Vectors and matrices

James Wood

2014-10-11

Notations and conventions

The notation used here is under development. Details can be found at `http://fancyfahu.blogspot.co.uk`. Particularly, the arguments of subtraction and division are swapped.

1 Complex numbers

1.1 Basic properties

If z is a complex number, it can be expressed in the form $a + i \cdot b$, where $a, b \in \mathbb{R}$ and ${}^2\!\!\cdot i = 1^-$. Complex numbers are closed under addition, subtraction, multiplication, division and exponentiation.

1.2 Argand diagram

Complex numbers hold the same information as real 2-vectors ($\mathbb{R} \times \mathbb{R}$), and hence complex numbers can be represented as 2-vectors visually. Also, addition of complex numbers corresponds to addition of 2-vectors. From this representation, the triangle inequality is obvious.

$$\operatorname{abs}(z_0 + z_1) \leq \operatorname{abs} z_0 + \operatorname{abs} z_1$$

The fact that the triangle inequality holds is the characteristic of complex numbers that makes them form a metric space.

The following inequality also holds, and is identical:

$$\operatorname{abs}(z_0 - z_1) \geq \operatorname{abs}(\operatorname{abs} z_0 - \operatorname{abs} z_1)$$

This is proved as follows. $\operatorname{abs} z_0 = \operatorname{abs}(z_1 + z_1 - z_0)$. By the triangle inequality, $\operatorname{abs}(z_1 + (z_1 - z_0)) \leq \operatorname{abs} z_1 + \operatorname{abs}(z_1 - z_0)$. Hence, $\operatorname{abs} z_1 - \operatorname{abs} z_0 \leq \operatorname{abs}(z_1 - z_0)$. A similar argument gives $\operatorname{abs} z_1 \leq \operatorname{abs} z_0 + \operatorname{abs}(z_0 - z_1)$, and hence $\operatorname{abs} z_0 - \operatorname{abs} z_1 \leq \operatorname{abs}(z_0 - z_1)$. $\operatorname{abs}(z_1 - z_0) = \operatorname{abs}(z_0 - z_1)$, so $\operatorname{abs}(z_0 - z_1)^- \leq \operatorname{abs} z_0 - \operatorname{abs} z_1; \leq \operatorname{abs}(z_0 - z_1)$. In other words, $\operatorname{abs}(\operatorname{abs} z_0 - \operatorname{abs} z_1) \leq \operatorname{abs}(z_0 - z_1)$, QED.

1.3 Complex exponential

The Taylor series of the $\exp : \mathbb{R} \to \mathbb{R}$ function can be used to extend its domain (and, incidentally, codomain).

$$\begin{aligned} \exp &= 1 + \iota + \frac{2!}{{}^{2}_{\cdot}\iota} + \ldots; \\ &= \sum_{\infty}^{0} z : \frac{\iota!}{{}^{\iota}_{\cdot}z} \end{aligned}$$

This function converges for all $\mathbb{C}$.

1.3.1 Multiplication

Here, we prove that all$(\exp z_0 \cdot \exp z_1 = \exp(z_0 + z_1))$ $(\mathbb{C} \times \mathbb{C})$. Let $f\, z\, n = \frac{n!}{{}^{n}_{.}z}$

