M-Matrix Strategies for Pinning-Controlled Leader-Following Consensus in Multiagent Systems With Nonlinear Dynamics

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***Abstract*—This paper considers the leader-following consensus problem for multiagent systems with inherent nonlinear dynamics. Some M-matrix strategies are developed to address several challenging issues in the pinning control of multiagent systems by using algebraic graph theory and the properties of nonnegative matrices. It is shown that second-order leader-following consensus in a nonlinear multiagent system can be reached if the virtual leader has a directed path to every follower and a derived quantity is greater than a positive threshold. In particular, this paper analytically proves that leader-following consensus may be easier to be achieved by pinning more agents or increasing the pinning feedback gains. A selective pinning scheme is then proposed for nonlinear multiagent systems with directed network topologies. Numerical results are given to verify the theoretical analysis.**

***Index Terms*—Directed spanning tree, leader-following consensus, M-matrix, multiagent system, pinning control.**

I. Introduction

The COOPERATIVE control of multiagent systems has received increasing attention in recent years due to its wide applications in many fields, such as flocking of unmanned air vehicles, formation control of mobile robots, attitude control of satellites, and so on. Among the main research issues on multiagent systems, the consensus problem plays a basic role in investigating the collective behaviors of multiagent systems. Substantial effort has been devoted to developing distributed consensus protocols for multiagent systems such that all the agents can reach an agreement on a common value by using local neighboring information.

The first-order consensus problem of multiagent systems with single-integrator dynamics has been intensively studied [1]–[7], and it is now well known that first-order consensus in a multiagent system with a dynamic topology can be reached if and only if the interaction digraph contains a directed spanning tree jointly as the system evolves over time [2]. Recently, the second-order consensus problem of multiagent systems with double-integrator dynamics has been widely investigated, and some remarkable results have been obtained [8]–[15]. In sharp contrast to first-order consensus, second-order consensus depends on not only the network topology but also the network parameters [8], [12]. Ren and Atkins [8] have shown that containing a directed spanning tree is just a necessary condition for reaching second-order consensus, i.e., second-order consensus may fail to be achieved even if the network topology has a directed spanning tree.

The consensus problem of first-order multiagent systems with nonlinear dynamics can be treated as the synchronization problem of complex networks described by first-order nonlinear differential equations [16]–[19]. In most studies on second-order consensus, multiagent systems are usually assumed to be composed of a set of interacting double integrators, where all agents finally agree on a constant velocity. Nevertheless, some second-order dynamical systems in the real world, such as mass–spring mechanical systems [20], pendulums [21], and harmonic oscillators [22], [23], are governed by complicated dynamics. Hence, it is necessary to consider the intrinsic nonlinear dynamics of each individual agent in second-order multiagent systems. Recently, some progress has been made in the study of second-order consensus problem for multiagent systems with general dynamics, in which all agents may finally agree on a time-varying velocity. Ren [22] and Su *et al.* [23] addressed second-order consensus in a network of harmonic oscillators. Yu *et al.* [24], Song *et al.* [25], and Su *et al.* [26] studied the second-order consensus problem for multiagent systems with nonlinear dynamics, which is quite challenging because the linear control theory can not be directly applied for investigating consensus of nonlinear multiagent systems.

For a large-scale multiagent system, it has been shown that some leader-following consensus algorithms can be designed based on pinning control approach [16]–[19] such that all agents can asymptotically follow the virtual leader, where only

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a small fraction of agents can access the information of the virtual leader. Ren [3] and Chen *et al.* [5] considered first-order consensus in multiagent systems with single-integrator dynamics under pinning control. Hu and Hong [9], Ren [10], Tian and Liu [11], and Meng *et al.* [14] addressed second-order consensus in multiagent systems with double-integrator dynamics by pinning a subset of agents. By adopting a pinning control strategy and the Lyapunov functional method, Song *et al.* [25] and Su *et al.* [26] studied the second-order leader-following consensus problem for multiagent systems with nonlinear dynamics, which is more difficult than that of multiagent systems with double-integrator dynamics.

Recently, Song *et al.* [27] proposed an M-matrix approach [28], [29] to investigate first-order leader-following consensus in pinning-controlled nonlinear multiagent systems. This paper further extends the results in [27] to study second-order leader-following consensus in multiagent systems with inherent nonlinear dynamics by using the M-matrix theory, algebraic graph theory, and pinning control strategy. The main contributions of this paper are threefold. First, a simple quantity, which is the minimum real part of eigenvalues of the matrix formed by the Laplacian matrix and the matrix composed by pinning feedback gains, is derived to analyze leader-following consensus in multiagent systems. If this derived quantity is greater than some threshold, it is shown that second-order leader-following consensus in a nonlinear multiagent system can be achieved. Second, this paper significantly improves the results in [25]. According to the theoretical analysis in [25], the followers whose out-degrees are bigger than their in-degrees should be chosen as pinned candidates, although the numerical studies in [25] have shown that one only needs to pin a subset of the followers. Moreover, the pinning feedback gains in [25] were designed based on matrix decomposition, which is not very convenient for practical use. In this paper, the conditions for reaching second-order leader-following consensus are more general and less conservative than those in [25], and the pinning feedback gains can be easily designed without involving matrix decomposition. Third, some novel M-matrix strategies are developed to solve several challenging problems in the pinning control of multiagent systems, including what kind of agents must be pinned, what kind of agents can be chosen as pinned candidates, and what kind of agents may be left unpinned. In particular, this paper analytically shows that leader-following consensus may be easier to be achieved by pinning more agents or increasing the pinning feedback gains, which greatly improves our preliminary results in [27].

The remainder of this paper is organized as follows. Section II provides some necessary preliminaries, including M-matrix and algebraic graph theories. Section III derives some sufficient conditions for reaching second-order leader-following consensus in multiagent systems under pinning control. Section IV develops some M-matrix strategies to address several challenging issues in the pinning control of multiagent systems and proposes a selective pinning scheme for reaching leader-following consensus in multiagent systems with nonlinear dynamics. Numerical results are given in Section V to verify the theoretical analysis. Finally, some concluding remarks are stated in Section VI.

II. Preliminaries

A. Notations

The standard notations are used throughout this paper. $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. For a symmetric matrix $M\in\mathbb{R}^{N\times N}$, let $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ be its minimum and maximum eigenvalues, respectively. For $z\in\mathbb{C}$, let $\bar{z}$, $|z|$, and $\mathrm{Re}(z)$ represent its conjugate, modulus, and real part, respectively. Let I_n be the n-dimensional identity matrix, $1_n\in\mathbb{R}^n$ $(0_n\in\mathbb{R}^n)$ be the vector of all ones (zeros), and $O_{m\times n}$ be the matrix with all zero entries. For a square matrix $A\in\mathbb{C}^{n\times n}$, let $A(\mathcal{I})$ be its principal submatrix formed by the row–column pairs indexed by the set $\mathcal{I}\subset\{1,\ldots,n\}$, $\lambda_i(A)$ be the ith eigenvalue, $\rho(A)=\max_{1\le i\le n}|\lambda_i(A)|$ be the spectral radius, and $\mathcal{J}(A)=\min_{1\le i\le n}\mathrm{Re}(\lambda_i(A))$ be the minimum real part of all eigenvalues. For two Hermitian matrices A and B, write $A\succ 0$ if A is positive definite, and write $A\succ B$ if $A-B\succ 0$. A real matrix X is called nonnegative (positive) and denoted as $X\ge 0$ $(X>0)$ if all the entries of X are nonnegative (positive), and write $X\ge Y$ if $X-Y\ge 0$. The symbol $\otimes$ denotes the Kronecker product [28].

B. M-Matrix Theory

In this section, some results on M-matrices are provided to investigate the pinning control of multiagent systems.

Definition 1 [28]: Let $Z_N=\{A=(a_{ij})\in\mathbb{R}^{N\times N}: a_{ij}\le 0 \text{ if } i\ne j, i,j=1,\ldots,N\}$ denote the set of real matrices whose off-diagonal elements are all nonpositive.

Definition 2 [28]: A nonsingular matrix A is called an M-matrix if $A\in Z_N$ and all the eigenvalues of A have positive real parts.

