Inferences on the Reliability Function for Progressive Type II Censored Data from Lomax Distribution using General and Classical Variable Method

Danush K. Wijekularathna, Bikal Lamichhane, Nikesh Shrestha

July 2019

Abstract

Lomax distribution has appealed to many scholars, statisticians, and business organizations in recent years. In this paper, we present a popular problem of hypothesis testing and interval estimation for the reliability function of progressively type II censored sample with fixed removal data from Lomax distribution. Two methods are compared: Classical variable method and Generalized variable method. Different algorithms presented in this paper are used to derive the confidence intervals and probability values numerically. We present an illustrative example along with a simulated example using Monte Carlo simulation for comparison of the reliability function under general and classical models. Finally, we conclude the paper comparing results from both methods.

Keywords

Lomax distribution, Reliability engineering, Progressive type II censored sample, Fixed removal, Reliability function, Classical variable method, General variable method, Confidence interval

1 Introduction

Reliability function gives the probability of a product, software, or patient's ability to operate for a particular length of time without facing failure. Reliability function is also known as survivor function or complementary cumulative distribution function. In today's world, quality, which defines an item's ability to stand to its specification, is a top priority for the consumers. Thus, reliability function has become one of the most important measurements of an item's quality. Reliability function is given by :

$$R(Y \leq y) = 1 - F(Y) = P(Y > y)$$

Reliability engineering is the sub-field of system engineering that foregrounds dependability of a product's life-cycle management while being cost-effective. The field deals with the evaluation, prevention, and administration of lifetime engineering uncertainty and risks of failure. Reliability engineering concentrates on the failure cost when there are various errors with the product such as failure's cost, repair cost and equipment, and so on. Even if the randomly selected parameters for a reliability function may define some parts of reliability of a product, the entire reliability of the product cannot be determined by the calculating statistics of the model. Nevertheless, many technical and scholarly fields around the globe use reliability function as a part of reliability engineering. Reliability is essential for a company's cost-effectiveness and overall development of a company.

Reliability theory mostly developed for use in the electronic industry as products tend to fail with little to no warning in the industry. A Hazard function is continuous for such a failed condition. There might be situations where hazard function is time-dependent - increasing or decreasing with time. Developing a lifetime model might make it feasible for component failure such as maintenance, part replacement, etc.

The Lomax distribution, also known as Pareto type II distribution, is a two-parameter heavy-tailed probability distribution first proposed by Lomax (1987). Lomax [ ] used this distribution to analyze business failure lifetime data. With a closer look, one can tell that Lomax is essentially a Pareto distribution which is adjusted such that its support begins at zero. The Lomax distribution is also considered as a mixture of the exponential gamma distribution. The distribution is recognized and used effectively in many fields such as economics, business, internet traffic modeling, actuarial science, medicine, engineering, and biological science. Lomax distribution has also received special attention in the theoretical researches.

The Lomax distribution is related to many distributions. Lomax distribution is a particular case of generalized Pareto distribution. Lomax distribution is also related to other distribution such as F distribution, logistic distribution, and Gamma-exponential mixture distribution. Lomax distribution provides a different perspective for lifetime data analysis compared to some common other distributions such as weibull, rayleigh, exponential or gamma distribution as it has a heavy tail and can assume a heavy-tailed population distribution. Dozens of scholarly articles have used various mixtures of the Lomax distribution. To name a few, a version of power lomax model

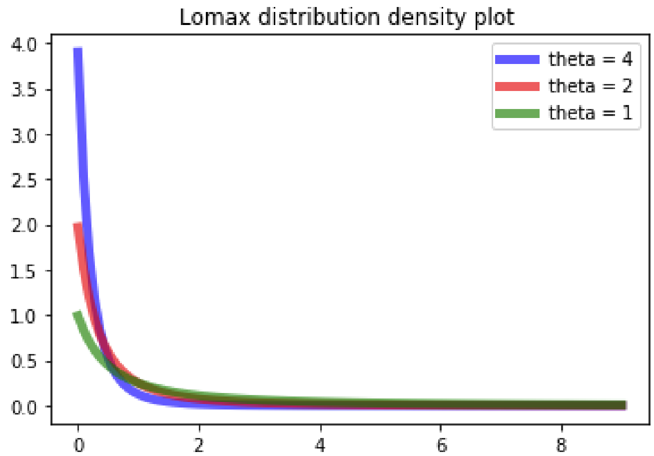

Figure 1: PDF of Lomax distribution

has been extensively used in the study for remission times of bladder cancer data in the paper by El-Houssainy A. Rady, W. A. Hassanein and T.A. Elhaddad in their paper [The power Lomax distribution with an application to bladder cancer data]. T. H. M. Abouelmagd introduced a Bur X Lomax model for its use on failure data.[A New Flexible Version of the Lomax Distribution with Applications]. Balakrishnan and Ahsanullah introduced some recurrence relations between the moments of record values from Lomax distribution.

