Containment of Higher-Order Multi-Leader Multi-Agent Systems: A Dynamic Output Approach

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***Abstract*—This technical note addresses the distributed containment control problem for a linear multi-leader multi-agent system with a directed communication topology. A new class of distributed observer-type containment protocols based only on the relative output measurements of the neighboring agents is proposed, removing the impractical assumption in some of the existing approaches that the observers embedded in the multiple dynamic agents have to share information with their neighbors. Under the mild assumption that, for each follower, there exists at least one leader having a directed path to that follower, some sufficient conditions are derived to guarantee the states of the followers to asymptotically converge to a convex hull formed by those of the dynamic leaders. Finally, some numerical simulations on containment of a multi-vehicle system are given to verify the effectiveness of the theoretical results.**

***Index Terms*—Communication, containment control, multi-leader system, multi-vehicle system, networked system, observer-type protocol.**

I. Introduction

There has been tremendous amount of research to date on consensus control of networked multi-agent systems, due to its potential applications in distributed sensor networks, unmanned air vehicles (UAVs), automated highway systems, and so forth (see [1]–[4], and the references therein). Many researchers have contributed to the understanding of this topic with fruitful results, including consensus in first- and second-order multi-agent systems [5]–[8], higher-order consensus [9]–[11] and consensus of networked agents with general linear node dynamics [12]–[14], to name just a few.

The above-mentioned works are primarily concerned with consensus in networks of agents without any leader. In practice, the introduction of a single leader or even multiple leaders can broaden the applications by guaranteeing all the individual dynamics converge onto a desired trajectory or to enter a prescribed region. Consensus tracking problem with a single stationary leader under switching undirected communication topologies was first investigated in [15] by using tools from algebraic graph theory. In [16] and [17], a local controller together with a neighbor-based state-estimation rule were designed to solve the consensus tracking problem with a dynamic leader. In [18], consensus tracking with delayed transmissions and a time-varying leader was addressed. Some novel distributed consensus tracking algorithms for second- and higher-order multi-agent systems with a single dynamic leader were developed in [19] and [20], respectively. In the presence of multiple leaders, a stop-and-go strategy was proposed in the pioneering work [21] to solve the so-called containment control problem under a fixed undirected network topology, where all the followers are eventually evolving into the convex hull spanned by the leaders. In [22], containment control for first-order multi-agent systems was studied for cases with stationary or dynamic leaders under directed switching communication topologies. Containment control for multi-agent systems with double integrator dynamics under a directed network topology was introduced and considered in [23]. By using tools from nonsmooth control theory, finite-time containment control problem for a group of rigid bodies described by second-order Euler-Lagrange equations under a time-invariant undirected topology was addressed in [24]. Most of the aforementioned references are previously assumed that the relative state measurements of neighboring agents can be used for coordination. However, there may exist some situations where the full state information of dynamic agents are unavailable. Hence, it is practically important to solve the containment problem via an output feedback approach. Motivated by this observation, containment control for a group of second-order multi-agent systems has been addressed in [25] by using only position measurements. Some new distributed observer-based containment protocols for multi-agent systems with general linear node dynamics were constructed in [26] and [27]. However, it is always assumed in most of the above-mentioned works that the observers embedded in the following agents can communicate with each other as the system evolves. Although these protocols are distributed, the states of the observers embedded in multiple agents are internal information which might be sometimes difficult or even impossible to detect and use in practice. It is thus interesting and important to explore some observer-type protocols relying only on the relative output measurements of neighboring agents.

Motivated by the aforementioned works, this technical note considers the containment control problem for linear multi-leader multi-agent systems under a directed communication topology from a dynamic output approach. To achieve containment, a new class of distributed observer-type containment protocols are constructed based only on the relative output measurements of neighboring agents, removing the common assumption that the observers embedded in the followers have to share information with each other. Under the assumption that each agent is stabilizable and detectable, it is theoretically proved by using tools from $\mathcal{H}_\infty$ control theory that the states of the followers will eventually evolve into the convex hull spanned by the states of the leaders if, for each follower, there exists at least one leader that has a directed path to that follower. Some simulations are finally performed to verify the analytical results.

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The rest of this brief is organized as follows. Some preliminaries and the model formulation are given in Section II. The main theoretical results are provided and proved in Section III. Simulations are presented in Section IV to verify the analytical results. Conclusions are finally drawn in Section V.

II. Preliminaries and Model Formulation

Some basic preliminaries and the problem formulation are provided in this section.

A. Preliminaries

The communication topology among N agents is represented by a weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, with a set of nodes $\mathcal{V} = \{1, 2, \ldots, N\}$, a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$, where $a_{ij} \geq 0$ for all $i, j \in \mathcal{V}$. Note that $(i, j) \in \mathcal{E}$ if and only if $a_{ij} > 0$. An edge $(i, j) \in \mathcal{E}$ in graph $\mathcal{G}$ means that agent j can receive information from agent i but not necessarily conversely. Furthermore, self-loops are not allowed which indicates that each agent can not obtain any absolute information about its states or outputs. The Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$ of graph $\mathcal{G}$ is defined as $l_{ii} = \sum_{j\neq i} a_{ij}$ and $l_{ij} = -a_{ij}, i \neq j$. A directed path from node i to j in $\mathcal{G}$ is a sequence of edges $(i, i_1), (i_1, i_2), \ldots, (i_k, j)$ in the directed network with distinct nodes i_s, $s = 1, 2, \ldots, k$ [28].