Lemma 1 [28], [29]: If $A\in Z_N$, the following statements are equivalent.

1) A is an M-matrix.
2) Matrix A can be expressed by $A=\gamma I_N-B$, where $B\ge 0$ and $\gamma>\rho(B)$.
3) A^{-1} exists, and $A^{-1}\ge 0$.
4) $\mathrm{Re}(\lambda_i(A))>0$ for all $i=1,\ldots,N$, which implies that $\mathcal{J}(A)>0$;
5) There exists a positive definite diagonal matrix $\Xi=\mathrm{diag}(\xi_1,\cdots,\xi_N)$ such that $\Xi A+A^{\mathrm{T}}\Xi\succ 0$.
6) There is a positive vector $x\in\mathbb{R}^N$ such that $Ax>0$.

Remark 1: In this paper, Lemma 1 plays a key role in analyzing the leader-following consensus of multiagent systems under pinning control.

Lemma 2 [29]: For nonnegative square matrices $A\ge 0$ and $B\ge 0$, one has the following.

1) $\rho(A)$ is an eigenvalue of A.
2) $\rho(\widetilde{A})\le\rho(A)$, where $\widetilde{A}$ is a principal submatrix of A.
3) If $0\le B\le A$, then $\rho(B)\le\rho(A)$.

Lemma 3: For an M-matrix $A\in\mathbb{R}^{N\times N}$, let $A=\gamma I_N-B$, where $B\ge 0$ and $\gamma>\rho(B)$. One has $\mathcal{J}(A)=\gamma-\rho(B)=1/\rho(A^{-1})$.

Proof: Let $\lambda_i(A)$ be the ith eigenvalue of A. By [28, Lemma 2.5.2.1], we know that $\gamma-\rho(B)$ is a *real* eigenvalue of A and $\mathcal{J}(A)=\min_{1\le i\le N}\mathrm{Re}(\lambda_i(A))=\gamma-\rho(B)>0$. Then, it follows that $\min_{1\le i\le N}|\lambda_i(A)| = \gamma - \rho(B)$. Hence, $\mathcal{J}(A) = \gamma - \rho(B) = \min_{1\le i\le N}|\lambda_i(A)|$.

Since A is an M-matrix, A^{-1} exists, and $A^{-1} \ge 0$ according to Lemma 1. Let $\lambda_i(A^{-1})$ be the ith eigenvalue of A^{-1}. Obviously, $\lambda_i(A) = 1/\lambda_i(A^{-1})$ is the ith eigenvalue of A. Then, considering $|\lambda_i(A^{-1})| \le \rho(A^{-1})$, we have $|\lambda_i(A)| \ge 1/\rho(A^{-1})$. By Lemma 2, we know that $\rho(A^{-1})$ is an eigenvalue of A^{-1}, which implies that $1/\rho(A^{-1})$ is an eigenvalue of A. Hence, $\min_{1\le i\le N}|\lambda_i(A)| = 1/\rho(A^{-1})$.

Now, we have $\mathcal{J}(A) = \gamma - \rho(B) = \min_{1\le i\le N}|\lambda_i(A)| = 1/\rho(A^{-1})$ and thus finish the proof. ■

Lemma 4: Let $A, B \in Z_N$, and assume that A is an M-matrix. If $B \ge A$, then B is an M-matrix with $\mathcal{J}(B) \ge \mathcal{J}(A)$.

Proof: Since A is an M-matrix, by Lemma 1, there is a positive vector x such that $Ax > 0$ holds. Considering $B \ge A$, we have $Bx = Ax + (B - A)x \ge Ax > 0$, which indicates that B is an M-matrix according to Lemma 1. Considering that both A and B are M-matrices, we have $A^{-1} \ge 0$ and $B^{-1} \ge 0$. In view of $B \ge A$, we know that $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$ is the product of three nonnegative matrices and hence is nonnegative, which indicates that $A^{-1} \ge B^{-1}$. By Lemmas 2 and 3, we have $\mathcal{J}(B) = 1/\rho(B^{-1}) \ge 1/\rho(A^{-1}) = \mathcal{J}(A)$. ■

Lemma 5: For an M-matrix $A \in \mathbb{R}^{N\times N}$, let $\widetilde{A}$ be any principal submatrix of A. One has $\mathcal{J}(\widetilde{A}) \ge \mathcal{J}(A)$, which indicates that $\widetilde{A}$ is also an M-matrix.

Proof: Let $\widetilde{A} = A(\mathcal{I})$, where $\mathcal{I}$ is a subset of the set $\{1, \ldots, N\}$. In view of Lemma 1, M-matrix A can be written as $A = \gamma I_N - B$ with $B \ge 0$ and $\gamma > \rho(B)$. It is easy to see that $\widetilde{A} = \gamma I_m - \widetilde{B}$, where m is the number of the elements in $\mathcal{I}$ and $\widetilde{B} = B(\mathcal{I}) \ge 0$ is the principal submatrix of B. Then, by Lemma 2, we have $\rho(\widetilde{B}) \le \rho(B)$. Hence, $\gamma > \rho(\widetilde{B})$ holds. According to Lemma 1, we can conclude that $\widetilde{A}$ is also an M-matrix. By Lemma 3, we have $\mathcal{J}(\widetilde{A}) = \gamma - \rho(\widetilde{B}) \ge \gamma - \rho(B) = \mathcal{J}(A)$. ■

Remark 2: In this paper, Lemmas 3, 4, and 5 are very important for developing M-matrix strategies to theoretically analyze several challenging problems in the pinning control of multiagent systems.

C. Graph Theory

Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ be a digraph with a node set $\mathcal{V} = \{1, \ldots, N\}$, an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = (a_{ij})_{N\times N}$ with nonnegative elements. A directed edge denoted by (j, i) means that node i has access to node j, i.e., node i can receive information from node j. The elements of the adjacency matrix $\mathcal{A}$ are defined as follows: If there is a directed link from node j to i ($j \ne i$), then $a_{ij} > 0$; otherwise, $a_{ij} = 0$. We assume that $a_{ii} = 0$ for all $i \in \mathcal{V}$. The Laplacian matrix $L = (l_{ij})_{N\times N}$ associated with the adjacency matrix $\mathcal{A}$ is defined as [1]–[3], [6], [8], [10], [12], [24], [25]

$$\begin{aligned} l_{ij} &= -a_{ij} \le 0, \qquad i \ne j \\ l_{ii} &= \sum_{j=1, j\ne i}^{N} a_{ij}, \qquad i = 1, \ldots, N \end{aligned} \tag{1}$$

which ensures that $\sum_{j=1}^{N} l_{ij} = 0$. For an undirected graph, one has $L = L^{\mathrm{T}}$. Usually, the Laplacian matrix of a directed graph is asymmetric.

The in-degree and out-degree of node i can be defined as follows [1], [11], [25]:

$$\deg_{\mathrm{in}}(i) = \sum_{j=1, j\ne i}^{N} a_{ij} \qquad \deg_{\mathrm{out}}(i) = \sum_{j=1, j\ne i}^{N} a_{ji}. \tag{2}$$

A directed path from node j to node i is a sequence of edges $(j, i_1), (i_1, i_2), \ldots, (i_l, i)$ in the directed graph $\mathcal{G}$ with distinct nodes i_k, $k = 1, \ldots, l$. A directed graph is strongly connected if, for any two distinct nodes j and i, there exists a directed path from node j to node i. A digraph with N nodes is called a directed tree if it has $N - 1$ edges and there exists a root node with directed paths to every other node. A directed spanning tree is a directed tree that includes all the nodes of the digraph. A digraph is said to have or contain a directed spanning tree if there is at least one node having a directed path to every other node [2], [3], [8], [10]–[12], [24], [25].

Lemma 6: For any digraph $\mathcal{G}$, let $\mathcal{V} = \{1, \ldots, N\}$ and $L = (l_{ij})_{N\times N}$ be the node set and the Laplacian matrix, respectively. Let node 0 be the leader node, and assume that there is no directed link from any node in $\mathcal{V}$ to node 0. For node $i \in \mathcal{V}$, let d_i be the linking weight from node 0 to i, and one says that node i is pinned if $d_i > 0$ and unpinned if $d_i = 0$. The matrix $L + D$ is an M-matrix, which is equivalent to $\mathcal{J}(L + D) > 0$, if and only if the leader node has a directed path to every other node $i \in \mathcal{V}$, where $D = \mathrm{diag}(d_1, \ldots, d_N)$.

