Designing Distributed Specified-Time Consensus Protocols for Linear Multiagent Systems Over Directed Graphs

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***Abstract*—This technical note addresses the distributed specified-time consensus protocol design problem for multiagent systems with general linear dynamics over directed graphs. By using motion planning approaches, a novel class of distributed consensus protocols are developed. With a prespecified settling time, the proposed protocols solve the consensus problem of linear multiagent systems over directed graphs containing a directed spanning tree. In particular, the settling time can be offline prespecified, according to task requirements. Compared with the existing results for multiagent systems, to our best knowledge, this is the first time specified-time consensus problems are raised and solved for general linear multiagent systems over directed graphs. Extensions to the specified-time formation flying are further studied for multiple satellites described by Hill equations.**

***Index Terms*—Directed spanning tree, distributed protocol, linear multiagent system, specified-time consensus.**

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I. Introduction

Over the past few years, the coordinative control problems of multiagent systems have been of great interest to various scientific and engineering communities [1]–[6] due to the broad applications in such fields as spacecraft formation flying, distributed sensor networks, automated highway systems, and so forth [7]–[12]. Compared with the traditional monolithic systems, the coordination control reduces the systems cost, breaches the size constraints, and prolongs the life span of the systems. One interesting and important issue arising from the coordination control of multiagent systems is to design distributed protocols based only on the local relative measurements to guarantee that the states of all agents reach an agreement, known as the consensus problem. According to the convergence rate, existing consensus problems can be roughly categorized into two classes, namely, asymptotic consensus [13]–[17] and finite-time consensus [18]–[22]. For the consensus problem, a key task is to design appropriate distributed controllers, which are usually called consensus protocols. Due to the practical engineering requirements of networked agents, designing finite-time consensus protocols has been a hot research topic in the area of consensus problem.

Previous works of study on the finite-time consensus problem were first presented in [18] for first-order multiagent systems, where the signed gradient flows of a differential function and discontinuous protocols were used. Since then, a variety of finite-time consensus protocols were proposed to solve the consensus problem under different scenarios; see [19]–[21], [23] and references therein. Then, over directed graphs having a spanning tree, the finite-time consensus protocol designing problems were studied in [24]–[26]. Further, a class of finite-time consensus protocols for second-order multiagent systems were given in [22], [27], [28], and [33]. Recently, finite-time consensus problems for multiple nonidentical second-order nonlinear systems were studied in [34] with the settling time estimation, where the settling time estimation depends on the initial states of all agents. It is failed in practical applications if the knowledge of initial conditions is unavailable in advance. Therefore, the notion of *fixed-time consensus* was defined in [29]. The fixed-time consensus problem is solved if and only if the consensus can be solved in a finite settling time, and if the settling time is uniformly bounded for any initial conditions of the systems. Furthermore, authors in [30]–[32] presented some protocols for solving fixed-time consensus problems of multiagent systems.

However, in practical applications, there are several shortcomings for existing finite- and fixed-time consensus protocols presented in [18]–[34] and [36]. First of all, the settling time for achieving consensus was estimated to be a conservative upper bound in existing literatures [18]–[23], [27]–[34]. It is difficult to point out the exact moment to achieve consensus. A new notion, *specified-time consensus*, is introduced and studied in this note. Specifically, the specified-time consensus problem is solved if the fixed-time consensus can be solved, and if the moment of the settling time can be specified exactly in advance, according to mission requirements. Compared with fixed-time consensus, the specified-time consensus is more difficult to achieve in theory but has a more extensive potential in engineering applications, such as in collaborative guidance and attacks. Second, many existing literatures on finite- and fixed-time consensus [18]–[23], [27]–[34] were studied under an assumption that the topology among agents was undirected. Although, there are a few papers [24]–[26], where the authors studied the finite-time consensus problems for first-order multiagent systems over directed graphs, it is difficult to extend the finite-time protocols in [24]–[26] to higher order multiagent systems over directed graphs. In real practice, many physical systems are modeled by higher order linear dynamics, and the topology among agents is described by a directed graph. Therefore, it is very significant and challenging to solve the specified-time consensus problem for higher order multiagent systems over directed graphs.

Motivated by the aforementioned observations, in this technical note, the specified-time consensus problem of general linear multiagent systems is investigated over directed graphs. Main contributions of this technical note are threefold. First, by using motion planning approaches, a novel framework is introduced to solve continuous-time consensus problems. In this framework, for general linear multiagent systems considered in this technical note, a novel class of distributed protocols are designed to solve specified-time consensus problems. Compared with the existing finite- and fixed-time protocols in [18]–[20], [22], [23], and [27]–[34], an important distinction of this note is that the consensus of the multiagent system can be achieved at an accurate prespecified finite settling time offline, according to task requirements, without conservative estimation. This has great application potential for collaborative guidance. Second, the measuring-networked topologies among the multiple agents are directed with a directed spanning tree. Compared with finite-time consensus protocols over undirected graphs in [18]–[23], [27]–[29], and [31]–[34], the topologies over directed graphs studied here are more general in theory and more effective when applied to practical problems. Besides, compared with finite-time results over directed graphs for first-order multiagent systems in [24]–[26], where the protocols are difficult to be extended for solving the same problem of higher order multiagent systems, the designed protocols here can guarantee that general higher order linear multiagent systems achieve finite-time consensus over a general directed graph with a spanning tree. Furthermore, compared with the existing results on finite-time consensus problems in [18]–[22], [27]–[29], and [31]–[34], where the topologies are all continuous-time, in this note, the agents only need to communicate with each other at the sampling time. Thus, it reduces the cost of the network measurement.

Notations: Let $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times n}$ be the sets of real numbers, real vectors, and real matrices, respectively. The superscript T denotes the transpose for real matrices. I_n represents the identity matrix of dimension n. Denote by $\mathbf{1}_n$ a column vector with all entries equal to one. $\mathrm{diag}(A_1,\cdots,A_N)$ represents a block-diagonal matrix with matrices A_i, $i=1,\cdots,N$, on its diagonal. The matrix inequality $A>(\geq)B$ means that $A-B$ is positive (semi) definite. Denote by $A\otimes B$ the Kronecker product [37] of matrices A and B. For a vector $x\in\mathbb{R}^n$, let $\|x\|$ denote the 2-norm of x. For a set V, $|V|$ represents the number of elements in V.