$$\begin{aligned}
\exp z_0 \cdot \exp z_1 &= \left(\sum_{\infty}^{0} z_0, z_1 : f\, z_0 \right) \cdot \left(\sum_{\infty}^{0} z_0, z_1 : f\, z_1 \right); \\
&= \sum_{\infty}^{0} z_0, z_1 : \left(\sum_{\infty}^{0} \kappa : f\, z_0\, \iota \cdot f\, z_1\, \kappa \right) \iota; \\
&= \sum_{\infty}^{0} z_0, z_1 : \left(\sum_{\leq |0,\iota| \geq} \kappa : f\, z_0\, (\iota - \kappa) \cdot f\, z_1\, \iota \right) \iota; \quad \text{taking sums along minor diagonals} \\
&= \sum_{\infty}^{0} z_0, z_1 : \left(\sum_{\leq |0,\iota| \geq} \kappa : \frac{(\iota - \kappa)!}{{}^{\iota-\kappa}_{.}z_0} \cdot \frac{\iota!}{{}^{\iota}_{.}z_1} \right) \iota; \\
&= \sum_{\infty}^{0} z_0, z_1 : \frac{\iota!}{1} \cdot \left(\sum_{\leq |0,\iota| \geq} \kappa : \frac{(\iota - \kappa)! \cdot \iota!}{\kappa!} \cdot {}^{\iota-\kappa}_{.}z_0 \cdot {}^{\iota}_{.}z_1 \right) \iota; \\
&= \sum_{\infty}^{0} z_0, z_1 : \frac{\iota!}{{}^{\iota}_{.}(z_0 + z_1)}; \quad \text{reversing the binomial expansion} \\
&= \exp(z_0 + z_1)
\end{aligned}$$

2 The complex exponential and Argand diagram

2.1 Trigonometric functions

The basic trigonometric functions can be defined using the complex exponential function and splitting the result into real and imaginary parts. On the Argand diagram (for real argument θ), this is represented by plotting the point $\exp(i \cdot \theta)$ and measuring the x (for cos) and y (for sin) parts of its coördinates.

$$
\begin{aligned}
\exp(i \cdot z) &= \sum_{\infty}^{0} \frac{\iota!}{{}^{\iota}_{\cdot}i \cdot {}^{\iota}_{\cdot}z} \\
&= \sum_{\infty}^{0} \frac{(2 \cdot \iota)!}{{}^{2\cdot\iota}_{\cdot}i \cdot {}^{2\cdot\iota}_{\cdot}z} + \sum_{\infty}^{0} \frac{(1 + 2 \cdot \iota)!}{{}^{1+2\cdot\iota}_{\cdot}i \cdot {}^{1+2\cdot\iota}_{\cdot}z} \\
&= \sum_{\infty}^{0} \frac{(2 \cdot \iota)!}{{}^{\iota}_{\cdot}(1^{-}) \cdot {}^{2\cdot\iota}_{\cdot}z} + i \cdot \sum_{\infty}^{0} \frac{(1 + 2 \cdot \iota)!}{{}^{\iota}_{\cdot}(1^{-}) \cdot {}^{1+2\cdot\iota}_{\cdot}z} \\
&= \cos z + i \cdot \sin z
\end{aligned}
$$

2.2 Roots of unity

The equation ${}^{n}_{\cdot}\iota = 1$ (with $n \in \mathbb{N}$), being a polynomial equation of order n, has n values that satisfy it. The obvious root is 1, but there are others.

$$
\begin{aligned}
&\text{Let } \exp(\tau \cdot i \cdot k) = 1; = {}^{n}_{\cdot}z \\
&\exp \frac{n}{\tau \cdot i \cdot k} = z \\
&z \in \exp \frac{n}{\tau \cdot i \cdot \mathbb{Z}}
\end{aligned}
$$

Though $\mathbb{Z}$ is infinite, $\exp(\frac{n}{\tau \cdot i} \cdot \mathbb{Z}) = \exp(\frac{n}{\tau \cdot i} \cdot (n + \mathbb{Z}))$, due to the latter being a complete rotation around the Argand diagram of the former.

The roots of unity for a given n form a regular polygon about 0 when adjacent roots are connected on the Argand diagram. This is related to the fact that the sum of all roots (for $n > 1$) is equal to 0:

$$
\begin{aligned}
&\text{Let } {}^{\leq|0,n|>}_{\cdot}\omega \text{ be the solutions to } 1 - {}^{n}_{\cdot}\iota = 0 \\
&1 - {}^{n}_{\cdot}\omega = 0 \quad \text{by definition of } \omega \\
&1 - {}^{n}_{\cdot}\omega = (1 - \omega) \cdot \sum_{n}^{0} ({}^{\iota}_{\cdot}\omega) \\
&0 = (1 - \omega) \cdot \sum_{n}^{0} ({}^{\iota}_{\cdot}\omega) \\
&\omega \neq 1 \quad \therefore \quad \sum_{n}^{0} ({}^{\iota}_{\cdot}\omega) = 0
\end{aligned}
$$