The probability density function (pdf) for Lomax(θ,ζ) distribution model is given as:

$$f(y \leq Y) = \theta\zeta^{\theta}(y+\zeta)^{-(\theta+1)}$$

and the cumulative distribution function (cdf) is given as:

$$F(y \leq Y) = 1 - \left(\frac{\zeta}{y+\zeta}\right)^{\theta}$$

where θ is shape parameter and ζ is scale parameter. The reliability function R(t) at time t for the Lomax distribution is given by

$$R(y \leq Y) = \left(\frac{\zeta}{y+\zeta}\right)^{\theta}$$

Using the transformation $H = \log(\frac{\zeta+y}{\zeta})$, the lomax distribution can be converted to exponential distribution with rate parameter ζ. Thus, $H \sim \exp(\zeta)$. The density function of this transformed distribution H can be given by,

$$f(H,\zeta) = \zeta e^{-\zeta H}$$

And the reliability function of this transformed distribution H can be given by,

$$R(H,\zeta) = e^{-\zeta H}$$

2 Progressive type II censoring scheme

While conducting various real-life experiments on the reliability of products, many scenarios which includes failure of products, unintentional loss, etc. might occur in the process. Unintentional loss may be an accidental breakage of a studied unit or complete halt of the study due to exhaustion of funds and so on. Sometimes, the removal of units from the experiment is deliberate for saving time and cost. Progressive censoring, namely progressive type II censoring is studied as a part of such intentional experiment in this paper. Progressive type II right censoring has been studied by many scholarly authors such as Harter (1970), Mann, Schaferand Singpurwalla (1974), Bain (1978), Lawless (1982), Nelson (1982) just to name a few.

In the process of progressive type II censoring scheme, there are n units placed on the test. The person conducting the experiment decides the number of failures m that is supposed to be observed. After the failure of the first item, R_1 items of the remaining $n-1$ items that remained surviving are randomly removed from the experiment. After the second failed item, R_2 of the remaining $n - R_1 - 1$ items are randomly withdrawn from the experiment. Similarly, for the m-th failure of the item, the remaining surviving items $R_m = n - m - R_1 - R_2 - ... - R_{m-1}$ are withdrawn from the experiment.

The algorithm for computing progressive type II right censoring for Lomax distribution is given below:

Algorithm 1

1: Generate random variables U_i from independent uniform random distribution i.e. $U(0,1)$.
2. Simulate m independent exponential random variables Z_1, Z_2,...,Z_m using the inverse function $Z_i = -\log(1-U_i)$
3. Set

$$X_{i,m,n} = \frac{Z_1}{n} + \frac{Z_2}{n-R_1-1} + \frac{Z_3}{n-R_1-R_2-2} + ... + \frac{Z_i}{n-R_1-R_2-...-R_{i-1}-i+1}$$

where i = 1,2,...,m.
4. Use the function $Y_{i,m,n} = F^{-1}[1 - e^{(-X_{i,m,n})}]$. The $F^{-1}(x)$ is the inverse cdf of Lomax distribution. The $Y_{i,m,n}$ is the required progressively Type II right censored sample from Lomax distribution.

3 MLE and Reliability function

Let Y denote the lifetime of a product with Lomax density function. According to the progressive type II censoring n units are considered for the test. Here, $Y_{1,n} \leq Y_{2,n} \leq, ..., \leq Y_{m,n}$ are corresponding progressively type II right censored sample, with $R = (R_1, R_2, R_3, \ldots, R_m)$ censoring schema. Here, m denotes the number of failures observed before termination from n items which are on test. Order statistics of the data is $y_{1,n} \leq y_{2,n} \leq \ldots \leq y_{m,n}$. R_i is considered to be the total number of units removed at the i-th failure time such that $0 \leq R_1 \leq n-1$ and $R_m = n - \sum_{j=1}^{n-1} R_{j-i}$ where $i = 2, \ldots, m-1$ where R_i and m are pre-specified integers.

Now, we consider joint density of $Y_{1,n}, Y_{2,n}, \ldots, Y_{m,n}$ be

$$\prod_{i=1}^{m} K_i f_Y(y_{i,n}; \theta, \zeta)[1 - F_Y(y_{i,n}; \theta, \zeta)]^{R_i}$$

where

$$K_i = \sum_{j=1}^{i=1} (R_j + 1)$$

Likelihood function now can be derived as,

$$L(\theta, \zeta) = k\theta^m \zeta^{m\theta} \prod_{i=1}^{m} \zeta^{R_i \theta (y+\zeta)^{-(\theta(R_i+1)+1)}}$$

where k is a constant. The corresponding log-likelihood function is given by:

$$L(\theta, \zeta) = \log k + m \log \theta + m\theta \log \zeta + \sum_{i=1}^{m} \theta R_i \log \zeta - \sum_{i=1}^{m} (\theta(R_i + 1) + 1) \log(x + \zeta)$$

Computing partial derivative with respect to θ and ζ and equalizing the result to zero, we get

$$\frac{\partial L(\theta, \zeta)}{\partial \theta} = \frac{m}{\theta} + m \log \zeta + \sum_{i=1}^{m} R_i \log \zeta - \sum_{i=1}^{m} (R_i + 1) \log(x + \zeta) = 0$$

Therefore,

$$\hat{\theta} = \frac{m}{\sum_{i=1}^{m} (R_i + 1) \log(x + \hat{\zeta}) - \sum_{i=1}^{m} (R_i \log \zeta - m \log \hat{\zeta})}$$

Also,

$$\frac{\partial L(\theta, \zeta)}{\partial \zeta} = \frac{m\theta}{\zeta} + \sum_{i=1}^{m} \frac{\theta R_i}{\zeta} - \sum_{i=1}^{m} \frac{(\theta(R_i + 1) + 1)}{x + \zeta} = 0$$