Proof: According to [9, Lemma 4] or [30, Lemma 1.6], we know that all eigenvalues of $L + D$ have positive real parts if and only if the leader node has a directed path to every other node $i \in \mathcal{V}$. Then, by Lemma 1, we can complete the proof considering $L + D \in Z_N$. ■

III. Pinning-Controlled Leader-Following Consensus in Second-Order Multiagent Systems With Nonlinear Dynamics

For a multiagent system, let L and D be the Laplacian matrix and the matrix composed by pinning feedback gains, respectively. By utilizing the properties of M-matrices, Song *et al.* [27] showed that first-order leader-following consensus in a pinning-controlled nonlinear multiagent system can be achieved if the virtual leader has a directed path to every follower and $\mathcal{J}(L + D)$ is greater than a positive threshold.

This section further extends the theoretical results in [27] to investigate second-order leader-following consensus in multiagent systems with nonlinear dynamics via pinning control. It is shown that $\mathcal{J}(L + D)$ can also be used to analyze the second-order leader-following consensus problem.

Consider a second-order multiagent system with nonlinear dynamics [24], [25]

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= f(t, x_i(t), v_i(t)) + u_i(t), \qquad i = 1, \ldots, N \end{aligned} \tag{3}$$

where $x_i = (x_{i1}, \ldots, x_{\mathrm{in}})^{\mathrm{T}}$ and $v_i = (v_{i1}, \ldots, v_{\mathrm{in}})^{\mathrm{T}}$ are the position and velocity states of agent i, respectively, $f(t, x_i, v_i) = (f_1(t, x_i, v_i), \ldots, f_n(t, x_i, v_i))^{\mathrm{T}} \in \mathbb{R}^n$ is a vector-valued continuous function describing the inherent dynamics of agent i, and u_i is the control input to be designed.

Note that multiagent system (3) has double-integrator dynamics when $f \equiv 0_n$ [8], [10]–[12].

The virtual leader for multiagent system (3) is given by

$$\begin{aligned}\dot{x}_r(t) &= v_r(t)\\ \dot{v}_r(t) &= f(t, x_r(t), v_r(t))\end{aligned} \tag{4}$$

where $x_r \in \mathbb{R}^n$ and $v_r \in \mathbb{R}^n$ are the position and velocity states of the virtual leader, respectively. Note that, when $f \neq 0_n$, the velocity of the virtual leader is time varying.

Definition 3: Multiagent system (3) is said to achieve second-order leader-following consensus if $x_i(t) \to x_r(t)$ and $v_i(t) \to v_r(t)$, $i = 1, \ldots, N$, as $t \to \infty$ for any initial condition, where x_r and v_r are the position and velocity states of the virtual leader (4), respectively.

To reduce the number of controllers, we can adopt the pinning control strategy to solve the leader-following consensus problem for multiagent system (3). Let $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{V}_{\text{pin}} = \{i_1, \ldots, i_l\} \subset \mathcal{V}$ denote the sets of total agents and pinned agents, respectively, where $1 \leq l < N$. Let $\mathcal{A} = (a_{ij})_{N\times N}$ be the adjacency matrix of multiagent system (3). Consider the following pinning control algorithm to achieve second-order leader-following consensus of multiagent system (3):

$$\begin{aligned}\dot{x}_i &= v_i\\ \dot{v}_i &= f(t, x_i, v_i) + \sigma \sum_{j=1}^{N} a_{ij}\left[(x_j - x_i) + (v_j - v_i)\right]\\ &\quad + \sigma d_i\left[(x_r - x_i) + (v_r - v_i)\right], \qquad i = 1, \ldots, N\end{aligned} \tag{5}$$

where $\sigma > 0$, and the feedback gains are defined as $d_i > 0$ if $i \in \mathcal{V}_{\text{pin}}$ and $d_i = 0$ if $i \in \mathcal{V} \setminus \mathcal{V}_{\text{pin}}$.

The following assumption is needed to investigate the consensus problem of multiagent system (5).

Assumption 1: For the nonlinear function f in (5), there exist two constant *nonnegative* matrices $W = (w_{ij})_{n\times n}$ and $M = (m_{ij})_{n\times n}$ such that

$$\begin{aligned}&|f_j(t, x, v) - f_j(t, y, z)|\\ &\leq \sum_{k=1}^{n}\left(w_{jk}|x_k - y_k| + m_{jk}|v_k - z_k|\right), \quad j = 1, \ldots, n\end{aligned} \tag{6}$$

where $x = (x_1, \ldots, x_n)^{\mathrm{T}}$, $y = (y_1, \ldots, y_n)^{\mathrm{T}}$, $v = (v_1, \ldots, v_n)^{\mathrm{T}}$, and $z = (z_1, \ldots, z_n)^{\mathrm{T}}$.

By Assumption 1, let

$$\rho = \max\{\rho_1, \rho_2 + 1\} \tag{7}$$

in which $\rho_1 = (1/2)\max_{1\leq j\leq n}\sum_{k=1}^{n}(w_{jk} + m_{jk} + 2w_{kj})$ and $\rho_2 = (1/2)\max_{1\leq j\leq n}\sum_{k=1}^{n}(m_{jk} + w_{jk} + 2m_{kj})$ are nonnegative.

The following lemmas will be used to derive the theoretical results to study consensus in multiagent system (5).

Lemma 7 (Schur Complement) [31]: The following linear matrix inequality:

$$\begin{bmatrix} Q(x) & S(x)\\ S^{\mathrm{T}}(x) & R(x)\end{bmatrix} \succ 0$$

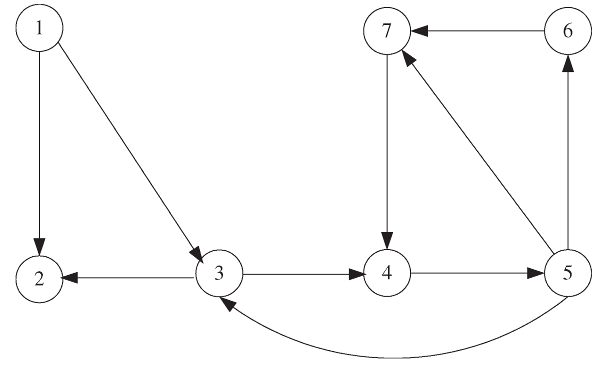

Fig. 1. Interaction digraph of seven agents.

where $Q(x) = Q^{\mathrm{T}}(x)$, $R(x) = R^{\mathrm{T}}(x)$, is equivalent to either of the following conditions.

1) $Q(x) \succ 0, R(x) - S^{\mathrm{T}}(x)Q^{-1}(x)S(x) \succ 0$.
2) $R(x) \succ 0, Q(x) - S(x)R^{-1}(x)S^{\mathrm{T}}(x) \succ 0$.

Lemma 8 [28]: For matrices A, B, C, and D with appropriate dimensions, one has the following.

1) $(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}$.
2) $(A + B) \otimes C = A \otimes C + B \otimes C$;.
3) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

Lemma 9: For real vectors $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ and $y = (y_1, \ldots, y_n)^{\mathrm{T}}$ and a nonnegative matrix $A = (a_{jk})_{n\times n}$ with $a_{jk} \geq 0$, one has $x^{\mathrm{T}}Ay \leq (1/2)\sum_{j=1}^{n}\sum_{k=1}^{n}(a_{jk}x_j^2 + a_{kj}y_j^2)$.

Proof: Please see Appendix A. ■

Theorem 1: Under Assumption 1, the pinning-controlled multiagent system (5) achieves second-order leader-following consensus if the virtual leader (4) has a directed path to every follower and

$$\mathcal{J}(L + D) > \frac{\rho}{\sigma} \tag{8}$$

where ρ is defined in (7).