Throughout this technical note, let $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ be the $n\times m$ real and complex matrices space, respectively. The superscript T means transpose for real matrices and $*$ means conjugate transpose for complex matrices. Let I_n be the n-dimensional identity matrix and $0_{n\times m}$ be the $n\times m$ matrix with each entry being 0. Moreover, matrices are assumed to have compatible dimensions if not explicitly stated. A square matrix is said to be stable if and only if all of its eigenvalues have negative real parts. Notation $\otimes$ represents the Kronecker product. For a set $\Xi = \{\xi_1, \ldots, \xi_m\}$ with $\xi_i \in \mathbb{R}^n$, $i = 1, \ldots, m$, its convex hull $\mathrm{col}(\Xi)$ is defined as $\mathrm{col}(\Xi) = \{\sum_{i=1}^{m} p_i\xi_i | \xi_i \in \Xi, \sum_{i=1}^{m} p_i = 1, p_i \geq 0\}$.

B. Model Formulation

Consider a group of N agents with identical linear dynamics, which may be regarded as the linearized model of some nonlinear systems. The dynamics of agent i are described as

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t) \\ y_i(t) &= Cx_i(t) \end{aligned} \tag{1}$$

where $x_i(t) = (x_{i1}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n$ is the state, $u_i(t) \in \mathbb{R}^p$ is the control input, and $y_i(t) \in \mathbb{R}^q$ is the measured output. It is assumed that (A, B, C) is stabilizable and detectable.

Suppose that there are M $(1 < M < N)$ leaders and $N - M$ followers in system (1). Without loss of generality, it is assumed that the agents labeled by $1, \ldots, M$, are leaders, while those labeled by $M + 1, \ldots, N$, are followers. Denote the leader and follower sets by $\mathbb{L} \triangleq \{1, \ldots, M\}$ and $\mathbb{F} \triangleq \{M + 1, \ldots, N\}$, respectively. As the leaders always take the role of command generators that provide reference states for the followers, it is assumed that the states of each leader evolve without being effected by those of other leaders or followers, i.e., each leader has no neighbor. Then, the Laplacian matrix associated with $\mathcal{G}$ can be partitioned as

$$\mathcal{L} = \begin{bmatrix} 0_{M\times M} & 0_{M\times(N-M)} \\ \mathcal{L}_1 & \mathcal{L}_2 \end{bmatrix} \tag{2}$$

where $\mathcal{L}_1 \in \mathbb{R}^{(N-M)\times M}$ and $\mathcal{L}_2 \in \mathbb{R}^{(N-M)\times(N-M)}$.

It is further assumed that each agent can only access the relative output measurements between itself and its neighbors. The relative measurements of other agents with respect to each agent are synthesized into a single signal in the following form:

$$\begin{aligned} \varrho_i(t) &= \sum_{j=1}^{N} a_{ij} \left(y_i(t) - y_j(t)\right), \quad i \in \mathbb{F} \\ \varrho_i(t) &= 0, \quad i \in \mathbb{L}. \end{aligned} \tag{3}$$

As each leader i has no neighbor, it has no input, i.e., $u_i(t) \equiv 0, i \in \mathbb{L}$. The control goal here is to design distributed containment protocols to make the states of the followers converge asymptotically to the convex hull formed by those of the leaders. To do this, a distributed controller for each follower is proposed as

$$\begin{aligned} \dot{\upsilon}_i(t) &= (A + FC + BK)\upsilon_i(t) + F\varrho_i(t) \\ u_i(t) &= -K\upsilon_i(t), \quad i \in \mathbb{F} \end{aligned} \tag{4}$$

where $\upsilon_i(t) \in \mathbb{R}^n$ is the state of the observer embedded at agent i, $F \in \mathbb{R}^{n\times q}$, and $K \in \mathbb{R}^{p\times n}$ are the feedback gain matrices to be determined.

Remark 1: From (4), one can see that the present observer for each follower i, $i \in \mathbb{F}$, depends only on the states of this observer and the relative output measurements between follower i and the neighbors. It is thus easier to implement than those presented in some existing literature where the observers embedded at followers should communicate with each other. Furthermore, $\upsilon_i(t)$ is available to follower i if there is an internal local information transmission loop between the observer embedded at follower i and the actuator of follower i. Thus, one has no need to measure the feedback signal $\upsilon_i(t)$ by employing any external sensing devices. Since it is not hard to add such an information transmission loop between the observer and the actuator in practice, it is assumed in the present technical note that each follower i can utilize this local information for feedback. Note that this is also a common assumption in the existing works on distributed output feedback control of linear multi-agent systems [13], [27], [29].

Assumption 1: For each follower i, $i \in \mathbb{F}$, there exists at least one leader j, $j \in \mathbb{L}$ that has a directed path from it to that follower.

Lemma 1: [23]. Suppose that Assumption 1 holds. Then, all the eigenvalues of $\mathcal{L}_2$ have positive real parts, each entry of $-\mathcal{L}_2^{-1}\mathcal{L}_1$ is nonnegative, and each row of $-\mathcal{L}_2^{-1}\mathcal{L}_1$ has a sum equal to one.