Therefore,

$$\frac{1}{\hat{\zeta}} = \frac{1}{m} \Bigg[\sum_{i=1}^{m} \frac{(R_i + 1) + (\sum_{i=1}^{m} (R_i + 1) \log(x + \hat{\zeta}) - \sum_{i=1}^{m} (R_i \log \hat{\zeta} - m \log \hat{\zeta}) m^{-1}}{(x + \hat{\zeta})} - \sum_{i=1}^{m} \frac{R_i}{\hat{\zeta}} \Bigg]$$

Use of Newton-Raphson iteration can be used to solve the value for $\hat{\theta}$ and $\hat{\zeta}$. But, we assume ζ to be known. Then, by property of invariance of MLE, the MLE of R can be written as

$$\hat{R} = \left[\frac{\hat{\zeta}}{y+\hat{\zeta}}\right]^{\theta}$$

Suppose,

$$W = \sum_{i=1}^{m}(1+R_i)\log(y+\hat{\zeta}) + \sum_{i=1}^{m}(R_i\log\hat{\zeta} - m\log\hat{\zeta})$$

then, MLE of R can be written in the following way,

$$\hat{R} = \left[\frac{\zeta}{y+\zeta}\right]^{\frac{m}{W}}.$$

Also, we can show that, $V = 2\theta W = 2m\frac{\zeta}{\hat{\zeta}} \sim \chi^2_{2(m+1)}$

4 Classical Inference

For finding whether the reliability function cohere with the required level, the following algorithm is proposed. In the procedure, a pivotal quantity, $V = 2\theta W$ is presented where $W = \sum_{i=1}^{m}(1+R_i)\log(x+\hat{\zeta}) + \sum_{i=1}^{m}(R_i\log\hat{\zeta} - m\log\hat{\zeta})$. The alternative hypothesis is stated as the required reliability being larger than r^*, the target value, and the null hypothesis is stated as the required reliability being less than or equal to r^*. i.e. $H_0 : R \leq r^*$ and $H_a : R > r^*$. Rejection region is derived using the equation $\hat{R}|\hat{R} > R_0^*$ where R_0^* is the critical value and $\hat{R}$ is the MLE for R. For the given value of significance level α, the critical value R_0^* is calculated using the following:

$$\text{Sup } Pr(\hat{R} > R_0^*) = \alpha$$
$$Pr(\hat{R} > R_0^*|R = r^*) = \alpha$$
$$Pr((\tfrac{\zeta}{y+\zeta})^{\frac{m}{W}} > R_0^*|(\tfrac{\zeta}{y+\zeta})^{\theta}) = \alpha$$
$$Pr(\tfrac{m}{W}\log(\tfrac{\zeta}{y+\zeta}) > R_0^*|\theta\log(\tfrac{\zeta}{y+\zeta})) = \alpha$$
$$Pr(W < m\log(\tfrac{\zeta}{y+\zeta})/\log R_0^*|\theta = \log r^*/\log(\tfrac{\zeta}{y+\zeta}) = \alpha$$
$$Pr(2\theta W < 2\theta m\log(\tfrac{\zeta}{y+\zeta})/\log R_0^*|\theta = \log r^*/\log(\tfrac{\zeta}{y+\zeta}) = \alpha$$
$$Pr(V < 2m\log r^*/\log R_0^*) = \alpha$$

where, $V = 2\theta W \sim \chi^2_{2(m+1)}$.

Utilizing of the chi-square function, which is $\chi^2_{2(m+1),1-\alpha}$, the lower $100(1-\alpha)$th percentile of $\chi^2_{2(m+1)}$ can be derived i.e. $2m\log r^*/\log R_0^* = \chi^2_{2(m+1),1-\alpha}$. Thus, using the obtained function, critical value i.e.

$$R_0^* = e^{\left(\frac{2m\log r^*}{\chi^2_{2(m+1),1-\alpha}}\right)}$$

Here, r^* is the target value, α and m are target value, significance value and the total number of observed failures.

5 Lower bound for reliability using classical method

The provided pivotal quantity which is $V \sim \chi^2_{2(m+1)}$ and $\chi^2_{2(m+1),1-\alpha}$, characterizes the lower $100(1-\alpha)\%$ of $\chi^2_{2(m+1)}$. Steps for deriving the $100(1-\alpha)\%$ one sided confidence interval for reliability function R is given below:

$$Pr\Bigg(2\theta W \geq \chi^2_{2(m+1),1-\alpha}\Bigg) = 1-\alpha$$

$$\text{Here, } R = \left(\tfrac{\zeta}{y+\zeta}\right)^{\theta} \text{ and , } \hat{R} = \left(\tfrac{\zeta}{y+\zeta}\right)^{m/W}$$

$$Pr\Bigg(\frac{2\log R}{\log\left(\frac{\zeta}{y+\zeta}\right)} \frac{m2\left[\frac{2\log R}{\log\frac{\zeta}{\zeta+y}}\right]}{\log\hat{R}} \geq \chi^2_{2(m+1),1-\alpha}\Bigg) = 1-\alpha$$

$$Pr\Bigg(2m\frac{\log R}{\log\hat{S}} \geq \chi^2_{2(m+1),1-\alpha}\Bigg) = 1-\alpha$$

$$Pr\Bigg(R \geq e^{(\chi^2_{2(m+1),1-\alpha})\frac{\log\hat{S}}{2m}}\Bigg) = 1-\alpha$$

$$\text{i.e. } R \geq e^{\left(\chi^2_{2(m+1),1-\alpha}\right)\frac{\log\hat{R}}{2m}}$$

This equation is the $100(1-\alpha)\%$ one-sided confidence interval, i.e. $100(1-\alpha)\%$ classical lower limit for reliability function R. This can be written as follows:

$$LB_C(y) = e^{\left(\chi^2_{2(m+1)}\right)\frac{\hat{R}}{2m}}$$

Here, $\hat{R}$ denotes the MLE of R, α is the significance level and m denotes the number of observed failures before terminating the procedure.