II. Preliminaries

A. Graph Theory

For systems with N agents, a directed graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ is developed to model the interaction among these agents, where $\mathcal{V}=\{1,2,\cdots,N\}$ is the vertex set and $\mathcal{E}\subset\{(i,j):i,j\in\mathcal{V}\}$ is the edge set, where an edge (i,j) is an ordered pair of vertices in $\mathcal{V}$, which means that agent j can receive information from agent i. If there is a directed edge from i to j, i is defined as the parent vertex and j is defined as the child vertex. The parent-vertex set of vertex i is denoted by $\mathcal{N}_i=\{j\in\mathcal{V}|(j,i)\in\mathcal{E}\}$, and $|\mathcal{N}_i|$ presents the number of elements in the parent-vertex set $\mathcal{N}_i$ of vertex i. A directed tree is a directed graph, where every node has exactly one parent, except the root, which has no parent. A directed spanning tree of a directed graph $\mathcal{G}$ is a directed tree that connects all the vertices in the graph $\mathcal{G}$.

The adjacency matrix A associated with $\mathcal{G}$ is defined such that $a_{ij}=1$ if $(j,i)\in\mathcal{E}$, and $a_{ij}=0$ otherwise. Furthermore, assume $a_{ii}=0$ for $i\in\mathcal{V}$. The Laplacian matrix of the graph associated with the adjacency matrix A is given as $\mathcal{L}=(l_{ij})$, where $l_{ii}=\sum_{j=1}^{N}a_{ij}$ and $l_{ij}=-a_{ij}$, $i\neq j$.

Assumption 1: Suppose that the graph $\mathcal{G}$ of the communication topology is directed and has a spanning tree.

Lemma 1: [38] Under Assumption 1, zero is a simple eigenvalue of $\mathcal{L}$ with $\mathbf{1}$ as an eigenvector and all of the nonzero eigenvalues are in the open right-half plane.

Lemma 2: [2] Let $M=[m_{ij}]\in M_N(\mathbb{R})$ be a stochastic matrix, where $M_N(\mathbb{R})$ represents the set of all $N\times N$ real matrices. An indecomposable and aperiodic stochastic matrix M is called SIA. If M has an eigenvalue $\lambda=1$ with algebraic multiplicity equal to one, and all the other eigenvalues satisfy $|\lambda|<1$, then M is SIA. Further, $\lim_{k\to\infty}M^k=\mathbf{1}\xi^T$, where $\xi=(\xi_1,\xi_2,\cdots,\xi_N)^T\in\mathbb{R}^N$, $\xi_i\geq 0, i=1,2,\cdots,N$, $M^T\xi=\xi$, and $\mathbf{1}^T\xi=1$.

B. Pontryagin's Maximum Principle

Consider the finite-time optimal control problem, we have

$$\dot{x}=f(x,u,t),\ x(t_0)=x_0,\ g(x(T),T)=0$$

$$J=\frac{1}{2}\int_{t_0}^{T}L(x,u,t)dt\to\min$$

where t_0 and T are the initial time and terminal time, respectively, $x\in\mathbb{R}^n$ and $u\in\mathbb{R}^r$ are the state vector and control vector, respectively, $f=[f_1,f_2,\cdots,f_n]^T$ is a vector function, $g(x(T),T)=0$ are the terminal constraints, and J is a cost function. In general, the minimization of the above-mentioned cost function is to minimize the geodesic length or the control energy.

Let $p\in\mathbb{R}^n$ be the costate. For the above-mentioned optimal control problem, the Hamiltonian function can be constructed as follows:

$$H(x,p,u,t)=\frac{1}{2}L(x,u,t)+p^Tf(x,u,t).$$

Lemma 3: [39] Let $u^*(t)$ be the optimal control solution to the above-mentioned problem with a given time T, and $x^*(t)$ be the corresponding optimal trajectory. Then, there exists a constant vector μ and a vector function $p(t)\in\mathbb{R}^n$, which are not equal to zero, such that

$$\dot{p}(t)=-\frac{\partial H(x,p,u,t)}{\partial x}$$

$$p(T)=\frac{\partial g(x(T),T)}{\partial x(T)}\mu$$

$$H(x^*(t),p(t),u^*(t),t)=\min_{u(t)\in\mathbb{R}^r}H(x^*(t),p(t),u(t),t).$$

III. Distributed Specified-Time Consensus Control for Linear Multiagent Systems Over Directed Graphs

In this section, the distributed specified-time consensus problem for linear multiagent systems over directed graphs is studied. Consider a multiagent system having N agents with general linear dynamics, which may be regarded as the linearized model of some nonlinear systems [5], [7]. The dynamics of each agent in networks is described by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i = 1, 2, \ldots, N \tag{1}$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t)]^T \in \mathbb{R}^n$ is the state of agent i, $u_i \in \mathbb{R}^m$ is its control input, A and B are constant matrices with compatible dimensions, and (A, B) is controllable.

Definition 1: The specified-time consensus problem for multiagent system (1) is said to be solved if and only if, for an offline prespecified settling time $T_s > 0$, the states of the multiagent system (1) satisfy $\|x_i(t) - x_j(t)\| = 0$, $\forall i, j \in \mathcal{V}$, for all $t \geq T_s$ with any initial conditions.