Lemma 2: (Bounded Real Lemma [30]). Suppose that $G(s) = C(sI - A)^{-1}B$ with A being stable, and $\gamma > 0$ is a constant. Then, the following conditions are equivalent:

1) $\|G(s)\|_\infty < \gamma$;
2) There exists a $W > 0$ such that the algebraic Riccati inequality (ARI) holds: $A^TW + WA + C^TC + (1/\gamma^2)WBB^TW < 0$;
3) The Hamiltonian matrix H has no eigenvalues on the imaginary axis, where $H = \begin{bmatrix} A & (\frac{1}{\gamma^2})BB^T \\ -C^TC & -A^T \end{bmatrix}$.

III. Main Results

For notational brevity, let $x_l(t) = (x_1^T(t), \ldots, x_M^T(t))^T$ and $x_f(t) = (x_{M+1}^T(t), \ldots, x_N^T(t))^T$, respectively. Furthermore, let $\varepsilon_l(t) = (\varepsilon_1^T(t), \ldots, \varepsilon_M^T(t))^T$, $\varepsilon_f(t) = (\varepsilon_{M+1}^T(t), \ldots, \varepsilon_N^T(t))^T$, where $\varepsilon_i(t) = (x_i^T(t), 0_n^T)^T$, $i \in \mathbb{L}$, and $\varepsilon_i(t) = (x_i^T(t), \upsilon_i^T(t))^T$, $i \in \mathbb{F}$. Then, the closed-loop dynamics of system (1) with protocol (4) can be written in the following compact form:

$$\begin{aligned} \dot{\varepsilon}_l(t) &= (I_M \otimes \mathcal{T})\varepsilon_l(t) \\ \dot{\varepsilon}_f(t) &= (I_{N-M} \otimes \mathcal{S} + \mathcal{L}_2 \otimes \mathcal{R})\varepsilon_f(t) + (\mathcal{L}_1 \otimes \mathcal{R})\varepsilon_l(t) \end{aligned} \tag{5}$$

where $\mathcal{T} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$, $\mathcal{S} = \begin{bmatrix} A & -BK \\ 0 & A+FC+BK \end{bmatrix}$, and $\mathcal{R} = \begin{bmatrix} 0 & 0 \\ FC & 0 \end{bmatrix}$.

Before moving on, let $\delta_i(t) = \sum_{j\in\mathbb{F}\cup\mathbb{L}} a_{ij}(\varepsilon_i(t) - \varepsilon_j(t))$, $i \in \mathbb{F}$, and $\delta(t) = (\delta_{M+1}^T(t), \ldots, \delta_N^T(t))^T$. Then,

$$\delta(t) = (\mathcal{L}_1 \otimes I_{2n})\varepsilon_l(t) + (\mathcal{L}_2 \otimes I_{2n})\varepsilon_f(t). \tag{6}$$

It thus follows from (2), (5), and (6) that:

$$\begin{aligned}\dot{\delta}(t) &= (\mathcal{L}_1 \otimes I_{2n})\dot{\varepsilon}_l(t) + (\mathcal{L}_2 \otimes I_{2n})\dot{\varepsilon}_f(t)\\ &= (\mathcal{L}_1 \otimes I_{2n})(I_M \otimes \mathcal{T})\varepsilon_l(t) + (\mathcal{L}_2 \otimes I_{2n})\\ &\quad\times [(I_{N-M} \otimes \mathcal{S} + \mathcal{L}_2 \otimes \mathcal{R})\varepsilon_f(t) + (\mathcal{L}_1 \otimes \mathcal{R})\varepsilon_l(t)]\\ &= (\mathcal{L}_1 \otimes \mathcal{T} + \mathcal{L}_2\mathcal{L}_1 \otimes \mathcal{R})\varepsilon_l(t) + \left(\mathcal{L}_2 \otimes \mathcal{S} + \mathcal{L}_2^2 \otimes \mathcal{R}\right)\\ &\quad\times \left[\left(\mathcal{L}_2^{-1} \otimes I_{2n}\right)\delta(t) - \left(\mathcal{L}_2^{-1}\mathcal{L}_1 \otimes I_{2n}\right)\varepsilon_l(t)\right]\\ &= (I_{N-M} \otimes \mathcal{S} + \mathcal{L}_2 \otimes \mathcal{R})\delta(t).\end{aligned} \tag{7}$$

Lemma 3: For multi-agent system (1) with directed communication topology $\mathcal{G}$ satisfying Assumption 1, the states of the followers under controller (4) will asymptotically converge to the convex hull formed by the states of the leaders if the matrices $\mathcal{S} + \lambda_i\mathcal{R}$, $i = 1, \ldots, N - M$, are Hurwitz, where λ_i are the eigenvalues of the matrix $\mathcal{L}_2$. Specifically, $\lim_{t\to\infty}(x_f(t) - \varpi_{x_l}(t)) = 0$, where

$$\varpi_{x_l}(t) = \left(-\mathcal{L}_2^{-1}\mathcal{L}_1 \otimes I_n\right)\begin{pmatrix} x_1(t)\\ \vdots \\ x_M(t)\end{pmatrix}. \tag{8}$$