2.3 Complex logarithm

The ln function is defined as the inverse of the exp function:

$$\begin{aligned}
\iota &= r \cdot \exp(i \cdot \theta) \\
\ln &= \ln r + \ln(\exp(i \cdot \theta)) \quad r \in \mathbb{R}\text{, so we can calculate } \ln r \\
\ln &= \ln r + i \cdot \theta \quad \text{by definition of } \ln \\
\ln &= \ln \operatorname{abs} + i \cdot \arg \quad \text{explicitly}
\end{aligned}$$

Since, for complex input, there are infinite classes of values which can be passed to exp to give the same output, the range of ln has to be restricted. Given the above expression, the obvious way is to restrict the range of arg to a τ-sized half-open interval.

2.4 Complex power function

The complex power function is derived as follows:

$$\begin{aligned}
{}^{\alpha}_{\cdot}z &= \exp(\ln({}^{\alpha}_{\cdot}z)) \\
&= \exp(\alpha \cdot \ln z)
\end{aligned}$$

With the multivalued (list-producing) ln function, we note the following:

$$\alpha \in \mathbb{Q} \iff \operatorname{card} {}^{\alpha}_{\cdot}z < \infty$$

2.5 Lines and circles in $\mathbb{C}$

2.5.1 Lines

Lines can be expressed in a parametric way, as they would be with vectors.

$$\iota = z + \lambda \cdot w,\ \lambda \in \mathbb{R}$$

Compare to $\iota = \mathbf{a} + \lambda \cdot \mathbf{d}$. However, with complex numbers, the λ can be removed by the following process:

$$\begin{aligned}
\lambda &= \frac{w}{z - \iota} \\
\overline{\lambda} &= \overline{\left(\frac{w}{z - \iota}\right)} \\
\lambda &= \frac{\overline{w}}{\overline{z} - \overline{\iota}} \quad \text{N.B: } \lambda \in \mathbb{R} \\
\frac{w}{z - \iota} &= \frac{\overline{w}}{\overline{z} - \overline{\iota}} \\
\overline{w} \cdot (z - \iota) &= w \cdot (\overline{z} - \overline{\iota})
\end{aligned}$$

2.5.2 Circles

Circles are specified with a centre and a radius. By definition, every distance from a point on the circle to the centre is the radius.

$$\mathrm{abs}(c - \iota) = \rho$$

This can be expressed in an alternative form:

$$\begin{aligned}
{}^{2}_{.}\mathrm{abs}(c - \iota) &= {}^{2}_{.}\rho \quad \text{N.B: both abs}(c - \iota) \text{ and } \rho \text{ are positive.} \\
(c - \iota) \cdot (\overline{c} - \overline{\iota}) &= {}^{2}_{.}\rho \\
c \cdot \overline{\iota} - \iota \cdot \overline{c} - \iota \cdot \overline{\iota} &= c \cdot \overline{c} - {}^{2}_{.}\rho
\end{aligned}$$

3 Vectors

3.1 Definition and basic properties

A vector is defined by its magnitude ($\leq|0, \infty>$) and direction (in n dimensions). In any vector space, there exists the vector $\mathbf{0}$, such that $\mathrm{mgn}\, \mathbf{v} = 0 \iff \mathbf{v} = \mathbf{0}$. Also, the function $\hat{\iota}$ is such that $\mathrm{mgn}\, \hat{\mathbf{v}} = 1 \wedge \mathbf{v} \parallel \hat{\mathbf{v}}$.

A vector field is a function mapping the points of a space to vectors. These are found often in physics, where forces (like gravity and electromagnetism) have associated fields. These fields map points in 3D space to 3D force vectors.