Proof: Please see Appendix B. ■

Remark 3: Let $\overline{G} = (G + G^{\mathrm{T}})/2$ for notational brevity, where $G = -L$. Under some conditions, Song *et al.* [25] showed that leader-following consensus in multiagent system (5) can be achieved if $\lambda_{\max}(\overline{G}_l) < -\rho/\sigma$, where $\overline{G}_l$ is the minor matrix of $\overline{G}$ by removing its row–column pairs indexed by $\mathcal{V}_{\text{pin}}$. Then, we should first find a set of pinned agents such that $\lambda_{\max}(\overline{G}_l) < 0$ is satisfied. If the virtual leader has a directed path to every follower, we always have $\mathcal{J}(L + D) > 0$ in view of Lemma 6, but $\lambda_{\max}(\overline{G}_l) < 0$ may not hold. Here, we give an example of illustration. Note that agent 1 is the only root of the digraph shown in Fig. 1. If agent 1 is pinned with control gain $d_1 = 1.0$, we have $\mathcal{J}(L + D) = 0.1168$, but $\lambda_{\max}(\overline{G}_l) = 0.0537 > 0$. Hence, the consensus condition (8) is less conservative than the counterpart in [25]. When agents 1 and 5 are pinned with $d_1 = d_5 = 1.0$, we have $\lambda_{\max}(\overline{G}_l) = -0.7506 < 0$. In Section IV-D, we will discuss what conditions can ensure that $\lambda_{\max}(\overline{G}_l) < 0$ always holds.

Remark 4: In [25], the pinning feedback gains for multiagent system (5) should be larger than a threshold obtained by matrix decomposition. In this paper, one can take relatively lower pinning feedback gains according to condition (8), which is convenient for practical implementations. It is worth mentioning that the design of pinning feedback gains will be further discussed in Section IV. From Remark 3 and the aforementioned discussions, one knows that the theoretical results for leader-following consensus of multiagent system (5) in this paper significantly improve the results in [25].

Obviously, multiagent system (3) has double-integrator dynamics when $f \equiv 0_n$ [8], [10]–[12]

$$\dot{x}_i(t) = v_i(t) \qquad \dot{v}_i(t) = u_i(t), \qquad i = 1, \dots, N \tag{9}$$

and the virtual leader is given by

$$\dot{x}_r(t) = v_r(t) \qquad \dot{v}_r(t) = 0 \tag{10}$$

which indicates that the virtual leader has a constant velocity.

From algorithm (5), we have the following pinning control protocol for multiagent system (9):

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= \sigma \sum_{j=1}^{N} a_{ij} \left[(x_j - x_i) + (v_j - v_i) \right] \\ &\quad + \sigma d_i \left[(x_r - x_i) + (v_r - v_i) \right], \quad i = 1, \dots, N. \end{aligned} \tag{11}$$

When $f \equiv 0_n$, it is easy to see that $W = M = O_{n \times n}$ can satisfy Assumption 1. So, we have $\rho = 1$ in view of (7). From Theorem 1, we have the following result.

Corollary 1: The second-order leader-following consensus in the pinning-controlled multiagent system (11) can be reached if the virtual leader (10) has a directed path to every follower and

$$\mathcal{J}(L + D) > \frac{1}{\sigma}. \tag{12}$$

IV. M-Matrix Strategies for Pinning-Controlled Multiagent Systems With Directed Topologies

Up to this point, we know that $\mathcal{J}(L + D)$ plays an important role in analyzing leader-following consensus in both first- and second-order multiagent systems with nonlinear dynamics. Obviously, $\mathcal{J}(L + D) > 0$ should hold in order to satisfy condition (8). For a given σ, the leader-following consensus will be easier to be achieved if $\mathcal{J}(L + D)$ is larger. In this section, some M-matrix strategies are developed to theoretically analyze the effects of the number of pinned agents and the pinning feedback gains on $\mathcal{J}(L + D)$. A selective pinning scheme is proposed for achieving leader-following consensus in multiagent system (5).

Given a multiagent system, let $\mathcal{V} = \{1, \dots, N\}$ and $\mathcal{V}_{\text{pin}} \subset \mathcal{V}$ denote the sets of total agents and pinned agents, respectively, $D = \text{diag}(d_1, \dots, d_N)$, where $d_i > 0$ if $i \in \mathcal{V}_{\text{pin}}$ and, otherwise, $d_i = 0$, and $\mathcal{G}$ and L be the interaction digraph and the Laplacian matrix, respectively.

A. Minimum Number of Pinned Agents

In this section, we discuss what kind of agents must be pinned such that $\mathcal{J}(L + D) > 0$ always holds.

Proposition 1: Assume that the multiplicity of the zero eigenvalue of L is equal to m. Then, the interaction digraph $\mathcal{G}$ can be partitioned into m components, where each component has a directed tree. To ensure that $\mathcal{J}(L + D) > 0$, at least one root node in each component should be pinned, which means that the minimum number of pinned agents is m.

Proof: By [27, Prop. 2] and condition 4 in Lemma 1 of this paper, one can complete the proof. ■

Remark 5: Proposition 1 is an intuitive yet nontrivial result. The augmented digraph formed by the leader and the followers should contain a directed spanning tree such that the leader can directly or indirectly affect all followers. Otherwise, there must exist a subsystem which is not influenced by the leader, and generally speaking, the multiagent system will not agree on a homogeneous state.

Remark 6: Chen *et al.* [17] and Lu *et al.* [18] showed that one may pin a network with a minimum number controllers by induction based on the properties of Frobenius form. In this paper, Proposition 1 provides an M-matrix approach to analyze the pinning control of networked systems.

B. Effect of Pinning Feedback Gains

Now, we study the effect of pinning feedback gains on $\mathcal{J}(L + D)$.

Proposition 2: Suppose that the pinned-agent set $\mathcal{V}_{\text{pin}}$ is designed such that $L + D$ is an M-matrix. Then, $\mathcal{J}(L + D)$ does not decrease when any pinning feedback gain d_i $(i \in \mathcal{V}_{\text{pin}})$ is increased. Moreover, $\mathcal{J}(L + D) \leq \mathcal{J}(L(\mathcal{V} \setminus \mathcal{V}_{\text{pin}}))$ always holds even if the pinning feedback gains are chosen to be sufficiently large.

Proof: Let $D = \text{diag}(d_1, \dots, d_N)$ and $\widehat{D} = \text{diag}(\widehat{d}_1, \dots, \widehat{d}_N)$, where $d_i > 0$ and $\widehat{d}_i > 0$ if $i \in \mathcal{V}_{\text{pin}}$ and, otherwise, $d_i = \widehat{d}_i = 0$. Assume that $\widehat{D} \geq D$ holds. Then, it follows that $L + \widehat{D} \in Z_N$ and $L + \widehat{D} \geq L + D$. Considering that $L + D$ is an M-matrix, we have $\mathcal{J}(L + \widehat{D}) \geq \mathcal{J}(L + D)$ by Lemma 4. Hence, we can conclude that $\mathcal{J}(L + D)$ does not decrease when any pinning feedback gain d_i $(i \in \mathcal{V}_{\text{pin}})$ is increased. It is easy to see that $L(\mathcal{V} \setminus \mathcal{V}_{\text{pin}})$, which is the Laplacian submatrix formed by unpinned agents, is a principal submatrix of $L + D$. Since $L + D$ is an M-matrix, by Lemma 5, we know that $L(\mathcal{V} \setminus \mathcal{V}_{\text{pin}})$ is an M-matrix and $\mathcal{J}(L + D) \leq \mathcal{J}(L(\mathcal{V} \setminus \mathcal{V}_{\text{pin}}))$ always holds. ■

Remark 7: From Proposition 2, we know that it may be easier to pin a network by increasing pinning feedback gains. Up to date, this result is confirmed by some previous work *only* through simulation studies, and now, we give its analytical proof in Proposition 2. Note that $\mathcal{J}(L(\mathcal{V} \setminus \mathcal{V}_{\text{pin}}))$ is a positive constant determined by the Laplacian submatrix formed by *unpinned* agents. By Proposition 2, $\mathcal{J}(L + D) \leq \mathcal{J}(L(\mathcal{V} \setminus \mathcal{V}_{\text{pin}}))$ always holds. Hence, choosing too large pinning feedback gains will not significantly improve the control performance. Instead, we can design relatively lower pinning feedback gains to satisfy consensus condition (8).