Here and throughout, it is assumed that only relative sampling measurement information can be used to develop the distributed consensus protocols. Moreover, the ith agent can only obtain the consensus error value $\sum_{j\in\mathcal{N}_i}(x_i(t_k) - x_j(t_k))$ from its parent vertices. The objective is to design a distributed control law u_i based on the abovementioned relative information such that the states of all the agents in networks reach a specified-time consensus over directed graphs. In order to achieve the control objective in this technical note, the following protocol is proposed:

$$u_i(t) = -\frac{1}{|\mathcal{N}_i| + 1} B^T P \sum_{j\in\mathcal{N}_i} (x_i(t_k) - x_j(t_k)) \tag{2}$$

where $P = e^{-A^T(t-t_k)}\Phi^{-1}e^{A(t_{k+1}-t_k)}$, $\Phi = \begin{pmatrix} I_n & 0_n \end{pmatrix} e^{M(t_{k+1}-t_k)} \begin{pmatrix} 0_n \\ I_n \end{pmatrix}$, $M = \begin{pmatrix} A & BB^T \\ 0_{n\times n} & -A^T \end{pmatrix}$, $t_k \leq t < t_{k+1}$, and $i = 1, 2, \cdots, N$, with the sampling time sequence given by

$$S = \left\{ t_k = t_{k-1} + T_k \,|\, t_0 = 0, T_k = \frac{\delta_k}{\Delta} T_s, k = 1, 2, \cdots \right\} \tag{3}$$

where $T_s > 0$ is an offline prespecified settling time, according to task requirements, and $\Delta = \sum_{k=1}^{\infty} \delta_k$ is the sum of an infinite power series $\{\delta_k, \ k = 1, 2, \cdots\}$ with convergence at a polynomial rate. In particular, one can choose a special case of S, such as $S = \{t_k = t_{k-1} + T_k \,|\, t_0 = 0, T_k = \frac{6}{(\pi k)^2} T_s\}$.

The above-mentioned specified-time consensus protocol is designed by considering the following Hamiltonian function:

$$H_{i,k} = -\frac{1}{2}\sum_{i=1}^{N} u_i^T(t)u_i(t) + \sum_{i=1}^{N} p_i^T(t)(Ax_i(t) + Bu_i(t)) \tag{4}$$

where $p_i(t) \in \mathbb{R}^n$ represents the costate. Then, (4) is the Hamiltonian function of the following cost function:

$$J_{i,k} = \frac{1}{2}\int_{t_k}^{t_{k+1}} \sum_{i=1}^{N} u_i^T(t) R_i u_i(t) dt \tag{5}$$

where t_k and t_{k+1} can be seen as the initial and terminal times, respectively. Then, according to Pontryagin's principle [39], the necessary condition for optimality can be written as follows:

$$\begin{cases} \dot{x}_i(t) = \frac{\partial H_{i,k}}{\partial p_i(t)} = Ax_i(t) + Bu_i(t) & \text{(6a)} \\ \dot{p}_i(t) = -\frac{\partial H_{i,k}}{\partial x_i(t)} = -A^T p_i(t). & \text{(6b)} \end{cases}$$

Besides, according to the extremal condition $\frac{\partial H_{i,k}}{\partial u_i(t)} = -u_i(t) + B^T p_i(t) = 0$, one has

$$u_i(t) = B^T p_i(t). \tag{7}$$

Therefore, the determination of the optimal control (7) boils down to computing $p_i(t)$.

Substituting (7) into (6a) and (6b), one gets

$$\begin{pmatrix} \dot{x}_i(t) \\ \dot{p}_i(t) \end{pmatrix} = \begin{pmatrix} A & BB^T \\ 0_{n\times n} & -A^T \end{pmatrix} \begin{pmatrix} x_i(t) \\ p_i(t) \end{pmatrix}. \tag{8}$$

Integrating (8) from t_k to t_{k+1}, one gets

$$\begin{pmatrix} x_i(t_{k+1}) \\ p_i(t_{k+1}) \end{pmatrix} = e^{M(t_{k+1}-t_k)} \begin{pmatrix} x_i(t_k) \\ p_i(t_k) \end{pmatrix}. \tag{9}$$

Since $M = \begin{pmatrix} A & BB^T \\ 0_{n\times n} & -A^T \end{pmatrix}$, one has

$$(M(t_{k+1} - t_k))^0 = \begin{pmatrix} I_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & I_{n\times n} \end{pmatrix}$$

and

$$(M(t_{k+1} - t_k))^s = \begin{pmatrix} A^s & \sum_{\alpha=0}^{s-1} A^{s-1-\alpha} BB^T (-A^T)^\alpha \\ 0_{n\times n} & (-A^T)^s \end{pmatrix} \times (t_{k+1} - t_k)^s$$

for $s = 1, 2, \cdots$. Then, it follows that

$$\begin{aligned} e^{M(t_{k+1}-t_k)} &= \sum_{s=0}^{\infty} \frac{(M(t_{k+1} - t_k))^s}{s!} \\ &= \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \end{aligned} \tag{10}$$

where $M_{11} = \sum_{s=0}^{\infty} \frac{A^s(t_{k+1}-t_k)^s}{s!} = e^{A(t_{k+1}-t_k)}$, $M_{12} = \sum_{s=1}^{\infty} \frac{(t_{k+1}-t_k)^s \sum_{\alpha=0}^{s-1} A^{s-1-\alpha} BB^T(-A^T)^\alpha}{s!} = \begin{pmatrix} I_n & 0_n \end{pmatrix} e^{M(t_{k+1}-t_k)} \begin{pmatrix} 0_n \\ I_n \end{pmatrix} = \Phi$, $M_{21} = 0_{n\times n}$, and $M_{22} = \sum_{s=0}^{\infty} \frac{(-A^T)^s (t_{k+1}-t_k)^s}{s!} = e^{-A^T(t_{k+1}-t_k)}$. Further, design the terminal condition of (4) over $[t_k, t_{k+1}]$ as follows:

$$\begin{aligned} x_i(t_{k+1}) &= \frac{1}{|\mathcal{N}_i| + 1} e^{A(t_{k+1}-t_k)} \left[\sum_{j\in\mathcal{N}_i} x_j(t_k) + x_i(t_k) \right] \\ i &= 1, 2, \cdots, N. \end{aligned} \tag{11}$$

Substituting (11) into (9), one obtains

$$\begin{aligned} &e^{A(t_{k+1}-t_k)} \frac{1}{|\mathcal{N}_i| + 1} \left[\sum_{j\in\mathcal{N}_i} x_j(t_k) + x_i(t_k) \right] \\ &= \begin{pmatrix} I_n & 0_n \end{pmatrix} e^{M(t_{k+1}-t_k)} \begin{pmatrix} I_n \\ 0_n \end{pmatrix} x_i(t_k) \\ &\quad + \begin{pmatrix} I_n & 0_n \end{pmatrix} e^{M(t_{k+1}-t_k)} \begin{pmatrix} 0_n \\ I_n \end{pmatrix} p_i(t_k) \\ &= e^{A(t_{k+1}-t_k)} x_i(t_k) + \Phi p_i(t_k). \end{aligned}$$

Thus, if Φ is invertible, it follows that

$$p_i(t_k) = \Phi^{-1} e^{A(t_{k+1}-t_k)} \frac{1}{|\mathcal{N}_i| + 1} \sum_{j\in\mathcal{N}_i} [x_j(t_k) - x_i(t_k)].$$

Therefore, for the time sequence t_k, one has the distributed protocol (2).