6 Power for testing reliability using Classical approach

Power of an hypothesis testing is defined as the probability of rejecting the null hypothesis H_0 in a case when null hypothesis is actually false. Power is inversely related to type II error β. Type II error β arises by the acceptance of false null hypothesis. Thus, power can also be defined as avoiding type II error or false null hypothesis. For this paper, null hypothesis is $H_0 : R \leq r^*$ and $H_a : R > r^*$. Here, type II error occurs when R is more than r^*, but the null hypothesis, $H_0 : R \leq R^*$, is accepted as the truth. From the lower bound equation we get,

$$\hat{R}|\hat{R} > R_0^* = e^{\left(\frac{2m\log r^*}{\chi^2_{2(m+1),1-\alpha}}\right)}$$

The power $P(r_1^*)$ of the test at the point $R = r_1^* > r^*$ is given by:

$$Po(r_1^*) = Pr(\hat{R} > R_0^*|R = r_1^*)$$
$$Pr\Bigg(\big(\tfrac{\zeta}{\zeta+y}\big)^{\frac{m}{W}} > S_0^*|\big(\tfrac{\zeta}{\zeta+y}\big)^{\theta} = r_1^*\Bigg)$$
$$Pr\big(\tfrac{m}{W}\log(\tfrac{\zeta}{\zeta+y}) > \log R_0^*|\theta\log(\tfrac{\zeta}{\zeta+y}) = \log r_1^*\big)$$
$$Pr(W < m\log\big(\tfrac{\zeta}{\zeta+y}\big)/\log R_0^*|\theta = \frac{\log r_1^*}{\log\big(\frac{\zeta}{\zeta+y}\big)}$$
$$Pr(2\theta W < 2\theta m\log\big(\tfrac{\zeta}{\zeta+y}\big)/\log R_0^*|\theta = \frac{\log r_1^*}{\log\big(\frac{\zeta}{\zeta+y}\big)}$$
$$Pr\Bigg(2\theta W < 2m\frac{\log r_1^*}{\log R_0^*}\Bigg)$$
$$Po(r_1^*) = Pr\Bigg(V < \frac{\log r_1^*}{\log r^* \chi^2_{2(m+1),1-\theta}}\Bigg)$$

where $V \sim \chi^2_{2m}$

Algorithm to construct power using Classical approach $P(r_1^*)$

Algorithm 2

1: Use Algorithm 1 to generate a progressive Type II sample from Lomax distribution.
2. Calculate the value of $\hat{R} = (\frac{\zeta}{\zeta+X})^{\hat{\theta}}$, where $\hat{\theta} = \frac{m}{\sum_{i=1}^{m}(R_i+1)\log(x_i+\hat{\zeta})-\sum_{i=1}^{m}(R_i\log\zeta)-m\log\hat{\zeta}}$
3. Calculate $R_0^* = e^{\left(\frac{2m\log r^*}{\chi^2_{2(m+1),1-\alpha}}\right)}$
4. If $\hat{R} > R^{0*}$, then count = 1, else count = 0
5. Step 1, 2, 3 and 4 are repeated 1000 times and estimated power $P(r_1^*)$ is calculated as $\hat{P}(r_1^*)$ = Total count / 1000
6. Step 5 is repeated 100 times and $\hat{P}_i(r_1^*)$ are obtained. Here i = 1, 2,.., 100. Mean is calculated for $\hat{P}(r_1^*)$ as $\overline{\hat{P}(r_1^*)} = \frac{\sum_{i=1}^{100}\hat{P}_i(r_1^*)}{100}$
7. Mean square error (MSE) for $\hat{P}_i(r_1^*)$ is calculated using $MSE = \frac{\sum_{100}^{i=1}[\hat{P}_i(r_1^*)-P(r_1^*)]^2}{100}$

7 Generalized variable approach

The dominant system used in the statistician world is the classical approach. Bayesian framework has also had a tremendous impact in many fields. Both of these frameworks are well-established, studied and employed in all kind of fields such as medicine, business, education etc. A newly developed framework, known as the generalized variable method, was recently developed by Tsui and Weerahandi [27]. Weerahandi also presented a generalized confidence interval in his paper [26] as an alternative to classical approach for statistical inference. The generalized inferences have been extensively studies and used by many scholars for solving numerous statistical problems ever since the frameworks development. The newly developed general variable approach have been used in many scholarly articles and is used for various analysis such as Analysis of Variance (ANOVA), Multivariate Analysis of Variance (MANOVA), Analysis of Covariance (ANCOVA), Multivariate Analysis of Variance (MANCOVA) etc.

8 Lower Bound for reliability using Generalized Approach

For the derivation of one-sided generalized $100(1-\alpha)\%$ confidence interval for reliability function R, the following algorithm is proposed.

