A Framework for Substructural Type Systems

James Wood[1]   Robert Atkey[1]

[1]University of Strathclyde

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Mechanisation of type systems

- Boring syntactic operations and lemmas
- Learn how to do syntax well [AR99; McB05; BHKM12].
- Substructural systems are even harder!

Frameworks for type theories

- A framework for substructural type systems in a general-purpose proof assistant (Agda)
- Prior work: simple types [AACMM21]
  - Renaming and substitution for *all* syntaxes
  - Renaming/substitution fusion laws
  - Scope checking, type checking
  - Printing, optimisation passes, NbE
- This work: similar for *linear* and *modal* type systems.
  - DILL [Bar96], modal S4 [PD99], graded [OLE19]
  - Semiring-annotated usage-aware calculi

Linearity is not simple

- Key invariant of [AACMM21]: contexts are only ever...
  - added to (binding a variable)
  - accessed by the variable rule
- Motivates an abbreviated notation for typing rules.

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- Relying on details of the context: [Wad92]

$$\frac{!A_1, \ldots, !A_n \vdash B}{!A_1, \ldots, !A_n \vdash !B}$$

Semiring usage annotations

- Enrich the judgemental syntax to account for substructurality.
- $r_1 x_1 : A_1, \ldots, r_n x_n : A_n \vdash B$
- $r_1, \ldots, r_n \in \mathscr{R}$, where $\mathscr{R}$ is a (partially ordered) semiring.
- Annotations can appear in types via $!r$.
- Generalises beyond linearity: [OLE19; AB20; WA21]
  - Linearity: $\mathscr{R} = (\{0, 1, \omega\}, \ldots)$
  - S4 modal logic: $\mathscr{R} = (\{\mathsf{unused}, \mathsf{true}, \mathsf{valid}\}, \ldots)$
  - Monotonicity: $\mathscr{R} = (\{\equiv, \uparrow, \downarrow, ?\}, \ldots)$
  - Exact usage-counting: $\mathscr{R} = (\mathbb{N}, =, +, \times)$

Tensor-introduction

$$\frac{r_1 x_1 : A_1, \ldots, r_n x_n : A_n \vdash B \qquad s_1 x_1 : A_1, \ldots, s_n x_n : A_n \vdash C}{(r_1 + s_1) x_1 : A_1, \ldots, (r_n + s_n) x_n : A_n \vdash B \otimes C}$$

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$$\frac{\vdash B \quad * \quad \vdash C}{\vdash B \otimes C}$$

With-introduction

$$\frac{r_1 x_1 : A_1, \ldots, r_n x_n : A_n \vdash B \qquad r_1 x_1 : A_1, \ldots, r_n x_n : A_n \vdash C}{r_1 x_1 : A_1, \ldots, r_n x_n : A_n \vdash B \mathbin{\&} C}$$

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With-introduction

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$$\frac{\mathcal{R}\gamma \vdash B \qquad \mathcal{R}\gamma \vdash C}{\mathcal{R}\gamma \vdash B \mathbin{\&} C}$$

$$\frac{\vdash B \quad \dot{\times} \quad \vdash C}{\vdash B \mathbin{\&} C}$$

Bang-introduction

$$\frac{s_1 x_1 : A_1, \ldots, s_n x_n : A_n \vdash B}{r s_1 x_1 : A_1, \ldots, r s_n x_n : A_n \vdash\, !r B}$$

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$$\frac{r \cdot (\vdash A)}{\vdash !rA}$$

Plus-elimination

$$\frac{\mathcal{P}\gamma \vdash A \oplus B \qquad \mathcal{Q}\gamma, 1A \vdash C \qquad \mathcal{Q}\gamma, 1B \vdash C}{(\mathcal{P} + \mathcal{Q})\gamma \vdash C}$$

$$\frac{\vdash A \oplus B \;\; * \;\; \left(1A \vdash C \;\; \dot{\times} \;\; 1B \vdash C\right)}{\vdash C}$$

Syntax descriptions

$$
\begin{array}{llcl}
\text{Premises} & ps, qs & ::= & \Delta \vdash A \\
 & & \mid & \dot{1} \quad \mid \quad ps \mathbin{\dot{\times}} qs \\
 & & \mid & I^* \quad \mid \quad ps * qs \\
 & & \mid & \color{green}{r} \cdot ps \\
 & & & \\
\text{Rule} & r & ::= & \dfrac{ps}{\vdash A} \\
 & & & \\
\text{System} & s & := & \Sigma(\mathit{Label} : \mathrm{Set}).\ \mathit{Label} \to \mathrm{Rule}
\end{array}
$$

Resembles bunched logic [RBKV20].