Proof: Let $U \in \mathbb{C}^{(N-M)\times(N-M)}$ be a unitary matrix such that $U^*\mathcal{L}_2U = \Lambda$, where Λ is an upper-triangular matrix with λ_i, $i = 1, \ldots, N-M$, being its diagonal entries. Here, λ_i, $i = 1, \ldots, N-M$, are the eigenvalues of $\mathcal{L}_2$. Define $\tilde{\delta}(t) \triangleq (\tilde{\delta}_1^T(t), \ldots, \tilde{\delta}_{N-M}^T(t))^T = (U^* \otimes I_{2n})\delta(t)$. It thus follows from (7) that:

$$\dot{\tilde{\delta}}(t) = (I_{N-M} \otimes \mathcal{S} + \Lambda \otimes \mathcal{R})\tilde{\delta}(t). \tag{9}$$

Note that Λ is upper-triangular. Hence, the zero equilibrium point of (9) is asymptotically stable if and only if the zero equilibrium points of systems

$$\dot{\tilde{\delta}}_i(t) = (\mathcal{S} + \lambda_i\mathcal{R})\tilde{\delta}_i(t), \quad i = 1, \ldots, N - M \tag{10}$$

are simultaneously asymptotically stable. According to the condition that the matrices $\mathcal{S} + \lambda_i\mathcal{R}$, $i = 1, \ldots, N - M$, are stable, one has that $\|\tilde{\delta}\|$ will converges to zero asymptotically, where symbol $\|\cdot\|$ indicates the Euclidean norm. On the other hand, it follows from Assumption 1 and Lemma 1 that all the eigenvalues of $\mathcal{L}_2$ have positive real parts. Then, $\|\varepsilon_f(t) + (\mathcal{L}_2^{-1}\mathcal{L}_1 \otimes I_{2n})\varepsilon_l(t)\| \to 0$, as $t \to \infty$, leading to (8) directly. By Lemma 1, the states of the followers asymptotically converge to the convex hull formed by the states of the leaders; that is, the containment problem is solved. This completes the proof. ■

By Lemma 3, the containment control problem of (1) with controller (4) has been converted to the simultaneously asymptotical stability of zero equilibrium points for the following $N - M$ systems:

$$\begin{aligned}\dot{\tilde{\delta}}_i(t) &= \begin{bmatrix} A & -BK\\ \lambda_iFC & A + BK + FC\end{bmatrix}\tilde{\delta}_i(t)\\ &= \left\{\begin{bmatrix} A & -BK\\ \lambda_iFC & A + BK + \lambda_iFC\end{bmatrix}\right.\\ &\quad\left.+\begin{bmatrix} 0 & 0\\ 0 & (1-\lambda_i)FC\end{bmatrix}\right\}\tilde{\delta}_i(t)\end{aligned} \tag{11}$$

where λ_i, $i = 1, \ldots, N - M$, are the eigenvalues of the matrix $\mathcal{L}_2$. By taking the nonsingular linear transformation $\zeta_i(t) = \begin{bmatrix} I & 0\\ I & I\end{bmatrix}\tilde{\delta}_i(t)$, it follows from (11) that:

$$\dot{\zeta}_i(t) = \left\{\begin{bmatrix} A+BK & -BK\\ 0 & A+\lambda_iFC\end{bmatrix} + (1-\lambda_i)\begin{bmatrix} 0 & 0\\ -FC & FC\end{bmatrix}\right\}\zeta_i(t) \tag{12}$$

where $i = 1, \ldots, N - M$. Set $\mathcal{C} = [-C \ \ C] \in \mathbb{R}^{q\times 2n}$ and $\mathcal{F} = \begin{bmatrix} 0\\ F\end{bmatrix} \in \mathbb{R}^{2n\times q}$, systems (12) can be rewritten as

$$\dot{\zeta}_i(t) = \left\{\begin{bmatrix} A + BK & -BK\\ 0 & A + \lambda_iFC\end{bmatrix} + (1 - \lambda_i)\mathcal{FC}\right\}\zeta_i(t) \tag{13}$$

for $i = 1, \ldots, N - M$.

For notational brevity, let

$$G_i(s) = \mathcal{C}\left[sI - \begin{pmatrix} A + BK & -BK\\ 0 & A + \lambda_iFC\end{pmatrix}\right]^{-1}\mathcal{F} \tag{14}$$

and λ_i, $i = 1, \ldots, N - M$, are the eigenvalues of $\mathcal{L}_2$, s is a complex variable.

Based on the above analysis, one may directly get the following theorem by using Lemma 3.

Theorem 1: Suppose that Assumption 1 holds and $\lambda_i = 1$, for $i = 1, \ldots, N-M$, the states of the followers driven by the controller (4) will asymptotically converge to the convex hull formed by the states of the leaders, if and only if the matrices $A+BK$, $A+FC$ are stable. Furthermore, $\lim_{t\to\infty}(x_f(t) - \varpi_{x_l}(t)) = 0$, of which $\varpi_{x_l}(t)$ is given by (8).

Remark 2: A necessary and sufficient condition to ensure the existence of feedback gain matrices K and F such that $A + BK$ and $A + FC$ are stable is that the triplet (A, B, C) is stabilizable and detectable. In this case, the feedback gain matrices K and F in controller (4) for achieving containment in multi-agent system (1) can be designed by using the pole placement method in linear control theory. Suppose that Assumption 1 holds and $\lambda_i = 1$, for $i = 1, \ldots, N - M$, containment in multi-agent system (1) with controller (4) can be achieved with an arbitrarily given exponential convergence rate if the triplet (A, B, C) is completely controllable and observable, and the feedback gain matrices K and F are appropriately designed.