Summarizing the aforementioned, with the specified-time control protocol (2), the states $x_i(t_k)$ in (1) can be controlled to a predicted average state $x_i(t_{k+1}) = \frac{e^{A(t_{k+1}-t_k)}}{|\mathcal{N}_i|+1}(\sum_{j\in\mathcal{N}_i} x_j(t_k) + x_i(t_k))$ over the interval $[t_k, t_{k+1})$. Intuitively, after enough multiple steps, the states of all agents in system (1) will achieve consensus.

Note that the above-mentioned proposed protocol exists only if Φ is invertible. Thus, before moving on, the following lemma is given.

Lemma 4: Φ is invertible if and only if (A, B) is controllable.

Proof: It follows from (10) that

$$\Phi = \sum_{s=1}^{\infty} \frac{(t_{k+1} - t_k)^s \sum_{\alpha=0}^{s-1} A^{s-1-\alpha} BB^T (-A^T)^\alpha}{s!}.$$

First, the proof of the sufficiency is given as follows. Let

$$\Pi(t) = \sum_{s=1}^{\infty} \frac{t^s \sum_{\alpha=0}^{s-1} A^{s-1-\alpha} BB^T (-A^T)^\alpha}{s!}.$$

By reductio ad absurdum, assume that $\Pi(t)$ is singular. Thus, there exists at least one nonzero vector $\alpha \in \mathbb{R}^n$ which makes that

$$\alpha^T \Pi(t) = 0 \tag{12}$$

for any $t \in \mathbb{R}$. Taking the kth-order derivatives $(k = 1, 2, \cdots, n-1)$ on both sides of (12) with respect to t, one has

$$\alpha^T \sum_{s=k}^{\infty} \frac{t^{s-k} \sum_{\alpha=0}^{s-1} A^{s-1-\alpha} BB^T (-A^T)^\alpha}{(s-k)!} = 0 \tag{13}$$

for all $t \in \mathbb{R}$. Owing to the arbitrary nature of t, (13) holds for $t = 0$. Thus, the following equations hold:

$$\alpha^T \sum_{\alpha=0}^{k-1} A^{k-1-\alpha} BB^T (-A^T)^\alpha = 0, \;\; k = 1, 2, \cdots, n-1. \tag{14}$$

Simplifying the above-mentioned equations, one has $\alpha^T A^{k-1} BB^T = 0, k = 1, 2, \cdots, n$. Thus, $\alpha^T BB^T = 0, \alpha^T ABB^T = 0, \alpha^T A^2 BB^T = 0, \alpha^T A^3 BB^T = 0, \cdots, \alpha^T A^{n-1} BB^T = 0$. Let $Q = [BB^T \; ABB^T \; A^2BB^T \; \cdots \; A^{n-1}BB^T]$. Then, one has $\alpha^T Q = 0$. It follows from $\alpha \neq 0$ that the matrix Q is linearly dependent. Note that (A, BB^T) is controllable if and only if (A, B) is controllable. This contradicts with the condition that (A, B) is controllable. Thus, it is proved that Π is nonsingular. Further, Φ is invertible.

Similarly, the proof of the necessity is given. By reductio, it is assumed that (A, B) is uncontrollable. Thus, there is at least one nonzero vector $\beta \in \mathbb{R}^n$ such that $\beta^T Q = 0$. It follows that $\beta^T BB^T = 0, \beta^T ABB^T = 0, \beta^T A^2 BB^T = 0, \beta^T A^3 BB^T = 0, \cdots, \beta^T A^{n-1} BB^T = 0$. Then, it is obtained that $\beta^T \Pi(t) = 0$. This contradicts with the condition that Φ is invertible. Thus, it is proved that (A, B) is controllable. The proof is completed. ■

Remark 1: From Lemma 4, one has that the proposed protocol exists if (A, B) is controllable, which is a fundamental requirement for control of linear systems.

Theorem 1: Suppose that Assumption 1 holds. For an offline prespecified settling time T_s, the distributed protocol (2) can solve the specified-time consensus problem of the linear multiagent system (1) if (A, B) is controllable.

Proof: First, we prove that at the sampling time series $\{t_k\}$, the states of system (1) with (2) will achieve consensus. Substituting (2) into (1), one gets

$$\dot{x}_i(t) = Ax_i(t) - \frac{1}{|\mathcal{N}_i| + 1} BB^T e^{-A^T(t-t_k)} \cdot \Phi^{-1} e^{A(t_{k+1}-t_k)} \sum_{j \in \mathcal{N}_i} \big(x_i(t_k) - x_j(t_k)\big). \tag{15}$$

Integrating (15) from t_k to t_{k+1}, $k = 0, 1, \cdots$, one gets

$$x_i(t_{k+1}) - e^{A(t_{k+1}-t_k)} x_i(t_k) = e^{A(t_{k+1}-t_k)} \frac{1}{|\mathcal{N}_i| + 1} \sum_{j \in \mathcal{N}_i} \big(x_j(t_k) - x_i(t_k)\big).$$

Let $X(t_k) = (x_1^T(t_k), x_2^T(t_k), \cdots, x_N^T(t_k))^T$. It follows that

$$X(t_{k+1}) - I_N \otimes e^{A(t_{k+1}-t_k)} X(t_k) = -(\mathcal{N} \otimes e^{A(t_{k+1}-t_k)})(\mathcal{L} \otimes I_n) \cdot X(t_k)$$

where $\mathcal{N} = \mathrm{diag}(\frac{1}{|\mathcal{N}_1|+1}, \frac{1}{|\mathcal{N}_2|+1}, \ldots, \frac{1}{|\mathcal{N}_N|+1})$. Then, we have

$$\begin{aligned} X(t_{k+1}) &= (I_N - \mathcal{N}\mathcal{L}) \otimes e^{A(t_{k+1}-t_k)} X(t_k) \\ &= (I_N - \mathcal{N}\mathcal{L})^{k+1} \otimes e^{A(t_{k+1}-t_0)} X(t_0). \end{aligned}$$