Let Y be the lifetime of the product with Lomax distribution model. For the test, n units are considered. The progressively type II right censoring for the Lomax model is given as $Y_{1,n}, Y_{2,n}, ..., Y_{m,n}$ with censoring schema $R = (R_1, R_2, ..., R_m)$. Here, m denotes total number of failed observed items. The order statistics for observed items considered is $y_{1,n}, ..., y_{m,n}$ such that $0 \leq R_1 \leq n-1$ and $R_m = n - \sum_{j=1}^{i-1} R_{j-i}$, where $i = 2, 3, ...m-1$.

A generalized pivotal statistics for θ i.e.$E^{\theta}(y;\mu)$ is introduced such that $\mu = (\theta, \zeta)$ satisfies the property which states observed value of E^{θ} is θ and the distribution function of E_{θ} is free of any unknown parameters. Thus, by following this, a generalized pivotal statistics for reliability function R i.e. E^R is defined such that $E^R = \left(\frac{\zeta}{\zeta+y}\right)^{E^R}$ for given ζ. The estimated value for θ is $\hat{\theta}_{obs} = \frac{m}{\sum_{i=1}^{m}(1+R_i)\log(y+\hat{\zeta})+\sum_{i=1}^{m}(R_i \log\hat{\zeta} - m\log\hat{\zeta})}$. Using $V = 2\theta W \sim \chi^2_{2(m+1)}$, it can be seen that E^{θ} is given by:

$$E^{\theta} = \frac{U\hat{\theta}_{obs}}{2m} = \frac{V}{2W}$$

Here,

$$W = \sum_{i=1}^{m}\sum_{i=1}^{m}(1+R_i)\log(y+\hat{\zeta}) + \sum_{i=1}^{m}(R_i \log\hat{\zeta} - m\log\hat{\zeta})$$

Thus, the generalized pivotal statistics for reliability function R can be written as: $E^R = e^{\frac{V\hat{\theta}_{obs}}{2m}\log\left(\frac{\zeta}{\zeta+y_{i,n}}\right)}$
Suppose, $E^R_{1-\alpha}(y_{m,n};\hat{\theta}_{obs})$ satisfies the following equality:

$$Po\big(E^R \leq E^R_{1-\alpha}(y_{m,n};\hat{\theta}_{obs})\big) = 1-\alpha$$

The $E^S_{1-\alpha}(y_{m,n};\hat{\theta}_{obs})$ is defined as the $100(1-\alpha)\%$ lower confidence limit for the reliability function S i.e. lower confidence limit is:

$$LB_G(y) = E^R_{1-\alpha}(y_{m,n};\hat{\theta}_{obs})$$

9 Generalized p-value and power for reliability function

For the statistical testing procedure using generalized procedure, a hypotheses testing can be conducted. For the testing, the null hypothesis is $H_0 : R \leq r^*$ and the alternative hypothesis is $H_a : R > r^*$. The general test variable $J^s(y;\phi) = E^s_{1-\alpha}(y;\phi) - R$, which is the one-sided hypothesis testing for reliability function R is acquired. The target value is r^* and the assumption is that required reliability function is larger than the threshold value. Here, the p-value from the general variable method or the generalized p-value denoted by Pr_g is constructed as:

$$Pr_G = Pr(e^{\left[\frac{V\hat{\theta}_{obsv}}{2m}\log\left(\frac{\zeta}{\zeta+x_{i,n}}\right)\right]} \geq s^*)$$

The probability value Pr_g can be approximated using the numerical integration. Monte Carlo simulation is much more accurate for approximation the accuracy of the procedure. Here, Pr_g measures the evidence in support of null hypothesis. This p-value can be used to reject the null hypothesis H_0 such that $Pr_g < \phi$, where ϕ is desired nominal level.

The following algorithm can be used to calculate p-value:

Algorithm 3

1: Use Algorithm 1 to generate a progressive Type II sample from Lomax distribution.
2. Calculate the value of $\hat{\theta}_{obsv} = \frac{m}{\sum_{i=1}^{m}(R_i+1)\log(x_{i,n}+\hat{\zeta}) - \sum_{i=1}^{m}(R_i\log\zeta) - m\log\hat{\zeta}}$
3. Generate $V \sim \chi^2_{2(m+1)}$
4. Compute $E^\theta = \frac{V\theta_{obsv}}{2m}$
5. Compute $E^R = e^{\left(E^\theta \log\left[\frac{\zeta}{\zeta+x_{i,n}}\right]\right)}$
6. If the value of E^R greater than r^*, add 1 to COUNT
7. Repeat step $4, 5$ and 6 for 100 times.
8. The mean of COUNT is the generalized p-value.

10 Numerical examples

For the purpose of illustrating an application of using the mentioned procedures for finding the confidence interval of reliability function, we propose two examples: a real-life data set and a simulated data set.