3.1.1 Addition

Vectors are added together using the parallelogram rule. Addition of vectors always has these properties:

$$\begin{aligned}
\mathbf{a}+\mathbf{b} &= \mathbf{b}+\mathbf{a} && \text{(commutative)}\\
\mathbf{a}+(\mathbf{b}+\mathbf{c}) &= (\mathbf{a}+\mathbf{b})+\mathbf{c} && \text{(associative)}\\
\mathbf{a}+\mathbf{0} &= \mathbf{a} && \text{(there is a unique } \mathbf{0}\text{)}\\
\mathbf{a}+(\mathbf{a}^{-}) &= \mathbf{0} && \text{(every element has an inverse)}\\
\mathbf{a}+\mathbf{b} &= \mathbf{c} && \text{(closure)}
\end{aligned}$$

These properties mean that an Abelian group can be formed from any vector space.

3.1.2 Multiplication by a scalar

For vectors **a** and **b**, and scalars λ and μ:

$$\begin{aligned}
\operatorname{mgn}\lambda\cdot\mathbf{a} &= \operatorname{mgn}\lambda\cdot\operatorname{mgn}\mathbf{a}\\
\lambda\cdot\mathbf{a} &\parallel \mathbf{a}\\
(\lambda+\mu)\cdot\mathbf{a} &= \lambda\cdot\mathbf{a}+\mu\cdot\mathbf{a} && \text{(distributive over addition of scalars)}\\
\lambda\cdot(\mathbf{a}+\mathbf{b}) &= \lambda\cdot\mathbf{a}+\lambda\cdot\mathbf{b} && \text{(distributive over addition of vectors)}\\
\lambda\cdot(\mu\cdot\mathbf{a}) &= (\lambda\cdot\mu)\cdot\mathbf{a} && \text{(associative with scalar multiplication)}\\
1\cdot\mathbf{a} &= \mathbf{a} && \text{(unit)}\\
\lambda\cdot\mathbf{a} &= \mathbf{b} && \text{(closure)}
\end{aligned}$$

3.2 Vector spaces

A vector space is a set of vectors which satisfy all of the previous properties. The obvious class of examples is ${}^{\mathbb{N}}_{\times}\mathbb{R}$. Given ${}^{n}_{\times}S = \{\mathbf{x}|\mathbf{x} = (x_0, x_1, \ldots, x_{1-n}), \text{all } x_{\leq 0,n>} \in S\}$ and the following definitions, the vector properties hold for all n.

$$\begin{aligned}
\mathbf{x}+\mathbf{y} &= (x_0+y_0,\ldots,x_{1-n}+y_{1-n})\\
\mathbf{0} &= (0,\ldots,0)\\
\mathbf{x}^{-} &= (x_0^{-},\ldots,x_{1-n}^{-})\\
\lambda\cdot\mathbf{x} &= (\lambda\cdot\mathbf{x}_0,\ldots,\lambda\cdot\mathbf{x}_{1-n})
\end{aligned}$$

A consequence of the properties is that any vector space embedded inside another space must contain the space's origin. For example, any line passing through the origin of a 2D plane can form a vector space, but not any other lines on the 2D plane.

3.3 Scalar product

The scalar product is defined by the expression

$$\mathbf{a} \cdot \mathbf{b} = \operatorname{mgn} \mathbf{a} \cdot \operatorname{mgn} \mathbf{b} \cdot \cos \theta$$

where θ is the angle between the two vectors. This angle is more difficult to define in higher-dimensional space, and turns out to be defined by rearranging the above expression:

$$\theta = {}^{1-}_{\circ}\cos \frac{\operatorname{mgn} \mathbf{a} \cdot \operatorname{mgn} \mathbf{b}}{\mathbf{a} \cdot \mathbf{b}}$$

The scalar product has the following properties:

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \mathbf{b} \cdot \mathbf{a} \\ \mathbf{a} \cdot \mathbf{a} &= {}^{2}_{.} \operatorname{mgn} \mathbf{a} \\ \mathbf{a} \cdot \mathbf{b} = 0 &\iff \mathbf{a} \perp \mathbf{b} \lor \mathbf{a} = \mathbf{0} \lor \mathbf{b} = \mathbf{0} \end{aligned}$$

The scalar product can be used to produce a projection of one vector up to another. We can define:

$$\operatorname{prj} \mathbf{a}\, \mathbf{b} = (\hat{\mathbf{a}} \cdot \mathbf{b}) \cdot \hat{\mathbf{a}}$$

This produces a vector parallel to **a** with a length such that if a perpendicular line is taken from its end, it will reach the end of **b**.