C. Effect of the Number of Pinned Agents

For a fixed pinned-agent set $\mathcal{V}_{\text{pin}}$, by Proposition 2, we know that $\mathcal{J}(L+D) \leq \mathcal{J}(L(\mathcal{V} \setminus \mathcal{V}_{\text{pin}}))$ always holds even if the pinning feedback gains are sufficiently large. Then, if $\mathcal{J}(L(\mathcal{V} \setminus \mathcal{V}_{\text{pin}}))$ is very small, a quite large σ will be needed to satisfy condition (8). To solve this problem, we can increase the number of pinned agents to achieve leader-following consensus with a relatively lower σ.

Proposition 3: Suppose that $L+D$ is an M-matrix. Then, $\mathcal{J}(L+D)$ does not decrease if more agents are pinned.

Proof: Let $\mathcal{D}^* = \{d_i^* > 0 : i \in \mathcal{V}\}$. Let $\mathcal{V}_{\text{pin}}$ and $\widehat{\mathcal{V}}_{\text{pin}}$ be two sets of pinned agents satisfying $\mathcal{V}_{\text{pin}} \subset \widehat{\mathcal{V}}_{\text{pin}}$. Define $D = \text{diag}(d_1, \ldots, d_N)$, where $d_i = d_i^* > 0$ if $i \in \mathcal{V}_{\text{pin}}$ and $d_i = 0$ if $i \in \mathcal{V} \setminus \mathcal{V}_{\text{pin}}$, and $\widehat{D} = \text{diag}(\widehat{d}_1, \ldots, \widehat{d}_N)$, where $\widehat{d}_i = d_i^* > 0$ if $i \in \widehat{\mathcal{V}}_{\text{pin}}$ and, otherwise, $\widehat{d}_i = 0$. Then, we have $L + \widehat{D} \geq L + D$ considering $\mathcal{V}_{\text{pin}} \subset \widehat{\mathcal{V}}_{\text{pin}}$. Since $L+D$ is an M-matrix, by Lemma 4, we know that $L+\widehat{D}$ is also an M-matrix and $\mathcal{J}(L+\widehat{D}) \geq \mathcal{J}(L+D)$ always holds. Hence, $\mathcal{J}(L+D)$ does not decrease if more agents are pinned. ■

Remark 8: Proposition 3 shows that it may be easier to control a network if we pin more agents. Until now, *no theoretical analysis* has been provided for this result. In this paper, we have solved this problem by using the M-matrix theory.

Remark 9: The consensus condition (8) is equivalent to $\sigma > \rho/\mathcal{J}(L+D)$. Then, a relatively lower σ for practical use may be obtained to achieve leader-following consensus by increasing the number of pinned agents.

D. Specific Selection of Pinned Agents

Section IV-A addresses what kind of agents must be pinned. Section IV-C shows that one may increase $\mathcal{J}(L+D)$ by pinning more agents. In this section, we discuss what kind of agents can be chosen as pinned candidates and what kind of agents may be left unpinned.

Lemma 10 [32]: Consider a matrix $A = (a_{ij}) \in Z_N$ satisfying $a_{ii} \geq \sum_{j=1, j\neq i}^{N} |a_{ij}|$ for all $i \in \mathcal{I}$ with the inequality strict for $i \in J$, where $\mathcal{I} = \{1, \ldots, N\}$ and $J = \{i \in \mathcal{I} : a_{ii} > \sum_{j=1, j\neq i}^{N} |a_{ij}|\} \neq \varnothing$. Matrix A is an M-matrix if, for each $i \in \mathcal{I} \setminus J$, there is a sequence of indices $i_1, i_2, \ldots, i_r$ such that $a_{i,i_1}, a_{i_1,i_2}, \ldots, a_{i_r,j}$ are all nonzero with $j \in J$.

Proposition 4: For a multiagent system, assume that the pinned-agent set and pinning feedback gains are designed according to the following two conditions.

1) For all $i \in \mathcal{V}_{\text{pin}}$, $\deg_{\text{out}}(i) > \deg_{\text{in}}(i)$ and $d_i = (\deg_{\text{out}}(i) - \deg_{\text{in}}(i))/2 + \epsilon/2 > 0$ with $\epsilon > 0$, and $\deg_{\text{in}}(i) \geq \deg_{\text{out}}(i)$ for all $i \in \mathcal{V} \setminus \mathcal{V}_{\text{pin}}$.
2) For each $i \in \mathcal{V} \setminus \mathcal{V}_{\text{pin}}$, one can always find agent $j \in \mathcal{V}_{\text{pin}}$ such that there is a directed path from agent j to agent i.

Define $G = -L$. Let $\overline{G} = (G + G^{\text{T}})/2$, and $\overline{G}_l$ be the minor matrix of $\overline{G}$ by removing its row–column pairs indexed by $\mathcal{V}_{\text{pin}}$. Then, one has $\mathcal{J}(L+D) > 0$ and $\lambda_{\max}(\overline{G}_l) < 0$.

Proof: We first show that $\mathcal{J}(L+D) > 0$. Let $M = L + D = (m_{ij})_{N\times N}$. It is easy to see that $m_{ij} = l_{ij} \leq 0$ for all $i \neq j$ and $m_{ii} = l_{ii} + d_i$, where $d_i > 0$ if $i \in \mathcal{V}_{\text{pin}}$ and $d_i = 0$ if $i \in \mathcal{V} \setminus \mathcal{V}_{\text{pin}}$. Hence, $M \in Z_N$. By condition 1 and (1), we have $m_{ii} = \sum_{j=1, j\neq i}^{N} |m_{ij}|$ for each $i \in \mathcal{V} \setminus \mathcal{V}_{\text{pin}}$ and $m_{ii} > \sum_{j=1, j\neq i}^{N} |m_{ij}|$ for each $i \in \mathcal{V}_{\text{pin}}$. From condition 2, for any agent $i \in \mathcal{V} \setminus \mathcal{V}_{\text{pin}}$, there always exists a sequence of indices $i_1, i_2, \ldots, i_r$ such that $l_{i,i_1}, l_{i_1,i_2}, \ldots, l_{i_r,j}$ are all nonzero elements with $j \in \mathcal{V}_{\text{pin}}$. This follows that $m_{i,i_1}, m_{i_1,i_2}, \ldots, m_{i_r,j}$ are all nonzero. Then, by Lemma 10, we can conclude that M is an M-matrix, which means that $\mathcal{J}(L+D) = \mathcal{J}(M) > 0$.

Now, we prove the second statement. Let $H = (L + D) + (L+D)^{\text{T}} = (h_{ij})_{N\times N}$. It is easy to see that $h_{ij} = l_{ij} + l_{ji} \leq 0$ for all $i \neq j$ and $h_{ii} = 2l_{ii} + 2d_i$. By (1) and (2), we have $h_{ii} = 2\deg_{\text{in}}(i) + 2d_i$ and $\sum_{j=1, j\neq i}^{N} |h_{ij}| = \deg_{\text{in}}(i) + \deg_{\text{out}}(i)$. Then, according to condition 1, we obtain $h_{ii} \geq \sum_{j=1, j\neq i}^{N} |h_{ij}|$ for each $i \in \mathcal{V} \setminus \mathcal{V}_{\text{pin}}$ and $h_{ii} - \sum_{j=1, j\neq i}^{N} |h_{ij}| = \epsilon > 0$ for each $i \in \mathcal{V}_{\text{pin}}$. In view of condition 2 and Lemma 10, we can show that H is an M-matrix following the same line in the proof for the first statement. Let $\widetilde{H} = H(\mathcal{V} \setminus \mathcal{V}_{\text{pin}})$, which is a principal submatrix of H. According to Lemma 5, we obtain $\mathcal{J}(\widetilde{H}) \geq \mathcal{J}(H) > 0$. It follows that $\widetilde{H}$ is positive definite since $\widetilde{H}$ is symmetric. Also, we can easily verify that $\widetilde{H} = (L + L^{\text{T}})_l$, which is the minor matrix of H by removing its row–column pairs indexed by $\mathcal{V}_{\text{pin}}$. Then, we know that $-\widetilde{H} = (G + G^{\text{T}})_l$ is negative definite considering $G = -L$, which indicates that $\lambda_{\max}(\overline{G}_l) < 0$. ■

Remark 10: From Proposition 4, we can conclude that the agents whose out-degrees are bigger than their in-degrees can be chosen as pinned candidates and the agents whose out-degrees are less than their in-degrees may be left *unpinned*.