10.1 Testing of electrical insulating fluid data set

The data set in section 10.1 is a real-life data set examined by Lawless [28]. The data set considers breaking down time of an insulating fluid between the electrodes at a $34kV$ voltage. Considering $n = 19$ data given in the table 1, and selecting a progressive type II censoring of size $m = 8$ with censoring schema as $R = (0, 0, 0, 0, 0, 0, 0, 11)$, we estimated the two parameters for the data set using Gini index, SSE and graphical procedure. The procedure for finding the estimated parameter for the data set is given below:

The ordered data list is given as:

0.19	0.78	0.96	1.31	2.78	3.16	4.15
4.67	4.85	6.50	7.35	8.01	8.27	12.06
31.75	32.52	33.91	36.71	72.89		

Progressively type II censored sample of size m = 8 for our procedure is given by:

0.19	0.78	0.96	1.31	2.78	3.16	4.15	4.67

Here, the data set follows a lomax distribution. The following algorithm is proposed for deriving the parameters of the distribution:

Algorithm 4

---

1. Consider Y to follow a lomax distribution with cdf

$$F(y) = 1 - \left(\frac{\zeta}{y+\zeta}\right)$$

where, $y > 0, \zeta > 0, \theta > 0$
2. Assume $Y_{1,n} \leq Y_{2,n} \leq ... \leq Y_{m,n}$ to be the progressively type II right censored sample data from the given data set with censoring schema to be $R = (R_1, R_2, ..., R_m)$. Then, we can assume the expectation value of $F_Y = 1 - \prod_{j=m-i+1}^{m} {}_{t_j} t_j + 1$ such that $i = 1, 2, ...m$ and $t_j = j + \sum_{i=m-j+1}^{m} R_i$
3. Compute $\theta = -\log\left(1 - \left(1 - \prod_{j=m-i+1}^{m}\left(\frac{t_j}{t_j+1}\right)\right)/log\left(\frac{\zeta}{\zeta+y}\right)\right)$
4. Lease square estimation method for θ is calculated using the formula $SSE = \sum_{i=1}^{m}\left(\theta_i - \hat{\theta}_i\right)$, where $\hat{\theta} = \frac{m}{\sum_{i=1}^{m}(R_i+1)\log(x+\hat{\zeta}) - \sum_{i=1}^{m}(R_i \log\zeta - m\log\hat{\zeta})}$

5. For the scale-free goodness-of-fit test of exponential distribution based on Gini Statistics, we consider the null hypothesis as $H_0 : Y \sim$ Lomax distribution with pdf $f(Y) = \theta 0.47^{\theta}(y + 0.47)^{-(\theta+1)}$.
6. The formula for Gini index is given by:

$$G = \sum_{m-1}^{i=1} \frac{iU_{(i+1)}}{m-1} \sum_{i=1}^{m} U_i$$

where, $U_i = (m - i + 1)(T_i - T_{(i-1)}), U_0 = 0, T_1 = nX_1, T_i = n - \sum_{i=1}^{i-1}(R_j + 1)(X_i - X_{(i-1)}) Also, i = 1, 2, ...m.$ and $X_i = \log(1 + \frac{y}{\zeta})$
7. In the process, we consider the values of m to be in-between 3 to 20. The two critical values are $\eta_{0.025}$ and $\eta_{0.975}$, which are the $100(\alpha/2)$ and $100(1-\alpha/2)$the percentile of gini index.

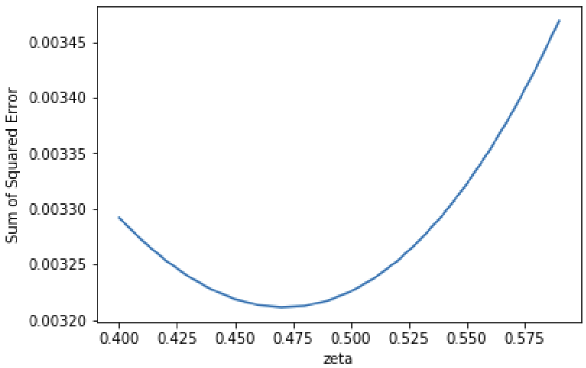

Figure 2: Minimum SSE for value of ζ

From our procedure, it is found that the minimum SSE is observed when $\zeta = 0.47$. Also, our estimated value for θ is given by $\hat{\theta} = \frac{m}{\sum_{i=1}^{m}(R_i+1)\log(x_i+\hat{\zeta})-\sum_{i=1}^{m}(R_i\log\zeta)-m\log\hat{\zeta}} = 0.1795$ using the algorithm 5 .
From our formula,

$$G = \sum_{m-1}^{i=1} \frac{iU_{(i+1)}}{m-1} \sum_{i=1}^{m} U_i = 0.344$$

Here, $\eta_{0.025} = 0.31232$ and $\eta_{0.975} = 0.68768$. Our value is 0.344 which is greater than $\eta_{0.025}$ and less than $\eta_{0.975}$. Thus, we do not reject null hypothesis H_0 at 0.05 significance level and conclude that the observed failure data is from Lomax distribution with pdf $f(Y) = \theta 0.47^{\theta}(y+0.47)^{-(\theta+1)}$. The 95% reliability interval estimate are compared and given in the table below:

	Point Estimates	Classical	Generalized
0	0.814907	0.7227 - 0.8762	0.7284 - 0.879
1	0.667439	0.5265 - 0.7702	0.5348 - 0.775
2	0.643684	0.4971 - 0.7523	0.5056 - 0.7575

Figure 3: P-value table

	x	s	Classical	General
0	1	0.81	0.054	0.019
1	4		0.029	0.009
2	5		0.026	0.017

Figure 4: A picture of the universe!

From the table, its clear that the generalized variable approach is much more reliable than the classical variable approach. When $x = 1$, we can see that classical method gives a value of 0.054, which is slightly greater than the significance value 0.05, but the general approach gives a p-value of less than the significance value. Also, we see that the p-value given by the general approach is always less than the p-value obtained using the classical approach. Hence, we can conclude that generalized approach is much more reliable and accurate than the classical method.

10.2 Simulated data Set

For the application of our general and classical approach on simulated data, we use algorithm 1 to generate progressively type II right censored lomax sample. The following algorithm gives testing procedure about reliability function R.

