N(z)}{D(z)} = M\dfrac{\prod_i(z - z_i)}{\prod_p(z - z_p)} \to$ uniquely determined by its zero-pole diagram. ROC is always bounded by poles.
- Right-sided signals: ROC includes exterior of circle $[\max(p_i), \infty)$. If $N < 0 \to X(z)$ contains positive powers of z $\to \infty \notin ROC$.
- Left-sided signals: ROC includes interior of circle $(0, \min(p_i)]$. If $N > 0 \to X(z)$ contains negative powers of z $\to 0 \notin ROC$.

- Signals of finite length converge on the entire z-plane (excl. 0 and ∞)
- If ROC contains $\infty/0$ i.e. is outside of the largest pole/inside of the smallest pole, x(n) is causal/anti-causal.
- If ROC includes unit circle, DTFT $\hat{x}(\theta)$ exists and system is stable.

Eigenfunctions of time-discrete LTI-Systems

$x[n] = z^n = r^n e^{2\pi i\theta n}$ $\quad y[n] = (Hx)[n] = H(z)x[n]$ with eigenvalues being the Z-transform of the impulse response.

Properties

- Initial Value Theorem: $x[0] = \lim_{z\to\infty} X(z)$
- Final Value Theorem: $\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$
- Differentiation in time: $\frac{dx[n]}{dn} = \frac{x[n]-x[n-1]}{n-(n-1)} \quad \circ\!\!-\!\!\bullet \quad (\frac{z-1}{z})X(z)$

6.1 Differential Equations

We consider a LTI-system with additional conditions:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m] \qquad H(z) := \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

The solution consists of a homogeneous and a known particular part.

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0$$

This condition can be interpreted as $y_h[n]$ being an eigenfunction of the LTI-System with impulse response $a[n]$ (limited to the intervall N) with eigenvalue 0.

We observe, that

$$A(z) = \sum_{k=0}^{N} a_k z^{-k}$$

is a polynome of degree N with zeros z_m. Thus, all solutions are of the form

$$y_h[n] = \sum_{m=1}^{N} A_m z_m^n \to \sum_{k=0}^{N} a_k y_h[n-k] = \sum_{k=0}^{N} a_k \sum_{m=1}^{N} A_m z_m^{n-k} = \sum_{m=1}^{N} A_m z_m^n \underbrace{\sum_{k=0}^{N} a_k z_m^{-k}}_{A(z_m)=0} = 0$$

The coefficients $A_m \in \mathbb{C}$ are determined through N values of $y[n]$.

$$y[n] = \sum_{m=0}^{M} \frac{b_m}{a_0} x[n-m] - \sum_{k=0}^{N} \frac{a_k}{a_0} y[n-k]$$

$$\begin{pmatrix} y[0]-y_p[0] \\ \vdots \\ \vdots \\ \vdots \\ y[N-1]-y_p[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_N \\ z_1^2 & z_2^2 & \dots & z_N^2 \\ \vdots & \vdots & \vdots & \vdots \\ z_1^{N-1} & z_2^{N-1} & \dots & z_N^{N-1} \end{pmatrix} \begin{pmatrix} A_1 \\ \vdots \\ \vdots \\ \vdots \\ A_N \end{pmatrix}$$

7 Signals and Systems

For a LTI system, the output signal can be represented as a convolution of the impulse response and the input signal. 0 is mapped to 0.