Furthermore, one can establish the following theorem based on the analysis given from (11) to (14) and by using [31, Theorem 2.7].

Theorem 2: Suppose that Assumption 1 holds and $\lambda_i \neq 1$, for some i $(1 \le i \le N-M)$, the states of the followers driven by the controller (4) will asymptotically converge to the convex hull formed by the states of the leaders, if there exist $F \in \mathbb{R}^{n\times q}$, $K \in \mathbb{R}^{p\times n}$ such that matrices $A+BK$, $A+\lambda_iFC$ are stable and $\|G_i(s)\|_\infty < 1/\kappa_1$, $i = 1, \ldots, N-M$, where $\kappa_1 = \max_{i=1,\ldots,N-M}\{|1-\lambda_i|\}$, λ_i are the eigenvalues of $\mathcal{L}_2$, symbol $|\cdot|$ represents the modulus function. Furthermore, $\lim_{t\to\infty}(x_f(t) - \varpi_{x_l}(t)) = 0$, of which $\varpi_{x_l}(t)$ is given by (8).

Remark 3: Containment in multi-agent system (1) with controller (4) will be achieved if and only if the zero equilibrium points of $N - M$ systems given in (13) are simultaneously asymptotically stable. Thus, it suffices to show the achievement of containment in multi-agent system (1) under controller (4) if there is a quadratic Lyapunov function for each subsystem of (13). By treating the terms $(1 - \lambda_i)\mathcal{FC}$ as uncertainties, each subsystem of (13) can be viewed as an uncertain linear system studied in the robust stability literature, see e.g., [31]. Motivated by the above analysis, Theorem 2 was established by using the quadratic stability theory for uncertain linear systems.

Next, an algorithm is presented to construct the feedback gain matrices in (4) to guarantee containment in multi-agent system (1).

Algorithm 1: Suppose that Assumption 1 holds and $\lambda_i \neq 1$, for some i $(1 \le i \le N - M)$. Then, the containment controller (4) can be constructed as follows:

(i) Choose $c_0 \ge 0$. Then, solve the following linear matrix inequality (LMI):

$$PA + A^TP - C^TC + 2c_0P < 0 \tag{15}$$

to get a real matrix $P > 0$. Then, take the feedback gain matrix $F = -(1/2)\alpha P^{-1}C^T$, where $\alpha \ge \alpha_{th}$, with

$$\alpha_{th} = \frac{1}{\min_{i=1,\ldots,N-M}\{\mathrm{Re}(\lambda_i)\}} \tag{16}$$

where $\lambda_i, i = 1, \ldots, N-M$, are the eigenvalues of $\mathcal{L}_2$, $\mathrm{Re}(\lambda_i)$ represents the real part of λ_i.

(ii) Solve the following LMI:

$$\begin{bmatrix} AQ + BV + (AQ+BV)^T & QC^T & I \\ CQ & -I & 0 \\ I & 0 & -\kappa_0^2 I \end{bmatrix} < 0 \tag{17}$$

to obtain real matrices V and $Q > 0$, where $\kappa_0 = 1/(\kappa_1\kappa_2)$, $\kappa_1 = \max_{i=1,\ldots,N-M}\{|1-\lambda_i|\}$, λ_i are the eigenvalues of $\mathcal{L}_2$, $\kappa_2 = \max_{i=1,\ldots,N-M}\{\|T_i(s)\|_\infty\}$, $T_i(s) = F + \lambda_i FC[sI - (A+\lambda_i FC)]^{-1}F$, $i = 1, \ldots, N-M$. Then, set $K = VQ^{-1}$.

Remark 4: Suppose that the pair (A, C) is completely observable, it can be obtained from the canonical projection lemma in $\mathcal{H}_\infty$ control theory [32] that there exists a $P > 0$ such that LMI (15) holds for an arbitrarily given $c_0 \geq 0$. For the case that the pair (A, C) is detectable but not completely observable, it can be derived that there exists a $P > 0$ such that LMI (15) holds for any given $c_0 \in [0, c_1)$ where $-c_1 < 0$ is the largest real part of the unobservable eigenvalues of (A, C). Note that a necessary but generally not sufficient condition for the solvability of LMI (17) is that the pair (A, B) is stabilizable. The above analysis indicates that the triplet (A, B, C) being stabilizable and detectable is only a necessary condition for utilizing Algorithm 1. Nevertheless, it is practically important to provide some sufficient conditions for the feasibility of Algorithm 1. To achieve this goal, the solvability of LMI (17) is further analyzed as follows. According to the Schur complement lemma, one knows that LMI (17) is feasible if and only if there exist $Q > 0$ and V such that

$$AQ + BV + (AQ+BV)^T + QC^TCQ + \kappa_0^{-2}I < 0. \tag{18}$$

Furthermore, it can be verified that ARI (18) is feasible if and only if there exist $Q > 0$ and K such that

$$(A+BK)Q + Q(A+BK)^T + QC^TCQ + \kappa_0^{-2}I < 0. \tag{19}$$

By Lemma 2, one gets that ARI (19) is feasible if and only of there exists a matrix K such that $A + BK$ is stable, and any of the following conditions holds:

1) $\|C[sI-(A+BK)]^{-1}\|_\infty < \kappa_0$, where κ_0 is defined in step (ii) of Algorithm 1;
2) The Hamilton matrix $H = \begin{bmatrix} A+BK & (\frac{1}{\kappa_0^2})I \\ -C^TC & -(A+BK)^T \end{bmatrix}$ has no eigenvalues on the imaginary axis.