Algorithm 5

---

1. The null hypothesis $H_0 : R \leq r^*$ and $H_a : R > r^*$ are given such that s^* is the determined lower reliability function.
2. Calculation of $\hat{R}$ is done with the help of the formula $\hat{R} = (\frac{\zeta}{y+\zeta})^{\frac{m}{W}}$,
where $W = \sum_{i=1}^{m}(1 + R_i)\log(y + \hat{\zeta}) + \sum_{i=1}^{m}(R_i \log \hat{\zeta} - m \log \hat{\zeta})$
3. The critical value R_0^* is calculated using the formula:
$R_0^* = e^{\left(\frac{2m \log r^*}{\chi^2_{2(m+1),1-\alpha}}\right)}$
where, α is significance level and r^* is the target value.
4. For $\hat{R} > R_0^*$, we conclude that the reliability function satisfies the required level.

	theta	zeta	r*	Generalized	Classical
0	2	1	0.50	0.1231	0.0634
1	3	2	0.71	0.1228	0.0601
2	3	2	0.21	0.1249	0.0674
3	4	3	0.54	0.1202	0.0690
4	2	1	0.50	0.1099	0.0660
5	3	2	0.71	0.1083	0.0713
6	3	2	0.21	0.1099	0.0676
7	4	3	0.54	0.1146	0.0658
8	2	1	0.50	0.1013	0.0730
9	3	2	0.71	0.1047	0.0703
10	3	2	0.21	0.0993	0.0671
11	4	3	0.54	0.0980	0.0703

	theta	zeta	r_star	Generalized	Classical
0	2	1	0.50	0.076	0.035
1	3	2	0.71	0.066	0.025
2	3	2	0.21	0.078	0.024
3	4	3	0.54	0.065	0.021
4	2	1	0.50	0.043	0.036
5	3	2	0.71	0.055	0.020
6	3	2	0.21	0.053	0.028
7	4	3	0.54	0.045	0.028
8	2	1	0.50	0.036	0.032
9	3	2	0.71	0.047	0.029
10	3	2	0.21	0.038	0.029
11	4	3	0.54	0.049	0.029

Figure 5: sdfsdf

For the coverage of confidence interval using classical and general approach, we propose $y = (0.01, 0.02, 0.04), \theta = (2, 3, 3, 4), \zeta = (1, 2, 2, 3)$. We used the proposed values to compute the type I error rate for reliability using both the methods. The type I error is define as the rejection of the null hypothesis when it is true. Here, $H_0 : R \leq r^*$ and $H_a : R > r^*$. So, the type I error is the selection of events when p-value is less than the significance level. We choose the significance level as $\alpha = 0.10$ and $\alpha = 0.05$. The following table gives the following results.

	zeta	theta	LBc(0.100000)	LBg(0.100000)	LBc(0.200000)	LBg(0.200000)	LBc(0.500000)	LBg(0.500000)
0	0.5	1.5	0.94	0.91	0.91	0.90	0.90	0.92
1	0.5	2.5	0.95	0.91	0.94	0.94	0.92	0.92
2	0.5	3.5	0.88	0.86	0.94	0.95	0.95	0.89
3	1.0	1.5	0.93	0.87	0.92	0.95	0.93	0.89
4	1.0	2.5	0.96	0.95	0.91	0.97	0.94	0.89
5	1.0	3.5	0.91	0.90	0.94	0.95	0.93	0.92
6	1.5	1.5	0.91	0.93	0.89	0.88	0.94	0.91
7	1.5	2.5	0.94	0.94	0.93	0.95	0.93	0.95
8	1.5	3.5	0.93	0.89	0.95	0.92	0.87	0.90

Figure 6: sdfsdf

The table clearly suggests that the generalized approach is far superior to the common classical approach. In this Monte Carlo simulation, the type I error rates, which is the actual size of the tests, for generalized variable approaches very close to 0.10 and 0.05, which is the significance level and the desired value. The classical method does not get the values as close as the generalized method does. The coverage of the confidence interval based on the lower bound using both generalized and classical approach shows that generalized method approaches the desired value in a much better manner as compared to the classical method. Therefore, we can conclude that the general variable approach is better for this case.

11 Conclusion

In the article, we put forward a simple, yet intuitive comparison between two methods for inferring statistics of reliability function of Progressively type II right censored sample from a Lomax distribution. Both the methods are quite reliable, but generalized variable method tend to be more definitive in some cases. For consistency, Monte Carlo simulation of a million runs is recommended. In the comparison of the p-values for both methods, we saw that p-values for generalized variable method are significantly lower than the classical method. The coverage of confidence interval based on the lower bound of reliability function is also better while using generalized approach compared to the classical method. Nevertheless, both methods can be employed to tackle real-life complicated computational problems.

References

[1] Montgomery DC, *"Introduction To Statistical Quality Control"*. John Wiley Sons, New York, NY, USA, 1985.