Remark 11: Song and Cao [19] showed that $\lambda_{\max}(\overline{G}_l) \leq 0$ if one pins the nodes whose out-degrees are bigger than their in-degrees. In Proposition 4, we provide two conditions to ensure that $\lambda_{\max}(\overline{G}_l) < 0$ always holds and hence improve the results in [19].

Remark 12: Condition 2 of Proposition 4 implies that the virtual leader has a directed path to every follower.

E. Selective Pinning Scheme

Here, we present the following selective pinning scheme for reaching leader-following consensus in multiagent system (5) such that the number of pinned agents is as small as possible and the parameter σ and the pinning feedback gains $d_i (i \in \mathcal{V}_{\text{pin}})$ are acceptable for practical use.

1) Partition the interaction digraph of the multiagent system into m components, where each component contains a directed tree. Initialize the pinned-agent set $\mathcal{V}_{\text{pin}}$ with m root nodes of these components.
2) Rearrange the remaining $N - m$ agents in descending order according to the differences of their out-degrees and in-degrees.
3) Compute $\sigma^* = \rho/\mathcal{J}(L(\mathcal{V} \setminus \mathcal{V}_{\text{pin}}))$ for multiagent system (5). If σ^* is too large, add more agents to the pinned-agent set $\mathcal{V}_{\text{pin}}$ until σ^* is satisfactory. Let $\sigma = \sigma^* + \epsilon$ with $\epsilon > 0$, and choose appropriate pinning feedback gains.
4) If consensus condition (8) is not satisfied, either gradually increase the pinning feedback gains or continually add more agents to $\mathcal{V}_{\text{pin}}$ until the consensus condition holds.

Remark 13: The M-matrix strategies developed in this section can be used to analyze the pinning control of both first- and second-order nonlinear multiagent systems and greatly improve our preliminary results in [27].

V. Numerical Results

In this section, a simulation example is given to verify our pinning control techniques for achieving leader-following consensus in second-order nonlinear multiagent systems.

Consider a multiagent system consisting of seven agents described by

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= f(t, x_i, v_i) + u_i, \qquad i = 1, \ldots, 7 \end{aligned} \tag{13}$$

where $x_i = (x_{i1}, x_{i2})^{\mathrm{T}}$ and $v_i = (v_{i1}, v_{i2})^{\mathrm{T}}$ are the position and velocity states of agent i, respectively, the nonlinear function f is given by

$$f(t, x_i, v_i) = Ag(x_i) + Bh(v_i) - v_i \tag{14}$$

in which

$$A = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -0.1 \\ -4 & 3 \end{pmatrix}$$

$$g(x_i) = 0.5\left(|x_{i1}+1| - x_{i1} - 1|, |x_{i2}+1| - |x_{i2}-1|\right)^{\mathrm{T}}$$

$$h(v_i) = \left(\tanh(v_{i1}), \tanh(v_{i2})\right)^{\mathrm{T}}.$$

After some simple calculations, we obtain the following two matrices such that the nonlinear function f in (14) satisfies Assumption 1:

$$M = \begin{pmatrix} 1.5 & 0.1 \\ 0.2 & 2 \end{pmatrix} \qquad W = \begin{pmatrix} 1 & 0.1 \\ 4 & 2 \end{pmatrix}.$$

By (7), we get $\rho_1 = 6.2$, $\rho_2 = 6.35$, and $\rho = 7.35$. The interaction digraph of multiagent system (13) is shown in Fig. 1 (see Remark 3 in Section III). It is easy to verify that Laplacian matrix L has a simple zero eigenvalue, which indicates that $m = 1$ by Proposition 1. Note that agent 1 is the *only* root node and thus should be pinned. Hence, we can let $\mathcal{V}_{\mathrm{pin}} = \{1\}$, which ensures that the virtual leader has a directed path to every follower. Let $d_1 = 1.0$ be the pinning feedback gain and, choose $\sigma = 65$. We can see that the consensus condition (8) holds when agent 1 is pinned

$$\mathcal{J}(L + D) = 0.1168 > \rho/\sigma = 7.35/65 = 0.1131.$$

With pinning control algorithm (5), the evolutions of positions and velocities of seven agents are shown in Figs. 2 and 3, respectively. Note that the multiagent system composed of seven nonlinear agents reaches second-order leader-following consensus by *only* pinning a single agent.

Remark 14: According to Remark 3, at least two agents, such as agents 1 and 5, need to be pinned in order to satisfy the leader-following consensus condition in [25] for multiagent system (13) with consensus algorithm (5). In this paper, consensus condition (8) holds if agent 1 is pinned, which improves the consensus condition in [25].

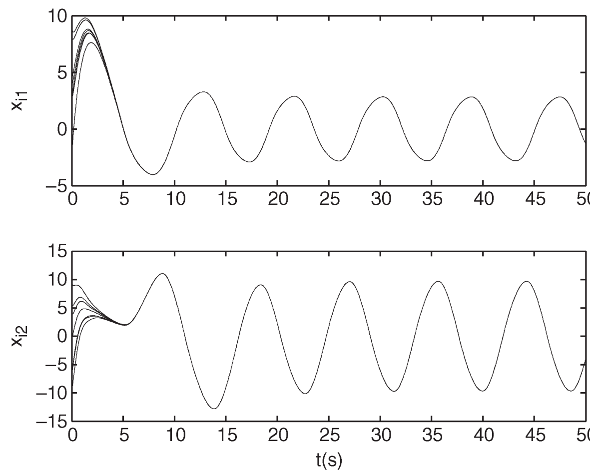

Fig. 2. Position trajectories of seven agents with nonlinear dynamics.

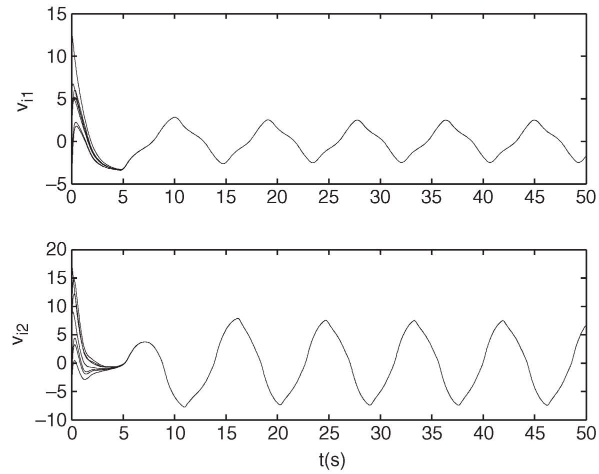

Fig. 3. Velocity trajectories of seven agents with nonlinear dynamics.

VI. Conclusion

In this paper, we have derived some simple conditions for reaching leader-following consensus in second-order multiagent systems with nonlinear dynamics. Some M-matrix strategies have been developed to provide theoretical analysis for several challenging issues in the pinning control of multiagent systems. By partitioning the interaction digraph into a minimum number of components, a selective pinning scheme has been presented to solve the leader-following consensus problem for nonlinear second-order multiagent systems with directed topologies. Numerical simulations have validated the effectiveness of the proposed techniques. It is worth mentioning that the theoretical results in this paper significantly improve the results in [19], [25], and [27] and generalize some previous results on pinning control of networked systems. In the future, we will consider extending the theoretical analysis in this paper to study the consensus problem for higher order multiagent systems with more general dynamics.