Under Assumption 1, the directed graph $\mathcal{G}$ has a spanning tree. Thus, according to Lemma 1, the stochastic matrix $I_N - \mathcal{N}\mathcal{L}$ has an eigenvalue $\lambda_1 = 1$ with algebraic multiplicity equals to one, and all the other eigenvalues satisfy $|\lambda_i| < 1, i = 2, \cdots, N$. Then, it follows from Lemma 2 that for matrix $I_N - \mathcal{N}\mathcal{L}$, there exists a column vector ξ such that

$$\lim_{k\to\infty} (I_N - \mathcal{N}\mathcal{L})^k = \mathbf{1}\xi^T. \tag{16}$$

Besides, since $t_k \to T_s$ as $k \to \infty$, we obtain that $t_{k-1} - t_0$ is bounded, which ensures that each item of matrix $e^{A(t_{k+1}-t_0)}$ is bounded. It follows that

$$\lim_{k\to\infty} \Big[(I_N - \mathcal{N}\mathcal{L})^k - \mathbf{1}\xi^T\Big] \otimes e^{A(t_{k+1}-t_0)} = 0. \tag{17}$$

Let $x^*(t) = \sum_{i=1}^N \xi_i e^{A(t-t_0)} x_i(t_0)$ and $X^*(t) = \mathbf{1} \otimes x^*(t)$. One has $X^*(t) = [\mathbf{1}\xi^T \otimes e^{A(t-t_0)}]X(t_0)$. Thus, we have

$$\begin{aligned} &\lim_{k\to\infty} (X(t_{k+1}) - X^*(t_{k+1})) \\ &= \lim_{k\to\infty} \Big[\Big((I_N - \mathcal{N}\mathcal{L})^{k+1} - \mathbf{1}\xi^T\Big) \otimes e^{A(t_{k+1}-t_0)}\Big] X(t_0) \\ &= 0. \end{aligned}$$

Note that $X^*(t_k) = \mathbf{1} \otimes [\sum_{i=1}^N \xi_i e^{A(t_k-t_0)} x_i(t_0)]$. Thus, it can be concluded that the discrete states $x_i(t_k)$ will achieve consensus at an exponential rate as $k \to \infty$, i.e., $\lim_{k\to\infty} \|x_i(t_k) - x_j(t_k)\| = 0$.

Second, for the offline prespecified settling time T_s, we will prove that the discrete states $x_i(t_k)$ can achieve specified-time consensus as $t_k \to T_s$. According to the sampling sequence S with convergence at a polynomial rate, one has $\lim_{k\to\infty} t_k = T_s$. Thus, $\lim_{t_k \to T_s} \|x_i(t_k) - x_j(t_k)\| = \lim_{k\to\infty} \|x_i(t_k) - x_j(t_k)\| = 0, \; i, j = 1, 2, \cdots, N$. Therefore, for the prespecified settling time T_s, the discrete states $x_i(t_k)$ will achieve specified-time consensus at an exponential rate as $t_k \to T_s$.

Finally, we will prove that the continuous states $x_i(t)$ can achieve specified-time consensus as $t \to T_s$. Integrating (15) from t_k to t, we obtain

$$x_i(t) - e^{A(t-t_k)} x_i(t_k) = -\frac{1}{|\mathcal{N}_i| + 1} \int_{t_k}^{t} e^{A(t-\tau)} BB^T e^{-A^T(\tau-t_k)} d\tau \cdot \Phi^{-1} e^{A(t_{k+1}-t_k)} \sum_{s \in \mathcal{N}_i} \big(x_i(t_k) - x_s(t_k)\big)$$

$$t_k \leq t < t_{k+1}.$$

Let $\delta_i(t) = -\frac{1}{|\mathcal{N}_i|+1} \int_{t_k}^{t} e^{A(t-\tau)} BB^T e^{-A^T(\tau-t_k)} d\tau \cdot \Phi^{-1} e^{A(t_{k+1}-t_k)} \sum_{s\in\mathcal{N}_i} (x_i(t_k) - x_s(t_k))$. Thus, $x_i(t) - x_j(t) = e^{A(t-t_k)}(x_i(t_k) -$

$x_j(t_k)) + \delta_i(t) - \delta_j(t)$. Further, we have

$$\|x_i(t) - x_j(t)\| \leq \| e^{A(t_{k+1}-t_k)} \| \cdot \| x_i(t_k) - x_j(t_k) \| + \| \delta_i(t_k) \| + \| \delta_j(t_k) \| .$$

Besides, note that

$$\| \delta_i(t) \| \leq \frac{\| e^{A(t_{k+1}-t_k)} \|}{|\mathcal{N}_i|+1} \left\| \int_{t_k}^{t_{k+1}} BB^T e^{-A^T(\tau - t_k)} d\tau \right\| \cdot \| \Phi^{-1} \| \| e^{A(t_{k+1}-t_k)} \| \cdot \sum_{s\in\mathcal{N}_i} \| x_i(t_k) - x_s(t_k) \| .$$