[2] Sumith Gunasekera, *"Inference for the reliability function based on progressively type II censored data from the Pareto model, The generalized variable approach"*. Journal of Computational and Applied Mathematics (2018), vol. 343, 275-288

[3] N. Balakrishnan, R. Aggarwala , *"Progressive Censoring Theory, Methods and Applications"*. Birkhäuser Publisher (2000), Boston

[4] N. Balakrishnan, R. A. Sandhu, *"A Simple Simulation Algorithm for Generating Progressive Type II Censored Samples"*. American Statistician (1995), 49, no.2, pages 229-230

[5] Mahmoud Mohamed Abdul Wahab and El-Sagheer Rashad Mohamed and Soliman Ahmed Abo-Elmagd and Abd Ellah Ahmed Hamed,*"Inferences of the Lifetime Performance Index with Lomax Distribution Based on Progressive Type-II Censored Data"*. Stochastics and Quality Control, De Gruyter (2014), vol. 29(1), pages 1-13, June.

[6] Lee WC, Wu JW, Hong ML, Lin LS, Chan RL, *"Assessing the lifetime performance index of Rayleigh products based on the Bayesian estimation under progressive type-II right censored samples"*. J Comput Appl Math, (2011), page 235, 1676–1688.

[7] Jianhua Leng, *"Bayesian shrinkage estimation of shape parameter of Lomax distribution based on record values"*. International Conference on Multimedia Technology (2011), 2414-2417

[8] Lee WC, Wu JW, Hong CW. *"Assessing the lifetime performance index of products from progressively type-II right censored data using Burr XII model"*. Math Comput Simulat (2014), 79(3), 2167–2179

[9] Sumith Gunasekera, *"Inference for the Burr XII Reliability inder Progressice Censoring with Random Removals"*. Math Compute Simulation (2018), vol. 144, 181-195

[10] E.Cramer, A. B. Schmiedt, *"Progressively Type-II censored competing risks data from Lomax distributions"*. Journal of Computational and Applied Mathematics (2011), 55, 1285-1303

[11] M.H. Gail, J.L. Gastwirth, *"A scale-free goodness-of-fit test for exponential distribution based on the Gini statistic"*. Journal of the Royal Statistical Society, Series B, 40 (1978), 350-357

[12] Sumith Gunasekera, *"Classical, Bayesian, and generalized inferences of the reliability of a multicomponent system with censored data"*. Journal of Statistical Computation and Simulation (2018), 18, 3455-3501

[13] Gholizadeh, R, M Khalilpor, and M Hadian, *"Bayesian estimations in the Kumaraswamy distribution under progressively type II censoring data"*. International Journal of Engineering, Science and Technology (2011), Vol. 3, No. 9, pp. 47-65

[14] K.S. Lomax, *"Business Failures: Another Example of the Analysis of Failure Data"*. Journal of the American Statistical Association (1954), Vol. 49, No.268, pp. 847-852

[15] Wen-Chuan Lee, Jong-Wuu Wu, Ching-Wen Hong, and Shie-Fan Hong, *"Evaluating the Lifetime Performance Index Based on the Bayesian Estimation for the Rayleigh Lifetime Products with the Upper Record Values"*. Journal of Applied Mathematics (2013), Article ID 547209, 13 pages, http://dx.doi.org/10.1155/2013/547209.

[16] Lee WC, Wu JW, Hong CW, *"Assessing the lifetime performance index of products with the exponential distribution under progressively type-II right censored samples"*. J Comput Appl Math 2009,231,648–656

[17] Ahmadi MV, Doostparast M, Ahmadi J, *"Estimating the lifetime performance index with Weibull distribution based on progressive first-failure censoring scheme"*. J Comput Appl Math 2013, 239, 93–102

[18] Habibullah M, Ahsanullah M, *"Estimation of parameters of a Pareto distribution by generalized order statstics"*. Commun Stat Theory 2000,29,1597–1609

[19] H.K. Iyer, C.M. Wang, T. Mathew, *"Models and Confidence Intervals for True Values in Interlaboratory Trials"*, Journal of the American Statistical Association (2004). vol. 99, 1060-1071

[20] Sumith Gunasekera, *"Generalized Inference of R = Pr(X > Y) for Pareto Distribution"*. Stat. Pap. (Née Stat. Hefte) (2015) vol.56 (2), 333–351. http://dx.doi.org/10.1007/s00362-014-0584-8.

[21] Masood Anwar and Jawaria Zahoor, *"The Half-Logistic Lomax Distribution for Lifetime Modeling"*. Journal of Probability and Statistics, vol. 2018, Article ID 3152807, 12 pages, 2018. https://doi.org/10.1155/2018/3152807.

[22] Abdul-Moniem, Ibrahim and F Abdel-Hameed, H, *"On exponentiated Lomax distribution"*. International Journal of Mathematical Education (2012), vol. 33, 1-7

[23] Sumith Gunasekera, *"Generalized inferences for the expected winning bids"*. Amer. J. Math. Management Sci. (2016), vol. 35, 333-351

[24] Sumith Gunasekera and M.M.A. Ananda, *"Generalized variable method inference for the location parameter of the general half-normal distribution"*. J. Stat. Comput. Simul. (2015), vol. 85, no.10, 2115-2132

[25] S. Weerahandi, *"Exact Statistical Methods for Data Analysis"*. Springer Verlag, New york (1995)

[26] S. Weerahandi, *"Generalized Confidence Intervals"*. J. Amer. Statist. Assoc. (1993), vol. 88, no. 423, 899-905

[27] K. Tsui and S. Weerahandi, *"Generalized p-values in significance testing of hypothesis in the presence of nuisance parameters"*. J. Amer. Statist. Assoc. (1989), vol. 84, no. 406, 602-607

[28] Lawless JF, *"Statistical models and methods for lifetime data"*. NewYork, Wiley, 1982