Thus, a sufficient condition for the existence of a distributed controller (4) designed by Algorithm 1 is that the triplet (A, B, C) is stabilizable and detectable, and there exists a feedback gain matrix K such that $A + BK$ is stable and either of conditions 1) and 2) holds.

Remark 5: It can be seen from the analysis given in Remark 4 that, for a given multi-agent system (1) with a fixed topology satisfying Assumption 1, the larger the parameter κ_0 is, the easier the conditions 1) and 2) to be satisfied. Furthermore, the LMI (15) is always feasible by appropriately selecting the parameter c_0 under the condition that the pair (A, C) is detectable and Assumption 1 holds. Roughly speaking, there exists a feedback gain matrix K such that conditions 1) and 2) given in Remark 4 hold if κ_0 defined in (17) is sufficiently large and the pair (A, B) is stabilizable. Suppose that the triplet (A, B, C) is stabilizable and detectable, and Assumption 1 holds. The question whether Algorithm 1 is applicable for solving the containment problem of such a multi-agent system (1) can be answered by performing the following tasks:

1) Check whether the pair (A, C) is completely observable. If it is not, go to step 2). If it holds, solve the following optimization problem: minimize κ_2, subject to $PA + A^TP - C^TC + 2c_0P < 0$, $c_0 \geq 0$, $F = -(1/2)\alpha P^{-1}C^T$, $\alpha \geq \alpha_{th}$, and then set $\kappa_0 = 1/(\kappa_1\kappa_2)$, where κ_1 and κ_2 are defined in step (ii) of Algorithm 1. Then, go to step 3).
2) Solve the following optimization problem: minimize κ_2, subject to $PA + A^TP - C^TC + 2c_0P < 0$, $0 \leq c_0 \leq c_1$, $F = -(1/2)\alpha P^{-1}C^T$, $\alpha \geq \alpha_{th}$, and then set $\kappa_0 = 1/(\kappa_1\kappa_2)$, where κ_1 and κ_2 are defined in step (ii) of Algorithm 1, and $-c_1 < 0$ is the largest real part of the unobservable eigenvalues of (A, C). Then, go to step 3).
3) Check whether LMI (17) is feasible. If it is feasible, then Algorithm 1 is applicable for the multi-agent system; otherwise, Algorithm 1 is inapplicable.

Note that the above two optimization problems are nonlinear optimization problems. It is generally difficult to get their optimal solutions. Nevertheless, one may obtain their suboptimal solutions by employing available numerical methods.

Theorem 3: Suppose that Assumption 1 holds and $\lambda_i \neq 1$, for some i $(1 \leq i \leq N-M)$, the states of the followers under the controller (4) constructed by Algorithm 1 will asymptotically converge to the convex hull formed by the states of the leaders. Specifically, $\lim_{t\to\infty}(x_f(t) - \varpi_{x_l}(t)) = 0$, where $\varpi_{x_l}(t)$ is given by (8).

Proof: By step (i) in Algorithm 1, one gets that there exists a $P > 0$ satisfying

$$\begin{aligned} &P(A+\lambda_i FC) + (A+\lambda_i FC)^*P \\ &= PA + A^TP - \alpha\mathrm{Re}(\lambda_i)C^TC \\ &\leq PA + A^TP - C^TC < -2c_0P, \quad i = 1, \ldots, N-M. \end{aligned} \tag{20}$$

That is, $A + \lambda_i FC$, $i = 1, \ldots, N-M$, are Hurwitz stable. Furthermore, by the fact of $K = VQ^{-1}$, it follows from (17) that $A + BK$ is Hurwitz stable. Based on the above analysis, it can be yielded from (14) that the transfer function matrices $G_i(s)$, $i = 1, \ldots, N-M$, are stable, i.e., all of them are analytic and bounded in the open right-half plane. On the other hand, some tedious calculations give that

$$\left[sI - \begin{pmatrix} A+BK & -BK \\ 0 & A+\lambda_i FC \end{pmatrix}\right]^{-1} = \begin{bmatrix} \Xi_1 & \Xi_2 \\ 0 & \Xi_3 \end{bmatrix}$$

where $\Xi_1 = [sI-(A+BK)]^{-1}$, $\Xi_2 = -[sI-(A+BK)]^{-1} BK[sI-(A+\lambda_i FC)]^{-1}$, $\Xi_3 = [sI-(A+\lambda_i FC)]^{-1}$, $i=1, \ldots, N-M$. Based on the above analysis and by (14), one gets

$$\begin{aligned} G_i(s) &= C\,[sI-(A+BK)]^{-1}\,BK\,[sI-(A+\lambda_i FC)]^{-1}\,F \\ &\quad + C\,[sI-(A+\lambda_i FC)]^{-1}\,F \\ &= C\left\{[sI-(A+BK)]^{-1}BK+I\right\}[sI-(A+\lambda_i FC)]^{-1}\,F \\ &= C\,[sI-(A+BK)]^{-1}\,[sI-(A+\lambda_i FC)+\lambda_i FC] \\ &\quad \times [sI-(A+\lambda_i FC)]^{-1}\,F \\ &= C\,[sI-(A+BK)]^{-1}\,T_i(s) \end{aligned} \tag{21}$$