Since the length of the time interval $t_k - t_{k-1} = T_k, k = 1, 2, \ldots\}$ is upper bounded, one has that $\| e^{A(t_{k+1}-t_k)} \|$ and $\| \int_{t_k}^{t_{k+1}} BB^T e^{-A^T(\tau-t_k)} d\tau \|$ are bounded. Furthermore, since (A, B) is controllable, it follows from Lemma 4 that Φ is invertible and $\Phi^{-1} = \frac{\Phi^*}{|\Phi|}$, where $|\Phi| \neq 0$. Thus, by assuming that $|\Phi| = \sum_{s=0}^{\infty} f_s(t_{k+1} - t_k)^s$, where $f_s, s = 0, 1, \cdots$, are coefficients, there exists at least a finite constant m such that coefficient $f_m \neq 0$ and $f_0 = f_1 = \cdots = f_{m-1} = 0$. Thus, $|\Phi|$ can be rewritten as $|\Phi| = f_m(t_{k+1} - t_k)^m + \circ(t_{k+1} - t_k)$, when $t_{k+1} - t_k$ is very small, where $\circ(t_{k+1} - t_k)$ presents the higher order infinitesimal of $t_{k+1} - t_k$. Since $\lim_{k\to\infty}(t_k - t_{k-1}) = \lim_{k\to\infty} T_k = \lim_{k\to\infty} \frac{\delta_k}{\Delta} T_s = 0$ converges at a polynomial rate, one has $\lim_{k\to\infty} \frac{\|x_i(t_k)-x_j(t_k)\|}{|\Phi|} = 0$. Since $\lim_{k\to\infty} t_k = T_s$, $\lim_{t_k\to T_s} \| \Phi^{-1} \| \| x_i(t_k) - x_j(t_k) \| = 0$. It is obtained that $\lim_{t\to T_s} \| \delta_i(t) \| = 0$. Similarly, one gets $\lim_{t\to T_s} \| \delta_j(t) \| = 0$. Therefore, $\lim_{t\to T_s} \|x_i(t) - x_j(t)\| = 0$.

To sum up, under the protocol (2), the linear multiagent system (1) can achieve consensus in the prespecified settling time T_s. The proof is completed. ■

Remark 2: The sampling time sequence S in (3) plays an important role in ensuring the specified-time consensus. A time-scale transformation is used to map the infinite convergence time to a limit time T_s. The reason is that the protocol (2) is decoupled from the sampling step, which means that the sampling step can be designed independently, according to the control objective. Thus, by designing S with the time-scale transformation in (3), one achieves the specified-time consensus. By designing S without constraints, one achieves asymptotical consensus.

Remark 3: Compared with the existing works [19], [22], [27], [34] on finite- and fixed-time consensus problems, in this technical note, the settling time can be offline prespecified, according to task requirements, without conservatively estimating the upper bound of the settling time. Thus, the proposed protocols in this technical note not only realize the consensus in the state space but also control the settling time in the time axis.

Remark 4: It is worth mentioning that the specified-time protocols designed in this technical note are based only on sampling measurements of the relative state information among its neighbors, which greatly reduce the cost of network communication [40]–[42].

Remark 5: Considering time-varying topologies, a key in Theorem 1 is the convergence rate of the discrete states $x_i(t_k)$. Under a time-invariant topology, due to (13), the discrete state $x_i(t_k)$ converges to consensus at an exponential rate. Under general time-varying topologies, one has $X(t_{k+1}) = (I_N - \mathcal{N}(t_k)\mathcal{L}(t_k)) \otimes e^{A(t_{k+1}-t_k)} X(t_k) = \prod_{s=1}^{k+1}[I_N - \mathcal{N}(t_s)\mathcal{L}(t_s)] \otimes e^{A(t_{k+1}-t_0)} X(t_0)$, with $\mathcal{N}(t_k) = \mathrm{diag}(\frac{1}{|\mathcal{N}_1(t_k)|+1}, \frac{1}{|\mathcal{N}_2(t_k)|+1}, \cdots, \frac{1}{|\mathcal{N}_N(t_k)|+1})$, and $\mathcal{L}(t_k)$ is the Laplacian at time t_k. Thus, a new assumption is required for the case with time-varying topologies. That is, the limit of $\prod_{s=1}^{k}[I_N - \mathcal{N}(t_s)\mathcal{L}(t_s)]$ converges to consensus at an exponential rate as $k \to \infty$. In particular, the case that the time-varying topologies are periodical with a fixed joint spanning tree in each period is feasible.

Remark 6: Notice that the protocol (2) with an infinite sequence S in (3) will generate the Zeno phenomenon as $t \to T_s$. Thus, S in (3) can be seen as an ideal sampling time sequence in theory. However, it cannot be applied to practical engineering. In order to avoid the Zeno phenomenon and to apply theoretical results to real practice, the ideal sampling time sequence S in (3) needs to be modified to satisfy practical constraints.

In engineering, one usually allows a consensus error bound d for the consensus. It means that the norm of the error between the state of every agent and the final consensus state is less than or equal to the given boundary d. Thus, after some calculations, one obtains

$$k_d \geq \log_{|\lambda_2|} \frac{d}{\|(I_N \otimes e^{AT_s})X(t_0)\|} \tag{18}$$

where k_d is referred to as the *sampling truncation* based on the consensus error bound d, and λ_2 is the eigenvalue of $I_N - \mathcal{N}\mathcal{L}$ with the largest modulus, except 1. Based on the ideal sequence S in (3), a modified sampling time sequence S_m is defined by

$$S_m = \left\{ t_k = t_{k-1} + T_k | \, t_0 = 0 \quad T_k = \left\{ \begin{array}{ll} \frac{\delta_k}{\Delta_m} T_s, & k \leq k_d \\ \varepsilon, & k > k_d \end{array} \right. \right\} \tag{19}$$

where $\Delta_m = \sum_{k=1}^{k_d} \delta_k$, and $\varepsilon > 0$ is an allowable small sampling step. Then, one has the following corollary.

Corollary 1: Suppose that Assumption 1 holds. For an offline prespecified settling time T_s and an error bound d, the distributed protocol (2) with the modified sampling time sequence S_m in (19) can guarantee that the states of the agents in (1) move into the d-neighborhood centered at the final consensus state within the prespecified time T_s, if (A, B) is controllable.

Remark 7: Notice that, by using the modified sampling time sequence S_m in (19), the distributed protocol (2) can ensure the specified-time-bounded error consensus with the boundary d. After the prespecified time T_s, with the allowable small sampling step ε, the agents will converge to the consensus asymptotically.

For example, consider the following typical multiagent systems.

