PSET - 06 - (HW 02) (20171049)

Given that: For BG cell, distribution is $N(\mu_1, \sigma_1^2)$
and for FG cell, distribution is $N(\mu_2, \sigma_2^2)$.
Also, given that, if $x_{ij} < 0$, then BG, else FG.

(a) If $P(BG) = P(FG)$ & $\sigma_1 = \sigma_2$,

So, $\cancel{P(BG)}.\ P(x|BG) = \cancel{P(FG)}.\ P(x|FG)$

$$\Rightarrow \cancel{\frac{1}{(\sqrt{2\pi})\sigma_1}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1}} = \cancel{\frac{1}{(\sqrt{2\pi})\sigma_2}} e^{-\frac{(x-\mu_2)^2}{2\sigma_2}}$$

$$\Rightarrow \frac{(x-\mu_1)^2}{2\sigma_1^2} = \frac{(x-\mu_2)^2}{2\sigma_2^2} \Rightarrow (x-\mu_1)^2 = (x-\mu_2)^2$$

$$\Rightarrow (x-\mu_1) = \pm(x-\mu_2) \Rightarrow \underline{\mu_1 = \mu_2} \text{ or } \boxed{x = \frac{\mu_1+\mu_2}{2}}$$

Hence, the optimal value of θ is $\boxed{\theta^* = \frac{\mu_1+\mu_2}{2}}$.

(b) For $\theta^* = \frac{\mu_1+\mu_2}{2}$, Putting $x = \left(\frac{\mu_1+\mu_2}{2}\right)$

$$\therefore P(BG).P(x|BG) = \frac{P(BG)}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{\left(\frac{\mu_1+\mu_2}{2}-\mu_1\right)^2}{2\sigma_1^2}} \quad \text{--- (1)}$$

$$\text{Also, } P(FG).P(x|FG) = \frac{P(FG)}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{\left(\frac{\mu_1+\mu_2}{2}-\mu_2\right)^2}{2\sigma_2^2}} \quad \text{--- (2)}$$

Also, (1) = (2)

$$\frac{P(BG)}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{(\mu_2-\mu_1)^2}{8\sigma_1^2}} = \frac{P(FG)}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{(\mu_1-\mu_2)^2}{2\sigma_2^2}}$$

$$\Rightarrow \frac{P(BG)}{P(FG)} = \frac{\sigma_1}{\sigma_2} \cdot e^{-\frac{(\mu_1-\mu_2)^2}{8}\left[\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right]}$$

$$\Rightarrow \boxed{\frac{P(BG)}{P(FG)} = \frac{\sigma_1}{\sigma_2} \cdot e^{\frac{(\mu_1-\mu_2)^2}{8}\left(\frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}\right)}}$$

(c) Given $\mu_1 = 100$, $\mu_2 = 200$ & $\sigma_1 = \sigma_2$. {Let $\sigma_1 = \sigma_2 = \sigma$}.
Now, $P(BG) = 4 \times P(FG)$.

$$\Rightarrow P(BG).P(x|BG) = P(FG).P(x|FG).$$

$$\Rightarrow 4\,P(x|BG) = P(x|FG).$$

$$\Rightarrow \frac{4}{(\sqrt{2\pi})^2\sigma} e^{-\frac{(\theta^*-100)^2}{2\sigma^2}} = \frac{1}{(\sqrt{2\pi})^2\sigma} e^{-\frac{(\theta^*-200)^2}{2\sigma^2}}$$

$$\Rightarrow 4e^{-\frac{(\theta^*-100)^2}{2\sigma^2}} = e^{-\frac{(\theta^*-200)^2}{2\sigma^2}}$$

$$\Rightarrow \ln 4 - \frac{(\theta^*-100)^2}{2\sigma^2} = -\frac{(\theta^*-200)^2}{2\sigma^2}$$

$$\Rightarrow 2\sigma^2 \ln 4 = (\theta^*-100)^2 - (\theta^*-200)^2$$

$$\Rightarrow (2\theta^*-300)(100) = 2\sigma^2 \ln 4$$

$$\Rightarrow (\theta^*-150) = \frac{\sigma^2 \ln 4}{100} \Rightarrow \boxed{\theta^* = \frac{\sigma^2 \ln 4}{100} + 150}$$