Numerical Simulation of One-dimensional d'Alembert Equation

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Abstract

In this report we numerically study the wave propagation in one-dimensional strings. Both the Dirichlet boundary condition and mixed boundary condition is presented. In the simulation of standing waves, we consider the strings are released without initial velocity, or punched under a local impulse. And then we show the open string vibration with one end fixed. Finally, the effect of time step length is demonstrated.

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I. The discretization of dAlembert equation

The wave propagation can be described by the dAlembert equation:

$$\partial_t^2 y(x,t) - v^2 \partial_x^2 y(x,t) = 0 \tag{I.1}$$

with boundary condition given, we can numerically solve the equation above and simulate the wave propagation and excitation. To achieve this we first discretize the spacetime. The string of length L is meshed into 100 points, i.e., the spatial step is $\Delta x = L/100$. The temporal step is set as Δt. The spacetime (t,x) is then meshed as $(j\Delta t, k\Delta x)$ with $j,k = 1,2,\cdots$. The derivatives are then replaced by the finite difference:

$$\begin{aligned} \partial_t y(x,t_j) =& \frac{y(x,t_j) - y(x,t_{j-1})}{\Delta t} \\ \partial_x y(x_k,t) =& \frac{y(x_k,t) - y(x_{k-1},t)}{\Delta x} \end{aligned} \tag{I.2}$$

And for the 2nd-order derivatives:

$$\begin{aligned} \partial_t^2 y(x,t_j) =& \frac{1}{\Delta t^2}\left[y(x,t_{j+1}) + y(x,t_{j-1}) - 2y(x,t_j)\right] \\ \partial_x^2 y(x_k,t) =& \frac{1}{\Delta x^2}\left[y(x_{k+1},t) + y(x_{k-1},t) - 2y(x_k,t)\right] \end{aligned} \tag{I.3}$$

Then at the spacetime (j,k) we have

$$\begin{aligned} \partial_t^2 y(x_k,t_j) = v^2 \partial_x^2 y(x_k,t_j) \Rightarrow & \frac{1}{\Delta t^2}\left[y(x_k,t_{j+1}) + y(x_k,t_{j-1}) - 2y(x_k,t_j)\right] \\ =& \frac{v^2}{\Delta x^2}\left[y(x_{k+1},t_j) + y(x_{k-1},t_j) - 2y(x_k,t_j)\right] \end{aligned} \tag{I.4}$$

Then the time evolution is given by the iterative equation

$$y(x_k,t_{j+1}) = \frac{v^2\Delta t^2}{\Delta x^2}\left[y(x_{k+1},t_j) + y(x_{k-1},t_j)\right] + 2\left(1 - \frac{v^2\Delta t^2}{\Delta x^2}\right) y(x_k,t_j) - y(x_k,t_{j-1}) \tag{I.5}$$

The rest thing is to determine the boundary conditions.

II. Simulation of Standing waves

For the standing waves we fix the boundary point $y(0,t) = y(L,t) = 0$. And the string is released at rest, which means that $\partial_t y(x,t)\Big|_{t=0} = 0$. These are interpreted as

$$\begin{aligned} y(x_1,t_j) =& y(x_{101},t_j) = 0 \\ y(x_k,t_2) =& y(x_k,t_1) \end{aligned} \tag{II.1}$$

We consider two kinds of initial wave shape. The first is a half-wave with amplitude A:

$$y(x,0) = A\sin\frac{\pi x}{L} \tag{II.2}$$

The second is a propagating wave

$$y(x,0) = \begin{cases} A\sin\dfrac{2\pi x}{L} & 0 < x < L/2 \\ 0 & L/2 < x < L \end{cases} \tag{II.3}$$

When the string is at rest and suffers a sudden impulse, the vibration will propagate in the string. In the simulation we approximate the $\delta(x)$-type impulse as

$$\delta_a(x-x_0) = \frac{1}{a\sqrt{\pi}}\mathrm{e}^{-\left(\frac{x-x_0}{a}\right)^2} \tag{II.4}$$

where a is a parameter determining the width of the peak, and x_0 is the place suffering the impulse. The total impulse is

$$I_0 = \int \rho v(x,0)\mathrm{d}x = \int \partial_t y(x,0)\mathrm{d}x \tag{II.5}$$

and we have ($\rho = 1$ for simplicity)

$$\partial_t y(x,0) = y(x,t_2) = \frac{I_0\Delta t}{a\sqrt{\pi}}\mathrm{e}^{-\left(\frac{x-x_0}{a}\right)^2} \tag{II.6}$$

III. Open string vibration

For the string with one end fixed and another free, we have to add an additional iterative equation

$$y(x_{101},t_{j+1}) = \frac{v^2\Delta t^2}{\Delta x^2}\times y(x_{100},t_j) + 2\left(1-\frac{v^2\Delta t^2}{\Delta x^2}\right)y(x_{100},t_j) - y(x_{100},t_{j-1}) \tag{III.1}$$

IV. Effect of time step

In this section we choose time step $\Delta t = 0.005$, 0.01 and 0.02, and see the results. A smaller time step will require more computer memory and time to solve the iterative equation Eq. (I.5), and can get a more coherent and detailed picture, and vise versa. Besides, a large time step may cause the simulation fail because the coefficient $\dfrac{v^2\Delta t^2}{\Delta x^2}$ is too large. In conclusion, the scale $v\Delta t$ should be close to Δx.