1) For single-integrator dynamics, $\dot{\theta}_i = \omega_i, i = 1, 2, \cdots, N$, one has $A = 1$ and $B = 1$, which are obviously controllable. Thus, using (2), one obtains the following specified-time consensus protocol:

$$\omega_i = -\frac{1}{(t_{k+1} - t_k)(|\mathcal{N}_i| + 1)} \sum_{j\in\mathcal{N}_i} (\theta_i(t_k) - \theta_j(t_k)) \tag{20}$$

with S and S_m given by (3) and (19), respectively.

2) For double-integrator dynamics, $\dot{p}_i = q_i, \dot{q}_i = a_i, i = 1, 2, \cdots, N$, one has $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, which are obviously controllable. Thus, from (2), one obtains the following specified-time consensus protocol:

$$a_i = -\frac{6(t_{k+1} + t_k - 2t)}{(t_{k+1} - t_k)^3(|\mathcal{N}_i| + 1)} \sum_{j\in\mathcal{N}_i} \left(p_i(t_k) - p_j(t_k)\right) - \frac{2(2t_{k+1} + t_k - 3t)}{(t_{k+1} - t_k)^2(|\mathcal{N}_i| + 1)} \sum_{j\in\mathcal{N}_i} \left(q_i(t_k) - q_j(t_k)\right) \tag{21}$$

with S and S_m given by (3) and (19), respectively.

Corollary 2: Under Assumption 1, for single-integrator and double-integrator dynamics, the specified-time consensus problems can be solved by using protocols (20) and (21), respectively.

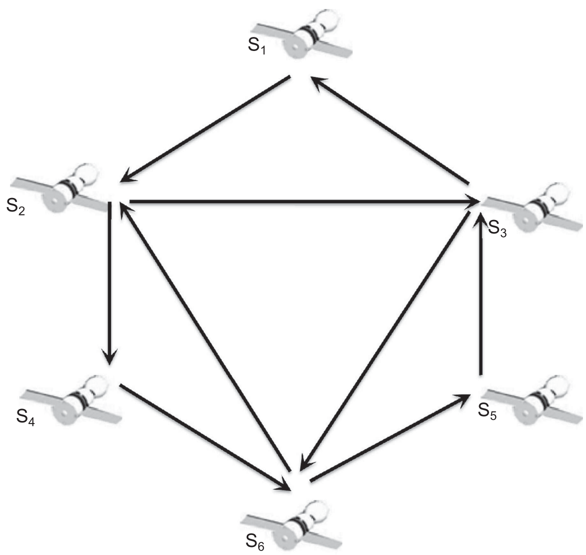

Fig. 1. Topology with six spacecrafts over a directed graph having a directed spanning tree.

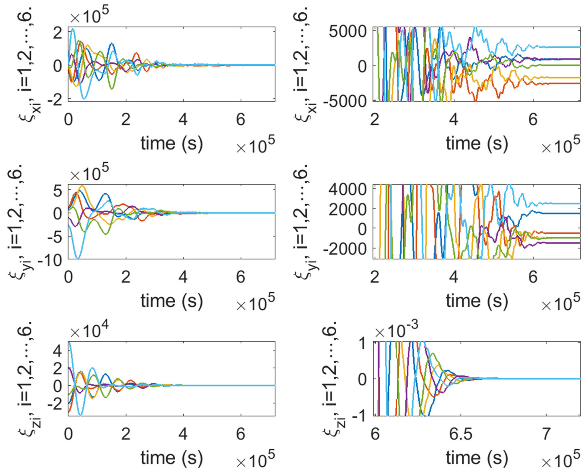

Fig. 2. Relative positions of the spacecrafts. The figures on the left side show the evolvements of the relative positions of the spacecrafts over a time interval of $[0, 200]$ h. The ones on the right side show the zoomed-in views.

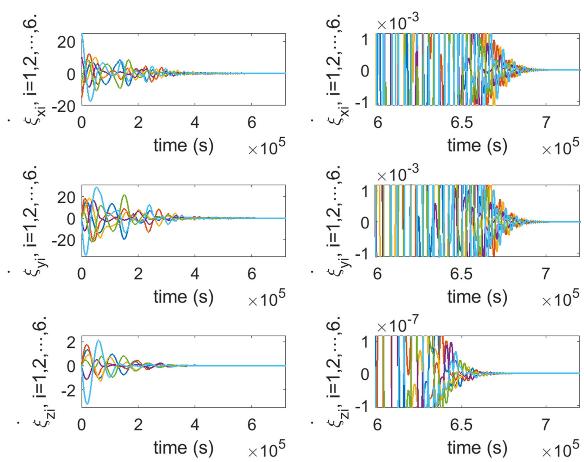

Fig. 3. Relative velocities of the spacecrafts. The figures on the left side show the evolvements of the velocities of the spacecrafts over a time interval of [0,200] h. The ones on the right side show the zoomed-in views.

Remark 8: The control inputs are affected by different settling times. From (3) and (19), one can see that a smaller T_s results in a smaller $T_k = t_k - t_{k-1}$. Obviously, from (20) and (21), a smaller $T_k = t_k - t_{k-1}$ results in larger control inputs. Thus, it is concluded that the shorter the specified settling time is, the larger the control inputs are. In real applications, under the constraints of control input saturation, the settling time T_s cannot be arbitrarily short. The smallest T_s with input saturation constraints is determined by the initial states, the topology, and the dynamics of agents.

IV. Application to Spacecraft Formation Flying

Consider the following formation flying of six satellites described by Hill's equations [35] with respect to a circular reference orbit of orbital radius $R_0 = 4.224 \times 10^7$ m:

$$\begin{bmatrix} \dot{r}_i \\ \ddot{r}_i \end{bmatrix} = \begin{bmatrix} 0 & I_3 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} r_i \\ \dot{r}_i \end{bmatrix} + \begin{bmatrix} 0 \\ I_3 \end{bmatrix} u_i, i = 1, 2, \cdots, 6$$

where $r_i = [r_{xi}, r_{yi}, r_{zi}]^T$ are three-dimensional (3-D) position coordinates of the ith agent, and

$$A_1 = \begin{bmatrix} 3{n_r}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -{n_r}^2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 2n_r & 0 \\ -2n_r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Based on protocol (2), a distributed specified-time formation control law for spacecraft i is given by