APPENDIX

A. Proof of Lemma 9

Proof: Let $z = (y_1^2, \ldots, y_n^2)^{\mathrm{T}}$. Recall that A is nonnegative. By the fact that $2ab \le a^2 + b^2, \forall a, b \in \mathbb{R}$, we have

$$\begin{aligned}
x^{\mathrm{T}} A y &= \sum_{j=1}^{n}\sum_{k=1}^{n} x_j a_{jk} y_k \\
&\le \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} x_j^2 + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} y_k^2 \\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} x_j^2 + \frac{1}{2} 1_n^{\mathrm{T}}(Az) \\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} x_j^2 + \frac{1}{2} z^{\mathrm{T}}(A^{\mathrm{T}} 1_n) \\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} x_j^2 + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{kj} y_j^2 \\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \left(a_{jk} x_j^2 + a_{kj} y_j^2\right).
\end{aligned}$$

The proof is thus completed. ∎

B. Proof of Theorem 1

Proof: Let $\widehat{x}_i = x_i - x_r, \widehat{v}_i = v_i - v_r, i = 1, \ldots, N$. From (4) and (5), we have the following error dynamical system:

$$\begin{aligned}
\dot{\widehat{x}}_i(t) &= \hat{v}_i \\
\dot{\widehat{v}}_i(t) &= f(t, x_i, v_i) - f(t, x_r, v_r) \\
&\quad - \sigma \sum_{j=1}^{N} l_{ij}(\widehat{x}_j + \widehat{v}_j) - \sigma d_i(\widehat{x}_i + \widehat{v}_i), \quad i = 1, \ldots, N
\end{aligned} \tag{15}$$

where l_{ij} is the (i, j)th entry of the Laplacian matrix L defined by (1).

Let

$$\begin{aligned}
\widehat{x} &= \left(\widehat{x}_1^{\mathrm{T}}, \ldots, \widehat{x}_N^{\mathrm{T}}\right)^{\mathrm{T}} \qquad \widehat{v} = \left(\widehat{v}_1^{\mathrm{T}}, \ldots, \widehat{v}_N^{\mathrm{T}}\right)^{\mathrm{T}} \\
F(t, x, v) &= \left(f^{\mathrm{T}}(t, x_1, v_1), \ldots, f^{\mathrm{T}}(t, x_N, v_N)\right)^{\mathrm{T}}.
\end{aligned}$$

Rewrite (15) as

$$\begin{aligned}
\dot{\widehat{x}}(t) &= \widehat{v} \\
\dot{\widehat{v}}(t) &= F(t, x, v) - 1_N \otimes f(t, x_r, v_r) \\
&\quad - \sigma\left((L + D) \otimes I_n\right)(\widehat{x} + \widehat{v}).
\end{aligned} \tag{16}$$

Let λ be any eigenvalue of $L + D$. If the virtual leader has a directed path to every follower, then $L + D$ is an M-matrix according to Lemma 6. By Lemma 1, $\mathrm{Re}(\lambda) > 0$ holds. It is easy to see that $\sigma\lambda - \rho$ is an eigenvalue of $\sigma(L + D) - \rho I_N$. From condition (8), we have $\mathrm{Re}(\sigma\lambda - \rho) > 0$, which indicates that $\sigma(L + D) - \rho I_N$ is an M-matrix by Lemma 1. Then, there exists a positive definite diagonal matrix $\Xi = \mathrm{diag}(\xi_1, \ldots, \xi_N) > 0$ such that

$$\Xi\left(\sigma(L + D) - \rho I_N\right) + \left(\sigma(L + D) - \rho I_N\right)^{\mathrm{T}} \Xi \succ 0. \tag{17}$$

Let $\widehat{y}(t) = \left(\widehat{x}^{\mathrm{T}}(t), \widehat{v}^{\mathrm{T}}(t)\right)^{\mathrm{T}}$. Construct the following Lyapunov functional candidate:

$$V(t) = \frac{1}{2}\widehat{y}^{\mathrm{T}}(t)(\Omega \otimes I_n)\widehat{y}(t) \tag{18}$$

where $\Omega = \begin{pmatrix} \sigma(\Xi(L + D) + (L + D)^{\mathrm{T}}\Xi) & \Xi \\ \Xi & \Xi \end{pmatrix}$.

Now, we show that the Lyapunov functional candidate in (18) is valid, which means that Ω is positive definite, i.e., $\Omega \succ 0$. It is obvious that Ω is symmetric. By Lemma 7, we know that $\Omega \succ 0$ is equivalent to $\sigma(\Xi(L + D) + (L + D)^{\mathrm{T}}\Xi) - \Xi \succ 0$. From (7) and (17), we have $\sigma(\Xi(L + D) + (L + D)^{\mathrm{T}}\Xi) \succ 2\rho\Xi \succ \Xi$ considering $\rho \ge 1$. Hence, we can conclude that $\Omega \succ 0$.

Rewrite (18) as

$$\begin{aligned}
V(t) &= \frac{\sigma}{2}\widehat{x}^{\mathrm{T}}(t)\left[\left(\Xi(L + D) + (L + D)^{\mathrm{T}}\Xi\right) \otimes I_n\right]\widehat{x}(t) \\
&\quad + \widehat{x}^{\mathrm{T}}(t)(\Xi \otimes I_n)\widehat{v}(t) + \frac{1}{2}\widehat{v}^{\mathrm{T}}(t)(\Xi \otimes I_n)\widehat{v}(t).
\end{aligned} \tag{19}$$

In view of Lemma 8, the time derivative of $V(t)$ along the trajectory (16) is given by

$$\begin{aligned}
\dot{V}(t) &= \sigma\widehat{x}^{\mathrm{T}}\left[\left(\Xi(L + D) + (L + D)^{\mathrm{T}}\Xi\right) \otimes I_n\right]\widehat{v} \\
&\quad + \widehat{v}^{\mathrm{T}}(\Xi \otimes I_n)\widehat{v} \\
&\quad + \widehat{x}^{\mathrm{T}}(\Xi \otimes I_n)\left[F(t, x, v) - 1_N \otimes f(t, x_r, v_r)\right. \\
&\qquad \left. - \sigma\left((L + D) \otimes I_n\right)(\widehat{x} + \widehat{v})\right] \\
&\quad + \widehat{v}^{\mathrm{T}}(\Xi \otimes I_n)\left[F(t, x, v) - 1_N \otimes f(t, x_r, v_r)\right. \\
&\qquad \left. - \sigma\left((L + D) \otimes I_n\right)(\widehat{x} + \widehat{v})\right] \\
&= -\sigma\widehat{x}^{\mathrm{T}}\left[(\Xi(L + D)) \otimes I_n\right]\widehat{x} \\
&\quad - \widehat{v}^{\mathrm{T}}\left[(\sigma\Xi(L + D) - \Xi) \otimes I_n\right]\widehat{v} \\
&\quad + \sum_{i=1}^{N} \xi_i\left(\widehat{x}_i^{\mathrm{T}} + \widehat{v}_i^{\mathrm{T}}\right)\left(f(t, x_i, v_i) - f(t, x_r, v_r)\right).
\end{aligned} \tag{20}$$

Let $\widehat{p}_i = (|\widehat{x}_{i1}|, \ldots, |\widehat{x}_{\mathrm{in}}|)^{\mathrm{T}}$ and $\widehat{q}_i = (|\widehat{v}_{i1}|, \ldots, |\widehat{v}_{\mathrm{in}}|)^{\mathrm{T}}$. By (7), Assumption 1, and Lemma 9, we have

$$\begin{aligned}
&\sum_{i=1}^{N} \xi_i\left(\widehat{x}_i^{\mathrm{T}} + \widehat{v}_i^{\mathrm{T}}\right)\left(f(t, x_i, v_i) - f(t, x_r, v_r)\right) \\
&\le \sum_{i=1}^{N} \xi_i\left(\widehat{p}_i^{\mathrm{T}} W \widehat{p}_i + \widehat{p}_i^{\mathrm{T}} M \widehat{q}_i + \widehat{q}_i^{\mathrm{T}} W \widehat{p}_i + \widehat{q}_i^{\mathrm{T}} M \widehat{q}_i\right) \\
&\le \frac{1}{2}\sum_{i=1}^{N} \xi_i\left[\sum_{j=1}^{n}\sum_{k=1}^{n}(w_{jk} + m_{jk} + 2w_{kj})\widehat{p}_{ij}^2\right] \\
&\quad + \frac{1}{2}\sum_{i=1}^{N} \xi_i\left[\sum_{j=1}^{n}\sum_{k=1}^{n}(m_{jk} + w_{jk} + 2m_{kj})\widehat{q}_{ij}^2\right] \\
&\le \rho_1\sum_{i=1}^{N} \xi_i \sum_{j=1}^{n} \widehat{x}_{ij}^2 + \rho_2\sum_{i=1}^{N} \xi_i \sum_{j=1}^{n} \widehat{v}_{ij}^2.
\end{aligned} \tag{21}$$