- Linearity: every term contains x(t) exactly once
- Time-Invariance: I/O characteristics of the system do not change with time-shifting. No time-scaling, coefficients do not contain t and t appears only in x(t)
- Causality: y(t) cannot exist before x(t) i.e. time argument of x $\leq$ y $h(t) = 0 \forall t < 0$
- Invertibility: bijective mapping (e.g. differentiator not invertible, integrator invertible)
- Memory: $h(t) = a\delta(t) \to$ causal
- Stability: $h(t) \in L_1 \to$ BIBO-stable

Overview of Transforms

Time Domain	Transformation	Frequency Domain
T-periodic & continuous	Fourier Series	c_k aperiodic & discrete
aperiodic & continuous	Fourier/Laplace	$\hat{x}(f)$ aperiodic & continuous
aperiodic & discrete	DTFT/Z-Transform	$\hat{x}(\theta)$ 1-periodic & continuous
N-periodic & discrete	DFT/FFT	$\hat{x}[k]$ N-periodic & discrete

8 Sampling Theorem

Sampling function: $sinc(t) = \frac{sin(\pi t)}{\pi t}$ $\quad \int_{-\infty}^{\infty} \frac{sin(wt)}{t} = \pi \quad \int_{0}^{\infty} \frac{sin(wt)}{t} = \frac{\pi}{2}$
Sample in frequency domain $\to$ Periodization of $x(t)$:

$$\mathcal{F}^{-1}\{\hat{x}(f)\sum_{k=-\infty}^{\infty}\delta(f-\frac{k}{T})\}(t) = (x*(T\sum_{k=-\infty}^{\infty}\delta(t-kT)))(t) = T\sum_{k=-\infty}^{\infty}x(t-kT)$$

Sample in time domain $\to$ Periodization of $\hat{x}(f)$ with expansion coefficients $x(kT)$:

$$\hat{x}_s(f) = \mathcal{F}\{x(t)\sum_{k=-\infty}^{\infty}\delta(t-kT)(t)\} = \frac{1}{T_s}\sum_{k=-\infty}^{\infty}\hat{x}(f-\frac{k}{T_s}) = \sum_{k=-\infty}^{\infty}x(kT)e^{-2\pi ikfT}$$

The original band-limited signal can only be reconstructed if the periodized spectra do not overlap each other. The sampling rate $f_s = 1/T_s \geq 2f_g$, where f_g is the maximum frequency or the one-sided width of a bandpass signal. The sampled signal frequencies appear at $nf_s \pm f_g$. Then it follows:

$$\hat{x}(f) = \hat{x}_a(f)T\hat{h}_{Id}(f) \quad \text{where } f_g \leq f_c \leq f_s - f_g$$

$$x(t) = (x_a * (Th_{id}))(t) = T\sum_{k=-\infty}^{\infty}x(kT)h_{Id}(t-kT) = \sum_{k=-\infty}^{\infty}x(kT)\frac{sin(\frac{\pi}{T}(t-kT)}{\frac{\pi}{T}(t-kT)}$$

Properties of Nyquist Rate

1. $x(t) \to f_s$
2. $x(t \pm t_0) \to f_s$
3. $x(at) \to af_s$
4. $x(t)^n \to nf_s$
5. integration or derivation of $x(t) \to f_s$
6. $x_1(t)x_2(t) \to f_{s1} + f_{s2}$

Nyquist-Theorem *A band-limited signal can be displayed in discrete time without loss of information if the sampling frequency is at least twice the highest frequency component of the signal.*

9 Trivia

- Half-wave symmetry: $c_k = 0 \quad \forall k$ gerade, if $x(t) = -x(t-\frac{T}{2})$
- Delta-Comb: $\sum_{n=0}^{N-1} e^{-i2\pi\frac{n}{N}k} = N\delta_N[k] \quad \delta_T = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{i2\pi\frac{k}{T}t}$
- Inverse DFT: $x[n] = \frac{1}{N}DFT_n\{\hat{x}[N-n]\}$
- Decimation in frequency: $\hat{x}[2k] = \sum_0^{N/2-1}(y[n]+z[n])w_{N/2}^{kn} \quad \hat{x}[2k+1] = \sum_0^{N/2-1}(y[n]-z[n])w_N^n w_{N/2}^{kn}$
- $log(1+z) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}z^n}{n} \quad |z| < 1$
- $(Exp_N x)[n] = \begin{cases} x[n/N] & N|n \\ 0 & \text{sonst} \end{cases} \quad \circ\!\!-\!\!\bullet \quad \hat{x}(N\theta)$
- $(Dez_N x)[n] = x[nN] \quad \circ\!\!-\!\!\bullet \quad \frac{1}{N}\sum_{k=0}^{N-1}\hat{x}(\frac{\theta-k}{N}) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{N}\sum_{n=0}^{N-1}X(z^{1/N}e^{2\pi in/N})$
- Rate of change in time domain $|x'(t)| \leq 4\pi W \sup|x(t)|$
- Bandwidth:

  $x_1 + x_2 \to w = \max\{w_1, w_2\}$

  $x_1 * x_2 \to w \leq w_1 + w_2$

  $x_1 x_2 \to w = \min\{w_1, w_2\}$
- For time-discrete periodic signals: $N = \min\{\frac{nk}{N}\} \in \mathbb{Z} = \frac{N}{ggT(N,k)} = \frac{kgV(N,k)}{k}$
- Derivative at discontinuities: $(Dx)(t) = x'(t) + (x(t_0^+) - x(t_0^-))\delta(t-t_0)$
- $\int_{-\infty}^{\infty}\delta'(t) = 0 \quad \int_{-\infty}^{t}\sigma(t) = t\sigma(t) \quad f(t)*\sigma(t) = \int_{-\infty}^{t}f(x)dx$
  $f[n]*\sigma[n] = \sum_{k=-\infty}^{n}f[k] \quad \sigma[n]*\sigma[n] = (n+1)\sigma[n]$
  $x(t) = f'(t) * \int g(t) = \int f(t) * g'(t)$
- Analysematrix: $c = \begin{bmatrix} c_1 \\ \vdots \\ c_M \end{bmatrix} = \begin{bmatrix} e_1^H \cdot x \\ \vdots \\ e_M^H \cdot x \end{bmatrix} = \begin{bmatrix} < x, e_1 > \\ \vdots \\ < x, e_M > \end{bmatrix} = Tx$ T quadratic & regular
- Synthesematrix: $\tilde{T}^H = T^{-1}$ dual basis with same coefficients, for unitary matrices $\tilde{T}^H = T^H$
- Gram-Schmidt: Für $j = 1, 2, ..., n$: $v_j' = w_j - \sum_{i=1}^{j-1}\frac{<v_i,w_j>}{<v_i,v_i>}v_i \quad v_j = \frac{v_j'}{||v_j'||} \quad j \to j+1$
- Young'sche Ungleichung: $x_1 \in L_2, x_2 \in L_2 \to |(x_1 * x_2)(t)| \leq ||x_1||_2||x_2||_2$
- Riemann-Lebesgue Lemma: Es sei x ein absolut integrierbares Signal, d.h. $x \in L_1$. Dann ist $(Fx)(f) = \hat{x}(f)$ stetig und $\lim_{|f|\to\infty}\hat{x}(f) = 0$.
- $h[n] = \sum_{m=0}^{M-N}B_m\delta[n-m] + \sum_{k=1}^{N}A_k d_k^n\sigma[n] \to$ FIR(no feedback,stable)+IIR
  $H(z) = \sum_{m=0}^{M-N}B_m z^{-m} + \sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}}$
- $|1 - ae^{-jw}| = \sqrt{1 - 2acosw + a^2}$
- $sign[n] \quad \circ\!\!-\!\!\bullet \quad \frac{1+e^{-2\pi i\theta}}{1-e^{-2\pi i\theta}}$
- Convolution of 2 rectangular signals: $A_{max} = A_1A_2\min\{T_1,T_2\}$, asc. $-(t_1+t_2)+\min\{T_1,T_2\}$, const. $-(t_1+t_2)+\max\{T_1,T_2\}$, desc. $-(t_1+t_2)+T_1+T_2$
- $\int_{-\infty}^{\infty}f(x)\frac{d^n\delta(x-a)}{dx^n}dx = (-1)^n\frac{d^n f(x)}{dx}|_{x=a}$