which trivially rearranges to give the recurrence relation (18.115).

To obtain the recurrence relation (18.116), we begin by differentiating the generating function (18.114) with respect to x, which yields

$$\frac{\delta G}{\delta x} = -\frac{he^{-xh/(1-h)}}{(1-h)^2} = \sum L'_n h^n,$$

and thus we have

$$-h\sum L_n h^n = (1-h)\sum L'_n h^n,$$

Equating coefficients of h^n on each side then gives

$$-L_{n-1} = L'_n - L'_{n-1}$$

which immediately simplifies to give (18.116).◄

18.8 Associated Laguerre functions

The associated Laguerre equation has the form

$$xy'' + (m+1-x)y' + ny = 0; \tag{18.118}$$

it has a regular singularity at x = 0 and an essential singularity at $x = \infty$. We restrict our attention to the situation in which the parameters n and m are both non-negative integers, as is the case in nearly all physical problems. The associated Laguerre equation occurs most frequently in quantum-mechanical applications.Any solution of (18.118) is called an *associated Laguerre function.*

Solution of (18.118) for non-negative integers $n$and m are given by the *associated Laguerre polynomials*

$$L_n^m(x) = (-1)^m \frac{d^m}{dx^m} L_{n+m}(x),$$

where $L_n(x)$ are the ordinary Laguerre polynomials.[§]

►Show that the functions L(x) defined in (18.119) are solutions of (18.118).

Since the Laguerre polynomials $L_n(x)$ are solutions of Laguerre's equation (18.107), we have

$$xL''_{n+m} + (1-x)L'_{n+m} + (n+m)L_{n+m} = 0$$

Differentiating this equation m times using Leibnitz' theorem and rearranging, we find

$$xL_{n+m}^{(m+2)} + (m+1-x)L_{n+m}^{(m+1)} + nL_{n+m}^{(m)} = 0.$$

On multiplying through by $(-1)^m$ and setting $L_n^m = (-1)^m L_{n+m}^{(m)}$, in accord with (18.119), we obtain

$$x(L_n^m)'' + (m+1-x)(L_n^m)' + nL_n^m = 0,$$

which shows that the functions L_n^m are indeed solutions of (18.118).◄

[§] Note that some authors define the associated Laguerre polynomials as $L_n^m(x) - (d^m/dx^m)L_n(x)$,, which is thus related to our expression (18.110) by $L_n^m(x) - (-1)^m L_{n+m}^m(x)$.