From (20) and (21), we obtain

$$\begin{aligned}
\dot{V}(t) &\le -\sigma\widehat{x}^{\mathrm{T}}\left[(\Xi(L + D)) \otimes I_n\right]\widehat{x} \\
&\quad - \widehat{v}^{\mathrm{T}}\left[(\sigma\Xi(L + D) - \Xi) \otimes I_n\right]\widehat{v} \\
&\quad + \rho_1\widehat{x}^{\mathrm{T}}(\Xi \otimes I_n)\widehat{x} + \rho_2\widehat{v}^{\mathrm{T}}(\Xi \otimes I_n)\widehat{v} \\
&= -\widehat{x}^{\mathrm{T}}\left[(\sigma\Xi(L + D) - \rho_1\Xi) \otimes I_n\right]\widehat{x} \\
&\quad - \widehat{v}^{\mathrm{T}}\left[(\sigma\Xi(L + D) - (\rho_2 + 1)\Xi) \otimes I_n\right]\widehat{v} \\
&= -\frac{1}{2}\widehat{x}^{\mathrm{T}}(P \otimes I_n)\widehat{x} - \frac{1}{2}\widehat{v}^{\mathrm{T}}(Q \otimes I_n)\widehat{v}
\end{aligned} \tag{22}$$

where $P = \sigma(\Xi(L+D) + (L+D)^{\mathrm{T}}\Xi) - 2\rho_1\Xi$ and $Q = \sigma(\Xi(L+D) + (L+D)^{\mathrm{T}}\Xi) - 2(\rho_2+1)\Xi$.

From (17), we know that $\sigma(\Xi(L+D) + (L+D)^{\mathrm{T}}\Xi) - 2\rho\Xi \succ 0$. Recall that $\rho = \max\ \{\rho_1, \rho_2+1\}$. Therefore, matrices P and Q are both positive definite. In view of (22), we know that $\dot{V}(t) \leq 0$ and $\dot{V}(t) \equiv 0$ holds if and only if $\widehat{x}(t) = 0_{Nn}$ and $\widehat{v}(t) = 0_{Nn}$. By LaSalle's invariance principle [20], it follows that $\widehat{x}(t) \to 0_{Nn}$ and $\widehat{v}(t) \to 0_{Nn}$ as $t \to \infty$. Hence, all agents in multiagent system (5) globally asymptotically approach virtual leader (4), i.e., $x_i(t) \to x_r(t)$ and $v_i(t) \to v_r(t)$, $i = 1, \ldots, N$, as $t \to \infty$. ■

REFERENCES

[1] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[2] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[3] W. Ren, "Multi-vehicle consensus with a time-varying reference state," *Syst. Control Lett.*, vol. 56, no. 7/8, pp. 474–483, Jul. 2007.

[4] M. Porfiri and D. J. Stilwell, "Consensus seeking over random weighted directed graphs," *IEEE Trans. Autom. Control*, vol. 52, no. 9, pp. 1767–1773, Sep. 2007.

[5] F. Chen, Z. Chen, L. Xiang, Z. Liu, and Z. Yuan, "Reaching a consensus via pinning control," *Automatica*, vol. 45, no. 5, pp. 1215–1220, May 2009.

[6] L. Wang and F. Xiao, "Finite-time consensus problems for networks of dynamic agents," *IEEE Trans. Autom. Control*, vol. 55, no. 4, pp. 950–955, Apr. 2010.

[7] W. Yu, G. Chen, and M. Cao, "Consensus in directed networks of agents with nonlinear dynamics," *IEEE Trans. Autom. Control*, vol. 56, no. 6, pp. 1436–1441, Jun. 2011.

[8] W. Ren and E. Atkins, "Distributed multi-vehicle coordinated control via local information exchange," *Int. J. Robust Nonlin. Control*, vol. 17, no. 10/11, pp. 1002–1033, Jul. 2007.

[9] J. Hu and Y. Hong, "Leader-following coordination of multi-agent systems with coupling time delays," *Phys. A, Stat. Mech. Appl.*, vol. 374, no. 2, pp. 853–863, Feb. 2007.

[10] W. Ren, "On consensus algorithms for double-integrator dynamics," *IEEE Trans. Autom. Control*, vol. 53, no. 6, pp. 1503–1509, Jul. 2008.

[11] Y. Tian and C. Liu, "Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations," *Automatica*, vol. 45, no. 5, pp. 1347–1353, May 2009.

[12] W. Yu, G. Chen, and M. Cao, "Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems," *Automatica*, vol. 46, no. 6, pp. 1089–1095, Jun. 2010.

[13] W. Zhu and D. Cheng, "Leader-following consensus of second-order agents with multiple time-varying delays," *Automatica*, vol. 46, no. 12, pp. 1994–1999, Dec. 2010.

[14] Z. Meng, W. Ren, Y. Cao, and Z. You, "Leaderless and leader-following consensus with communication and input delays under a directed network topology," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 41, no. 1, pp. 75–88, Feb. 2011.

[15] J. Qin, W. X. Zheng, and H. Gao, "Coordination of multiple agents with double-integrator dynamics under generalized interaction topologies," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 42, no. 1, pp. 44–57, Feb. 2012.

[16] X. F. Wang and G. Chen, "Pinning control of scale-free dynamical networks," *Phys. A, Stat. Mech. Appl.*, vol. 310, no. 3/4, pp. 521–531, Jul. 2002.

[17] T. Chen, X. Liu, and W. Lu, "Pinning complex networks by a single controller," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 54, no. 6, pp. 1317–1326, Jun. 2007.

[18] J. Lu, D. W. C. Ho, and Z. Wang, "Pinning stabilization of linearly coupled stochastic neural networks via minimum number of controllers," *IEEE Trans. Neural Netw.*, vol. 20, no. 10, pp. 1617–1629, Oct. 2009.

[19] Q. Song and J. Cao, "On pinning synchronization of directed and undirected complex dynamical networks," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 57, no. 3, pp. 672–680, Mar. 2010.

[20] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Englewood Cliffs, NJ: Prentice-Hall, 2002.

[21] P. Amster and M. C. Mariani, "Some results on the forced pendulum equation," *Nonlin. Anal., Theory, Methods, Appl.*, vol. 68, no. 7, pp. 1874–1880, Apr. 2008.

[22] W. Ren, "Synchronization of coupled harmonic oscillators with local interaction," *Automatica*, vol. 44, no. 12, pp. 3195–3200, Dec. 2008.

[23] H. Su, X. F. Wang, and Z. Lin, "Synchronization of coupled harmonic oscillators in a dynamic proximity network," *Automatica*, vol. 45, no. 10, pp. 2286–2291, Oct. 2009.

[24] W. Yu, G. Chen, M. Cao, and J. Kurths, "Second-order consensus for multiagent systems with directed topologies and nonlinear dynamics," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 40, no. 3, pp. 881–891, Jun. 2010.

[25] Q. Song, J. Cao, and W. Yu, "Second-order leader-following consensus of nonlinear multi-agent systems via pinning control," *Syst. Control Lett.*, vol. 59, no. 9, pp. 553–562, Sep. 2010.

[26] H. Su, G. Chen, X. F. Wang, and Z. Lin, "Adaptive second-order consensus of networked mobile agents with nonlinear dynamics," *Automatica*, vol. 47, no. 2, pp. 368–375, Feb. 2011.

[27] Q. Song, F. Liu, J. Cao, and W. Yu, "Pinning-controllability analysis of complex networks: An M-matrix approach," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 59, no. 11, pp. 2692–2701, Nov. 2012.

[28] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1991.

[29] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. Philadelphia, PA: SIAM, 1994.

[30] W. Ren and Y. Cao, *Distributed Coordination of Multi-agent Networks: Emergent Problems, Models, and Issues*. London, U.K.: Springer-Verlag, 2011.

[31] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM, 1994.

[32] P. N. Shivakumar and K. H. Chew, "A sufficient condition for nonvanishing of determinants," *Proc. Amer. Math. Soc.*, vol. 43, no. 1, pp. 63–66, Mar. 1974.

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