Vectors and matrices

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2014-10-11

Notations and conventions

The notation used here is under development. Details can be found at `http://fancyfahu.blogspot.co.uk`. Particularly, the arguments of subtraction and division are swapped.

1 Complex numbers

1.1 Basic properties

If z is a complex number, it can be expressed in the form $a + i \cdot b$, where $a, b \in \mathbb{R}$ and ${}^{2}_{.}i = 1^{-}$. Complex numbers are closed under addition, subtraction, multiplication, division and exponentiation.

1.2 Argand diagram

Complex numbers hold the same information as real 2-vectors ($\mathbb{R} \times \mathbb{R}$), and hence complex numbers can be represented as 2-vectors visually. Also, addition of complex numbers corresponds to addition of 2-vectors. From this representation, the triangle inequality is obvious.

$$\operatorname{abs}(z_0 + z_1) \leq \operatorname{abs} z_0 + \operatorname{abs} z_1$$

The fact that the triangle inequality holds is the characteristic of complex numbers that makes them form a metric space.

The following inequality also holds, and is identical:

$$\operatorname{abs}(z_0 - z_1) \geq \operatorname{abs}(\operatorname{abs} z_0 - \operatorname{abs} z_1)$$

This is proved as follows. $\operatorname{abs} z_0 = \operatorname{abs}(z_1 + z_1 - z_0)$. By the triangle inequality, $\operatorname{abs}(z_1 + (z_1 - z_0)) \leq \operatorname{abs} z_1 + \operatorname{abs}(z_1 - z_0)$. Hence, $\operatorname{abs} z_1 - \operatorname{abs} z_0 \leq \operatorname{abs}(z_1 - z_0)$. A similar argument gives $\operatorname{abs} z_1 \leq \operatorname{abs} z_0 + \operatorname{abs}(z_0 - z_1)$, and hence $\operatorname{abs} z_0 - \operatorname{abs} z_1 \leq \operatorname{abs}(z_0 - z_1)$. $\operatorname{abs}(z_1 - z_0) = \operatorname{abs}(z_0 - z_1)$, so $\operatorname{abs}(z_0 - z_1)^- \leq \operatorname{abs} z_0 - \operatorname{abs} z_1 ; \leq \operatorname{abs}(z_0 - z_1)$. In other words, $\operatorname{abs}(\operatorname{abs} z_0 - \operatorname{abs} z_1) \leq \operatorname{abs}(z_0 - z_1)$, QED.

1.3 Complex exponential

The Taylor series of the $\exp : \mathbb{R} \to \mathbb{R}$ function can be used to extend its domain (and, incidentally, codomain).

$$\begin{aligned} \exp &= 1 + \iota + \frac{2!}{{}^{2}_{.}\iota} + \ldots ; \\ &= \sum_{\infty}^{0} z : \frac{\iota!}{{}^{\iota}_{.}z} \end{aligned}$$

This funciton converges for all $\mathbb{C}$.

1.3.1 Multiplication

Here, we prove that all$(\exp z_0 \cdot \exp z_1 = \exp(z_0 + z_1))\,(\mathbb{C} \times \mathbb{C})$. Let $f\,z\,n = \frac{n!}{{}^{n}_{\cdot}z}$

$$
\begin{aligned}
\exp z_0 \cdot \exp z_1 &= \left(\sum_{\infty}^{0} z_0, z_1 : f\,z_0\right) \cdot \left(\sum_{\infty}^{0} z_0, z_1 : f\,z_1\right); \\
&= \sum_{\infty}^{0} z_0, z_1 : \left(\sum_{\infty}^{0} \kappa : f\,z_0\,\iota \cdot f\,z_1\,\kappa\right) \iota; \\
&= \sum_{\infty}^{0} z_0, z_1 : \left(\sum_{\leq|0,\iota|\geq} \kappa : f\,z_0\,(\iota - \kappa) \cdot f\,z_1\,\iota\right) \iota; \quad \text{taking sums along minor diagonals} \\
&= \sum_{\infty}^{0} z_0, z_1 : \left(\sum_{\leq|0,\iota|\geq} \kappa : \frac{(\iota - \kappa)!}{{}^{\iota-\kappa}_{\cdot}z_0} \cdot \frac{\iota!}{{}^{\iota}_{\cdot}z_1}\right) \iota; \\
&= \sum_{\infty}^{0} z_0, z_1 : \frac{\iota!}{1} \cdot \left(\sum_{\leq|0,\iota|\geq} \kappa : \frac{(\iota - \kappa)! \cdot \iota!}{\kappa!} \cdot {}^{\iota-\kappa}_{\cdot}z_0 \cdot {}^{\iota}_{\cdot}z_1\right) \iota; \\
&= \sum_{\infty}^{0} z_0, z_1 : \frac{\iota!}{{}^{\iota}_{\cdot}(z_0 + z_1)}; \quad \text{reversing the binomial expansion} \\
&= \exp(z_0 + z_1)
\end{aligned}
$$

2 The complex exponential and Argand diagram

2.1 Trigonometric functions

The basic trigonometric functions can be defined using the complex exponential function and splitting the result into real and imaginary parts. On the Argand diagram (for real argument θ), this is represented by plotting the point $\exp(i \cdot \theta)$ and measuring the x (for cos) and y (for sin) parts of its coördinates.