$$u_i(t) = -A_1 h_i - \frac{1}{|\mathcal{N}_i| + 1} B^T e^{-A^T(t-t_k)} \Phi^{-1} e^{A(t_{k+1}-t_k)} \times \sum_{j \in \mathcal{N}_i} \begin{bmatrix} r_i(t_k) - r_j(t_k) - h_i + h_j \\ \dot{r}_i - \dot{r}_j \end{bmatrix},$$

$$t_k \le t < t_{k+1}, \ i = 1, 2, \cdots, N \tag{22}$$

where $A = \left[\begin{smallmatrix} 0 & I_3 \\ A_1 & A_2 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 \\ I_3 \end{smallmatrix}\right]$. The natural frequency of the reference orbit is $n_r = 7.273 \times 10^{-5}$ s^{-1}. The directed communication topology between the spacecrafts is shown in Fig. 1. The desired formation is that the six satellites will maintain a regular hexagon with a separation of 1000 m. Thus, it is given that $h_1 = [0, 1000, 0]^T$ m, $h_2 = [-866, 500, 0]^T$ m, $h_3 = [-866, -500, 0]^T$ m, $h_4 = [0, -1000, 0]^T$ m, $h_5 = [866, -500, 0]^T$ m, and $h_6 = [866, 500, 0]^T$ m. The initial time is $t_0 = 0$ s. By choosing the sampling time sequence S_m in (19) with the offline prespecified formation time $T_s = 200$ h and $d = 1 \times 10^{-3}$ m, one has $k_d = 100$, $\delta_k = \frac{1}{(k+10)^2}, k = 1, 2, \cdots, k_d$, and $\Delta_m = 0.0861$.

Figs. 2 and 3 depict the relative positions ($\xi = [\xi_1^T, \xi_2^T, \cdots, \xi_6^T]^T = (\mathcal{L} \otimes I_3) r, \xi_i = [\xi_{xi}, \xi_{yi}, \xi_{zi}]^T, i = 1, 2, \cdots, 6$) and the relative velocities ($\dot{\xi} = [\dot{\xi}_1^T, \dot{\xi}_2^T, \cdots, \dot{\xi}_6^T]^T = (\mathcal{L} \otimes I_3)\dot{r}, \dot{\xi}_i = [\dot{\xi}_{xi}, \dot{\xi}_{yi}, \dot{\xi}_{zi}]^T$, $i = 1, 2, \cdots, 6$) among the spacecrafts over a time interval of $[0, 200]$ h, respectively. From the subgraphs on the right side of Fig. 2, one can see that the relative positions are kept constant. From the subgraphs on the right side of Fig. 3, one can see that the relative velocities converge to zero. The control forces are shown in Fig. 4. It can be seen that the

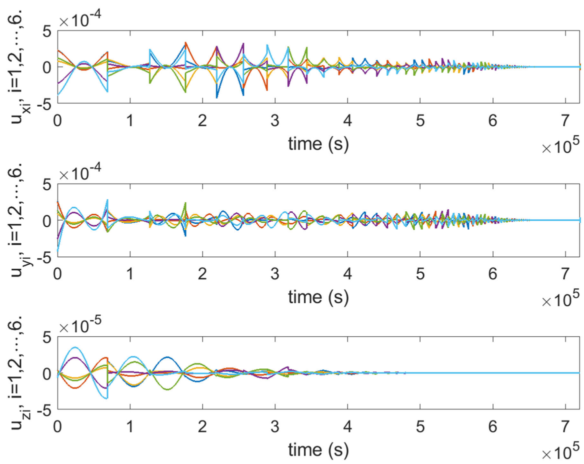

Fig. 4. Control inputs of the spacecrafts.

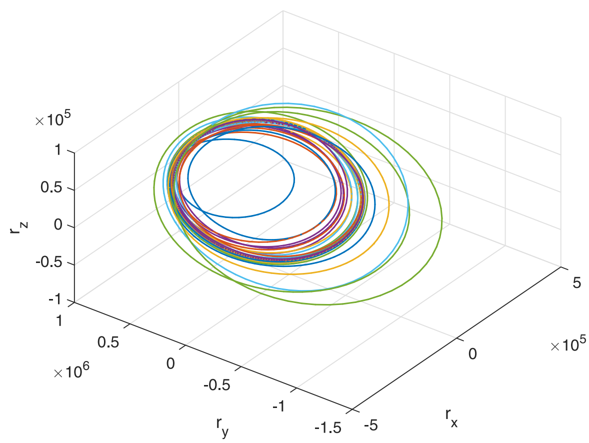

Fig. 5. Trajectories of the spacecrafts in 3-D space.

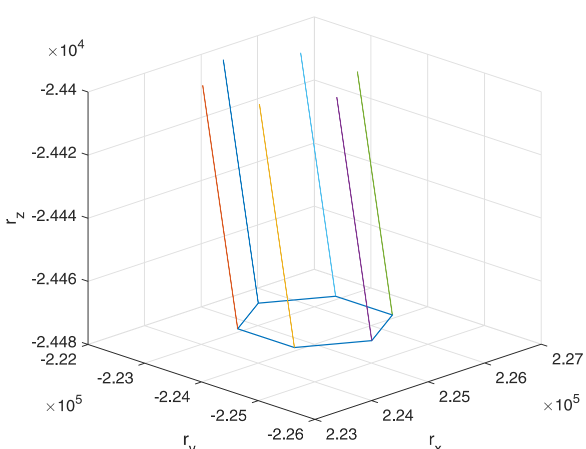

Fig. 6. Final formation configuration that is zoomed-in of the spacecrafts in 3-D space.

desired formation is derived at the specified settling time $T_s = 200$ h. The motion trajectories of these six spacecrafts in 3-D space are illustrated in Fig. 5. Fig. 6 shows the final formation configuration that is zoomed-in. The six agents form a regular hexagon, with about 1000 m on each side.

V. Conclusion

This technical note has studied the distributed specified-time consensus protocol design problem for multiagent systems with general continuous-time linear dynamics over directed graphs. A class of distributed specified-time consensus protocols have been developed. For linear multiagent systems, the proposed protocols solve the specified-time consensus problem for any directed graph containing a directed spanning tree. In particular, the settling time can be offline prespecified, according to task requirements.

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