Hexapod Kinematics with Geometric Algebra: A Practical Refactor

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1 Why Geometric Algebra (GA)?

Hexapod math based on per-joint rotation matrices and translations quickly accumulates frame bookkeeping. *Projective Geometric Algebra* (PGA) and *Conformal Geometric Algebra* (CGA) provide unified representations (motors/rotors) that act by a single sandwich product, reducing complexity and improving robustness.

PGA for SE(3). Rigid motions (rotations + translations) are represented by *motors*, which you can view as the geometric-algebraic analog of dual quaternions. Composition is associative; points are updated via $X' = MX\tilde{M}$.

CGA for incidence/distance. CGA encodes points as null vectors and treats planes, lines, circles, and spheres as native objects. Translations are rotors in CGA, and meet/join operations give concise formulas for projections, distances, and constraints.

2 Representations

2.1 PGA (practical)

We implement motors as dual quaternions $(r + \varepsilon d)$ with unit r. Given axis-angle $(\hat{\mathbf{u}}, \theta)$ and translation $\mathbf{t}$:

$$r = \cos\tfrac{\theta}{2} + \sin\tfrac{\theta}{2}\,(\hat{u}_x\mathbf{i} + \hat{u}_y\mathbf{j} + \hat{u}_z\mathbf{k}), \quad d = \tfrac{1}{2}\,\mathbf{t}\,r.$$

Composition is $(r_1, d_1)(r_2, d_2) = (r_1 r_2,\ r_1 d_2 + d_1 r_2)$. A point $\mathbf{p}$ transforms as

$$\mathbf{p}' = r\,\mathbf{p}\,r^* + 2\,\mathrm{vec}(d\,r^*).$$

This matches PGA motor action for SE(3).

2.2 CGA (parallel sketch)

CGA lifts $\mathbb{R}^3$ into a 5D space with two null directions e_0, e_∞. Points, planes, spheres, and circles become algebraic blades. Rotors (including translations) act by $X' = RX\tilde{R}$. We provide an API mirroring the PGA motor API; the current code routes to the PGA backend so you can wire usage now and swap later.

3 Forward and Inverse Kinematics

Forward Kinematics (FK). Define per-joint screw motions as motors and compose:

$$P_{\text{toe}} = M_B\, M_{\text{hip}}\, R_1\, M_{\text{thigh}}\, R_2\, M_{\text{knee}}\, R_3\, P_0\, \widetilde{(\cdot)}.$$

Our Go code builds these via `Screw` and `Translator`.

Analytic IK (per leg). Use line/point relations to recover joint angles: for a hip yaw, take the angle between the hip axis and the projected toe direction; for the planar thigh-knee pair, use cosine law on the triangle defined by link lengths and the hip→toe distance.

Numerical IK. Build Jacobian columns from twists. For a revolute joint about axis $(\mathbf{c}, \hat{\mathbf{u}})$, the instantaneous linear velocity of a point is $\hat{\mathbf{u}} \times (\mathbf{p} - \mathbf{c})$. Stack columns and solve $\Delta\theta = J^{\dagger}(\mathbf{x}^* - \mathbf{x})$ with damping.

4 Contact and Constraints

With CGA, the ground plane Π and toe point P satisfy $\Pi \wedge P = 0$ at contact. Projection is a meet/join expression. In the current package, you can emulate with vector math or upgrade the `cga` package later.

5 Migration Notes

Replace sequences of rotation matrices + translations with motors. Precompute static offsets as motors. FK becomes a product; IK uses either the analytic triangle plus axis-angle recovery or a Jacobian solver.

6 Complexity and Robustness

GA reduces trig and frame conversions, unifies operations, and offers a clear path to autodiff and MPC. Dual-quaternion motors are numerically stable and cheap to normalize.