where $T_i(s) = F + \lambda_i FC[sI-(A+\lambda_i FC)]^{-1}F$, $i=1, \ldots, N-M$. Since $A+\lambda_i FC$, $i=1, \ldots, N-M$, are stable, one has that $\|T_i(s)\|_\infty$, $i = 1, \ldots, N-M$, exist and are finite. According to (17) and by using the Schur complement lemma, there exists $Q > 0$ such that

$$(A+BK)Q + Q(A+BK)^T + QC^TCQ + \kappa_0^{-2}I < 0 \tag{22}$$

where $\kappa_0 = 1/(\kappa_1\kappa_2)$, $\kappa_1 = \max_{i=1,\ldots,N-M}\{|1-\lambda_i|\}$, $\kappa_2 = \max_{i=1,\ldots,N-M}\{\|T_i(s)\|_\infty\}$. On the other hand, pre- and post-multiplying (22) by Q^{-1} and its transpose, respectively, yields

$$W(A+BK) + (A+BK)^TW + C^TC + \kappa_0^{-2}W^2 < 0 \tag{23}$$

where $W = Q^{-1} > 0$. By Lemma 2, it follows from (23) that:

$$\left\|C\,[sI-(A+BK)]^{-1}\right\|_\infty < \kappa_0. \tag{24}$$

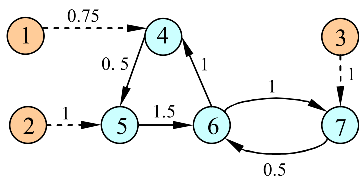

Fig. 1. Communication topology $\mathcal{G}$.

According to (21) and by using the submultiplicative property of the $\mathcal{H}_\infty$ norm, one gets

$$\|G_i(s)\|_\infty < \kappa_2 \left\| C\left[sI - (A+BK)\right]^{-1}\right\|_\infty \tag{25}$$

where $\kappa_2 = \max_{i=1,\ldots,N-M}\{\|T_i(s)\|_\infty\}$, and $T_i(s) = F + \lambda_i FC[sI - (A+\lambda_i FC)]^{-1}F$, $i = 1,\ldots,N-M$. Combining (24) and (25) gives $\|G_i(s)\|_\infty < 1/\kappa_1$, $i = 1,\ldots,N-M$. By Theorem 2, the states of the followers under the controller (4) will asymptotically converge to the convex hull formed by the states of the leaders. Specifically, $\lim_{t\to\infty}(x_f(t) - \varpi_{x_l}(t)) = 0$, where $\varpi_{x_l}(t)$ is given by (8). This completes the proof. ■

Remark 6: The observers embedded in the multiple agents could share their state information with each other, which is one of the critical assumptions on containment in linear multi-agent systems. However, the states of observers are difficult or even impossible to obtain since they are actually internal state information for the closed-loop systems, therefore hard to be detected. Motivated by the work in [27] and to overcome this shortcoming, a new class of containment protocols in the form of (4) is provided in this technical note. It is also worth mentioning that the techniques used in the proof of Lemma 3 is partly motivated by [33] where distributed consensus for a class of linear multi-agent systems has been studied by designing some dynamic consensus protocols.

Remark 7: In the above analysis, it is assumed that there exist at least two leaders in system (1), with which Theorems 1–3 provide some containment conditions for multi-agent system (1). For the case that there exists only one leader in system (1), i.e., $M = 1$, it follows from Lemma 1 that $-\mathcal{L}_2^{-1}\mathcal{L}_1 = (1,\ldots,1)^T \in \mathbb{R}^{N-1}$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are given in (2). In this scenario, the containment problem of system (1) reduces to the consensus tracking problem, that is, to construct a distributed controller such that the states of all followers track those of the leader asymptotically. Similarly to the proof of Theorems 1–3, some results on how to ensure consensus tracking for system (1) can be obtained.

IV. Numerical Example

Consider a group of seven agents, with three leader labeled from 1 to 3, and four followers, labeled from 4 to 7, where the communication topology $\mathcal{G}$ is shown in Fig. 1 (the weights are indicated on edges). It can be verified that $\mathcal{G}$ satisfies Assumption 1. According to (16), one has $\alpha_{th} = 1.4147$.

The dynamics of agents in (1) are taken the as the linearized dynamics of the Caltech multi-vehicle wireless testbed vehicles [34], with $x_i(t) = (x_{i1}(t),\ldots,x_{i6}(t))^T \in \mathbb{R}^6$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & -0.2003 & -0.2003 & 0 & 0\\ 0 & 0 & 0.2003 & 0 & -0.2003 & 0\\ 0 & 0 & 0 & 0 & 0 & -1.6129 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0.9441 & 0.9441 & -28.7097\\ 0 & 0 & 0 & 0.9441 & 0.9441, & 28.7097 \end{bmatrix}^T$$

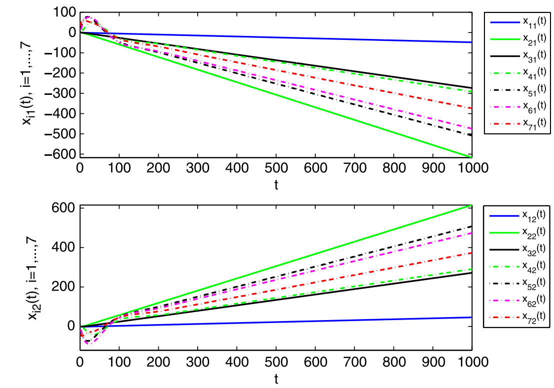

Fig. 2. State trajectories of agents.