3.4 Inner product

The scalar product is an example of an inner product. An inner product is a binary function (denoted $< x|y >$) that satisfies the following laws:

$$\begin{aligned} \langle \mathbf{x}|\mathbf{y}\rangle &= \langle \mathbf{y}|\mathbf{x}\rangle && \text{(symmetry)} \\ \langle \mathbf{x}|\lambda \cdot \mathbf{y} + \mu \cdot \mathbf{z}\rangle &= \lambda \cdot \langle \mathbf{x}|\mathbf{y}\rangle + \mu \cdot \langle \mathbf{x}|\mathbf{z}\rangle && \text{(linearity)} \\ \langle \mathbf{x}|\mathbf{x}\rangle &\geq \mathbf{0} \\ \langle \mathbf{x}|\mathbf{x}\rangle &= \mathbf{0} \iff \mathbf{x} = \mathbf{0} \\ {}^{\frac{2}{1}}_{.}\langle \mathbf{x}|\mathbf{x}\rangle &= \operatorname{mgn} \mathbf{x} && \text{(norm)} \end{aligned}$$

Inner products can be defined on things other than vectors. For example, let f and g be continuous functions of type $\leq|0,1|\geq \to \leq|0,1|\geq$. Then,

$$\langle f|g\rangle = \int_1^0 (f \cdot g)$$

3.5 Cauchy-Schwarz inequality

This inequality of vectors is as follows:

$$\mathbf{x} \cdot \mathbf{y} \leq \operatorname{mgn} \mathbf{x} \cdot \operatorname{mgn} \mathbf{y}$$

This is obvious in the definition involving $\cos\theta$, but can be proved without that. The proof is as follows, where λ is an arbitrary scalar:

$$\begin{aligned}
{}^{2}_{.}\operatorname{mgn}(\lambda \cdot \mathbf{x} - \mathbf{y}) &\geq 0 \\
(\lambda \cdot \mathbf{x} - \mathbf{y}) \cdot (\lambda \cdot \mathbf{x} - \mathbf{y}) &\geq 0 \\
{}^{2}_{.}\lambda \cdot \mathbf{x} \cdot \mathbf{x} + 2 \cdot \lambda \cdot \mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{y} &\geq 0
\end{aligned}$$

The last left-hand expression is a quadratic expression in λ. Because, for all values of λ, it is greater than 0, it has either 1 repeated root or no roots. This means:

$$\begin{aligned}
\Delta\,({}^{2}_{.}\iota \cdot \mathbf{x} \cdot \mathbf{x} + 2 \cdot \iota \cdot \mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{y}) &\leq 0 \\
4 \cdot (\mathbf{x} \cdot \mathbf{x}) \cdot (\mathbf{y} \cdot \mathbf{y}) - {}^{2}_{.}(2 \cdot \mathbf{x} \cdot \mathbf{y}) &\leq 0 \\
{}^{2}_{.}(2 \cdot \mathbf{x} \cdot \mathbf{y}) &\leq 4 \cdot (\mathbf{x} \cdot \mathbf{x}) \cdot (\mathbf{y} \cdot \mathbf{y}) \\
4 \cdot {}^{2}_{.}(\mathbf{x} \cdot \mathbf{y}) &\leq 4 \cdot {}^{2}_{.}\operatorname{mgn} \mathbf{x} \cdot {}^{2}_{.}\operatorname{mgn} \mathbf{y} \\
{}^{2}_{.}(\mathbf{x} \cdot \mathbf{y}) &\leq {}^{2}_{.}(\operatorname{mgn} \mathbf{x} \cdot \operatorname{mgn} \mathbf{y})
\end{aligned}$$

$\operatorname{mgn} \mathbf{x} \cdot \operatorname{mgn} \mathbf{y} \geq 0$, so the inequality follows.