$$
\begin{aligned}
\exp(i \cdot z) &= \sum_{\infty}^{0} \frac{\iota!}{{}^{\iota}_{.}i \cdot {}^{\iota}_{.}z} \\
&= \sum_{\infty}^{0} \frac{(2 \cdot \iota)!}{{}^{2\cdot\iota}_{.}i \cdot {}^{2\cdot\iota}_{.}z} + \sum_{\infty}^{0} \frac{(1 + 2 \cdot \iota)!}{{}^{1+2\cdot\iota}_{.}i \cdot {}^{1+2\cdot\iota}_{.}z} \\
&= \sum_{\infty}^{0} \frac{(2 \cdot \iota)!}{{}^{\iota}_{.}(1^{-}) \cdot {}^{2\cdot\iota}_{.}z} + i \cdot \sum_{\infty}^{0} \frac{(1 + 2 \cdot \iota)!}{{}^{\iota}_{.}(1^{-}) \cdot {}^{1+2\cdot\iota}_{.}z} \\
&= \cos z + i \cdot \sin z
\end{aligned}
$$

2.2 Roots of unity

The equation ${}^{n}_{.}\iota = 1$ (with $n \in \mathbb{N}$), being a polynomial equation of order n, has n values that satisfy it. The obvious root is 1, but there are others.

$$
\begin{aligned}
&\text{Let } \exp(\tau \cdot i \cdot k) = 1; = {}^{n}_{.}z \\
&\exp \frac{n}{\tau \cdot i \cdot k} = z \\
&z \in \exp \frac{n}{\tau \cdot i \cdot \mathbb{Z}}
\end{aligned}
$$

Though $\mathbb{Z}$ is infinite, $\exp(\frac{n}{\tau \cdot i} \cdot \mathbb{Z}) = \exp(\frac{n}{\tau \cdot i} \cdot (n + \mathbb{Z}))$, due to the latter being a complete rotation around the Argand diagram of the former.

The roots of unity for a given n form a regular polygon about 0 when adjacent roots are connected on the Argand diagram. This is related to the fact that the sum of all roots (for $n > 1$) is equal to 0:

$$
\begin{aligned}
&\text{Let } {}^{\leq|0,n|>}_{.}\omega \text{ be the solutions to } 1 - {}^{n}_{.}\iota = 0 \\
&1 - {}^{n}_{.}\omega = 0 \quad \text{by definition of } \omega \\
&1 - {}^{n}_{.}\omega = (1 - \omega) \cdot \sum_{n}^{0} ({}^{\iota}_{.}\omega) \\
&0 = (1 - \omega) \cdot \sum_{n}^{0} ({}^{\iota}_{.}\omega) \\
&\omega \neq 1 \quad \therefore \quad \sum_{n}^{0} ({}^{\iota}_{.}\omega) = 0
\end{aligned}
$$

2.3 Complex logarithm

The ln function is defined as the inverse of the exp function:

$$\begin{aligned}
\iota &= r \cdot \exp(i \cdot \theta) \\
\ln &= \ln r + \ln(\exp(i \cdot \theta)) \quad r \in \mathbb{R}\text{, so we can calculate } \ln r \\
\ln &= \ln r + i \cdot \theta \quad \text{by definition of } \ln \\
\ln &= \ln \operatorname{abs} + i \cdot \arg \quad \text{explicitly}
\end{aligned}$$

Since, for complex input, there are infinite classes of values which can be passed to exp to give the same output, the range of ln has to be restricted. Given the above expression, the obvious way is to restrict the range of arg to a τ-sized half-open interval.

2.4 Complex power function

The complex power function is derived as follows:

$$\begin{aligned}
{}^{\alpha}_{\cdot}z &= \exp(\ln({}^{\alpha}_{\cdot}z)) \\
&= \exp(\alpha \cdot \ln z)
\end{aligned}$$

With the multivalued (list-producing) ln function, we note the following:

$$\alpha \in \mathbb{Q} \iff \operatorname{card} {}^{\alpha}_{\cdot}z < \infty$$

2.5 Lines and circles in $\mathbb{C}$

2.5.1 Lines

Lines can be expressed in a parametric way, as they would be with vectors.

$$\iota = z + \lambda \cdot w,\ \lambda \in \mathbb{R}$$

Compare to $\iota = \mathbf{a} + \lambda \cdot \mathbf{d}$. However, with complex numbers, the λ can be removed by the following process:

$$
\begin{aligned}
\lambda &= \frac{w}{z - \iota} \\
\overline{\lambda} &= \overline{\left(\frac{w}{z - \iota}\right)} \\
\lambda &= \frac{\overline{w}}{\overline{z} - \overline{\iota}} \quad \text{N.B: } \lambda \in \mathbb{R} \\
\frac{w}{z - \iota} &= \frac{\overline{w}}{\overline{z} - \overline{\iota}} \\
\overline{w} \cdot (z - \iota) &= w \cdot (\overline{z} - \overline{\iota})
\end{aligned}
$$

2.5.2 Circles

Circles are specified with a centre and a radius. By definition, every distance from a point on the circle to the centre is the radius.

$$\mathrm{abs}(c - \iota) = \rho$$

This can be expressed in an alternative form:

$$
\begin{aligned}
{}^{2}_{\cdot}\mathrm{abs}(c - \iota) &= {}^{2}_{\cdot}\rho \quad \text{N.B: both } \mathrm{abs}(c - \iota) \text{ and } \rho \text{ are positive.} \\
(c - \iota) \cdot (\overline{c} - \overline{\iota}) &= {}^{2}_{\cdot}\rho \\
c \cdot \overline{\iota} - \iota \cdot \overline{c} - \iota \cdot \overline{\iota} &= c \cdot \overline{c} - {}^{2}_{\cdot}\rho
\end{aligned}
$$