Variables

Basis vector

$$\frac{}{\ldots, {\color{green}0} y_k : B_k, {\color{green}1} x : A, {\color{green}0} y_{k+1} : B_{k+1}, \ldots \vdash x : A}$$

$$\frac{|\gamma| \ni i \qquad \gamma_i = A}{(\langle i|)\gamma \vdash A}$$

$$\frac{\ni A}{\vdash A}$$

"System + variables = terms"

Instances

We have definitions (in Agda) of:

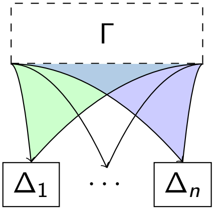
- Linear/quantitative STLC
- Core Granule
- Linear/non-linear type theory; $A ::= \mathit{lin}\, X \mid \mathit{int}\ Y$
  - Translations to and from L/nL and linear STLC
- Classical linear logic via $\mu\tilde{\mu}$;
  $A ::= \mathit{command} \mid \mathit{term}\, X \mid \mathit{coterm}\, X$

Moreover, we have *substitution* for any of these...

Substitution: intuitionistic

Let $\Delta = \Delta_1, \ldots, \Delta_n$.

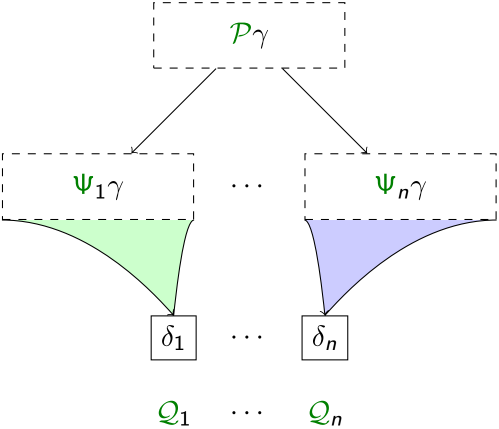
$$\begin{array}{c}\Gamma \\ \Downarrow \\ \Delta\end{array}$$



$$\forall A.\ \Delta \ni A \to \Gamma \vdash A$$

Substitution: linear

Let $\Gamma = \mathcal{P}\gamma$ and $\Delta = \mathcal{Q}_1\delta_1, \ldots, \mathcal{Q}_n\delta_n$.



$$\mathcal{P} = \sum_i \mathcal{Q}_i \Psi_i$$

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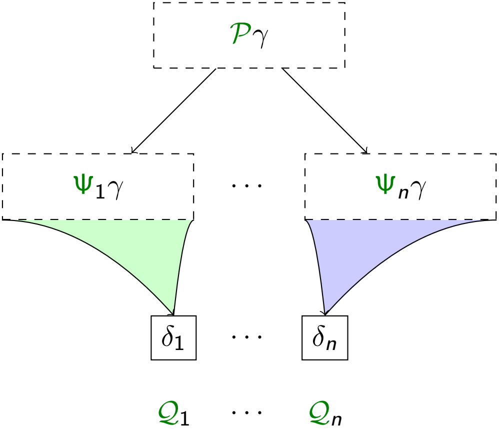

$$\mathcal{P} = \sum_i \mathcal{Q}_i \Psi_i$$

i.e.

$$\mathcal{P} = \mathcal{Q}\Psi$$

Substitution

- A substitution is based around a *linear* transformation.
- Using the linearity, substitutions commute with addition and scaling.
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Substitution generalises to *traversal* over terms. . .

- Replace $\vdash$ by $\ni$ to get *simultaneous renaming*.
- Replace $\vdash$ by semantic morphisms to get a denotational semantics.

Results

- Back in the room: AACMM-style semantic traversal follows.
  - Refine between $\dot{\times}$ and $*$, and $\dot{\to}$ and $-\!\!*$.
- Renaming and substitution for *all* syntaxes we can describe
  - E.g: DILL, S4 modal logic, core Granule, L/nL, classical $\mu\tilde{\mu}$
  - We add a $\Box$ connective for *duplicable* premises, e.g recursion.
- A generic usage elaborator for *all* these syntaxes
- Some tools for denotational semantics

Conclusions

`https://github.com/laMudri/generic-lr/tree/display`

- Lesson: we can adapt an intuitionistic framework when we understand…
  - the algebra of contexts, and
  - what it means to *derive* one context from another.
- Future work:
  - Explore expressibility of typing rules.
  - Try other substructural disciplines $\sim$ other interpretations of the bunched connectives.
  - Proofs about renaming/substitution (fusion laws)
  - A generalisation of multicategories