3.5.1 Example

Let $\mathbf{x} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Then, the Cauchy-Schwarz inequality states that:

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &\leq \operatorname{mgn} \mathbf{x} \cdot \operatorname{mgn} \mathbf{y} \\ \alpha + \beta + \gamma &\leq \sqrt{{}^{2}_{.}\alpha + {}^{2}_{.}\beta + {}^{2}_{.}\gamma} \cdot \sqrt{3} \\ {}^{2}_{.}(\alpha + \beta + \gamma) &\leq 3 \cdot \left({}^{2}_{.}\alpha + {}^{2}_{.}\beta + {}^{2}_{.}\gamma\right) \\ \left({}^{2}_{.}\alpha + {}^{2}_{.}\beta + {}^{2}_{.}\gamma\right) + 2 \cdot (\alpha \cdot \beta + \beta \cdot \gamma + \gamma \cdot \alpha) &\leq 3 \cdot \left({}^{2}_{.}\alpha + {}^{2}_{.}\beta + {}^{2}_{.}\gamma\right) \\ 2 \cdot (\alpha \cdot \beta + \beta \cdot \gamma + \gamma \cdot \alpha) &\leq 2 \cdot \left({}^{2}_{.}\alpha + {}^{2}_{.}\beta + {}^{2}_{.}\gamma\right) \\ \alpha \cdot \beta + \beta \cdot \gamma + \gamma \cdot \alpha &\leq {}^{2}_{.}\alpha + {}^{2}_{.}\beta + {}^{2}_{.}\gamma \end{aligned}$$

4 Triangle inequality and vector product

4.1 Triangle inequality

The triangle inequality for vectors is proved as follows:

$$\begin{aligned} {}^{2}_{.}\operatorname{mgn}(\mathbf{x} + \mathbf{y}) &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}); \\ &= \mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} + 2 \cdot \mathbf{x} \cdot \mathbf{y}; \\ &= {}^{2}_{.}\operatorname{mgn} \mathbf{x} + {}^{2}_{.}\operatorname{mgn} \mathbf{y} + 2 \cdot \mathbf{x} \cdot \mathbf{y}; \\ &\leq {}^{2}_{.}\operatorname{mgn} \mathbf{x} + {}^{2}_{.}\operatorname{mgn} \mathbf{y} + 2 \cdot \operatorname{abs}(\mathbf{x} \cdot \mathbf{y}); \\ &= {}^{2}_{.}\operatorname{mgn} \mathbf{x} + 2 \cdot \operatorname{abs} \mathbf{x} \cdot \operatorname{abs} \mathbf{y} + {}^{2}_{.}\operatorname{mgn} \mathbf{y}; \\ &= {}^{2}_{.}(\operatorname{mgn} \mathbf{x} + \operatorname{mgn} \mathbf{y}); \\ \operatorname{mgn}(\mathbf{x} + \mathbf{y}) &\leq \operatorname{mgn} \mathbf{x} + \operatorname{mgn} \mathbf{y} \end{aligned}$$

4.2 Vector product

The vector product is defined only for 3- and 7-vectors (and 1-vectors – scalars). It is defined as follows:

$$\mathbf{a} \times \mathbf{b} = \operatorname{mgn} \mathbf{a} \cdot \operatorname{mgn} \mathbf{b} \cdot \sin \theta \cdot \hat{\mathbf{n}}$$

where θ is the angle between the two vectors and $\hat{\mathbf{n}}$ is the unit vector perpendicular to both input vectors in the right-hand direction. The vector product satisfies these properties:

$$\begin{gathered}
\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}^{-} \\
\mathbf{a} \times \mathbf{a} = 0 \\
\mathbf{a} \times \mathbf{b} = 0 \implies \mathbf{a} = \mu \cdot \mathbf{b} \\
\mathbf{a} \times (\lambda \cdot \mathbf{b}) = \lambda \cdot (\mathbf{a} \times \mathbf{b}) \\
\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}
\end{gathered}$$