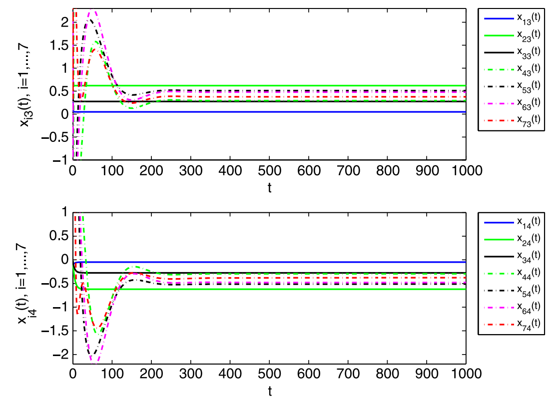

Fig. 3. State trajectories of agents.

where $x_{i1}(t)$ and $x_{i2}(t)$ are the positions of the i-th vehicle along the x and y coordinates, respectively; $x_{i3}(t)$ is the orientation of the i-th vehicle. Suppose that the output matrix $C = [I_5, 0_{5\times 1}]$. Then, it is can be verified that (A, B, C) is completely controllable and observable.

An observer-type containment controller in the form of (4) is designed as follows. Taking $c_0 = 0$, $\alpha = 1.5 > \alpha_{th} = 1.4147$ and according to step (i) of Algorithm 1 (with additional condition $P > 9I_n$), the feedback gain matrix F in (4) can be obtained as

$$F = \begin{bmatrix} -0.03713 & 0.03266 & 0.00107 & -0.00087 & 0.00123\\ 0.03266 & -0.03713 & -0.00107 & 0.00123 & -0.00087\\ 0.00107 & -0.00107 & -0.00016 & 0.00007 & -0.00007\\ -0.00087 & 0.00123 & 0.00007 & -0.00016 & 0.00008\\ 0.00123 & -0.00087 & -0.00007 & 0.00008 & -0.00016\\ 0.00006 & -0.00006 & 0.00012 & -0.00001 & 0.00001 \end{bmatrix}.$$

By the definition of $\mathcal{H}_\infty$ norm for stable transfer function matrix, one gets that $\kappa_1 = 1.2724$ and $\kappa_2 = 0.0721$. Simple calculations give that $\kappa_0 = 10.9050$. Then, solving LMI (17) gives that

$$K = -\begin{bmatrix} 5.1478 & 0.1483 & -0.7389 & 4.7529 & -1.6814 & -0.1565\\ 0.1483 & 5.1478 & 0.7389 & -1.6814 & 4.7529 & 0.1565 \end{bmatrix}.$$

According to Theorem 2, one gets that the states of followers under controller (4) with gain matrices F and K designed as above can asymptotically converge to the convex hull formed by those of the leaders. The state trajectories of seven agents are shown in Figs. 2–4. Using $Err(t) = (1/4)\sqrt{\sum_{i=4}^{7}\|x_i(t) - \sum_{j=1}^{3}\widehat{l}_{(i-3)j}x_j(t)\|^2}$ to denote the containment errors of the closed-loop multi-agent system, where $-\mathcal{L}_2^{-1}\mathcal{L}_1 = [\widehat{l}_{ks}] \in \mathbb{R}^{4\times 3}$. It thus can be concluded from Fig. 5 that the containment problem is indeed solved.

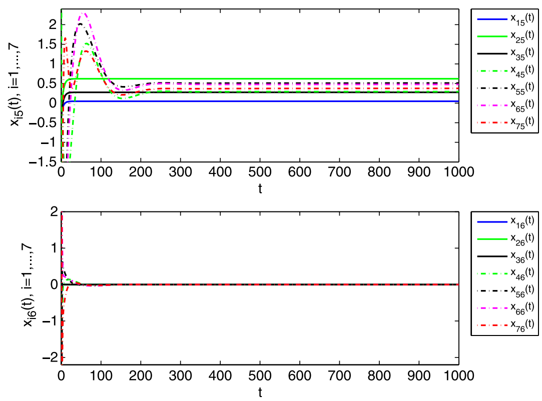

Fig. 4. State trajectories of agents.

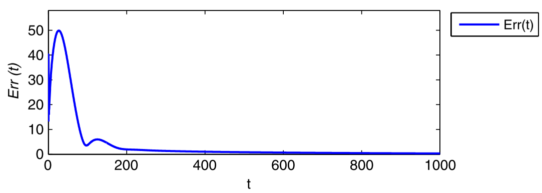

Fig. 5. Containment errors of the closed-loop multi-agent systems.

V. Conclusion

In this technical note, the distributed containment problem for a class of linear multi-leader multi-agent systems under a fixed topology has been studied. Without assuming that the dynamic observers embedded in followers have to share information with their neighbors, some new distributed observer-type containment protocols are designed based only on the relative output measurements of the neighboring agents. Under the assumption that, for each follower, there exists at least one leader that has a directed path to that follower, some sufficient conditions expressed in frequency and time domains are obtained to guarantee the states of the followers to converge to a convex hull formed by those of the leaders. Future work will focus on solving the distributed $\mathcal{H}_\infty$ containment problem of multi-agent systems with external disturbances and output measurements.

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