4.3 Area of a triangle

The area of the triangle subtended by two vectors can be calculated using their cross product. Imagine triangle OAB, with point N on OA and $BN \perp OA$. Taking OA as the base and BN as the height, we get:

$$\begin{aligned}
\text{area} &= \frac{2}{1} \cdot \operatorname{mgn} \mathbf{a} \cdot \operatorname{mgn} \overrightarrow{BN} \\
&= \frac{2}{1} \cdot \operatorname{mgn} \mathbf{a} \cdot \operatorname{mgn} \mathbf{b} \cdot \sin\theta \\
&= \frac{2}{1} \cdot \operatorname{mgn} (\mathbf{a} \times \mathbf{b})
\end{aligned}$$

The vector area of a triangle formed by two vectors is the same expression, but without mgn. The area of the parallelogram formed by the input vectors is simply $\operatorname{mgn} (\mathbf{a} \times \mathbf{b})$.

4.4 Scalar triple product

The scalar triple product, often written as $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ (but I shall use $\langle \mathbf{a} | \mathbf{b} | \mathbf{c} \rangle$), is defined by:

$$\langle \mathbf{a} | \mathbf{b} | \mathbf{c} \rangle = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

This gives the volume of the parallelepiped formed by the three input vectors. The proof is best stated graphically, so I shall defer to my handwritten notes.

Since volume is not affected by orientation, the arguments to the triple product can be reordered. However, if the arguments are reversed, the product will be negated.

The triple product can be used to prove whether 3 vectors are coplanar. It is self-evident that:

$$\text{coplanar}\,\mathbf{a}\,\mathbf{b}\,\mathbf{c} \implies \langle \mathbf{a}|\mathbf{b}|\mathbf{c}\rangle = 0$$

Also, if any two of the three vectors are mutually parallel, all three vectors will be coplanar. This is because they can be reärranged to form the cross product, which will yield $\mathbf{0}$.

4.5 Distributive law

The fact that the vector product distributes over addition can be proved by the fact that the scalar product distributes over addition:

$$\begin{aligned}
\text{Let } \mathbf{d} &= \mathbf{a}\times(\mathbf{b}+\mathbf{c}) - \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c} \\
\mathbf{d}\cdot\mathbf{d} &= \mathbf{d}\cdot\mathbf{a}\times(\mathbf{b}+\mathbf{c}) - \mathbf{d}\cdot(\mathbf{a}\times\mathbf{b}) + \mathbf{d}\cdot(\mathbf{a}\times\mathbf{c}) \\
&= \langle \mathbf{d}|\mathbf{a}|\mathbf{b}+\mathbf{c}\rangle - \langle \mathbf{d}|\mathbf{a}|\mathbf{b}\rangle + \langle \mathbf{d}|\mathbf{a}|\mathbf{c}\rangle \\
&= \langle \mathbf{b}+\mathbf{c}|\mathbf{d}|\mathbf{a}\rangle - \langle \mathbf{b}|\mathbf{d}|\mathbf{a}\rangle + \langle \mathbf{c}|\mathbf{d}|\mathbf{a}\rangle \\
&= (\mathbf{b}+\mathbf{c})\cdot(\mathbf{d}\times\mathbf{a}) - \mathbf{b}\cdot(\mathbf{d}\times\mathbf{a}) + \mathbf{c}\cdot(\mathbf{d}\times\mathbf{a}) \\
&= \mathbf{0}
\end{aligned}$$

4.6 Spanning sets and bases

A set of vectors is said to span a space if every point in the space can be translated to any other point in the space using a linear combination of vectors from the set. A basis is a spanning set such that if any element were to be removed from that set, it would no longer be a spanning set.

An example of a basis is $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}\}$ in 2D Euclidean space. If the vector $\hat{\mathbf{i}}+\hat{\mathbf{j}}$ is added to this set, it will still be a spanning set, but not a basis.